Dynamics on homogeneous spaces, exercises

Tel Aviv University, Fall 2013

1. Let $(X, \mathcal{B}, \mu)$ be a probability space, $T : X \to X$ measurable and measure preserving. Prove the following strengthenings of the Poincaré recurrence theorem:

(a) For any $A \in \mathcal{B}$ such that $\mu(A) > 0$, for a.e. every $x \in A$, there are $n_k \to \infty$ such that $T^{n_k} x \in A$.

(b) For any $A \in \mathcal{B}$ such that $\mu(A) > 0$ there is $n \in \left\{1, \ldots, \left\lfloor \frac{1}{\mu(A)} \right\rfloor + 1\right\}$ such that $\mu(A \cap T^{-n}(A)) > 0$.

(c) For any $a, \lambda \in (0,1)$ there is $c = c(a, \lambda)$ with the following property. For any $A \in \mathcal{B}$ with $\mu(A) = a$ there is $n \in \{1, \ldots, c\}$ such that $\mu(A \cap T^{-n}(A)) > \lambda \mu(A)^2$.

2. Let G be a Lie group acting continuously on a locally compact space X. Let μ be a regular Borel[1] probability measure on X and let $H = \{g \in G : g_* \mu = \mu\}$. Prove that H is a Lie subgroup of G.
Notation: $g_* \mu(A) = \mu(g^{-1} A)$.

3. Write the following spaces as homogeneous spaces, i.e. for each space X below, find a Lie group G with a closed subgroup H, such that X is G-equivariantly isomorphic to G/H.

(a) The space of lines through the origin in $\mathbb{R}^n$.

(b) The space of k-frames in $\mathbb{R}^n$.
Notation: A k-frame in $\mathbb{R}^n$ is a set $\vec{v}_1, \ldots, \vec{v}_k$ of linearly independent vectors in $\mathbb{R}^n$.

(c) For $1 \le k \le n$, the space of discrete subgroups $\mathbb{Z}\vec{v}_1 \oplus \cdots \oplus \mathbb{Z}\vec{v}_k$ of $\mathbb{R}^n$, with $\vec{v}_1, \ldots, \vec{v}_k$ a k-frame.

(d) For $1 \le k \le n$, the space of discrete subgroups $\Lambda = \mathbb{Z}\vec{v}_1 \oplus \cdots \oplus \mathbb{Z}\vec{v}_k$ as in (c), up to *homothety*, i.e. up to the equivalence $\Lambda \sim \Lambda'$ iff there is $c \in \mathbb{R}$, $c \neq 0$ such that $\Lambda' = c\Lambda$.

(e) The space of nondegenerate bilinear forms in n variables of signature (p, q).
Notation: A *bilinear form* in n variables is a function $B(\vec{x}, \vec{y}) = \sum_{i,j=1}^{n} a_{ij} x_i y_j$; it is said to be *nondegenerate* if for all $\vec{x} \neq 0$ there is $\vec{y}$ such that $B(\vec{x}, \vec{y}) \neq 0$; it is said to be *of signature* (p, q) if after a change of coordinates $\vec{u} = A\vec{x}, \vec{v} = A\vec{y}$, it can be written as $B(\vec{u}, \vec{v}) = u_1 v_1 + \cdots + u_p v_p - u_{p+1} v_{p+1} - \cdots - u_n v_n$, where $p + q = n$.

[1] From now on all measures on a locally compact space will always be assumed to be Borel regular measures.

(f) The space of full flags in $\mathbb{R}^n$.
Notation: A full flag in $\mathbb{R}^n$ is a collection of nested linear subspaces $\{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{R}^n$.

4. Prove that the following Lie groups are unimodular:

- Any abelian Lie group.
- Any compact Lie group.
- Any semisimple Lie group.
- $\mathrm{GL}_d(\mathbb{R})$.
- Any Lie group containing a lattice.

5. Prove that the group of 2×2 upper triangular matrices of determinant one is not unimodular.

6. Prove that $\mathrm{SL}_2(\mathbb{R})$ is a simple Lie group (hint: Jordan form). Deduce from this that $\mathrm{SL}_n(\mathbb{R})$ is simple for all $n \geq 2$.

