CONVEX BODIES WHICH TILE SPACE BY TRANSLATION

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Abstract. It is shown that a convex body K tiles E^d by translation if, and only if, K is a centrally symmetric d-polytope with centrally symmetric facets, such that every belt of K (consisting of those of its facets which contain a translate of a given $(d-2)$-face) has four or six facets. One consequence of the proof of this result is that, if K tiles E^d by translation, then K admits a face-to-face, and hence a lattice tiling.

§1. *Introduction.* We say that the convex body (compact convex set with non-empty interior) K *tiles* d-dimensional euclidean space E^d *by translation* if there is some family T of translation vectors, such that the family $\mathscr{K} = \{K+t \mid t \in T\}$ of translates of K covers E^d, while two distinct translates $K+t_1$ and $K+t_2$ $(t_1 \neq t_2)$ have disjoint interiors, so that $\mathscr{K}$ is also a packing of E^d. We shall call K, and its translates $K+t$, (*convex translation*) *tiles*, or, occasionally, if we wish to emphasize their dimension, *d-tiles*. It is known (we shall give references later) that tiles must satisfy various conditions; it is the object of this paper to prove that these conditions completely characterize tiles. The actual characterization is given by

THEOREM 1. *The convex body K is a tile if, and only if, K is a centrally symmetric polytope with centrally symmetric facets, such that each belt of K contains four or six facets.*

If $d \geqslant 3$, then Aleksandrov [1933] (for $d = 3$) and Shephard [1967] have shown that, if each facet of a d-polytope K is centrally symmetric, then K itself is centrally symmetric (see McMullen [1976] for a shorter proof). A *belt* of such a polytope K is the collection of its facets which contain a translate of a given subfacet (that is, $(d-2)$-face) G of K (and also, of course, of $-G$). (In E^3, belts are more familiarly known as *zones*, but in higher dimensions, the term zone is restricted to *zonotopes*—that is, vector sums of line segments—and has a rather different meaning.)

The history of this problem is very rich, and we shall do no more, here and below, than mention a few salient contributions. It was the crystallographer Fedorov who first investigated tiles (naturally in the context of *lattice tilings*—by which we mean that T is a lattice, or discrete additive subgroup—in E^3) from about 1881, and characterized the five parallelohedra (as tiles are alternatively called) shortly thereafter (see Fedorov [1885]). The necessity of the conditions involving central symmetry was proved by Minkowski [1897]; we shall discuss this in more detail later. The origin of the final condition is more obscure. While it must clearly have been recognized early, at least in the context of lattice tilings (see, for example, Delaunay [1926] and Aleksandrov [1958]), it was Coxeter [1962] who first proposed that the condition might be sufficient for zonotopes (at least) to tile space.

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§2. *The necessity of the conditions.* While the necessity of the conditions of Theorem 1 is known, albeit in the context of lattice tilings (though the proofs are no different), for completeness of exposition we give a proof here.

Throughout this section, we take K to be a given tile; as above, T denotes the corresponding family of translation vectors. There is clearly no loss in generality in supposing that the zero vector $o \in T$.

LEMMA 1. *K is a polytope.*

We first observe that, if $t_1, t_2 \in T$ with $t_1 \neq t_2$, then $K+t_1$ meets $K+t_2$ if, and only if, $t_1 - t_2 \in \text{bd}(DK)$, where $DK = K - K$ is the difference body of K. Thus $\|t_1 - t_2\|$ is bounded both above and below by positive numbers, and hence only finitely many translates $K + t_i$ $(i = 1, \ldots, k$, say) touch K. On the other hand, each point of bd K must belong to a tile $K+t$ for some $t \in T \setminus \{o\}$. It follows that bd K is a finite union of sets $K \cap (K + t_i)$ $(i = 1, \ldots, k)$, and we deduce that K is a bounded polyhedral set; that is, K is a polytope.

