Exercise Sheet, Dynamics on homogeneous spaces, Fall 2019

Notations and assumptions. Unless stated otherwise, all measures on a topological space are regular Borel Radon measures. ‘lcsc’ stands for locally compact second countable Hausdorff.

1. Let $G = \text{SL}_2(\mathbb{R})$, $B$ the subgroup of upper triangular matrices. Prove that there is no measure on $G/B$ which is invariant under $G$. Strengthen this by proving that if $\Gamma \subset G$ is an unbounded subgroup (i.e. $\Gamma$ is not compact) and $\Gamma$ does not leave invariant a finite set of points in $G/B$, then there is no measure on $G/B$ which is invariant under $\Gamma$.

2. Let $G$ be a lcsc group and $X$ a lcsc space. Suppose $G$ acts continuously and transitively on $X$ and for $x_0 \in X$, let $H = \{g \in G : gx_0 = x_0\}$ and let $\pi : G/H \to X$, $\pi(gH) = gx_0$ and $\pi' : G \to X$, $\pi'(g) = \pi(gH)$.

(i) Prove that $\pi$, $\pi'$ are well-defined and continuous and that $\pi$ is a homeomorphism.

(ii) Prove that $K \subset X$ is compact if and only if there is a compact $K' \subset G$ such that $\pi'(K') = K$.

(iii) Suppose $L$ is a closed subgroup of $G$ and define the orbit map $\iota : L/L \cap H \to X$, $\iota(\ell(L \cap H)) = \ell x_0$. Prove that $\iota$ is injective, and prove that it is a homeomorphism onto its image if and only if the orbit $Lx_0$ is closed.

(iv) Are (ii) and / or (iii) true for general actions of topological groups, i.e. if one does not assume that $G$ and / or $X$ are lcsc?

3. Let $\Gamma$ be a discrete subgroup of $G = \text{SL}_2(\mathbb{R})$, let $\mathbb{H}^+ \overset{\text{def}}{=} \{z \in \mathbb{C} : \text{Im} z > 0\}$ and let $d(\cdot , \cdot)$ denote the hyperbolic metric on $\mathbb{H}^+$. Suppose $z_0 \in \mathbb{H}^+$ is not stabilized by any element of $\Gamma \setminus \{e\}$, and let $D = \{z \in \mathbb{H}^+ : \forall \gamma \in \Gamma \setminus \{e\}, d(\gamma z, z_0) \geq d(z, z_0)\}$. In items (ii)—(v) below, assume $\Gamma$ is cocompact.

(i) Prove that $D$ is a surjective set for the action of $\Gamma$ on $\mathbb{H}^+$ by Moebius transformations, and the interior of $D$ is an injective set (i.e., $D$ is measurable, contains a representative from each orbit, and no two points in the interior of $D$ are in the same orbit). Also prove that the boundary of $D$ consists of a locally finite collection of geodesic arcs (a geodesic arc is a path which minimizes the hyperbolic distance of any two points on it; local finiteness means that any compact subset of $\mathbb{H}^+$ contains finitely many segments from $\partial D$).

(ii) Deduce that if $\Gamma$ is cocompact then $D$ has finitely many sides and $\Gamma$ is finitely generated.
(iii) Deduce that if \( \alpha \) is any geodesic which intersects the interior of \( \mathbb{D} \), then \( \mathbb{D} \setminus \alpha \) has two connected components.

(iv) Let \( B_T \overset{\text{def}}{=} \{ z \in \mathbb{H}^+ : d(z, z_0) \leq T \} \) and let \( \partial, \text{int} \) denote respectively boundary and interior. For each \( \gamma \in \Gamma \) and each \( x \in \text{int}(\gamma \mathbb{D}) \), let \( \alpha_x \) be the intersection of \( \gamma \mathbb{D} \) with the geodesic through \( x \) perpendicular to the geodesic from \( x \) to \( z_0 \), let \( \beta_x \) be the intersection of \( \partial B_{d(x,z_0)} \) with \( \gamma \mathbb{D} \) and let \( D_x \) be the maximal distance between a point of \( \alpha_x \) and the nearest point of \( \beta_x \). Prove that \( D_x \to 0 \) uniformly as \( d(x, z_0) \to \infty \), that is, 
\[
\sup \{ D_x : x \in \text{int}(\gamma \mathbb{D}) \cap \partial B_T \text{ for some } \gamma \in \Gamma \} \to_{T \to \infty} 0.
\]