7. Let $X = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SL}_2(\mathbb{Z})$ and let $x \in X$, $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. Let $\pi : \mathrm{SL}_2(\mathbb{R}) \to X$ be the projection and let $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ so that $x = \pi(h)$. Prove:

- The orbit $\{g_t x : t \in \mathbb{R}\}$ is compact $\iff$ there is $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$ with $|\mathrm{tr}(\gamma_0)| > 2$ and with eigenvectors $(d, -c)$, $(-b, a)$.
- The orbit $\{g_t x : t \in \mathbb{R}\}$ is closed but not compact $\iff g_t x \to_{t \to \pm\infty} \infty$ $\iff a/b$ and c/d belong to $\mathbb{Q} \cup \{\infty\}$.

8. Let $X_d \overset{\mathrm{def}}{=} \mathrm{SL}_d(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z})$ be the space of unimodular lattices in $\mathbb{R}^d$, and let $\lambda_1(\Lambda)$ denote the length of the shortest nonzero vector of Λ. Show that $\lambda_1 : X_d \to (0, \infty)$ is a continuous proper function (a function is proper if the pre-image of a compact set is compact).

9. Let $\|\cdot\|$ denote the Euclidean norm, and for $\Lambda \in X_d$, let

$$\mathrm{covrad}(\Lambda) \overset{\mathrm{def}}{=} \inf\{r > 0 : \forall y \in \mathbb{R}^d \, \exists x \in \Lambda \text{ such that } \|x - y\| < r\}.$$

Prove that $\mathrm{covrad} : X_d \to (0, \infty)$ is a continuous proper function.

10. For a vector $\vec{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, let $\Lambda_{\vec{x}} \in X_{d+1}$ denote the lattice spanned by the standard basis vectors $\vec{e}_1, \ldots, \vec{e}_d$ and $(x_1, \ldots, x_d, 1)$. For the following diophantine properties of $\vec{x}$, state and prove necessary and sufficient conditions for the property to hold in terms of the orbit of $\Lambda_{\vec{x}}$ under certain subgroups or subsemigroups of $\mathrm{SL}_{d+1}(\mathbb{R})$.

- $\vec{x}$ is **VWA**, i.e. there is $\varepsilon > 0$ and infinitely many solutions $\vec{p} \in \mathbb{Z}^d, q \in \mathbb{N}$ to the inequality $\|q\vec{x} - \vec{p}\| < \frac{1}{q^{1/d+\varepsilon}}$.
- $\vec{x}$ is **MVWA**, i.e. there is $\varepsilon > 0$ and infinitely many solutions $\vec{p} = (p_1, \ldots, p_d) \in \mathbb{Z}^d, q \in \mathbb{N}$ to the inequality $\prod_1^d |qx_i - p_i| < \frac{1}{q^{1+\varepsilon}}$.

- $\vec{x}$ is **singular**, i.e. for every $\varepsilon > 0$ there is Q_0 so that for every $Q > Q_0$, there is a solution $\vec{p} \in \mathbb{Z}^d, q \in \mathbb{N}$ to the inequalities $\|q\vec{x} - \vec{p}\| < \frac{\varepsilon}{Q^{1/d}}$, $q < Q$.

11. Let G be an lcsc group acting continuously on a lcsc space X preserving a Borel probability measure μ. Show that the G-action is ergodic if and only if any $f \in L^2(X, \mu)$ which is G-invariant (i.e. $gf = f$ for all $g \in G$, as elements of $L^2(X, \mu)$), is constant a.e. Prove that the G-action is mixing if and only if for any $f_1, f_2 \in L^2(X, \mu)$ and any $g_n \in G, g_n \to \infty$ we have $\langle g_n f_1, f_2 \rangle \to \int_X f_1 d\mu \cdot \int_X f_2 d\mu$.

12. Let $g \in \mathrm{SL}_n(\mathbb{R})$ such that at least one of the eigenvalues λ of g satisfies $|\lambda| \neq 1$. Prove that the sets

$$U^{\pm} = \{h \in \mathrm{SL}_n(\mathbb{R}) : g^{-n} h g^n \to_{n \to \pm\infty} e\}$$

are closed subgroups of $\mathrm{SL}_n(\mathbb{R})$ of positive dimension, and that G is generated by U^+, U^-. These groups are called respectively the *expanding and contracting horospherical subgroups* of g.