LEMMA 2. *K is centrally symmetric.*

The tile K meets only finitely many sets $-(K+t_i)$ with $t_i \in T$, say for $i = 1, \ldots, k$. Then K is the union of the centrally symmetric polytopes $K \cap (-K - t_i)$ $(i = 1, \ldots, k)$, whose interiors are disjoint. From a result of Minkowski [1897], it then follows that K is itself centrally symmetric.

From now on we suppose, again without loss of generality, that the centre of K is the origin o.

LEMMA 3. *Each facet F of K is centrally symmetric.*

The facet F can only be covered by the translates of $-F$ in those tiles $K + t_i$ $(t_i \in T)$ which meet K in F. Thus we can express F as a union of centrally symmetric $(d-1)$-polytopes $F \cap (-F + t_i)$ (with $i = 1, \ldots, k$, say), whose interiors, relative to the affine hull aff F of F, are disjoint. Hence, as above, we conclude that F is itself centrally symmetric.

Lemmas 1, 2 and 3 are all due to Minkowski [1897]; his arguments are directed at lattice tilings in E^3, but they generalize at once to arbitrary tilings by translation in E^d. We observe that we cannot expect to do better than Lemma 3 in general. For, as shown by McMullen [1970], if $d \geq 4$, then a d-polytope all of whose subfacets are centrally symmetric has all its faces of every dimension centrally symmetric, and so is a zonotope. However, the regular 24-cell $\{3, 4, 3\}$ tiles E^4, but is not a zonotope (see Coxeter [1973]).

LEMMA 4. *Each belt of K contains 4 or 6 facets.*

Let the belt of K determined by the subfacet G have m pairs of opposite facets; we shall suppose $m \geq 4$, and obtain a contradiction. We can choose $g \in G$ to lie on no j-face of any tile $K + t$ $(t \in T)$ with $j < d-2$, and we consider the tiles which contain g. The sum of the dihedral angles of K at its subfacets parallel to G (and so in the belt) is $2(m-1)\pi$. Thus, firstly, g cannot lie in the relative interior of a facet of any tile, for, since the dihedral angle at any subfacet is less than π, the sum of the dihedral

angles at two non-opposite subfacets of the belt is greater than $(m-1)\pi-(m-2)\pi = \pi$. Hence g lies in subfacets alone. But, similarly, the sum of the dihedral angles at three mutually non-opposite subfacets is greater than $(m-1)\pi-(m-3)\pi = 2\pi$, and so g cannot belong to three tiles. Thus $m \leqslant 3$, as was claimed.

A belt of K consisting of n facets will be called an *n-belt*. The argument used above in fact shows that, if the point g does lie in the relative interior of some facet of another tile, then $m = 2$, so that the belt is a 4-belt. Note that we have also used above the obvious fact that a facet F of K can only be covered by translates of its opposite facet $-F$. Hence, if a tile meeting K in (a part of) F actually meets g at a subfacet, its dihedral angle at g is the same as (because opposite to) the dihedral angle of K at the subfacet in F adjacent to G in the given belt.

§3. *The sufficiency of the conditions.* To prove the sufficiency of the conditions of Theorem 1, we shall see that we can confine our attention to *face-to-face tilings* $\mathscr{K}$, which are such that the intersection of two (distinct) tiles in $\mathscr{K}$ is empty or a common face of both tiles. For such a tiling, it is clear that the corresponding family T of translations is a lattice.

Given that K satisfies the conditions of Theorem 1, one might conceive of constructing a suitable tiling by K in the following way. If F is any facet of K, then the opposite facet $-F$ is a translate of F. Thus there is a translation vector t_F carrying $-F$ into F; then we have $(K+t_F) \cap K = F = -F+t_F$. We note that $t_{-F} = -t_F$. We have such translations for each facet of K, and by compounding them, we obtain the collection $\mathscr{K} = \{K+t \mid t \in T\}$ of translates of K, where $T = \{\sum_F n_F t_F \mid n_F \in Z\}$, with Z denoting the integers. This gives us our candidate for a tiling.