(v) Complete the details of the reduction sketched in class for Margulis’ result: let \( m_{G/\Gamma} \) denote the \( G \)-invariant Borel probability measure on \( G/\Gamma \), and suppose that for any \( f \in C(G/\Gamma) \), \( \frac{1}{\pi} \int_0^\pi f(g \tau \theta) d\theta \to_{t \to \infty} \int f \, dm_{G/\Gamma} \). Show that if \( m \) is the hyperbolic area measure on \( \mathbb{H}^+ \) and \( z_0 \in \mathbb{H}^+ \) has a trivial stabilizer under \( \Gamma \), then
\[
|\{ z \in \Gamma z_0 : d(z, z_0) \leq T \}| \sim \frac{m(B_T)}{m(\mathbb{D})}, \text{ as } T \to \infty.
\]

4. Give an example of an infinitely generated discrete subgroup of \( \text{SL}_2(\mathbb{R}) \).

5. Prove that \( G = \text{SL}_n(\mathbb{R}) \) is simple (as a topological group), i.e. if \( H \subset G \) is a closed normal subgroup then \( H \) is either discrete or equal to \( G \).

6. Show that the universal cover of a connected Lie group is a Lie group. Prove that the fundamental group of a connected Lie group is abelian. Show that \( \text{SL}_n(\mathbb{R}) \) is connected. Compute the fundamental group of \( \text{SL}_n(\mathbb{R}) \) for \( n \geq 2 \).

7. An element \( h \) in a group \( G \) is called central if \( hg = gh \) for all \( g \in G \), and a subgroup of \( G \) is called central if all its elements are central. Suppose \( G \) is a connected Lie group and \( H \) is a discrete normal subgroup. Show that \( H \) is central.

8. Show that if \( G \) is an lcsc group acting continuously and transitively on an lcsc space \( X \), and preserving a measure \( \nu \), then this measure is unique, that is if \( \nu_1, \nu_2 \) are two such nonzero measures on \( X \) such that for any Borel set \( A \) and \( g \in G \) we have \( \nu_i(gA) = \nu_i(A) \) \( (i = 1, 2) \) then there is a constant \( c > 0 \) such that \( \nu_2 = c\nu_1 \). Conclude that left and right Haar measures on an lcsc group \( G \) are unique up to scaling.

9. Show that a Lie group is unimodular if one of the following holds: \( G \) is simple (as a topological group); \( G \) is compact; \( G \) is nilpotent. Also
show that if a left Haar measure \( \nu_L \) on \( G \) satisfies \( \nu_L(G) < \infty \), then the same is true for a right Haar measure, and \( G \) is compact.

10. Let \( \mathcal{X}_n \) denote the collection of closed subsets of \( \mathbb{R}^n \), and define the Chabauty-Fell metric on \( \mathcal{X}_n \) as follows: \( d(A_0, A_1) \) is the minimum of 1 and

\[
\inf \left\{ r > 0 : \text{ for } i = 0, 1, \ B \left( 0, \frac{1}{r} \right) \cap A_i \subset \bigcup_{a \in A_{1-i}} B(a, r) \right\},
\]

with the convention \( \inf \emptyset = \infty \). Prove that \( d \) is a metric on \( \mathcal{X}_n \) and that with this metric, \( \mathcal{X}_n \) is compact and complete. Let \( \mathcal{L}_n = \text{SL}_n(\mathbb{R}) / \text{SL}_n(\mathbb{Z}) \), the space of covolume 1 lattices equipped with the quotient topology. Prove that the inclusion map \( \iota : \mathcal{L}_n \to \mathcal{X}_n \) is continuous, and deduce that for any \( i \), the maps \( \mathcal{L}_n \to \mathbb{R}, \Lambda \mapsto \lambda_i(\Lambda) \) and \( \Lambda \mapsto \bar{\lambda}_i \) are continuous, where

\[
\lambda_i(\Lambda) \overset{\text{def}}{=} \inf \{ r > 0 : \Lambda \cap B(0, r) \text{ contains } i \text{ linearly independent vectors} \},
\]

and

\[
\bar{\lambda}_i(\Lambda) \overset{\text{def}}{=} \inf \{ r > 0 : \Lambda \cap B(0, r) \text{ contains a primitive } i\text{-tuple} \}.
\]

Show that any element in the closure of \( \iota(\mathcal{L}_n) \) is a group. Show that the map \( \mathcal{L}_n \to \mathcal{X}_n \) which sends a lattice \( L \) to its Voronoi cell, is continuous.

