THE REL LEAF AND REAL-REL RAY OF THE ARNOUX-YOCCOZ SURFACE IN GENUS 3

W. PATRICK HOOPER AND BARAK WEISS

ABSTRACT. We analyze the rel leaf of the Arnoux-Yoccoz translation surface in genus 3. We show that the leaf is dense in the stratum $\mathcal{H}(2,2)^{\mathrm{odd}}$ but that the real-rel trajectory of the surface is divergent. On this real-rel trajectory, the vertical foliation of one surface is invariant under a pseudo-Anosov map (and in particular is uniquely ergodic), but the vertical foliations on all other surfaces are completely periodic.

1. INTRODUCTION

A translation surface is a compact oriented surface equipped with a geometric structure which is Euclidean everywhere except at finitely many singularities. A stratum is a moduli space of translation surfaces of the same genus whose singularities share the same combinatorial characteristics. In recent years, intensive study has been devoted to the study of dynamics of group actions and foliations on strata of translation surfaces. See §2 for precise definitions, and see [MaTa, Zo] for surveys.

Let x be a translation surface with $k > 1$ singularities. There is a local deformation of x obtained by moving its singularities with respect to each other while keeping the holonomies of closed curves on x fixed. This local deformation gives rise to a foliation of the stratum $\mathcal{H}$ containing x, with leaves of real dimension $2(k-1)$. In the literature this foliation has appeared under various names (see [Zo, §9.6], [Sch], [McM3] and references therein), and we refer to it as the *rel foliation*, since nearby surfaces in the same leaf differ only in their relative periods. A sub-foliation of this foliation, which we will refer to as the *real rel foliation*, is obtained by only varying the horizontal holonomies of vectors, keeping all vertical holonomies fixed. Although neither the rel or the real-rel leaves are given by a group action, the obstructions to flowing along the real rel foliation are completely understood (see [MW]) and in the special case $k = 2$ it makes sense to discuss the

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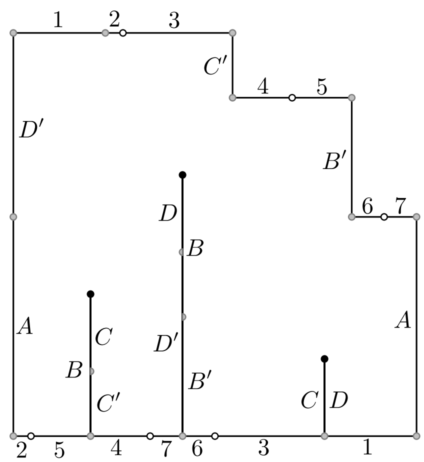

Label	Edge length
1	$1-\alpha$
2	$2\alpha-1$
3	α
4, 5	α^2
6, 7	α^3
A	2α
B	$\alpha+\alpha^3$
B'	$2\alpha^2$
C, D	$\alpha^2+\alpha^4$
C'	$2\alpha^3$
D'	$2\alpha^2+2\alpha^3$

Figure 1. The Arnoux-Yoccoz surface x_0 with distinguished singularities x_0. Edges with the same label are identified by translation, and their lengths are provided by the chart. Black and white points denote the two singularities of the surface, and grey dots denote regular points.

real-rel trajectories $\left\{\mathrm{Rel}_r^{(h)}x : r \in \mathbb{R}\right\}$ of surfaces x without horizontal saddle connections joining distinct singularities.

In genus 2, results of McMullen [McM2, McM4] give a detailed understanding of the closure of rel leaves in the eigenform locus. These results should be viewed as a companion to McMullen's classification of closed sets invariant under the action of $G = \mathrm{SL}_2(\mathbb{R})$ in genus 2 [McM1]. A recent breakthrough result of Eskin, Mirzakhani and Mohammadi [EMiMo] has shed light on the G-invariant closed subsets for arbitrary strata, and the purpose of this paper is to contribute to the study of the topology of closures of rel leaves. Here we focus on the rel leaf of the Arnoux-Yoccoz surface x_0 (see Figure 1), which lies in the connected component $\mathcal{H}(2,2)^{\mathrm{odd}}$ of the genus 3 stratum $\mathcal{H}(2,2)$. This surface, introduced by Arnoux and Yoccoz in [AY], is a source of interesting examples for the theory of translation surfaces. See in particular [HL], and the work of Hubert, Lanneau and Möller [HLM1] in which the orbit-closure $\overline{Gx_0}$ was determined. We denote by $\mathcal{H}$ the sub-locus of surfaces in $\mathcal{H}(2,2)^{\mathrm{odd}}$ whose area is the same as that of x_0.

We now state our results, referring to §2 for detailed definitions.

Theorem 1.1. *Suppose x_0 is as in Figure 1, so that the horizontal and vertical directions are fixed by the pseudo-Anosov map of* [AY].

Then the real rel trajectory of x_0 is divergent in $\mathcal{H}$. Moreover, for any $r \neq 0$, there is a vertical cylinder decomposition of $\mathrm{Rel}_r^{(h)} x_0$, and the circumferences of the vertical cylinders tend uniformly to zero as $r \to \pm\infty$.

The analogous statement replacing real with imaginary and vertical with horizontal also holds. See Appendix A.

It is well-known that some real-rel trajectories exit their stratum in finite time due to the collapse of horizontal saddle connections. In this case the real rel trajectory is not defined for all time. Theorem 1.1 shows that a real rel trajectory may be defined for all time and still be divergent in its stratum.

Theorem 1.1 has implications for the study of unique ergodicity of interval exchange transformations, which we now describe. Let $\mathbb{R}^d_+$ denote the vectors in $\mathbb{R}^d$ with positive entries. For a fixed permutation σ on d symbols, and $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{R}^d_+$, let $\mathcal{IE} = \mathcal{IE}_\sigma(\mathbf{a}) : I \to I$ be the interval exchange transformation obtained by partitioning the interval $I = [0, \sum a_i)$ into subintervals of lengths $a_1, \ldots, a_d$ and permuting them according to σ. Assuming irreducibility of σ, it is known that for almost all choices of $\mathbf{a}$, $\mathcal{IE}_\sigma(\mathbf{a})$ is uniquely ergodic, but nevertheless that the set of non-uniquely ergodic interval exchanges is not very small (see [MaTa] for definitions, and a survey of this intensively studied topic). It is natural to expect that all line segments in $\mathbb{R}^d_+$, other than some obvious counterexamples, should inherit the prevalence of uniquely ergodic interval exchanges. A conjecture in this regard was made by the second-named author in [W, Conj. 2.2], and partial positive results supporting the conjecture were obtained in [MW]. Namely, a special case of [MW, Thm. 6.1] asserts that for any uniquely ergodic $\mathcal{IE}_0 = \mathcal{IE}_\sigma(\mathbf{a}_0)$, there is an explicitly given hyperplane $\mathcal{L} \subset \mathbb{R}^d_+$ containing $\mathbf{a}_0$, such that for any line segment $\ell = \{\mathbf{a}_s : s \in I\} \subset \mathbb{R}^d_+$ with $\ell \not\subset \mathcal{L}$, there is an interval $I_0 \subset I$ containing 0, such that for almost every $s \in I_0$, $\mathcal{IE}_\sigma(s)$ is uniquely ergodic.

Nevertheless our results provide a strong counterexample to [W, Conj. 2.2]. Recall that a standard construction of an interval exchange transformation, is to fix a translation surface q with a segment γ transverse to vertical lines, and define $\mathcal{IE}(q, \gamma)$ to be the first return map to γ along vertical leaves on q (where we parametrize γ using the transverse measure dx). If $L = \{x(r) : r \in I\}$ is a sufficiently small straight line segment in a stratum of translations surfaces, γ can be chosen uniformly for all $r \in I$ and the interval exchanges $\mathcal{IE}(x(r), \gamma)$ can be written as $\mathcal{IE}_\sigma(\mathbf{a}(r))$ for some fixed permuation σ and some line

$\ell = \{\mathbf{a}(r) : r \in I\} \subset \mathbb{R}^d_+$ (see [MW] for more details). Thus, taking $x(r) = \mathrm{Rel}^{(h)}_r x_0$, Theorem 1.1 implies:

Corollary 1.2. *There is a uniquely ergodic self-similar interval exchange transformation $\mathcal{IE}_0$ and a line segment $\ell \subset \mathbb{R}^d_+$ such that $\mathcal{IE}_0 = \mathcal{IE}_\sigma(\mathbf{a}_0)$ with $\mathbf{a}_0$ in the interior of ℓ, and such that for all $\mathbf{a} \in \ell \smallsetminus \{\mathbf{a}_0\}$, $\mathcal{IE}_\sigma(\mathbf{a})$ is periodic.*

In contrast to the real-rel leaf, for the full rel leaf we have:

Theorem 1.3. *The rel leaf of x_0 is dense in $\mathcal{H}$.*

This is the first stratum in which an explicit dense rel leaf has been described. Although we only discuss the rel leaf of the Arnoux-Yoccoz surface in this paper, it is quite likely that the rel foliation is ergodic whenever $k > 1$, and thus almost all rel leaves are dense in all relevant strata. Our work establishes the existence of dense leaves in $\mathcal{H} = \mathcal{H}(2,2)^{\mathrm{odd}}$ but our arguments can be applied in greater generality. We hope to return to the general case in future work. Recently Calsamiglia, Deroin and Francaviglia [CDF] have announced a result which implies ergodicity and also exhibits many explicit dense rel leaves in principal strata, i.e. strata all of whose singularities are simple. Their method is very different from the one used in this paper.