We must show that $\mathscr{K}$ is simultaneously a covering and packing of E^d. We deal with the covering property first; as will become apparent, we shall not need to call upon the belt condition to do this. We find it convenient to introduce some notation, to use here and later. If G is any face of K, define the subset T_G of T recursively by $o \in T_G$, and $t \in T_G$ if there is some $t' \in T_G$, such that $K+t$ and $K+t'$ meet in a common facet which contains G. It is a natural convention that $T_\varnothing = T$; we also note that $T_K = \{o\}$. If $K' = K+t \in \mathscr{K}$ and G is a face of K', we write $\mathscr{K}_G = \{K'+t' \mid t' \in T_{G-t}\}$. Clearly it is appropriate to set $\mathscr{K}_\varnothing = \mathscr{K}$. We notice that, while the definition of T_G depends upon the initial choice of a tile of which G is a face, that of $\mathscr{K}_G$ does not, inasmuch as we recover $\mathscr{K}_G$ from any $K'' \in \mathscr{K}_G$ (see also the alternative description of $\mathscr{K}_G$ in the proof of Lemma 5 below). However, we cannot as yet exclude the possibility that G is also a face of some $K'' \notin \mathscr{K}_G$; in fact, our arguments will not use this assumption. We also see that, for $t \in T$, $\mathscr{K}_{G+t} = \mathscr{K}_G + t$ $(= \{K'+t \mid K' \in \mathscr{K}_G\})$. So, without loss of generality, in our next two lemmas we shall suppose that G is a face of K itself.

We say that $\mathscr{K}_G$ *surrounds* G if $\operatorname{relint} G \subseteq \operatorname{int} N(G)$, where we write $N(G) = \bigcup \mathscr{K}_G$. As will become clear below, the appropriate analogue for $\mathscr{K}$ itself is that $\mathscr{K}$ covers E^d.

LEMMA 5. *For each face G of K, $\mathscr{K}_G$ surrounds G.*

Let $\dim G = r$. The lemma is trivial for $r = d$; if $r = d-1$, then clearly $\mathscr{K}_G$ surrounds G (and $T_G = \{o, t_G\}$, in the notation introduced above). So, suppose that

$r < d-1$, and that the lemma holds for all faces of K of dimension at least $r+1$. Let $g \in \operatorname{relint} G$. We call $g' \in K$ *equivalent* to g, written $g' \sim g$, if there is a sequence $g = g_0, g_1, \ldots, g_n = g'$ in K, such that, for $k = 1, \ldots, n$, there is a facet F_k of K with $g_k = g_{k-1} + t_{F_k}$. It is easy to see that $\mathscr{K}_G = \{K - g' + g \mid g' \sim g\}$. Let S be a $(d-r-1)$-sphere centred at g and orthogonal to aff G, and of small enough radius that, for each $g' \sim g$, S meets only those facets of $K - g' + g$ which contain G. To show that $\mathscr{K}_G$ surrounds G, it is clearly enough to prove that S is covered by $\mathscr{K}_G$. For, in that case, if B is an r-ball in G centred at g, then $\operatorname{conv}(B \cup S)$ will be a closed neighbourhood of g in $N(G)$.

For convenience here and later, if $K' \in \mathscr{K}$ and G is a face of K' (possibly $G = \varnothing$), we write

$$C(K', G) = \bigcup \{N(F) \mid F \text{ a face of } K' \text{ with } G \subset F\},$$

where the inclusion in the definition is strict, and K' is always counted as a face of itself. We remark that, if $F_1 \subseteq F_2$, then $N(F_1) \supseteq N(F_2)$, so the union above could be taken over all faces F of K' of dimension $\dim G + 1$ which contain G.