11. Show that for any \( n \in \mathbb{N} \) there is \( c > 0 \) such that for any \( \Lambda \in \mathcal{L}_n \) there is a basis \( v_1, \ldots, v_n \) for \( \Lambda \) such that for all \( i = 1, \ldots, n \),

\[
\frac{1}{c} \|v_i\| \leq \lambda_i(\Lambda) \leq c\|v_i\|.
\]

12. The covering radius of a lattice \( \Lambda \) is defined as

\[
\text{covrad}(\Lambda) \overset{\text{def}}{=} \min \{ r > 0 : \forall x \in \mathbb{R}^n \exists y \in \Lambda \text{ such that } \|x - y\| \leq r \}.
\]

Show that the min in this definition is indeed attained, and that it is the minimal \( r \) so that the closed \( r \)-ball around 0 contains the Voronoi cell of \( \Lambda \). Show that for a sequence \( (\Lambda_j)_j \subset \mathcal{L}_n \), \( (\Lambda_j)_j \) has no convergent subsequences if and only if \( \text{covrad}(\Lambda_j) \to_{j \to \infty} \infty \).

13. Let \( E \subset \mathbb{R}^n \setminus \{0\} \) be a measurable set of Lebesgue measure \( V < \infty \). Let \( m \) be the \( \text{SL}_n(\mathbb{R}) \)-invariant probability measure on \( \mathcal{L}_n \). Show that \( \int_{\mathcal{L}_n} |\Lambda \cap E|d\mu(\Lambda) = V \). Deduce that there is \( c > 0 \) such that for every \( n \in \mathbb{N} \), there is a lattice \( \Lambda \in \mathcal{L}_n \) such that \( \lambda_1(\Lambda) \geq c\sqrt{n} \).

14. Let \( G \) be an lcsc group acting on an lcsc space \( X \). Suppose \( \mu \) is a Borel measure which is quasi-invariant (i.e. \( G \) maps sets of zero measure to sets of zero measure), ergodic (i.e. for any \( G \)-invariant set
A, either \( \mu(A) = 0 \) or \( \mu(X \setminus A) = 0 \), and with support equal to \( X \).

Show that almost every \( G \)-orbit is dense.

15. Suppose \( G \) is a Lie group and \( \Gamma \) is a lattice in \( G \), and let \( \pi : G \to G/\Gamma \) denote the quotient map. Show that for \( F \subset G \), \( \pi(F) \) is compact if and only if there is a neighborhood \( U \) of \( e \) in \( G \) such that for any \( \gamma \in \Gamma \setminus \{e\} \) and every \( g \in F \), \( g\gamma g^{-1} \notin U \). Show that if \( H \) is a closed subgroup of \( G \) and \( \Gamma_H = H \cap \Gamma \) is a lattice in \( H \), then the map \( H/\Gamma_H \to G/\Gamma \), \( h\Gamma_H \mapsto h\Gamma \), is proper.

16. Suppose \( \Gamma \) is a discrete subgroup of a unimodular Lie group \( G \), and \( N \) is a closed normal subgroup of \( G \). Let \( \pi : G \to G/N \) be the natural projection and suppose \( \pi(\Gamma) \) is a lattice in \( G/N \) and \( \Gamma \cap N \) is cocompact in \( N \). Show that \( \Gamma \) is a lattice in \( G \). Deduce that \( \text{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n \) is a lattice in \( \text{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n \).

17. Let \( G \) be a connected Lie group and let \( g \in G \). Let 
\[
U^+ \overset{\text{def}}{=} \{ h \in G : g^n h g^{-n} \to_{n \to +\infty} e \}, \quad U^- \overset{\text{def}}{=} \{ h \in G : g^n h g^{-n} \to_{n \to -\infty} e \}.
\]
Show that \( U^\pm \) are subgroups of \( G \), and that the closure of the group generated by \( U^+ \) and \( U^- \) is normal in \( G \).

18. Let \( G \) be an lcsc group acting on an lcsc space \( X \) and let \( \mu \) be a measure on \( X \). Show that if a measurable set \( A \) satisfies \( \forall g \in G, \mu(A \Delta gA) = 0 \) then there is a measurable set \( A' \) such that \( gA' = A' \) for all \( g \in G \) and \( \mu(A \Delta A') = 0 \).