We now briefly comment on the proofs of our results. Since there is a pseudo-Anosov map φ fixing x_0, there is a corresponding nontrivial diagonal element $\widetilde{g}$ with $\widetilde{g}x_0 = x_0$, and applying this element to the stratum defines a map which acts on the real-rel trajectory $\left\{\mathrm{Rel}^{(h)}_r x_0 : r \in \mathbb{R}\right\}$ of x_0 as an expansion moving points away from x_0. The proof of Theorem 1.1 relies on an explicit elementary computation, made in §3, which shows that deforming x_0 along its real-rel leaf introduces periodic cylinders, and these cylinders persist for a full period of the action of $\widetilde{g}$ on $\left\{\mathrm{Rel}^{(h)}_r x_0 : r > 0\right\}$. Our computation does not rely on any prior theory but to put it in context, we note that the surfaces $\mathrm{Rel}^{(h)}_r x_0$ have vanishing SAF invariant in the vertical direction, and the 'drift' for the interval exchange obtained by moving along the vertical direction on $\mathrm{Rel}^{(h)}_r x_0$ is sublinear and can be modeled arithmetically in the ring of integers in a certain cubic field. We refer to [Arn, LPV, McM4] for discussions of this fascinating topic.

In §4 we analyze the interaction of rel and the horocycle flow on surfaces which have a decomposition into parallel cylinders, where both rel and the horocycle flow fix the waist direction of the cylinders. It will be convenient to choose this direction to be vertical. It turns out that when there are no vertical saddle connections joining distinct singularities, both the horocycle and rel flows are given by linear flows on a torus defined by the twist parameters of the cylinders in a vertical cylinder decomposition. This parametrization in terms of twist parameters was used for the horocycle flow in [SmWe1] and is extended here to rel trajectories. In these coordinates, the horocycle flow orbit-closure is determined by the moduli of the cylinders while the rel orbit-closure is determined by their circumferences. This simple observation enables us to pick up additional invariance in the closure of the rel leaf. Namely, we find a family of 3-dimensional tori, $\{\mathcal{O}_r\}$ defined for all but a discrete set of $r > 0$, each of which is the closure of the vertical rel leaf of $\mathrm{Rel}_r^{(h)} x_0$.

Our explicit parameterization enables us to compute the behavior of the tangent planes T_r to $\mathcal{O}_r$, as $r \to 0+$. In §5 we show that the topological limit of the $\mathcal{O}_r$ contains the entire V-orbit of x_0, where $V = \left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} : s \in \mathbb{R} \right\}$ gives the vertical horocycle flow. From this, using [EMiMo], we deduce that the same topological limit contains the G-orbit of x_0, and hence, by [HLM1], also contains the entire hyperelliptic locus $\mathcal{L} \subset \mathcal{H}$. We then exploit the commutation relations between the rel foliation and the G-action, and ergodicity of the G-action, to conclude the proof of Theorem 1.3.

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2. Basics

2.1. **Translation surfaces, strata, G-action, cylinders.** In this section we define our objects of study and review their basic properties. We refer to [MaTa, Zo] for more information on translation surfaces and related notions, and for references for the statements given in this subsection.

Let S be a compact oriented surface of genus $g \geqslant 2$, let $\Sigma = \{\xi_1, \ldots, \xi_k\} \subset S$ and let $\mathbf{r} = (r_1, \ldots, r_k)$ be non-negative integers such that $\sum r_i = 2g-2$. A *translation atlas* of type $\mathbf{r}$ on (S, Σ) is an atlas of charts $(U_\alpha, \varphi_\alpha)$, where:

- For each α, the set $U_\alpha \subset S \smallsetminus \Sigma$ is open, and the map
$$\varphi_\alpha : U_\alpha \to \mathbb{R}^2$$
is continuous and injective.
- Whenever the sets U_α and U_β intersect, the transition functions are local translations, i.e., the maps
$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^2$$
are differentiable with derivative equal to the identity.
- around each $\xi_j \in \Sigma$ The charts glue together to form a cone point with cone angle $2\pi(r_j+1)$.

A *translation surface structure* on (S, Σ) of type $\mathbf{r}$ is an equivalence class of such translation atlases, where $(U_\alpha, \varphi_\alpha)$ and $(U'_\beta, \varphi'_\beta)$ are equivalent if there is an orientation preserving homeomorphism $h : S \to S$, fixing all points of Σ, such that $(U_\alpha, \varphi_\alpha)$ is compatible with $\left(h(U'_\beta), \varphi'_\beta \circ h^{-1}\right)$. A *marked translation surface structure* is an equivalence class of such atlases subject to the finer equivalence relation where $(U_\alpha, \varphi_\alpha)$ and $(U'_\beta, \varphi'_\beta)$ are equivalent if h can be taken to be isotopic to the identity via an isotopy fixing Σ. Thus, a marked translation surface $\mathbf{q}$ determines a translation surface q by *forgetting the marking*, and we write $q = \pi(\mathbf{q})$ to denote this operation. Note that our convention is that all singularities are labeled.

Pulling back dx and dy from the coordinate charts we obtain two well-defined closed 1-forms, which we can integrate along any path γ on S. If γ is a cycle or has endpoints in Σ (a relative cycle), then we define
$$\mathrm{x}(\gamma, \mathbf{q}) = \int_\gamma dx \quad \text{and} \quad \mathrm{y}(\gamma, \mathbf{q}) = \int_\gamma dy.$$
These integrals only depend on the homology class of γ in $H_1(S, \Sigma)$ and the pair of these integrals is the *holonomy* of γ,
$$\mathrm{hol}(\gamma, \mathbf{q}) = \begin{pmatrix} \mathrm{x}(\gamma, \mathbf{q}) \\ \mathrm{y}(\gamma, \mathbf{q}) \end{pmatrix} \in \mathbb{R}^2. \tag{2.1}$$
We let $\mathrm{hol}(\mathbf{q}) = \mathrm{hol}(\cdot, \mathbf{q})$ be the corresponding element of $H^1(S, \Sigma; \mathbb{R}^2)$, with coordinates $\mathrm{x}(\mathbf{q})$ and $\mathrm{y}(\mathbf{q})$ in $H^1(S, \Sigma; \mathbb{R})$. A *saddle connection* for a translation surface q is a straight segment which connects singularities and does not contain singularities in its interior.

The set of all (marked) translation surfaces on (S, Σ) of type $\mathbf{r}$ is called the *stratum of (marked) translation surface of type* $\mathbf{r}$ and is denoted by $\mathcal{H}(\mathbf{r})$ (resp. $\mathcal{H}_{\mathrm{m}}(\mathbf{r})$). The map $\mathrm{hol} : \mathcal{H}_{\mathrm{m}}(\mathbf{r}) \to H^1(S, \Sigma; \mathbb{R}^2)$

just defined gives local charts for $\mathcal{H}_{\mathrm{m}}(\mathbf{r})$, endowing it (resp. $\mathcal{H}(\mathbf{r})$) with the structure of an affine manifold (resp. orbifold).

Let $\mathrm{Mod}(S, \Sigma)$ denote the mapping class group, i.e. the orientation preserving homeomorphisms of S fixing Σ pointwise, up to an isotopy fixing Σ. The map hol is $\mathrm{Mod}(S, \Sigma)$-equivariant. The $\mathrm{Mod}(S, \Sigma)$-action on $\mathcal{H}_{\mathrm{m}}$ is properly discontinuous. Thus $\mathcal{H}(\mathbf{r}) = \mathcal{H}_{\mathrm{m}}(\mathbf{r})/\mathrm{Mod}(S, \Sigma)$ is a linear orbifold and $\pi : \mathcal{H}_{\mathrm{m}}(\mathbf{r}) \to \mathcal{H}(\mathbf{r})$ is an orbifold covering map. We have

$$\dim \mathcal{H}(\mathbf{r}) = \dim \mathcal{H}_{\mathrm{m}}(\mathbf{r}) = \dim H^1(S, \Sigma; \mathbb{R}^2) = 2(2g + k - 1). \quad (2.2)$$

There is an action of $G = \mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}(\mathbf{r})$ and on $\mathcal{H}_{\mathrm{m}}(\mathbf{r})$ by post-composition on each chart in an atlas. The projection $\pi : \mathcal{H}_{\mathrm{m}}(\mathbf{r}) \to \mathcal{H}(\mathbf{r})$ is G-equivariant. The G-action is linear in the homology coordinates, namely, given a marked translation surface structure $\mathbf{q}$ and $\gamma \in H_1(S, \Sigma)$, and given $g \in G$, we have

$$\mathrm{hol}(\gamma, g\mathbf{q}) = g \cdot \mathrm{hol}(\gamma, \mathbf{q}), \quad (2.3)$$

where on the right hand side, g acts on $\mathbb{R}^2$ by matrix multiplication.