For every $K' \in \mathscr{K}_G$, each point of $K' \cap S$ lies in the relative interior of some face F of K' strictly containing G. By our inductive assumption, each point of $\operatorname{relint} F \cap S$ has a neighbourhood in S contained in $N(F) \subseteq C(K', G)$. Since $K' \cap S$ is compact, it follows that there is some $\delta > 0$ such that the δ-neighbourhood of $K' \cap S$ in S is contained in $C(K', G)$. As $\mathscr{K}_G$ is finite, we can use the same δ for each $K' \in \mathscr{K}_G$. Hence the δ-neighbourhood in S of each point of S in

$$\bigcup \{C(K', G) \mid K' \in \mathscr{K}_G\} = N(G)$$

is contained in $N(G)$. Since the connected set S meets $N(G)$, we conclude that $S \subseteq N(G)$, as required.

Exactly the same argument shows that $\mathscr{K}$ covers E^d. Since $\mathscr{K}_G$ surrounds G for each non-empty face G of a tile $K' \in \mathscr{K}$, there is some δ-neighbourhood of K' contained in $C(K') = C(K', \varnothing)$, and as $C(K+t) = C(K)+t$ for $t \in T$, the same δ serves for each $K' \in \mathscr{K}$. Arguing as in the last paragraph, we conclude that $\mathscr{K}$ itself covers E^d.

We say that $\mathscr{K}_G$ *fits around* G if, for every $K_1, K_2 \in \mathscr{K}_G$, $K_1 \cap K_2$ is a face of each tile. For $G = \varnothing$, this condition implies that $\mathscr{K}$ is a packing of E^d.

LEMMA 6. *If G is a face of K, then $\mathscr{K}_G$ fits around G.*

We again use an inductive argument. The result is obvious if $\dim G = d$ or $d-1$. If $\dim G = d-2$, then the subfacets of K which are equivalent to G (under the equivalence relation induced by $\sim$) are all those in a 4-belt determined by G, or alternate subfacets in a 6-belt, as appropriate; in either case, the dihedral angles of K at these equivalent subfacets sum to 2π, and the four or three tiles (respectively) in $\mathscr{K}_G$ fit around G. So, we shall assume that $\dim G = r \leqslant d-2$, and that the lemma holds for all faces of dimension at least $r+1$.

Now, it is enough to show that, if $K', K'' \in \mathscr{K}_G$ are such that $\operatorname{int}(K' \cap K'') \neq \varnothing$, then $K' = K''$. For, suppose that $K' \cap K'' \supset G$ (the inclusion being strict), but $\operatorname{int}(K' \cap K'') = \varnothing$. Let F be a maximal (with respect to inclusion) face of K', with $(\operatorname{relint} F) \cap K'' \neq \varnothing$. By Lemma 5, $\operatorname{relint} F \subseteq \operatorname{int} N(F)$, and so there is some $K''' \in \mathscr{K}_F$, with $\operatorname{int}(K''' \cap K'') \neq \varnothing$. If $\operatorname{int}(K''' \cap K'') \neq \varnothing$ implies $K''' = K''$, it

would then follow that $K'' \in \mathscr{K}_F$ also, and that K' and K'' meet in a common face containing F. Since F is maximal, we would then deduce that $K' \cap K'' = F$, as required.

So, suppose that $K', K'' \in \mathscr{K}_G$ satisfy $\operatorname{int}(K' \cap K'') \neq \emptyset$. By definition, we can find $K' = K_0, K_1, \ldots, K_n = K''$ in $\mathscr{K}_G$, such that, for $i = 1, \ldots, n$, $K_{i-1} \cap K_i$ is a common facet of K_{i-1} and K_i, which also contains G. We can clearly suppose that $K_i \neq K_j$ for $i \neq j$. We call such a sequence of tiles a *strong chain* in $\mathscr{K}_G$. If $g \in \operatorname{relint} G$, and S is the small $(d-r-1)$-sphere centred at g, introduced in the proof of Lemma 5, then this strong chain in $\mathscr{K}_G$ gives rise to a strong chain of spherical $(d-r-1)$-polytopes $Q_i = K_i \cap S$ $(i = 0, \ldots, n)$. (We shall also write $Q' = K' \cap S$, and so on.)

