Exercise Sheet, Dynamics on homogeneous spaces, Fall 2019

Notations and assumptions. Unless stated otherwise, all measures on a topological space are regular Borel Radon measures. ‘lcsc’ stands for locally compact second countable Hausdorff.

1. Let $G = \text{SL}_2(\mathbb{R})$, $B$ the subgroup of upper triangular matrices. Prove that there is no measure on $G/B$ which is invariant under $G$. Strengthen this by proving that if $\Gamma \subset G$ is an unbounded subgroup (i.e. $\overline{\Gamma}$ is not compact) and $\Gamma$ does not leave invariant a finite set of points in $G/B$, then there is no measure on $G/B$ which is invariant under $\Gamma$.

2. Let $G$ be a lcsc group and $X$ a lcsc space. Suppose $G$ acts continuously and transitively on $X$ and for $x_0 \in X$, let $H = \{g \in G : gx_0 = x_0\}$ and let $\pi : G/H \to X$, $\pi(gH) = gx_0$ and $\pi' : G \to X$, $\pi'(g) = \pi(gH)$.
   (i) Prove that $\pi, \pi'$ are well-defined and continuous and that $\pi$ is a homeomorphism.
   (ii) Prove that $K \subset X$ is compact if and only if there is a compact $K' \subset G$ such that $\pi'(K') = K$.
   (iii) Suppose $L$ is a closed subgroup of $G$ and define the orbit map $\varphi = \pi \circ \iota$, where $\iota : L/L \cap H \to G/H$, $\iota(\ell(L \cap H)) = \ell H$. Prove that $\varphi$ is injective, and prove that it is a homeomorphism onto its image if and only if the orbit $Lx_0$ is closed.
   (iv) Are (ii) and / or (iii) true for general actions of topological groups, i.e. if one does not assume that $G$ and / or $X$ are lcsc?

3. Let $\Gamma$ be a discrete subgroup of $G = \text{SL}_2(\mathbb{R})$, let $\mathbb{H}^+ \overset{\text{def}}{=} \{z \in \mathbb{C} : \text{Im} \ z > 0\}$ and let $d(\cdot, \cdot)$ denote the hyperbolic metric on $\mathbb{H}^+$. Suppose $z_0 \in \mathbb{H}^+$ is not stabilized by any element of $\Gamma \setminus \{e\}$, and let $\mathbb{D} = \{z \in \mathbb{H}^+ : \forall \gamma \in \Gamma \setminus \{e\}, d(\gamma z, z_0) \geq d(z, z_0)\}$. In items (ii)—(v) below, assume $\Gamma$ is cocompact.
   (i) Prove that $\mathbb{D}$ is a fundamental domain for the action of $\Gamma$ on $\mathbb{H}^+$ by Möbius transformations, the boundary of $\mathbb{D}$ consists of a locally finite collection of geodesic arcs (a geodesic arc is a path which minimizes the hyperbolic distance of any two points on it; local finiteness means that any compact subset of $\mathbb{H}^+$ contains finitely many segments from $\partial \mathbb{D}$).
   (ii) Deduce that if $\Gamma$ is cocompact then $\mathbb{D}$ has finitely many sides and $\Gamma$ is finitely generated.
   (iii) Deduce that if $\alpha$ is any geodesic which intersects the interior of $\mathbb{D}$, then $\mathbb{D} \setminus \alpha$ has two connected components.
(iv) Let \( B_T \overset{\text{def}}{=} \{ z \in \mathbb{H}^+ : d(z, z_0) \leq T \} \) and let \( \partial, \text{int} \) denote respectively boundary and interior. For each \( \gamma \in \Gamma \) and each \( x \in \text{int}(\gamma \mathbb{D}) \), let \( \alpha_x \) be the intersection of \( \gamma \mathbb{D} \) with the geodesic through \( x \) perpendicular to the geodesic from \( x \) to \( z_0 \), let \( \beta_x \) be the intersection of \( \partial B_{d(x, z_0)} \) with \( \gamma \mathbb{D} \) and let \( D_x \) be the maximal distance between a point of \( \alpha_x \) and the nearest point of \( \beta_x \). Prove that \( D_x \to 0 \) uniformly as \( d(x, z_0) \to \infty \), that is, 
\[
\sup \{ D_x : x \in \text{int}(\gamma \mathbb{D}) \cap \partial B_T \text{ for some } \gamma \in \Gamma \} \to_{T \to \infty} 0.
\]

(v) Complete the details of the reduction sketched in class for Margulis’ result: let \( m_{G/\Gamma} \) denote the \( G \)-invariant Borel probability measure on \( G/\Gamma \), and suppose that for any \( f \in C(G/\Gamma) \), 
\[
\frac{1}{\pi} \int_{0}^{\pi} f(g t \theta) d\theta \to_{t \to \infty} \int f dm_{G/\Gamma}.
\]
Show that if \( m \) is the hyperbolic area measure on \( \mathbb{H}^+ \) and \( z_0 \in \mathbb{H}^+ \) has a trivial stabilizer under \( \Gamma \), then
\[
|\{ z \in \Gamma z_0 : d(z, z_0) \leq T \}| \sim \frac{m(B_T)}{m(\mathbb{D})}, \quad \text{as } T \to \infty.
\]

4. Give an example of an infinitely generated discrete subgroup of \( \text{SL}_2(\mathbb{R}) \).

5. Prove that \( G = \text{SL}_n(\mathbb{R}) \) is simple (as a topological group), i.e. if \( H \subset G \) is a closed normal subgroup then \( H \) is either discrete or equal to \( G \).