We will write

$$u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \qquad g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad v_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

Also we will denote

$$U = \{u_s : s \in \mathbb{R}\}, \quad A = \{g_t : t \in \mathbb{R}\},$$

$$V = \{v_s : s \in \mathbb{R}\}, \quad P = AU = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset G.$$

The connected components of strata $\mathcal{H}(\mathbf{r})$ have been classified by Kontsevich and Zorich. We will be interested in the particular connected component $\mathcal{H}(2,2)^{\mathrm{odd}}$ of $\mathcal{H}(2,2)$, since it is the component containing x_0. For any $\mathbf{r}$, the area of surfaces in $\mathcal{H}(\mathbf{r})$ is preserved by the action of G and we let $\mathcal{H}$ be a fixed-area sub-locus of a connected component of $\mathcal{H}(\mathbf{r})$. The convention usually adopted in the literature is to normalize area by setting $\mathcal{H}$ to be the locus of area-one surfaces, but it will be more convenient for us to fix the area equal to some constant, e.g. the area of x_0. There is a globally supported measure on $\mathcal{H}$ which is defined using Lebesgue measure on $H^1(S, \Sigma; \mathbb{R}^2)$ and a 'cone construction'. It was shown by Masur that the G-action is ergodic with respect to this measure, and in particular, almost every G-orbit is dense.

An *affine automorphism* of a translation surface q is a homeomorphism of q which leaves invariant the set of singular points and which is affine in charts. Some authors require affine automorphisms to preserve orientation but we will allow orientation reversing affine automorphisms. The derivative of an affine automorphism is a 2×2 real matrix of determinant ± 1. If this matrix is hyperbolic (i.e. has distinct real eigenvalues) then the affine automorphism is called a *Pseudo-Anosov map*. If the matrix is parabolic (i.e. is nontrivial and has both eigenvalues equal to 1) then the affine automorphism is called *parabolic*. The group of derivatives of orientation preserving affine automorphisms of q is called the *Veech group* of q.

Let $I \subset \mathbb{R}$ be a closed interval with interior, let $c > 0$ and let $\mathbb{R}/c\mathbb{Z}$ be the circle of circumference c. A *cylinder* on a translation surface is a subset homeomorphic to an annulus which is the image of $I \times \mathbb{R}/c\mathbb{Z}$ for some I and c as above, under a map which is a local Euclidean isometry, and which is maximal in the sense that the local isometry does not extend to $J \times \mathbb{R}/c\mathbb{Z}$ for an interval J properly containing I. The parameter c is called the *circumference* of the cylinder, and the image of $\{t\} \times \mathbb{R}/c\mathbb{Z}$ for some $t \in \mathrm{int}(I)$ is called a *core curve*. In this case the two boundary components of the cylinder are unions of saddle connections whose holonomies are all parallel to that of the core curve. If a translation surface q can be represented as a union of cylinders, which intersect along their boundaries, then the directions of the holonomies of the core curves of the cylinders are all the same, and we say that this direction is *completely periodic* and that q *has a cylinder decomposition* in that direction.

2.2. **Rel and Real Rel.** We describe the foliation rel as a foliation on $\mathcal{H}_{\mathrm{m}}(\mathbf{r})$ which descends to a well-defined foliation on $\mathcal{H}(\mathbf{r})$. We view our cohomology classes as linear maps from the associated homological spaces. Observe there is a restriction map

$$\mathrm{Res} : H^1(S, \Sigma; \mathbb{R}^2) \to H^1(S; \mathbb{R}^2)$$

which is obtained by mapping a cochain $H_1(S, \Sigma; \mathbb{R}) \to \mathbb{R}^2$ to its restriction to the 'absolute periods' $H_1(S; \mathbb{R}) \subset H_1(S, \Sigma; \mathbb{R})$. This restriction map is part of the exact sequence in cohomology,

$$H^0(S; \mathbb{R}^2) \to H^0(\Sigma; \mathbb{R}^2) \to H^1(S, \Sigma; \mathbb{R}^2) \xrightarrow{\mathrm{Res}} H^1(S; \mathbb{R}^2) \to \{0\}, \quad (2.4)$$

and we obtain a natural subspace

$$\mathfrak{R} = \ker \mathrm{Res} \subset H^1(S, \Sigma; \mathbb{R}^2),$$

consisting of the cohomology classes which vanish on $H_1(S; \mathbb{R}) \subset H_1(S, \Sigma; \mathbb{R})$. Since the sequence (2.4) is invariant under homeomorphisms in $\mathrm{Mod}(S, \Sigma)$, the subspace $\mathfrak{R}$ is $\mathrm{Mod}(S, \Sigma)$-invariant. Since hol is equivariant with respect to the action of the group $\mathrm{Mod}(S, \Sigma)$ on $\mathcal{H}_{\mathrm{m}}(\mathbf{r})$ and $H^1(S, \Sigma; \mathbb{R}^2)$, the foliation of $H^1(S, \Sigma; \mathbb{R}^2)$ by cosets of the subspace $\mathfrak{R}$ induces by pullback a foliation of $\mathcal{H}_{\mathrm{m}}(\mathbf{r})$, and descends to a well-defined foliation on $\mathcal{H}(\mathbf{r}) = \mathcal{H}_{\mathrm{m}}(\mathbf{r}) / \operatorname{Mod}(S, \Sigma)$. The area of a translation surface can be computed using the cup product pairing in absolute cohomology and hence the foliation preserves the area of surfaces, and in particular we obtain a foliation of a fixed area sublocus $\mathcal{H}$ (see [BSW] for more details). This foliation is called the *rel* foliation. Two nearby translation surfaces q and q' are in the same plaque if the integrals of the flat structures along all closed curves are the same on q and q'. Intuitively, q' is obtained from q by fixing one singularity as a reference point and moving the other singularity. Recall our convention that singularities are labeled, that is $\operatorname{Mod}(S, \Sigma)$ does not permute the singular points. Using this one can show that $\operatorname{Mod}(S, \Sigma)$ acts trivially on $\mathfrak{R} \cong H^0(\Sigma; \mathbb{R}) / H^0(S, \mathbb{R})$ and hence the leaves of the rel foliation are equipped with a natural translation structure, modeled on $\mathfrak{R}$. The leaves of the rel foliation have (real) dimension $2(k-1)$ (where $k = |\Sigma|$). In this paper we will focus on the case $k = 2$, so that rel leaves are 2-dimensional. We can integrate a cocycle $c \in \mathfrak{R}$ on any path joining distinct singularities and the resulting vector in $\mathbb{R}^2$ will be independent of the path, since any two paths differ by an element of $H_1(S)$. Thus in the case $k = 2$ we obtain an identification of $\mathfrak{R}$ with $\mathbb{R}^2$ by the map $u \mapsto u(\delta)$ for any path joining the singularities. Our convention for this identification will be that we take a path δ oriented from ξ_1 to ξ_2.

The existence of a translation structure on rel leaves implies that any vector $u \in \mathfrak{R}$ determines an everywhere-defined vector field on $\mathcal{H}$. We can apply standard facts about ordinary differential equations to integrate this vector field. This gives rise to paths $\psi(t) = \psi_{q,u}(t)$ such that $\psi(0) = q$ and $\frac{d}{dt}\psi(t) \equiv u$. We will denote the maximal domain of definition of $\psi_{q,u}$ by $I_{q,u}$. When $1 \in I_{q,u}$ we will say that $\mathrm{Rel}^u q$ is defined and write $\psi_{q,u}(1) = \mathrm{Rel}^u q$. Also, in the case $k = 2$ we will write

$$\mathrm{Rel}_r^{(h)} q = \mathrm{Rel}^u q \text{ when } u = (r, 0),$$

and

$$\mathrm{Rel}_s^{(v)} q = \mathrm{Rel}^u q \text{ when } u = (0, s).$$

These trajectories are called respectively the *real-rel* and *imaginary-rel* trajectories. We will use identical notations for $\mathbf{q} \in \mathcal{H}_{\mathrm{m}}(\mathbf{r})$, noting that since $\pi : \mathcal{H}_{\mathrm{m}}(\mathbf{r}) \to \mathcal{H}(\mathbf{r})$ is an orbifold covering map, $I_{\mathbf{q},u} = I_{q,u}$ and $\pi(\mathrm{Rel}^u \mathbf{q}) = \mathrm{Rel}^u q$.