We now make a fixed choice of a point $x \in \operatorname{int}(Q' \cap Q'')$ (here, interiors are taken relative to S), and consider spherical polygonal loops L based at x (that is, L is composed of finitely many arcs of great circles of S, and begins and ends at x). Such a loop L is *associated* with the strong chain $Q' = Q_0, Q_1, \ldots, Q_n = Q''$ if L passes successively from Q_{i-1} to Q_i $(i = 1, \ldots, n)$. Observe that we allow L to return to a given Q_i after leaving it, but only if L is at that point passing through a subsequent Q_j. Thus L is a union of $n+1$ subarcs $L_i \subseteq Q_i$ $(i = 0, \ldots, n)$, some of which possibly degenerate to points, such that L_{i-1} and L_i meet in an endpoint of each in $Q_{i-1} \cap Q_i$; of course, L_n and L_0 meet at x. We call L an *interior* loop if each L_i is contained in $\operatorname{int} Q_i \cup \operatorname{relint}(Q_{i-1} \cap Q_i) \cup \operatorname{relint}(Q_i \cap Q_{i+1})$ (with appropriate modifications for $i = 0, n$); thus L meets only the interiors of the Q_i and the relative interiors of the common facets $Q_{i-1} \cap Q_i$ through which it passes.

We denote by $\lambda(L)$ the spherical arc length of the loop L, and let λ be the infimum of $\lambda(L)$ over all (interior) loops based at x which are associated with strong chains from Q' to Q''. We shall show that $\lambda = 0$, which will imply that $Q' = Q''$. In this context, we remark that, if L is a loop associated with a given strong chain, then there is an interior loop L' associated with the same chain, such that $\lambda(L')$ approximates $\lambda(L)$ as closely as we wish.

To prove this, we use the fact that $\dim S = d-r-1 \geqslant 2$, so that S is simply connected, and hence loops in S based at x can be contracted over S to x. So, suppose that, in fact, $\lambda > 0$, and let $Q' = Q_0, Q_1, \ldots, Q_n = Q''$ be a strong chain from Q' to Q'', with which is associated a loop L_0 based at x of length $\lambda(L_0) = \lambda$. Note that, since the space of loops in S based at x is compact (in the Hausdorff metric topology; see after Conjecture 3 below), such a loop L_0 exists. We can slightly displace x in $\operatorname{int}(Q' \cap Q'')$ at the beginning, to ensure that no loop which is a single great circle can be minimal. It is then clear that L_0 cannot be an interior loop; necessarily, that part of L_0 in each Q_i consists of a single arc of a great circle, possibly degenerating to a single point.

Now L_0 has a non-straight angle at the relative interior of a face $F \cap S$ of some Q_i, where F is a face of the corresponding $K_i \in \mathscr{K}_G$, with $G \subset F$; clearly $\dim F \leqslant d-2$, so in fact F will be a common face of at least two K_i in the chain. Let L_0 meet F in y. Since $y \in \operatorname{relint} F \subseteq \operatorname{int} N(F)$ (by Lemma 5), there is some δ-neighbourhood W, say, of y in $N(F) \cap S$. We now choose $u, v \in L_0 \cap W$ to be separated on L_0 by y, and replace that part of L_0 between u and v by the great circle arc $uv \subset W \subseteq \operatorname{int}(N(F) \cap S)$, to obtain a new loop L_1, for which clearly $\lambda(L_1) < \lambda(L_0) = \lambda$. The proof will be completed (by contradicting the assumption that λ is minimal) if we show that L_1 is also associated with a strong chain from Q' to Q''.

