Exercise Sheet, Dynamics on homogeneous spaces, Fall 2019

Notations and assumptions. Unless stated otherwise, all measures on a topological space are regular Borel Radon measures. 'lcsc' stands for locally compact second countable Hausdorff.

1. Let $G = \mathrm{SL}_2(\mathbb{R})$, B the subgroup of upper triangular matrices. Prove that there is no measure on G/B which is invariant under G. Strengthen this by proving that if $\Gamma \subset G$ is an unbounded subgroup (i.e. $\overline{\Gamma}$ is not compact) and Γ does not leave invariant a finite set of points in G/B, then there is no measure on G/B which is invariant under Γ.

2. Let G be a lcsc group and X a lcsc space. Suppose G acts continuously and transitively on X and for $x_0 \in X$, let $H = \{g \in G : gx_0 = x_0\}$ and let $\pi : G/H \to X$, $\pi(gH) = gx_0$ and $\pi' : G \to X$, $\pi'(g) = \pi(gH)$.

(i) Prove that π, π' are well-defined and continuous and that π is a homeomorphism.

(ii) Prove that $K \subset X$ is compact if and only if there is a compact $K' \subset G$ such that $\pi'(K') = K$.

(iii) Suppose L is a closed subgroup of G and define the *orbit map* $\iota : L/L \cap H \to X$, $\iota(\ell(L \cap H)) = \ell x_0$. Prove that ι is injective, and prove that it is a homeomorphism onto its image if and only if the orbit Lx_0 is closed.

(iv) Are (ii) and / or (iii) true for general actions of topological groups, i.e. if one does not assume that G and / or X are lcsc?

3. Let Γ be a discrete subgroup of $G = \mathrm{SL}_2(\mathbb{R})$, let $\mathbb{H}^+ \stackrel{\mathrm{def}}{=} \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ and let $d(\cdot, \cdot)$ denote the hyperbolic metric on $\mathbb{H}^+$. Suppose $z_0 \in \mathbb{H}^+$ is not stabilized by any element of $\Gamma \smallsetminus \{e\}$, and let $\mathbb{D} = \{z \in \mathbb{H}^+ : \forall \gamma \in \Gamma \smallsetminus \{e\},\ d(\gamma z, z_0) \geq d(z, z_0)\}$. In items (ii)—(v) below, assume Γ is cocompact.

(i) Prove that $\mathbb{D}$ is a surjective set for the action of Γ on $\mathbb{H}^+$ by Möbius transformations, and the interior of $\mathbb{D}$ is an injective set (i.e., $\mathbb{D}$ is measurable, contains a representative from each orbit, and no two points in the interior of $\mathbb{D}$ are in the same orbit). Also prove that the boundary of $\mathbb{D}$ consists of a locally finite collection of geodesic arcs (a geodesic arc is a path which minimizes the hyperbolic distance of any two points on it; local finiteness means that any compact subset of $\mathbb{H}^+$ contains finitely many segments from $\partial\, \mathbb{D}$).

(ii) Deduce that if Γ is cocompact then $\mathbb{D}$ has finitely many sides and Γ is finitely generated.

(iii) Deduce that if α is any geodesic which intersects the interior of $\mathbb{D}$, then $\mathbb{D} \smallsetminus \alpha$ has two connected components.

(iv) Let $B_T \stackrel{\text{def}}{=} \{z \in \mathbb{H}^+ : d(z, z_0) \leq T\}$ and let ∂, int denote respectively boundary and interior. For each $\gamma \in \Gamma$ and each $x \in \text{int}(\gamma\mathbb{D})$, let α_x be the intersection of $\gamma\mathbb{D}$ with the geodesic through x perpendicular to the geodesic from x to z_0, let β_x be the intersection of $\partial B_{d(x,z_0)}$ with $\gamma\mathbb{D}$ and let D_x be the maximal distance between a point of α_x and the nearest point of β_x. Prove that $D_x \to 0$ uniformly as $d(x, z_0) \to \infty$, that is,

$$\sup\{D_x : x \in \text{int}(\gamma\mathbb{D}) \cap \partial B_T \text{ for some } \gamma \in \Gamma\} \to_{T\to\infty} 0.$$