Note that the trajectories need not be defined for all time, i.e. $I_{q,u}$ need not coincide with $\mathbb{R}$. For instance this will happen when a saddle connection on q is made to have length zero, i.e. if ‘singularities collide’.

It was shown in [MW] that this is the only obstruction to completeness of leaves. Namely, in the case $k = 2$, the following holds:

Proposition 2.1. *Let $\mathcal{H}$ be a stratum with two singular points, let $\mathbf{q} \in \mathcal{H}_{\mathrm{m}}$, and let $u \in \mathfrak{R}$. Then the following are equivalent:*

- $\mathrm{Rel}^u \mathbf{q}$ *is defined.*
- *For all saddle connections δ on $\mathbf{q}$, and all $s \in [0,1]$,*

$$\mathrm{hol}(\mathbf{q}, \delta) + s \cdot u(\delta) \neq 0.$$

Corollary 2.2. *If q has two singular points and no horizontal (resp. vertical) saddle connections joining distinct singularities, then $\mathrm{Rel}_r^{(h)} q$ (resp. $\mathrm{Rel}_s^{(v)} q$) is defined for all $r, s \in \mathbb{R}$.*

From standard results about ordinary differential equations we have that the map $(q, u) \mapsto \mathrm{Rel}^u q$ is continuous on its domain of definition, and

$$\mathrm{Rel}_{r_1}^{(h)}(\mathrm{Rel}_{r_2}^{(h)} q) = \mathrm{Rel}_{r_1+r_2}^{(h)}(q), \quad \mathrm{Rel}_{s_1}^{(v)}(\mathrm{Rel}_{s_2}^{(v)} q) = \mathrm{Rel}_{s_1+s_2}^{(v)}(q)$$

(where defined). On the other hand we caution the reader that the rel plane field need not integrate as a group action, i.e. it is easy to find examples for which

$$\mathrm{Rel}_r^{(h)} \left(\mathrm{Rel}_s^{(v)} q \right) \neq \mathrm{Rel}_s^{(v)} \left(\mathrm{Rel}_r^{(h)} q \right).$$

We let G act on the stratum $\mathcal{H}$ in the usual way and also let G act on $\mathbb{R}^2$ by its standard linear action. The action of G is equivariant for the map hol used to define the translation structure on rel leaves, which leads to the following result (see [BSW] for more details):

Proposition 2.3. *Let x be a surface with two singular points and let $u \in \mathfrak{R} \cong \mathbb{R}^2$. If $\mathrm{Rel}^u(x)$ is defined and $g \in G$ then $\mathrm{Rel}^{gu}(gx)$ is defined and $g(\mathrm{Rel}^u(x)) = \mathrm{Rel}^{gu}(gx)$. In particular, if q has no horizontal saddle connections joining distinct singularities, then for all $r, s, t \in \mathbb{R}$,*

$$g_t \mathrm{Rel}_r^{(h)} q = \mathrm{Rel}_{e^t r}^{(h)} g_t q. \tag{2.5}$$

2.3. **The Arnoux-Yoccoz surface and its symmetries.** Let α be the unique real solution to the polynomial equation

$$\alpha + \alpha^2 + \alpha^3 = 1. \tag{2.6}$$

This number α is approximately 0.5437. Its algebraic conjugates are complex and lie outside the unit circle. Hence, its multiplicative inverse, α^{-1}, is a Pisot number, i.e., its algebraic conjugates all lie within the unit circle.

In [AY], Arnoux and Yoccoz introduced the genus three translation surface $x_0 \in \mathcal{H}(2,2)^{\mathrm{odd}}$ which was depicted in Figure 1. The surface is built from a 2×2 square with three slits and a corner cut out as shown. Edge lengths are elements of the ring $\mathbb{Z}(\alpha)$ where α is as in (2.6). Our presentation is the surface of [Arn, pp. 496-498] scaled by a factor of two to remove the presence of fractions from edge lengths. It will be convenient for us to fix this particular scaling in our computations and thus in the remainder of the paper we let $\mathcal{H}$ denote the sublocus of $\mathcal{H}(2,2)^{\mathrm{odd}}$ consisting of surfaces whose area is the same as that of x_0.

Inspecting the figure and using the fact that α is cubic, one finds that the $\mathbb{Z}$-module $\mathrm{hol}(H_1(S,\Sigma;\mathbb{Z}), x_0)$ has rank 6. Arnoux and Yoccoz described a pseudo-Anosov automorphism φ of x_0, whose derivative is

$$\widetilde{g} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}. \tag{2.7}$$

The fact that x_0 admits such a pseudo-Anosov is somewhat challenging to see. We refer the reader to [Arn, p. 498] for an explanation, see also Remark 3.6.

The surface x_0 has two singularities each of cone angle 6π, which we distinguish as a black singularity and a white singularity; see Figure 1. The pseudo-Anosov $\varphi : x_0 \to x_0$ preserves these two singularities.

Aside from the pseudo-Anosov φ, the surface x_0 admits a few other symmetries. The surfaces admits two fixed-point free isometries whose derivatives are given by reflections of the plane in the x- and y-axes [Bow2]. (Technically, these are not affine automorphisms of x_0 under our definition, since they swap the singularities.) The composition of these maps gives an affine automorphism of derivative $-I$, which is the hyperelliptic involution of x_0.

Hubert and Lanneau [HL] showed that x_0 admits no parabolic affine automorphisms. This led to the natural question if the Veech group of x_0 is elementary, i.e., just a finite extension of $\langle \widetilde{g} \rangle$. This question was resolved negatively by Hubert, Laneau and Möller, who proved that there is another pseudo-Anosov automorphism of x_0 which does not commute with φ [HLM1, Theorem 1]. We will not have a use for this extra symmetry.

2.4. **Results of Hubert-Lanneau-Möller.** In this subsection we summarize the results of [HLM1].

A generic surface in $\mathcal{H}(2,2)^{\mathrm{odd}}$ does not have a hyperelliptic involution, but some surfaces do. Let $\mathcal{L} \subset \mathcal{H}$ denote the subset of surfaces with a hyperelliptic involution. We will need the following:

Proposition 2.4. *The subset $\mathcal{L}$ is of (real) codimension 2 in $\mathcal{H}$ and is transverse to rel leaves. In particular*

$$\{\mathrm{Rel}^v(z) : z \in \mathcal{L},\ v \in \mathfrak{R},\ \mathrm{Rel}^v(z) \textit{ is defined}\}$$

contains an open subset of $\mathcal{H}$.

Proof. The fact that $\mathcal{L}$ is of real codimension 2 follows from the explicit dimension computations in [HLM1]. To see transversality, both $\mathcal{L}$ and any rel leaf R intersecting $\mathcal{L}$ are linear in period coordinates, so it suffices to check that they intersect in a discrete set of points. The hyperelliptic involution fixes the two singular points (see [HLM1]) and hence its action fixes each rel leaf R passing through $\mathcal{L}$, acting on R by an affine automorphism A, such that a point on R belongs to $\mathcal{L}$ precisely when it is a fixed point for A. Since $D(A) = -\mathrm{Id}$ the fixed points for A on R are isolated. $\square$

The Arnoux-Yoccoz surface x_0 is contained in $\mathcal{L}$, and we have:

Theorem 2.5 ([HLM1, Theorem 1.3])**.** $\overline{Gx_0} = \mathcal{L}$.

2.5. **Results of Eskin-Mirzakhani-Mohammadi and some consequences.** Recent breakthrough results of Eskin, Mirzakhani and Mohammadi [EMi, EMiMo] give a wealth of information about orbit-closures for the actions of G and P on strata of translation surfaces. The following summarizes the results which we will need in this paper:

Theorem 2.6. *Let x be a translation surface in a stratum $\mathcal{H}$. Then*

$$\overline{Gx} = \overline{Px} = \mathcal{M},$$

where $\mathcal{M}$ is an immersed submanifold of $\mathcal{H}$ of even (real) dimension which is cut out by linear equations with respect to period coordinates, and $\mathcal{M}$ is the support of a finite smooth invariant measure μ. Moreover

$$\frac{1}{T}\int_0^T \int_0^1 (g_t u_s)_* \delta_x \, ds \, dt \to_{T\to\infty} \mu, \tag{2.8}$$

where the convergence is weak- convergence in the space of probability measures on $\mathcal{H}$.*

The following consequence will be crucial for us, and is of independent interest.