To this end, suppose s minimal and t maximal with, say, $u \in Q_s$ and $v \in Q_t$ (and, necessarily, $s \neq t$, since otherwise $uv \subseteq Q_s$, and then L_0 would not bend at y). Pick sequences $(u_k) \subset W \cap \operatorname{int} Q_s$ with $\lim u_k = u$ and $(v_k) \subset W \cap \operatorname{int} Q_t$ with $\lim v_k = v$, such that $u_k v_k \subset W$ is in general position with respect to $\mathscr{K}_F$, so that $u_k v_k$ meets no face of dimension less than $d-1$ of any tile in $\mathscr{K}_F$; this is possible, since $\mathscr{K}_F$ is finite. Again because $\mathscr{K}_F$ is finite, we can pass to a suitable subsequence of k's, if necessary, and suppose that $u_k v_k$ passes through the same sequence of tiles for each k, say $Q_s = \tilde{Q}_0, \tilde{Q}_1, \ldots, \tilde{Q}_m = Q_t$, where $\tilde{Q}_j = \tilde{K}_j \cap S$ and $\tilde{K}_j \in \mathscr{K}_F$ $(j = 0, \ldots, m)$. Our inductive assumption on $r = \dim G$ now enters. Since $\mathscr{K}_F$ fits around F, we see that, for $j = 1, \ldots, m$, $\tilde{Q}_{j-1}$ and $\tilde{Q}_j$ meet in a common $(d-r-2)$-face, which arises from the common facet $\tilde{K}_{j-1} \cap \tilde{K}_j$ (containing F, and hence also G) of $\tilde{K}_{j-1}$ and $\tilde{K}_j$. Finally, since $\lim (u_k v_k) = uv$, we conclude that, if we replace $Q_{s+1}, \ldots, Q_{t-1}$ in the original strong chain by $\tilde{Q}_1, \ldots, \tilde{Q}_{m-1}$, we obtain a new strong chain, with which L_1 is associated.

To complete the proof of Theorem 1, we can apply exactly the same argument as that of Lemma 6 to $\mathscr{K}$ itself, with S replaced by E^d, and using the simple connectedness of E^d. However, one further comment is needed. As we have seen, there is some $\delta > 0$, such that $C(K)$ contains a δ-neighbourhood of K. (We use the notation introduced above.) Thus, if $C(K)$ is composed of N tiles, then an interior loop (which by definition does not revisit any tile) can meet at most N tiles in an interval of length 2δ. It follows that the length n of our strong chain leading from K' to K'' is bounded above by a constant multiple of the supposed minimal length of interior loops based at x (roughly, $n \leqslant N\lambda/2\delta$).

We remark that Theorem 1 was proved in the special case of zonotopes by Shephard [1974] (for $d \leqslant 4$) and McMullen [1975].

§4. *Further results.* We shall note here that Theorem 1 has a number of consequences.

THEOREM 2. *Every convex translation tile admits a lattice tiling.*

For, as we now see, the family T of translation vectors associated with the tiling $\mathscr{K}$ constructed in §3 is a lattice.

This result is obvious in dimension $d = 2$, and Delaunay [1933] has sketched a proof for $d = 3$. Note that our methods yield no insight into whether a general tiling can be continuously distorted (through tilings) to the face-to-face tiling, or even some lattice tiling. Theorem 2 is also related to Problem XVIII of Hilbert [1902]. For, the same result does not hold for the slightly more general class of star-bodies, as was shown by Stein [1972] with a centrally symmetric polyhedral example in E^{10} (if we do not demand central symmetry, there is an example in E^5). We also remark that Kershner [1968] has answered Hilbert's original question by giving examples of convex pentagons which tile E^2 by rigid motions, but do not admit any tiling whose symmetry group is transitive on the tiles.