(v) Complete the details of the reduction sketched in class for Margulis' result: let $m_{G/\Gamma}$ denote the G-invariant Borel probability measure on G/Γ, and suppose that for any $f \in C(G/\Gamma)$, $\frac{1}{\pi}\int_0^\pi f(g_t r_\theta)d\theta \to_{t\to\infty} \int f\, dm_{G/\Gamma}$. Show that if m is the hyperbolic area measure on $\mathbb{H}^+$ and $z_0 \in \mathbb{H}^+$ has a trivial stabilizer under Γ, then

$$|\{z \in \Gamma z_0 : d(z, z_0) \leq T\}| \sim \frac{m(B_T)}{m(\mathbb{D})}, \quad \text{as } T \to \infty.$$

4. Give an example of an infinitely generated discrete subgroup of $\text{SL}_2(\mathbb{R})$.

5. Prove that $G = \text{SL}_n(\mathbb{R})$ is *simple (as a topological group)*, i.e. if $H \subset G$ is a closed normal subgroup then H is either discrete or equal to G.

6. Show that the universal cover of a connected Lie group is a Lie group. Prove that the fundamental group of a connected Lie group is abelian. Show that $\text{SL}_n(\mathbb{R})$ is connected. Compute the fundamental group of $\text{SL}_n(\mathbb{R})$ for $n \geq 2$.

7. An element h in a group G is called *central* if $hg = gh$ for all $g \in G$, and a subgroup of G is called central if all its elements are central. Suppose G is a connected Lie group and H is a discrete normal subgroup. Show that H is central.

8. Show that if G is an lcsc group acting continuously and transitively on an lcsc space X, and preserving a measure ν, then this measure is unique, that is if ν_1, ν_2 are two such nonzero measures on X such that for any Borel set A and $g \in G$ we have $\nu_i(gA) = \nu_i(A)$ $(i = 1, 2)$ then there is a constant $c > 0$ such that $\nu_2 = c\nu_1$. Conclude that left and right Haar measures on an lcsc group G are unique up to scaling.

9. Show that a Lie group is unimodular if one of the following holds: G is simple (as a topological group); G is compact; G is nilpotent. Also show that if a left Haar measure ν_L on G satisfies $\nu_L(G) < \infty$, then the same is true for a right Haar measure, and G is compact.

10. Let $\mathscr{X}_n$ denote the collection of closed subsets of $\mathbb{R}^n$, and define the *Chabauty-Fell metric* on $\mathscr{X}_n$ as follows: $d(A_0, A_1)$ is the minimum of 1 and

$$\inf \left\{ r > 0 : \text{ for } i = 0, 1, \ B\left(0, \frac{1}{r}\right) \cap A_i \subset \bigcup_{a \in A_{1-i}} B(a, r) \right\},$$

with the convention $\inf \varnothing = \infty$. Prove that d is a metric on $\mathscr{X}_n$ and that with this metric, $\mathscr{X}_n$ is compact and complete. Let $\mathcal{L}_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, the space of covolume 1 lattices equipped with the quotient topology. Prove that the inclusion map $\iota : \mathcal{L}_n \to \mathscr{X}_n$ is continuous, and deduce that for any i, the maps $\mathcal{L}_n \to \mathbb{R}$, $\Lambda \mapsto \lambda_i(\Lambda)$ and $\Lambda \mapsto \bar{\lambda}_i$ are continuous, where

$$\lambda_i(\Lambda) \stackrel{\text{def}}{=} \inf\{r > 0 : \Lambda \cap B(0, r) \text{ contains } i \text{ linearly indepedent vectors}\},$$

and

$$\bar{\lambda}_i(\Lambda) \stackrel{\text{def}}{=} \inf\{r > 0 : \Lambda \cap B(0, r) \text{ contains a primitive } i\text{-tuple}\}.$$

Show that any element in the closure of $\iota(\mathcal{L}_n)$ is a group. Show that the map $\mathcal{L}_n \to \mathscr{X}_n$ which sends a lattice L to its Voronoi cell, is continuous.