Proposition 2.7. *Suppose that x is a translation surface and $\{g_t x : t \in \mathbb{R}\}$ is a periodic trajectory for the geodesic flow. Then $\overline{Ux} = \overline{Vx} = \overline{Gx}$.*

Proof. We prove for U, the proof for V being similar. Suppose $g_{p_0} x = x, p_0 > 0$ where p_0 is the period for the closed geodesic. Let δ_x denote the Dirac measure on x. Let μ be the smooth G-invariant measure for which $\mathcal{M} = \overline{Gx} = \operatorname{supp}\mu$. By (2.8),

$$\begin{aligned}\mu &= \lim_{m\to\infty} \frac{1}{mp_0} \int_0^{mp_0} \int_0^1 (g_t u_s)_* \delta_x \, ds \, dt \\ &= \lim_{m\to\infty} \frac{1}{mp_0} \int_0^{p_0} \sum_{i=0}^{m-1} \int_0^1 (g_{ip_0+p} u_s)_* \delta_x \, ds \, dp.\end{aligned} \tag{2.9}$$

For each $p \in [0, p_0)$ we write

$$\nu_{p,m} = \frac{1}{m} \sum_{i=0}^{m-1} \int_0^1 (g_{ip_0+p} u_s)_* \delta_x \, ds = \frac{1}{m} \sum_{i=0}^{m-1} \frac{1}{e^{2(ip_0+p)}} \int_0^{e^{2(ip_0+p)}} (u_s g_p)_* \delta_x \, ds, \tag{2.10}$$

where we have used the commutation relations $g_\tau u_s = u_{e^{2\tau}s} g_\tau$ and the fact that $g_{ip_0} x = x$ for each i. Then the right hand-side of (2.9) is $\lim_{m\to\infty} \frac{1}{p_0} \int_0^{p_0} \nu_{p,m} dp$. Let $\nu_{0,m}$ be the measure corresponding to $p = 0$, then for any p, $\nu_{p,m} = g_{p*}\nu_{0,m}$. Take a subsequence $\{m_j\}$ along which ν_{0,m_j} converges to a measure ν on $\mathcal{H}$ (where $\nu(\mathcal{H}) \leqslant 1$). Then $\nu_{p,m_j} \to_{j\to\infty} g_{p*}\nu$. The right hand side of (2.10) shows that ν is U-invariant and therefore so is each $g_{p*}\nu$, and by (2.9) we have $\mu = \frac{1}{p_0} \int_0^{p_0} g_{p*}\nu \, dp$ (and in particular $\nu(\mathcal{H}) = 1$). Since μ is G-ergodic, by the Mautner property (see e.g. [EW]) it is U-ergodic. This implies that $g_{p*}\nu = \mu$ for almost every p, and (since μ is $\{g_t\}$-invariant), $\nu = \mu$. These considerations are valid for every convergent subsequence of the sequence $\nu_{0,m}$ and hence $\nu_{0,m} \to_{m\to\infty} \mu$. Since $\nu_{0,m}$ is obtained by averaging over the U-orbit of x, the orbit Ux is dense in $\operatorname{supp}\mu = \overline{Gx}$, i.e. $\overline{Ux} = \overline{Gx}$. □

Combining Theorem 2.5 and Proposition 2.7 we obtain:

Corollary 2.8. *With the above notations, we have $\overline{Vx_0} = \mathcal{L}$.*

3. Periodic vertical directions

In this section, we investigate real rel deformations of the Arnoux-Yoccoz surface x_0. Recall that Theorem 1.1 asserts that these surfaces admit vertical cylinder decompositions. In §3.1, we will state more detailed results about these cylinder decompositions. In particular, Theorem 1.1 follows directly from Lemma 3.4 and Remark 3.5. In §3.2, we will travel through the real rel leaf, and obtain explicit descriptions of $x_r = \mathrm{Rel}_r^{(h)} x_0$ for $1 \leqslant r < \alpha^{-1}$, which constitutes a full period under the action of $\tilde{g}$ on the real rel leaf. Our trip through the rel leaf results in formal proofs of the results stated in §3.1.

3.1. **Results on real rel deformations of the Arnoux-Yoccoz surface.** Recall that x_0 admits a pseudo-Anosov φ with derivative $\widetilde{g}$ as in (2.7). The map φ would have to preserves the finite set of horizontal saddle connection, multiplying their lengths by α^{-1}, and thus x_0 has no horizontal saddle connections. As a consequence of Corollary 2.2 we see that the horizontal rel deformation of x_0,

$$r \mapsto x_r = \mathrm{Rel}_r^{(h)} x_0$$

is defined for all $r \in \mathbb{R}$. Moreover this real-rel trajectory is not periodic, i.e. there is no $r > 0$ such that $x_r = x_0$; indeed if $x_r = x_0$ and $r > 0$ is the minimal number with this property, we can apply (2.5) to obtain $x_0 = x_{\alpha r}$, contradicting the minimality of r.

Let S be a genus three surface with two distinguished points, a black point and a white point, whose union is the set Σ. We use this surface to mark our translation surfaces. We take an arbitrary homeomorphism $S \to x_0$ respecting the colors of distinguished points. This also leads to a marking of horizontal rel deformations of x_0 via homotopy within the bundle $\bigcup_{r\in\mathbb{R}} x_r$ of surfaces over $\mathbb{R}$. In particular, we have identified the topological objects associated with our surfaces.

Each surface x_r determines a natural cohomology class $\mathrm{hol}(x_r) \in H^1(S, \Sigma; \mathbb{R}^2)$ via (2.1). Because of our conventions, $\mathrm{hol}(x_r)$ varies continuously in r. We take our horizontal rel flow to move the white distinguished point rightward relative to the black point. This means that if γ is the homology class of a curve moving from the black point to the white point, then the holonomies with respect to the different structures satisfy

$$\mathrm{hol}(\gamma, x_r) = \mathrm{hol}(\gamma, x_0) + (r, 0).$$

We show later in the section that the surface x_1 has a presentation as shown in Figures 2 and 5. Figure 2 shows some homology classes on the surface which will be important to us. Note that the surface admits a decomposition into three vertical cylinders, and some of the homology classes are clearly related to these cylinders. These homology classes belong to a pair of bi-infinite families of homology classes, $\{\beta_k\}, \{\gamma_k\} \subset H_1(S, \Sigma; \mathbb{Z})$. Several of these classes are shown in the figure, and we extend inductively according to the rules that for all $k \in \mathbb{Z}$, we have:

$$\begin{aligned} \beta_{k+4} &= \gamma_k - \beta_{k+2} - \gamma_{k+2} - 2\gamma_{k+3}, \\ \gamma_{k+4} &= \beta_k - \gamma_{k+2} - \beta_{k+2} - 2\beta_{k+3}. \end{aligned} \tag{3.1}$$

(The classes shown in Figure 2 satisfy these identities.)

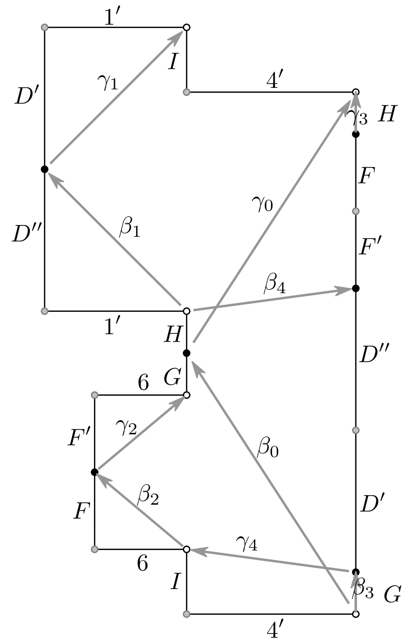


FIGURE 2. The surface $x_1 = \mathrm{Rel}_1^{(h)} x_0$ with some relative homology classes in $H_1(S, \Sigma; \mathbb{Z})$.

Note that $H_1(S, \Sigma; \mathbb{Z})$ is a $\mathbb{Z}$-module isomorphic to $\mathbb{Z}^7$. By inspecting the figure and using induction, one can verify the following:

Proposition 3.1. *For each $k \in \mathbb{Z}$, the collection of homology classes*

$$\{\beta_k, \gamma_k, \beta_{k+1}, \gamma_{k+1}, \beta_{k+2}, \gamma_{k+2}, \beta_{k+3}, \gamma_{k+3}\}$$

generates $H_1(S, \Sigma; \mathbb{Z})$, and are related by the identity

$$\beta_k + \gamma_k = \beta_{k+1} + \gamma_{k+1} + \beta_{k+2} + \gamma_{k+2} + \beta_{k+3} + \gamma_{k+3}.$$

In our trip through the rel-leaf, we will use the following result, whose proof will be given below:

Lemma 3.2. *The holonomies of the homology classes defined above are:*

$$\begin{aligned} \mathrm{hol}(\beta_k, x_r) &= \begin{pmatrix} \alpha^{3-k} - r \\ \alpha^k + \alpha^{k+2} \end{pmatrix}, \\ \mathrm{hol}(\gamma_k, x_r) &= \begin{pmatrix} r - \alpha^{3-k} \\ \alpha^k + \alpha^{k+2} \end{pmatrix}. \end{aligned} \tag{3.2}$$

This immediately gives an explicit relationship between these classes and the pseudo-Anosov $\varphi : x_0 \to x_0$.