Groemer [1962] has investigated the following closely related problem. We say K *tiles* E^d *by homothety*, or is a *homothety tile*, if there is some closed interval $[\alpha, \beta]$ of positive real numbers, such that some family $\mathscr{K}$ of homothetic copies $K' = \lambda K + t$ of K, with ratio of homothety $\lambda \in [\alpha, \beta]$, tiles E^d. Such a homothety tiling $\mathscr{K}$ is *proper* if $\mathscr{K}$ does not consist of translates of a single tile. Now Groemer showed that a

homothety tile is again a centrally symmetric polytope with centrally symmetric facets, and so satisfies the conditions of Lemmas 1 to 3 (the proof is more complicated than in our case). As the proof of Lemma 4 clearly demonstrates, the fact that each belt of a tile K contains four or six facets depends only on the shape of the tiles, and not on their relative sizes. As a result, we have:

THEOREM 3. *If the convex body K tiles E^d by homothety, then K admits a lattice tiling. In particular, K tiles E^d by translation.*

Groemer remarks that K can have at most $3^d - 3$ facets. But Minkowski [1897] showed (in E^3, but as usual his argument generalizes easily) that a lattice tile has at most $2(2^d - 1)$ facets. Hence:

COROLLARY. *A homothety tile in E^d has at most $2(2^d - 1)$ facets.*

In fact, Groemer [1964] proved rather more. His result consists of the cases $d \leqslant 4$ of the following.

THEOREM 4. *Let K be a proper homothety d-tile. Then K is a prism (with base a $(d-1)$-tile).*

To prove this, we first need:

LEMMA 7. *Let K be a tile, and F a facet of K. If every belt of K containing F is a 4-belt, then K is a prism with base F.*

For, let F' be the opposite facet of K; then $\operatorname{conv}(F \cap F') \subseteq K$. But if G is a subfacet of K lying in F, then the facet of K which meets F in G meets F' in the corresponding subfacet G'. It then follows that $\operatorname{conv}(F \cap F') \supseteq K$; hence $K = \operatorname{conv}(F \cap F')$ is a prism with base F, as stated. Finally, it is clear that F is itself a tile.

We now prove Theorem 4. Let G be a subfacet of a tile K in the proper homothety tiling $\mathcal{K}$, which determines a 6-belt. Each general point g of G lies in two other tiles (compare the argument of Lemma 4); we show that these tiles must be translates of K. For, let $K' = \lambda K + t$ be one of these tiles, and suppose, if possible, that $\lambda < 1$ (the case $\lambda > 1$ is the same, with the rôles of K and K' reversed). Then K and K' meet on part of a facet F of K and a facet F' of K'. Now K and K' are also related by an opposite homothety; if z is the centre of this homothety, so that $K' - z = -\lambda(K - z)$, then z lies in the hyperplane $\operatorname{aff} F = \operatorname{aff} F'$. Since $g \in F$, we see that $g_1 = -\lambda g + (\lambda + 1)z \in F'$. Similarly, since $g \in F'$, $g_2 = -\lambda^{-1} g + (\lambda^{-1} + 1)z \in F$. But now g_1 lies in the open line segment between g and g_2, and we conclude at once that the subfacet $-\lambda G + (\lambda + 1)z$ of K' meets $\operatorname{relint} F$. But the argument of Lemma 4 shows that this is impossible; hence $\lambda = 1$, as was claimed. We further see that, if G' is a subfacet of K' (necessarily in F') such that $\dim(G \cap G') = d - 2$, then the opposite subfacets of K in F and K' in F' meet in $-(G \cap G') + 2z$, which has the same volume as $G \cap G'$. It now follows at once that F is covered completely by such facets F' of tiles K', which meet K in this way. We finally conclude that, if $K' = \lambda K + t$ is a tile in $\mathcal{K}$ with $\lambda \neq 1$ which meets K in part of a facet F, then every belt of K containing F is a 4-belt. Thus, by Lemma 7, K is a prism with base F, and this proves Theorem 4.