11. Show that for any $n \in \mathbb{N}$ there is $c > 0$ such that for any $\Lambda \in \mathcal{L}_n$ there is a basis $v_1, \ldots, v_n$ for Λ such that for all $i = 1, \ldots, n$,

$$\frac{1}{c} \|v_i\| \le \lambda_i(\Lambda) \le c\|v_i\|.$$

12. The *covering radius* of a lattice Λ is defined as

$$\mathrm{covrad}(\Lambda) \stackrel{\text{def}}{=} \min\{r > 0 : \forall x \in \mathbb{R}^n \, \exists y \in \Lambda \text{ such that } \|x - y\| \le r\}.$$

Show that the min in this definition is indeed attained, and that it is the minimal r so that the closed r-ball around 0 contains the Voronoi cell of Λ. Show that for a sequence $(\Lambda_j)_j \subset \mathcal{L}_n$, $(\Lambda_j)_j$ has no convergent subsequences if and only if $\mathrm{covrad}(\Lambda_j) \to_{j \to \infty} \infty$.

13. Let $E \subset \mathbb{R}^n \smallsetminus \{0\}$ be a measurable set of Lebesgue measure $V < \infty$. Let m be the $\mathrm{SL}_n(\mathbb{R})$-invariant probability measure on $\mathcal{L}_n$. Show that $\int_{\mathcal{L}_n} |\Lambda \cap E| d\mu(\Lambda) = V$. Deduce that there is $c > 0$ such that for every $n \in \mathbb{N}$, there is a lattice $\Lambda \in \mathcal{L}_n$ such that $\lambda_1(\Lambda) \ge c\sqrt{n}$.

14. Let G be an lcsc group acting on an lcsc space X. Suppose μ is a Borel measure which is *quasi-invariant* (i.e. G maps sets of zero measure to sets of zero measure), *ergodic* (i.e. for any G-invariant set

A, either $\mu(A) = 0$ or $\mu(X \smallsetminus A) = 0$), and with support equal to X. Show that almost every G-orbit is dense.

15. Suppose G is a Lie group and Γ is a lattice in G, and let $\pi : G \to G/\Gamma$ denote the quotient map. Show that for $F \subset G$, $\overline{\pi(F)}$ is compact if and only if there is a neighborhood U of e in G such that for any $\gamma \in \Gamma \smallsetminus \{e\}$ and every $g \in F$, $g\gamma g^{-1} \notin U$. Show that if H is a closed subgroup of G and $\Gamma_H = H \cap \Gamma$ is a lattice in H, then the map $H/\Gamma_H \to G/\Gamma$, $h\Gamma_H \mapsto h\Gamma$, is proper.

16. Suppose Γ is a discrete subgroup of a unimodular Lie group G, and N is a closed normal subgroup of G. Let $\pi : G \to G/N$ be the natural projection and suppose $\pi(\Gamma)$ is a lattice in G/N and $\Gamma \cap N$ is cocompact in N. Show that Γ is a lattice in G. Deduce that $\mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ is a lattice in $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$.

17. Let G be a connected Lie group and let $g \in G$. Let

$$U^+ \stackrel{\text{def}}{=} \{h \in G : g^n h g^{-n} \to_{n \to +\infty} e\}, \;\; U^- \stackrel{\text{def}}{=} \{h \in G : g^n h g^{-n} \to_{n \to -\infty} e\}.$$

Show that $U^{\pm}$ are subgroups of G, and that the closure of the group generated by U^+ and U^- is normal in G.

18. Let G be an lcsc group acting on an lcsc space X and let μ be a measure on X. Show that if a measurable set A satisfies $\forall g \in G, \mu(A \triangle gA) = 0$ then there is a measurable set A' such that $gA' = A'$ for all $g \in G$ and $\mu(A \triangle A') = 0$.