Corollary 3.3. *For each k, $\varphi_*(\beta_k) = \beta_{k+1}$ and $\varphi_*(\gamma_k) = \gamma_{k+1}$.*

Proof. Observe that an absolute homology class in $H_1(S;\mathbb{Z})$ is determined by its holonomy on the surface x_0 since both $H_1(S;\mathbb{Z})$ and $\mathrm{hol}(H_1(S;\mathbb{Z}), x_0)$ are $\mathbb{Z}$-modules of rank 6. Also recall that the action of φ on x_0 preserves the two singularities. Fix k and consider the possible images of β_k. Observe that each β_k is representable as a path from the white singularity to the black singularity, so its image is as well. The difference of any two such homology classes is an absolute class. So, homology classes representable by paths from the white singularity to the black singularity are also determined by their holonomy on x_0. So, it suffices to observe that

$$\widetilde{g} \cdot \mathrm{hol}(\beta_k, x_0) = \mathrm{hol}(\beta_{k+1}, x_0).$$

Similar considerations hold for the classes γ_k. □

The following lemma describes all vertical cylinders in the surfaces x_r for $r > 0$.

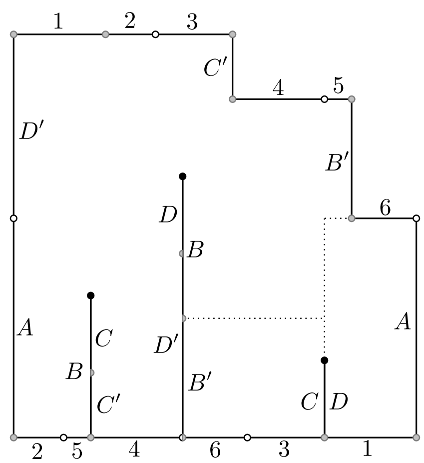
Lemma 3.4. *Let $r > 0$ and let $k \in \mathbb{Z}$ so that $\alpha^{-k} \leqslant r < \alpha^{-(k+1)}$. If $r = \alpha^{-k}$, then x_r admits a decomposition into three vertical cylinders C_k, C_{k+1}, and C_{k+2} whose core curves represent the homology classes $\beta_k+\gamma_k$, $\beta_{k+1}+\gamma_{k+1}$ and $\beta_{k+2}+\gamma_{k+2}$, respectively. If $\alpha^{-k} < r < \alpha^{-(k+1)}$, then x_r admits a decomposition into four vertical cylinders $C_k, \ldots, C_{k+3}$ whose core curves represent the homology classes $\beta_k + \gamma_k, \ldots, \beta_{k+3} + \gamma_{k+3}$, respectively.*

For each $j \in \mathbb{Z}$, and each r satisfying $\alpha^{-(j-3)} < r < \alpha^{-(j+1)}$, the homology class $\beta_j + \gamma_j$ is represented by a cylinder C_j in x_r. The circumference of this cylinder is $2\alpha^j + 2\alpha^{j+2}$, and its variable width is given by the equation

$$\mathrm{Width}(C_j, x_r) = \begin{cases} r - \alpha^{-(j-3)} & \text{for } \alpha^{-(j-3)} < r \leqslant \alpha^{-j}, \\ \alpha^{-(j+1)} - r & \text{for } \alpha^{-j} \leqslant r < \alpha^{-(j+1)}. \end{cases}$$

Remark 3.5. *A nearly identical statement holds for x_r with $r < 0$. Because x_0 admits a hyperelliptic involution preserving the singularities, we have $-Ix_r = x_{-r}$ (see §2.3).*

3.2. **A trip through the real rel leaf.** The goal of this subsection is to find the rel deformations of the surface $x_r = \mathrm{Rel}_r^{(h)} x_0$, with $1 \leqslant r < \alpha^{-1}$. The surface x_0 as presented has the white singularity located only on the top and bottom edges of our cut square. In this case, we can view action of $\mathrm{Rel}_r^{(h)}$ as simply changing the length of edges of the presentation for values of r close to zero. For positive r, we can view it this way until the edge labeled 7 collapses when $r = \alpha^3$. At this point the surface becomes as presented in Figure 3. (Checking that



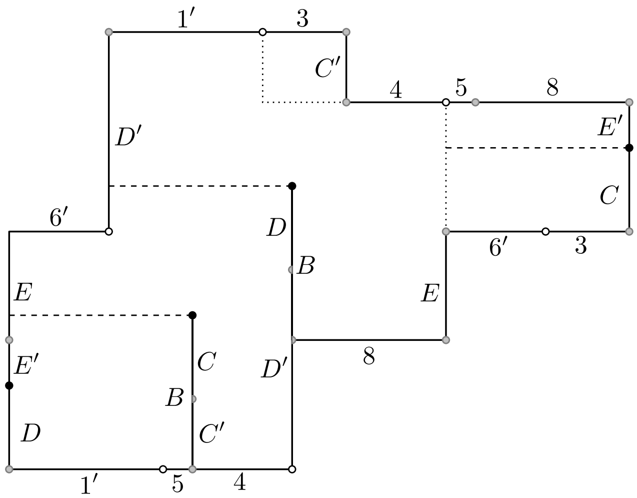
Label	Edge length
1	$1-\alpha$
2	$\alpha-\alpha^2$
3	$\alpha-\alpha^3$
4	$\alpha^2+\alpha^3$
5	$\alpha^2-\alpha^3$
6	$2\alpha^3$
A	2α
B	$\alpha+\alpha^3$
B'	$2\alpha^2$
$C,\ D$	$\alpha^2+\alpha^4$
C'	$2\alpha^3$
D'	$2\alpha^2+2\alpha^3$

FIGURE 3. The deformed Arnoux-Yoccoz surface $x_{\alpha^3} = \mathrm{Rel}^{(h)}_{\alpha^3} x_0$.

edge lengths change as indicated in the figure follows from the relation $\alpha^k = \alpha^{k+1} + \alpha^{k+2} + \alpha^{k+3}$. We omit the straightforward calculations.)

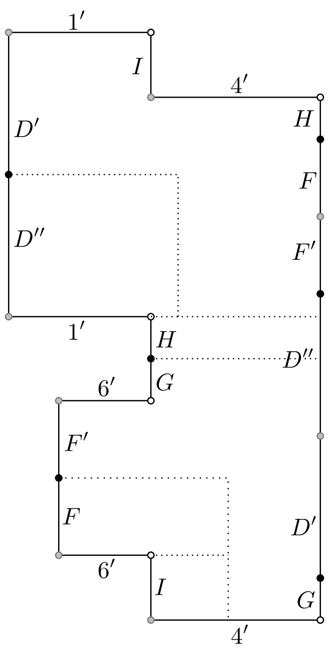
In order to understand further deformations, we will change the presentation of the surface x_{α^3}. This surface depicted in Figure 3 has dotted lines on it. We cut along these dotted lines and reattach the two rectangles along edges A and B'. The new presentation is shown in Figure 4. Our changes have the effect of creating some new edges in the figure (labeled 8, E and E') and enlarges the edge labeled 6 into a new edge we call $6'$. Also, both copies of the edges labeled 1 and 2 are adjacent in the new picture with a regular point between them, and we define $1'$ to be their union.

Figure 4 presents the surface x_{α^3} in a way which allows us to understand an additional rel deformation. We will further horizontally rel deform by $\alpha + \alpha^2$, which will give us a total rel deformation by $\alpha+\alpha^2+\alpha^3 = 1$. We think of sliding the black singularity left by $\alpha+\alpha^2$ rather than sliding the white singularity rightward (but this amounts to the same thing). Note that the black singularity has cone angle 6π, and so has three horizontal separatrices leaving leftward. We slit the surface along these leftward separatrices for length $\alpha+\alpha^2$. These three segments are shown as dashed lines in Figure 4. We identify edges of the slits so that in the position where the black singularity was, no singularities remain. Thus the black point is replaced by three regular points. This determines the gluings of the slits, and when this is done, the leftmost points of the three slits are identified to form a new 6π cone singularity. This is the image of the black singularity under the



Label	Edge length
$1'$	$1-\alpha^2$
3	$\alpha-\alpha^3$
4	$\alpha^2+\alpha^3$
5	$\alpha^2-\alpha^3$
$6'$	$1-\alpha$
8	$\alpha+\alpha^3$
B	$\alpha+\alpha^3$
C, D	$\alpha^2+\alpha^4$
C'	$2\alpha^3$
D'	$2\alpha^2+2\alpha^3$
E	$2\alpha-2\alpha^2$
E'	$\alpha^2-\alpha^4$

FIGURE 4. The surface x_{α^3}, presented differently.