§5. *Open questions.* It is possible that the ideas we have been exploring here may help to find solutions to some interesting problems on tiles. At least, our preliminary investigations, even though they have so far been unsuccessful, lead us to believe this, and embolden us to state the problems in the form of conjectures.

Our first problem is somewhat related to the result of Theorem 4.

CONJECTURE 1. *Let K admit a non-face-to-face tiling of E^d. Then K is a direct sum (of smaller dimensional tiles).*

This conjecture is straightforward to prove if $d \leqslant 3$.

Our remaining conjecture is an old one, and to elucidate the problem, we shall split it up into several equivalent conjectures. As traditionally defined, a *Voronoĭ polytope* (or *Dirichlet region*) is the set of points of E^d no farther from the origin o than from any other point of a lattice T. Because of the essentially affine nature of the problem, we find it convenient to generalize the definition. Let T be a lattice in E^d, and ϕ a positive definite quadratic form on E^d. Then the *Voronoĭ polytope* $V(T, \phi)$ is defined by

$$V(T, \phi) = \{x \in E^d \mid \phi(x) \leqslant \phi(x-t) \quad \text{for all } t \in T\}.$$

Clearly, by applying suitable linear transformations, we can normalize so that $T = Z^d$ or $\phi = \|x\|^2$ (not, of course, simultaneously). Then our main conjecture, bearing in mind our convention that a tile is centred at o, is:

CONJECTURE 2. *Every tile is a Voronoĭ polytope.*

It was Voronoĭ [1908–09] who first investigated this problem, in connexion with questions on the geometry of numbers. (See Gruber [1979] for a recent survey on this area.) To discuss Voronoĭ's partial solution, we need a definition. Let K be a d-tile. If $r = 0, \ldots, d-2$, we call K *r-primitive* if, in its face-to-face tiling $\mathscr{K}$, every r-face G of K lies in exactly $d-r+1$ tiles of $\mathscr{K}$. Of course, $d-r+1$ is the minimal number of tiles which can surround G. The special case $r = 0$ we call simply *primitive*. Now Voronoĭ himself showed that every primitive tile is a Voronoĭ polytope, and so Conjecture 2 would follow from:

CONJECTURE 3. *Every tile is a limit of primitive tiles.*

The limit here is in the Hausdorff metric topology on non-empty compact (convex) sets, where the distance $\rho(L, M)$ between L and M is given by

$$\rho(L, M) = \min\{\rho \geqslant 0 \mid L \subseteq M+\rho B, \quad M \subseteq L+\rho B\},$$

the sum is Minkowski addition, and B is the unit ball. In this context, we remark that Groemer [1971] has shown that, if a limit of Voronoĭ polytopes is full dimensional, it is again a Voronoĭ polytope.

In fact, Žitomirskiĭ [1929] has shown that every $(d-2)$-primitive tile is a Voronoĭ polytope. (Note that a d-tile is $(d-2)$-primitive if, and only if, each of its belts is a 6-belt.) So, Conjecture 2 would also follow from the (apparently) even weaker

CONJECTURE 4. *Every d-tile is a limit of* $(d-2)$*-primitive tiles.*

We also observe, by fixing the quadratic form ϕ, but varying the lattice T, that every Voronoï polytope $V(T, \phi)$ is a limit of primitive tiles, and so Conjecture 2 implies Conjecture 3. Hence these three conjectures are equivalent.

We further remark that Delaunay [1929], in the context of enumerating their 52 combinatorial types, has shown that every 4-tile is a Voronoï polytope. That every tile of smaller dimension is a Voronoï polytope can easily be established directly. Finally, McMullen [1975] has proved a slightly weaker result for zonotopes: every zonotope which tiles E^d is equivalent to a Voronoï polytope. By calling two zonotopes *equivalent*, we mean that one can be obtained from the other (up to affinity) by altering the lengths of its component line segments.

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52A45: *CONVEX SETS AND RELATED GEOMETRIC TOPICS; Packing, covering, tiling.*

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