6. Show that the universal cover of a connected Lie group is a Lie group. Prove that the fundamental group of a connected Lie group is abelian. Show that \( \text{SL}_n(\mathbb{R}) \) is connected. Compute the fundamental group of \( \text{SL}_n(\mathbb{R}) \) for \( n \geq 2 \).

7. An element \( h \) in a group \( G \) is called central if \( hg = gh \) for all \( g \in G \), and a subgroup of \( G \) is called central if all its elements are central. Suppose \( G \) is a connected Lie group and \( H \) is a discrete normal subgroup. Show that \( H \) is central.

8. Show that if \( G \) is an lcsc group acting continuously and transitively on an lcsc space \( X \), and preserving a measure \( \nu \), then this measure is unique, that is if \( \nu_1, \nu_2 \) are two such nonzero measures on \( X \) such that for any Borel set \( A \) and \( g \in G \) we have \( \nu_i(gA) = \nu_i(A) \) \( (i = 1, 2) \) then there is a constant \( c > 0 \) such that \( \nu_2 = c \nu_1 \). Conclude that left and right Haar measures on an lcsc group \( G \) are unique up to scaling.

9. Show that a Lie group is unimodular if one of the following holds: \( G \) is simple (as a topological group); \( G \) is compact; \( G \) is nilpotent. Also show that if a left Haar measure \( \nu_L \) on \( G \) satisfies \( \nu_L(G) < \infty \), then the same is true for a right Haar measure, and \( G \) is compact.
10. Let $\mathcal{X}_n$ denote the collection of closed subsets of $\mathbb{R}^n$, and define the Chabauty-Fell metric on $\mathcal{X}_n$ as follows:
\[
d(A_0, A_1) = \inf \left\{ r > 0 : \text{ for } i = 0, 1, \quad B \left( 0, \frac{1}{r} \right) \cap A_i \subset \bigcup_{a \in A_{1-i}} B(a, r) \right\}.
\]
Prove that $d$ is a metric on $\mathcal{X}_n$ and that with this metric, $\mathcal{X}_n$ is compact and complete. Let $\mathcal{L}_n = \text{SL}_n(\mathbb{R})/\text{SL}_n(\mathbb{Z})$, the space of covolume 1 lattices equipped with the quotient topology. Prove that the inclusion map $\iota : \mathcal{L}_n \to \mathcal{X}_n$ is continuous, and deduce that for any $i$, the maps $\mathcal{L}_n \to \mathbb{R}$, $\Lambda \mapsto \lambda_i(\Lambda)$ and $\Lambda \mapsto \bar{\lambda}_i$ are continuous, where
\[
\lambda_i(\Lambda) \overset{\text{def}}{=} \inf \{ r > 0 : \Lambda \cap B(0, r) \text{ contains } i \text{ linearly independent vectors} \},
\]
and
\[
\bar{\lambda}_i(\Lambda) \overset{\text{def}}{=} \inf \{ r > 0 : \Lambda \cap B(0, r) \text{ contains a primitive } i\text{-tuple} \}.
\]
Show that any element in the closure of $\iota(\mathcal{L}_n)$ is a group. Show that the map $\mathcal{L}_n \to \mathcal{X}_n$ which sends a lattice $\Lambda$ to its Voronoi cell, is continuous.

11. Show that for any $n \in \mathbb{N}$ there is $c > 0$ such that for any $\Lambda \in \mathcal{L}_n$ there is a basis $v_1, \ldots, v_n$ such that for all $i = 1, \ldots, n$,
\[
\frac{1}{c} \|v_i\| \leq \lambda_i(\Lambda) \leq c \|v_i\|.
\]

12. The covering radius of a lattice $\Lambda$ is defined as
\[
\text{covrad}(\Lambda) \overset{\text{def}}{=} \min \{ r > 0 : \forall x \in \mathbb{R}^n \exists y \in \Lambda \text{ such that } \|x - y\| \leq r \}.
\]
Show that the min in this definition is indeed attained, and that it is the minimal $r$ so that the closed $r$-ball around 0 contains the Voronoi cell of $\Lambda$. Show that for a sequence $(\Lambda_j)_j \subset \mathcal{L}_n$, $(\Lambda_j)_j$ has no convergent subsequences if and only if $\text{covrad}(\Lambda_j) \to j \to \infty \infty$.

13. Let $E \subset \mathbb{R}^n \setminus \{0\}$ be a measurable set of Lebesgue measure $V < \infty$. Let $m$ be the $\text{SL}_n(\mathbb{R})$-invariant probability measure on $\mathcal{L}_n$. Show that $\int_{\mathcal{L}_n} |\Lambda \cap E| \, d\mu(\Lambda) = V$. Deduce that there is $c > 0$ such that for every $n \in \mathbb{N}$, there is a lattice $\Lambda \in \mathcal{L}_n$ such that $\lambda_1(\Lambda) \geq c\sqrt{n}$.