13. Let $X_d = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$. Prove that for any $\varepsilon > 0$ there is a compact subset $K \subset X_d$ such that for any lattice $\Lambda \in X_d$, and any one-parameter unipotent subgroup $\{u_t : t \in \mathbb{R}\} \subset \mathrm{SL}_d(\mathbb{R})$, one of the following holds:

(i) $\liminf_{T \to \infty} \frac{1}{T} |\{t \in [0, T] : u_t \Lambda \in K\}| \geq 1 - \varepsilon$.
(ii) There exists $k \in \{1, \ldots, d-1\}$ and $v_1, \ldots, v_k \in \Lambda$ such that the subspace $\mathrm{span}(v_1, \ldots, v_k)$ is invariant under u_t.

Are (i) and (ii) mutually exclusive? Show by an example that one cannot omit the statement (ii) in the preceding statement.

14. A *symmetric box* in $\mathbb{R}^d$ is a set of the form $[-a_1, a_1] \times \cdots \times [-a_d, a_d]$ for $a_i > 0, i = 1, \ldots, d$. Let X_d be the space of unimodular lattices and for $\Lambda \in X_d$, say that a box B is *admissible for* Λ if $B \cap \Lambda = \{0\}$. Let

$$\kappa(\Lambda) = \frac{1}{2^d} \sup\{\mathrm{Vol}(B) : B \text{ an admissible box for } \Lambda\}.$$

(i) Is κ continuous?
(ii) Show that $\inf_{\Lambda \in X_d} \kappa(\Lambda)$ is attained, i.e. is equal to $\kappa(\Lambda_0)$ for some Λ.
(iii) Show that there is $\kappa_0 = \kappa_0(d)$ such that for a.e. $\Lambda \in X_d$ (w.r.t. the natural measure on the space of lattices), $\kappa(\Lambda) = \kappa_0$.
(iv) Compute κ_0.

15. Let $I \subset \mathbb{R}$ be an interval, $\varphi : I \to \mathbb{R}^d$ a polynomial function such that $\{\varphi(s) : s \in I\} \subset \mathbb{R}^d$ is not contained in an affine hyperplane. Prove that for almost every $s \in I$, $\varphi(s)$ is not singular (as defined in question 10).

16. Let G be a connected Lie group, let $m(\cdot)$ be a left-invariant Haar measure on G and let $\{F_n\}$ be a sequence of Folner sets in G, i.e. $m(F_n) < \infty$

for each n, and for each $g \in G$, $\frac{m(F_n \Delta g F_n)}{m(F_n)} \to_{n\to\infty} 0$. Suppose X is a compact G-space on which the G-action is uniquely ergodic. Denote the unique G-invariant Borel probability measure by μ.

- Prove that for any $f \in C(X)$ and any $x_0 \in X$,
$$\frac{1}{m(F_n)} \int_{F_n} f(gx_0)dm(g) \to_{n\to\infty} \int_X f d\mu.$$
- Show that the convergence in the previous assertion, is uniform with respect to x_0.
- Show by an example that the assumption that X is compact is essential.

17. Let $G = \mathrm{SL}_2(\mathbb{R})$ and let Γ be a cocompact lattice. Let g_t be as in Question 7 (the geodesic flow).

- Show that (in contrast to the case of the horocycle flow) there are infinitely many distinct closed orbits for the $\{g_t\}$-action on G/Γ.
- Show that for any two closed orbits $\mathcal{C}_i \stackrel{\mathrm{def}}{=} \{g_t x_i : t \in \mathbb{R}\}$, $i = 1, 2$, there is an orbit $\{g_t y : t \in \mathbb{R}\}$ such that for any sequence t_n tending to $+\infty$ (resp. $-\infty$), there is a convergent subsequence $g_{t_{n_k}} y \to_{k\to\infty} z$ with $z \in \mathcal{C}_1$ (resp. $z \in \mathcal{C}_2$).

18.

- Let Γ be a cocompact lattice in $\mathrm{SL}_d(\mathbb{R})$, acting by linear transformations on $\mathbb{R}^d$. Show that any Γ-orbit in $\mathbb{R}^d \smallsetminus \{0\}$ is dense.
- Show that for $\vec{v} \in \mathbb{R}^2 \smallsetminus \{0\}$, and for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, either the Γ-orbit of $\vec{v}$ is discrete and there is $c \in \mathbb{R} \smallsetminus \{0\}$ such that $c\vec{v} \in \mathbb{Q}^2$, or the Γ-orbit of $\vec{v}$ is dense in $\mathbb{R}^2$.