Label	Edge length
$1'$	$1-\alpha^2$
$4'$	$\alpha+\alpha^2$
$6'$	$1-\alpha$
D', D''	$1-\alpha^2$
F, F'	$\alpha-\alpha^3$
G, H	$\alpha^2-\alpha^4$
I	$2\alpha^3$

FIGURE 5. The surface x_1 admits a decomposition into three vertical cylinders. Dotted lines show the images of pieces from Figure 4.

deformation. In order to display the result, we also cut the surface up a bit more (along the dotted lines) and rearrange the pieces to show the result. Figure 5 shows the resulting surface x_1.

We observe that the surface x_1 admits a vertical cylinder decomposition into three cylinders. This is the beginning of the intervals worth of surfaces $\{x_r \ : \ 1 \leqslant r < \alpha^{-1}\}$ which we will analyze more closely.

Let us pause our trip to note that we have essentially proved the Lemma which provides the holonomies of our favorite homology classes.

Proof of Lemma 3.2. Figure 2 shows the homology classes β_k and γ_k for $k \in \{0, \dots, 4\}$. The presentation of x_1 shown in Figure 2 is the same as the one shown in Figure 5. The latter figure gives the dimensions of the shape, and we can read off the holonomies of these classes on the surface x_1 from the figure. We check that they agree with (3.2) when $r = 1$. This works when $k \in \{0, \dots, 4\}$, but we can use the inductive formula (3.1) to extend this and observe it holds for all $k \in \mathbb{Z}$ when $r = 1$. Furthermore because the paths representing the classes β_k move from a white singularity to a black singularity and the paths representing the classes γ_k do the opposite, they satisfy:

$$\mathrm{hol}(\beta_k, x_r) = \mathrm{hol}(\beta_k, x_1) + (1 - r, 0),$$

$$\mathrm{hol}(\gamma_k, x_r) = \mathrm{hol}(\gamma_k, x_1) + (r - 1, 0).$$

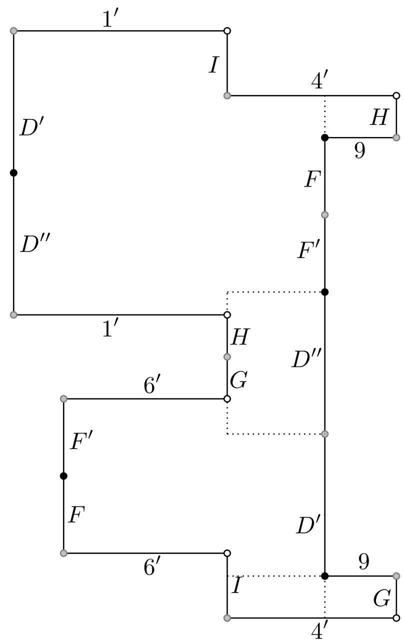
This observation allows us to verify (3.2) for all r. $\square$

We continue the deformation by sliding white singularities to the right relative to the black singularities. As r increases slightly beyond 1, the surface immediately develops a new vertical cylinder, see Figure 6. This figure describes surfaces x_r for $1 < r < \alpha^{-1}$.

Proof of Lemma 3.4. We begin with the first paragraph of the Lemma. Observe that the statement is true when $r = 1$ and when $1 < r < \alpha^{-1}$. From Figure 2, we can see that the upward-oriented core curves of the vertical cylinders represent the homology classes $\beta_0 + \gamma_0$, $\beta_1 + \gamma_1$ and $\beta_2 + \gamma_2$. By rel-deforming this picture into x_r with $1 < r < \alpha^{-1}$, we can see that when r is in this interval, we get a new cylinder whose core curve represents $\beta_3 + \gamma_3$. Now consider the case of general $r > 0$. We have $\alpha^{-k} \leqslant r < \alpha^{-(k+1)}$ for some integer k as stated in the lemma. Recall $\widetilde{g}(x_s) = x_{\alpha^{-1}s}$. Set $s = \alpha^k r$. Then, $1 \leqslant s < \alpha^{-1}$. Since $\widetilde{g}^k(x_s) = x_r$, we see that both x_s and x_r admit a cylinder decompositions into three cylinders if $r = \alpha^{-k}$ and four cylinders otherwise. Recall that the action of $\widetilde{g}$ on homology of these surfaces agrees with the action of φ. Using Corollary 3.3, we see that the homology classes of these cylinders are given by

$$\varphi_*^k(\beta_i + \gamma_i) = \beta_{i+k} + \gamma_{i+k}$$

for $i \in \{0, 1, 2\}$ if $r = \alpha^{-k}$ and for $i \in \{0, 1, 2, 3\}$ otherwise.



Label	Edge length
$1'$	$r - \alpha^2$
$4'$	$\alpha + \alpha^2$
$6'$	$r - \alpha$
9	$r - 1$
D', D''	$1 - \alpha^2$
F, F'	$\alpha - \alpha^3$
G, H	$\alpha^2 - \alpha^4$
I	$2\alpha^3$

FIGURE 6. The surface $x_r = \mathrm{Rel}_r^{(h)} x_0$ with $1 < r < \alpha^{-1}$ admits a decomposition into four vertical cylinders.

As in the lemma, we identify C_j as the cylinder whose core curve has homology class $\beta_j + \gamma_j$. From the prior paragraph, we note that this cylinder appears in x_r when $\alpha^{-k} \leqslant r < \alpha^{-(k+1)}$ and when $j = i + k$ for $i \in \{0, 1, 2, 3\}$ unless $r = \alpha^{-k}$ in which case $i = 3$ is not allowed. Thus, we get a cylinder whose core curve has homology class $\beta_j + \gamma_j$ when $\alpha^{-(j-3)} < r < \alpha^{-(j+1)}$. The second paragraph of the Lemma concerns the geometry of these cylinders. Their circumferences can be determined from the holonomies of the core curves, which can be computed using Lemma 3.2.

To compute the widths of the cylinders, observe that for $1 < r < \alpha^{-1}$ the widths of cylinders C_1, C_2, and C_3 are given by the horizontal holonomies of γ_1, γ_2, and γ_3, respectively. See Figure 6. The width of C_0 on the other other hand is given by the horizontal holonomy of $\gamma_0 - 2\gamma_3$. By applying the action of $\tilde{g}$ to these formulas, and using (2.5), we obtain the widths of all cylinders in the surface x_r for all $r > 0$. □

Remark 3.6. *If we were to continue our trip and compute the geometry of $x_{\alpha^{-1}}$ we would find that $x_{\alpha^{-1}} = \tilde{g} x_1$. This implies that $\tilde{g}$ fixes x_0, because we would have*

$$\tilde{g} x_0 = \tilde{g} \mathrm{Rel}_{-1}^{(h)} x_1 = \mathrm{Rel}_{-\alpha^{-1}}^{(h)} \tilde{g} x_1 = \mathrm{Rel}_{-\alpha^{-1}}^{(h)} x_{\alpha^{-1}} = x_0,$$

where we use (2.5) *in the second equality. This provides an independent proof that* x_0 *admits a pseudo-Anosov self-map with derivative* $\widetilde{g}$.

4. Minimal tori for the horocycle flow and vertical rel

In this section we give basic information about rel leaves of periodic surfaces. We will work with vertical rel as this is what we will need in other parts of the paper. While the results below are elementary, they are of independent interest and we formulate them in greater generality than we need.

4.1. **Twist coordinates.** Let $\mathbf{x}$ be a marked translation surface with a non-empty labeled singular set Σ, and suppose $\mathbf{x}$ is completely periodic in the vertical direction. Then $\mathbf{x}$ admits a decomposition into vertical cylinders, and the surface can be recovered by knowing some related combinatorial data and some geometric parameters. Denote the vertical cylinders by $C_1, \dots, C_m$. A *separatrix diagram* for a vertical cylinder decomposition consists of the ribbon graph of upward-pointing vertical saddle connections forming the union of boundaries of cylinders, whose vertices are given by Σ (with orientation at vertices induced by the translation surface structure on a neighborhood of the singular point), and an indication of which pairs of circles in the diagram bound each cylinder C_i. For more information see [KoZo], where this notion was introduced. We add more combinatorial information by selecting for each vertical cylinder C_i a rightward-oriented saddle connection σ_i joining the left side of C_i to the right side. The union of the separatrix diagram and the saddle connections σ_i still has a ribbon graph structure induced by the surface. The complete *combinatorial data* for our cylinder decomposition consists of a labeling of cylinders, and the ribbon graph which is the union of the separatrix diagram and the saddle connections $\sigma_i,\ i = 1, \dots, m$.