19. Let G be an lcsc group acting on an lcsc X, and let μ be a probability measure on X which is G-invariant, and suppose there are no other G-invariant probability measures on G (if this holds we say that the action of G on X is *uniquely ergodic*). Prove that μ is ergodic. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and let Γ be the abelian group generated by vectors $v_1, \ldots, v_d$ in $\mathbb{R}^n$, acting on $\mathbb{T}^n$ by the rule

$$\gamma\pi(x) = \pi(\gamma + x), \text{ where } \pi : \mathbb{R}^n \to \mathbb{T}^n \text{ is the projection.}$$

Prove that the Γ action on $\mathbb{T}^n$ is uniquely ergodic if and only if the group $\Gamma + \mathbb{Z}^n$ is dense in $\mathbb{R}^n$. Deduce that if Γ is the cyclic group generated by $v = (x_1, \ldots, x_n)$ then Γ acts ergodically if and only if $1, x_1, \ldots, x_n$ are linearly independent over $\mathbb{Q}$.

20. Let $k, \ell, n \in \mathbb{N}$ with $k + \ell = n$, let $G = \mathrm{SL}_n(\mathbb{R})$, let Γ be a lattice in G, let $X = G/\Gamma$ and m_X the G-invariant probability measure on X. Let $g_t = \mathrm{diag}(e^{\ell t} I_k, e^{-kt} I_\ell)$, where I_m is the $m \times m$ identity matrix. Let U be the matrices $(u_{ij})_{1 \le i,j \le n}$ satisfying $u_{ii} = 1$ for all i and $u_{ij} = 0$ when $i \neq j$ unless $i \le k$ and $j > k$. Let m_U be Haar measure on U and $\mathcal{O}$ a bounded open subset of U. For $x_0 \in X$ let ν_0 be the pushforward of the normalized restriction $m_U|_{\mathcal{O}}$, under the map $u \mapsto ux_0$. Show that $g_{t*}\nu_0$ converges to m_X in the weak-* topology, as $t \to \infty$.

21. Let U, G, g_t be as in the preceding question, let m_U be the Haar measure on U, let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, $X = G/\Gamma$, $\pi : G \to X$ the quotient map, and let m_X be the G-invariant probability measure on X. Show that the orbit $U\pi(e)$ is compact. Show that for m_U-a.e. u, the orbit $\{g_t\pi(u) : t \geq 0\}$ is equidistributed in X, i.e. $\forall f \in C_c(X)$, $\frac{1}{T}\int_0^T f(g_t\pi(u))dt \to_{T\to\infty} \int_X f\, dm_X$.

22. Suppose $(X, \mathcal{B}, \mu, T)$ is an ergodic p.p.s. and show that an estimate on the rate of decay of matrix coefficients gives an effective pointwise ergodic theorem. Namely, show that for any $\alpha > 0$ there is $\delta > 0$ so that that for $f \in L^2(\mu)$ with $\int_X f d\mu = 0$, if there is $C_1 > 0$ such that

$$|\langle f \circ T^n, f\rangle| \leq C_1 n^{-\alpha},$$

then there is $C_2 = C_2(f)$ such that for μ-a.e. $x \in X$,

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \right| \leq C_2 N^{-\delta}.$$

23. Let $X = \mathrm{SL}_{d+1}(\mathbb{R})/\mathrm{SL}_{d+1}(\mathbb{Z})$ and let $g_t = \mathrm{diag}(e^t, \ldots, e^t, e^{-dt})$. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let $\bar{x} = (x_1, \ldots, x_d, 1) \in \mathbb{R}^{d+1}$, and let $\Lambda_x = \mathrm{span}_{\mathbb{Z}}(\mathbf{e}_1, \ldots, \mathbf{e}_d, \bar{x}) \in X$. Say that $x \in \mathbb{R}^d$ is *singular* if for any $\varepsilon > 0$ there is Q_0 such that for all $Q > Q_0$ there are $p \in \mathbb{Z}^d, q \in \mathbb{N}$ such that $q < Q$ and $\|qx - p\| < \varepsilon Q^{-1/d}$. Show that x is singular if and only if $g_t\Lambda_x \to_{t\to\infty} \infty$. Show that for any polynomial map $p = (p_1, \ldots, p_d) : \mathbb{R} \to \mathbb{R}^d$ (i.e. p_i is a polynomial with real coefficients for every i), such that the image of p is not contained in a proper affine subspace of $\mathbb{R}^d$, for a.e. s, $p(s)$ is not singular.