19. Let \( G \) be an lcsc group acting on an lcsc \( X \), and let \( \mu \) be a probability measure on \( X \) which is \( G \)-invariant, and suppose there are no other \( G \)-invariant probability measures on \( G \) (if this holds we say that the action of \( G \) on \( X \) is uniquely ergodic). Prove that \( \mu \) is ergodic. Let \( \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \) and let \( \Gamma \) be the abelian group generated by vectors \( v_1, \ldots, v_d \) in \( \mathbb{R}^n \), acting on \( \mathbb{T}^n \) by the rule 
\[
\gamma \pi(x) = \pi(\gamma + x), \quad \text{where} \quad \pi : \mathbb{R}^n \to \mathbb{T}^n \quad \text{is the projection}.
\]
Prove that the \( \Gamma \) action on \( \mathbb{T}^n \) is uniquely ergodic if and only if the group \( \Gamma + \mathbb{Z}^n \) is dense in \( \mathbb{R}^n \). Deduce that if \( \Gamma \) is the cyclic group generated by \( v = (x_1, \ldots, x_n) \) then \( \Gamma \) acts ergodically if and only if \( 1, x_1, \ldots, x_n \) are linearly independent over \( \mathbb{Q} \).

20. Let \( k, \ell, n \in \mathbb{N} \) with \( k + \ell = n \), let \( G = \text{SL}_n(\mathbb{R}) \), let \( \Gamma \) be a lattice in \( G \), let \( X = G/\Gamma \) and \( m_X \) be the \( G \)-invariant probability measure on \( X \). Let \( g_t = \text{diag}(e^{\ell t} I_k, e^{-k t} I_\ell) \), where \( I_m \) is the \( m \times m \) identity matrix. Let \( U \) be the matrices \( (u_{ij})_{1 \leq i, j \leq n} \) satisfying \( u_{ii} = 1 \) for all \( i \) and \( u_{ij} = 0 \) when \( i \neq j \) unless \( i \leq k \) and \( j > k \). Let \( m_U \) be Haar measure on \( U \) and \( \mathcal{O} \) a bounded open subset of \( U \). For \( x_0 \in X \) let \( \nu_0 \) be the pushforward
of the normalized restriction $m_U|_O$, under the map $u \mapsto ux_0$. Show that $g_* \nu_0$ converges to $m_X$ in the weak-* topology, as $t \to \infty$.

21. Let $U, G, g_t$ be as in the preceding question, let $m_U$ be the Haar measure on $U$, let $\Gamma = \text{SL}_n(\mathbb{Z})$, $X = G/\Gamma$, $\pi : G \to X$ the quotient map, and let $m_X$ be the $G$-invariant probability measure on $X$. Show that the orbit $U\pi(e)$ is compact. Show that for $m_U$-a.e. $u$, the orbit $\{g_t \pi(u) : t \geq 0\}$ is equidistributed in $X$, i.e. $\forall f \in C_c(X), \frac{1}{T} \int_0^T f(g_t \pi(u))dt \to_{T \to \infty} \int_X f \, dm_X$.

22. Suppose $(X, \mathcal{B}, \mu, T)$ is an ergodic p.p.s. and show that an estimate on the rate of decay of matrix coefficients gives an effective pointwise ergodic theorem. Namely, show that for any $\alpha > 0$ there is $\delta > 0$ such that for $f \in L^2(\mu)$ with $\int_X f \, d\mu = 0$, if there is $C_1 > 0$ such that

$$ |(f \circ T^n, f) | \leq C_1 n^{-\alpha}, $$

then there is $C_2 = C_2(f)$ such that for $\mu$-a.e. $x \in X$,

$$ \left| \frac{1}{N} \sum_{n=0}^{N-1} f(T^nx) \right| \leq C_2 N^{-\delta}. $$

23. Let $X = \text{SL}_{d+1}(\mathbb{R})/\text{SL}_{d+1}(\mathbb{Z})$ and let $g_t = \text{diag}(e^t, \ldots, e^t, e^{-dt})$. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let $\bar{x} = (x_1, \ldots, x_d, 1) \in \mathbb{R}^{d+1}$, and let $\Lambda_x = \text{span}_\mathbb{Z}(e_1, \ldots, e_d, \bar{x}) \subseteq X$. Say that $x \in \mathbb{R}^d$ is singular if for any $\varepsilon > 0$ there is $Q_0$ such that for all $Q > Q_0$ there are $p \in \mathbb{Z}^d, q \in \mathbb{N}$ such that $q < Q$ and $\|qx - p\| < \varepsilon Q^{-1/d}$. Show that $x$ is singular if and only if $g_t \Lambda_x \to_{t \to \infty} \infty$. Show that for any polynomial map $p = (p_1, \ldots, p_d) : \mathbb{R} \to \mathbb{R}^d$ (i.e. $p_i$ is a polynomial with real coefficients for every $i$), such that the image of $p$ is not contained in a proper affine subspace of $\mathbb{R}^d$, for a.e. $s$, $p(s)$ is not singular.