14. Let $G$ be an lcsc group acting on an lcsc space $X$. Suppose $\mu$ is a Borel measure which is quasi-invariant (i.e. $G$ maps sets of zero measure to sets of zero measure), ergodic (i.e. for any $G$-invariant set $A$, either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$), and with support equal to $X$. Show that almost every $G$-orbit is dense.

15. Suppose $G$ is a Lie group and $\Gamma$ is a lattice in $G$, and let $\pi : G \to G/\Gamma$ denote the quotient map. Show that for $F \subset G$, $\pi(F)$ is
compact if and only if there is a neighborhood \( U \) of \( e \) in \( G \) such that for any \( \gamma \in \Gamma \setminus \{e\} \) and every \( g \in F \), \( g\gamma g^{-1} \notin U \). Show that if \( H \) is a closed subgroup of \( G \) and \( \Gamma_H = H \cap \Gamma \) is a lattice in \( H \), then the map \( H/\Gamma_H \to G/\Gamma, h\Gamma_H \mapsto h\Gamma \), is proper.

16. Suppose \( \Gamma \) is a discrete subgroup of a unimodular Lie group \( G \), and \( N \) is a closed normal subgroup of \( G \). Let \( \pi : G \to G/N \) be the natural projection and suppose \( \pi(\Gamma) \) is a lattice in \( G/N \) and \( \Gamma \cap N \) is cocompact in \( N \). Show that \( \Gamma \) is a lattice in \( G \). Deduce that \( SL_n(\mathbb{Z}) \ltimes \mathbb{Z}^n \) is a lattice in \( SL_n(\mathbb{R}) \ltimes \mathbb{R}^n \).

17. Let \( G \) be a connected Lie group and let \( g \in G \). Let
\[
U^+ \overset{\text{def}}{=} \{ h \in G : g^n h g^{-n} \to_{n \to +\infty} e \}, \quad U^- \overset{\text{def}}{=} \{ h \in G : g^n h g^{-n} \to_{n \to -\infty} e \}.
\]
Show that \( U^\pm \) are subgroups of \( G \), and that the closure of the group generated by \( U^+ \) and \( U^- \) is normal in \( G \).

18. Let \( G \) be an lcsc group acting on an lcsc space \( X \) and let \( \mu \) be a measure on \( X \). Show that if a measurable set \( A \) satisfies \( \forall g \in G, \mu(A \triangle gA) = 0 \) then there is a measurable set \( A' \) such that \( gA' = A' \) for all \( g \in G \) and \( \mu(A \triangle A') = 0 \).

19. Let \( G \) be an lcsc group acting on an lcsc \( X \), and let \( \mu \) be a probability measure on \( X \) which is \( G \)-invariant, and suppose there are no other \( G \)-invariant probability measures on \( G \) (if this holds we say that the action of \( G \) on \( X \) is uniquely ergodic). Prove that \( \mu \) is ergodic. Let \( T^n = \mathbb{R}^n/\mathbb{Z}^n \) and let \( \Gamma \) be the abelian group generated by vectors \( v_1, \ldots, v_d \) in \( \mathbb{R}^n \), acting on \( T^n \) by the rule
\[
\gamma \pi(x) = \pi(\gamma + x), \text{ where } \pi : \mathbb{R}^n \to T^n \text{ is the projection}.
\]
Prove that the \( \Gamma \) action on \( T^n \) is uniquely ergodic if and only if the group \( \Gamma + \mathbb{Z}^n \) is dense in \( \mathbb{R}^n \). Deduce that if \( \Gamma \) is the cyclic group generated by \( v = (x_1, \ldots, x_n) \) then \( \Gamma \) acts ergodically if and only if \( 1, x_1, \ldots, x_n \) are linearly independent over \( \mathbb{Q} \).

20. Let \( k, \ell, n \in \mathbb{N} \) with \( k + \ell = n \), let \( G = SL_n(\mathbb{R}) \), let \( \Gamma \) be a lattice in \( G \), let \( X = G/\Gamma \) and \( m_X \) the \( G \)-invariant probability measure on \( X \). Let \( g_t = \text{diag}(e^{\ell t} I_k, e^{-k t} I_\ell) \), where \( I_m \) is the \( m \times m \) identity matrix. Let \( U \) be the matrices \( (u_{ij})_{1 \leq i, j \leq n} \) satisfying \( u_{ii} = 1 \) for all \( i \) and \( u_{ij} = 0 \) when \( i \neq j \) unless \( i \leq k \) and \( j > k \). Let \( m_U \) be Haar measure on \( U \) and \( \mathcal{O} \) a bounded open subset of \( U \). For \( x_0 \in X \) let \( \nu_0 \) be the pushforward of the normalized restriction \( m_U|_{\mathcal{O}} \), under the map \( u \mapsto ux_0 \). Show that \( g_t \nu_0 \) converges to \( m_X \) in the weak-* topology, as \( t \to \infty \).