Fixing the number and labels of cylinders and the combinatorial data above, the marked translation surface structure on $\mathbf{x}$ is entirely determined by the following *parameters*: the circumferences $c_1, \dots, c_m \in \mathbb{R}_{>0}$ of the cylinders; the lengths $\ell_1, \dots, \ell_n \in \mathbb{R}_{>0}$ of the vertical saddle connections along the boundaries of the cylinders; and the holonomies

$$\mathrm{hol}(\sigma_i, \mathbf{x}) = (x_i, y_i) \in \mathbb{R}_{>0} \times \mathbb{R} \tag{4.1}$$

of each saddle connection σ_i for $i = 1, \dots, m$. Observe each x_i records the widths of the cylinder C_i.

We call the numbers y_i the *twist parameters*. Observe that we can vary the twist parameters at will. That is, given $\mathbf{x}$ as above, there is a map

$$\widetilde{\Phi} : \mathbb{R}^m \to \mathcal{H}_{\mathrm{m}}; \quad (\hat{y}_1, \ldots, \hat{y}_m) \mapsto \hat{\mathbf{x}}, \tag{4.2}$$

where $\hat{\mathbf{x}}$ is the surface built to have the same parameters as $\mathbf{x}$ except with twist parameters given by $\hat{y}_1, \ldots, \hat{y}_m$, and where $\mathcal{H}_{\mathrm{m}}$ is the space of marked translation surfaces structures modeled on the same surface and singularity set as $\mathbf{x}$. Let $\pi : \mathcal{H}_{\mathrm{m}} \to \mathcal{H}$ be the natural projection. The image of $\pi \circ \widetilde{\Phi}$ is the set of all translation surfaces which have a vertical cylinder decomposition with the same combinatorial data as $x = \pi(\mathbf{x})$, and the same parameters describing cylinder circumferences, lengths of vertical saddle connections, and widths of cylinders. We refer to this set as the *vertical twist space at* x, and denote it by $\mathcal{VT}_x$. We wish to explicitly parameterize this space.

Let $y_1, \ldots, y_m$ denote the twist parameters for $\mathbf{x}$, and choose a second set of twist parameters $\hat{y}_1, \ldots, \hat{y}_m$. Then $\hat{\mathbf{x}}$ can be obtained from $\mathbf{x}$ by slicing each cylinder C_i along a geodesic core curve and regluing so that the right side has moved upward by $\hat{y}_i - y_i$. Thus for each $\gamma \in H_1(S, \Sigma; \mathbb{Z})$,

$$\mathrm{hol}(\gamma, \hat{\mathbf{x}}) = \mathrm{hol}(\gamma, \mathbf{x}) + \sum_{i=1}^{m} \big(0, (\hat{y}_i - y_i)(\gamma \cap C_i)\big) \tag{4.3}$$

where $\cap$ from $\gamma \cap C_i$ denotes the algebraic intersection pairing,

$$\cap : H_1(S, \Sigma; \mathbb{Z}) \times H_1(S \smallsetminus \Sigma; \mathbb{Z}) \to \mathbb{Z},$$

taken between γ and a core curve of the cylinder C_i. Writing $C_i^* \in H^1(S, \Sigma; \mathbb{Z})$ to denote the cohomology class defined by

$$C_i^*(\gamma) = \gamma \cap C_i, \tag{4.4}$$

we see the following:

Proposition 4.1. *The map $\widetilde{\Phi}$ is an affine map whose derivative is*

$$D\widetilde{\Phi}\left(\frac{\partial}{\partial y_i}\right) = (0, C_i^*) \in H^1(S, \Sigma; \mathbb{R}^2), \quad i = 1, \ldots, m. \tag{4.5}$$

Recall that $\widetilde{\Phi}$ defined in (4.2) maps vectors to elements of $\mathcal{H}_{\mathrm{m}}$, i.e. *marked* surfaces. Since holonomies of the saddle connections σ_i distinguish surfaces in the image, $\widetilde{\Phi}$ is injective. However, the map $\pi \circ \widetilde{\Phi} : \mathbb{R}^m \to \mathcal{H}$ is certainly not injective. In the space $\mathcal{H}$ we consider translation surfaces equivalent if they differ by the action of an element of $\mathrm{Mod}(S, \Sigma)$. Let $\mathrm{M}(\mathbf{x}) \subset \mathrm{Mod}(S, \Sigma)$ denote the subgroup consisting of equivalence classes of orientation preserving homeomorphisms of $\mathbf{x}$ such that:

- Each cylinder C_i is mapped to a cylinder C_j of the same circumference and width.
- Each upward-pointing vertical saddle connection is mapped to an upward-pointing vertical saddle connection of the same length, respecting the orientation.

Note that each element of $\mathrm{M}(\mathbf{x})$ preserves the image of $\widetilde{\Phi}$, and that distinct twist parameters yield the same surface in $\mathcal{H}$ if and only if they differ by an element of $\mathrm{M}(\mathbf{x})$. In light of Proposition 4.1, $\mathrm{M}(\mathbf{x})$ pulls back to an affine action on the twist parameters, and we obtain an affine homeomorphism

$$\Phi : \mathbb{R}^m/\mathrm{M}(\mathbf{x}) \to \mathcal{VT}_x \subset \mathcal{H}. \tag{4.6}$$

Some elements of the subgroup $\mathrm{M}(\mathbf{x})$ are clear. Each Dehn twist $\tau_i \in \mathrm{Mod}(S, \Sigma)$ in each vertical cylinder C_i lies in $\mathrm{M}(\mathbf{x})$. The action of τ_i on twist parameters just affects the twist parameter y_i of C_i and has the effect of adding c_i, the circumference of C_i. The *multi-twist subgroup* $\mathrm{M}_0(\mathbf{x}) = \langle \tau_1, \ldots, \tau_m \rangle$ of $\mathrm{M}(\mathbf{x})$ is isomorphic to $\mathbb{Z}^m$. Moreover, $\mathrm{M}(\mathbf{x})$ acts by permutations on the vertical saddle connections, and $\mathrm{M}_0(\mathbf{x})$ is the kernel of this permutation action, and hence is normal and of finite index in $\mathrm{M}(\mathbf{x})$. Thus we have the following short exact sequence of groups

$$\{1\} \to \mathrm{M}_0(\mathbf{x}) \to \mathrm{M}(\mathbf{x}) \to \Delta \to \{1\}, \tag{4.7}$$

where Δ is a subgroup of the group of permutations of the vertical saddle connections. We set $\mathbb{T} = \mathbb{R}^m/\mathrm{M}_0(\mathbf{x}) \cong \prod_{i=1}^m \mathbb{R}/c_i\mathbb{Z}$ (an m-dimensional torus). By normality, the action of $\mathrm{M}(\mathbf{x})$ on $\mathbb{R}^m$ descends to an action on $\mathbb{T}$ which factors through Δ via (4.7). Thus we have the sequence of covers

$$\mathbb{R}^m \to \mathbb{T} \to \mathbb{R}^m/\mathrm{M}(\mathbf{x}) \cong \mathbb{T}/\Delta.$$

We see that $\mathcal{VT}_x$ is isomorphic to the quotient of a torus by an finite group of linear automorphisms. When Δ is trivial, we actually have $\mathbb{R}^m/\mathrm{M}(\mathbf{x}) = \mathbb{T}$. Thus the following holds:

Proposition 4.2. *Suppose that the vertical cylinders of x have distinct circumferences or distinct widths, and that each cylinder has a saddle connection on its boundary whose length is distinct from the lengths of other saddle connections on the boundary. Then $\mathrm{M}(\mathbf{x}) = \mathrm{M}_0(\mathbf{x})$ and therefore $\Phi : \mathbb{T} \to \mathcal{VT}_x$ is an isomorphism of affine manifolds.*

Remark 4.3. *The above discussion equips the quotient $\mathcal{VT}_x \cong \mathbb{T}/\Delta$ with the structure of an affine orbifold, since it is the quotient of $\mathbb{T}$ by the action of a finite group of affine automorphisms Δ. For an* *example in which Δ is nontrivial and $\mathcal{VT}_x$ is not a torus, let x be the* Escher staircase surface *obtained by cyclically gluing $2m$ squares (see e.g.* [LW, Figure 3]*). The surface has m parallel cylinders, the torus $\mathbb{T}$ is isomorphic to $(\mathbb{R}/\mathbb{Z})^m$, and the group Δ contains the group of cyclic permutations of the coordinates, realized by homeomorphisms of the surface which go up and down the staircase.*

Remark 4.4. *The maps* (4.2) *and* (4.6) *were used in* [SmWe1] *in order to analyze the horocycle flow on completely periodic surfaces, but the case in which Δ is nontrivial was overlooked. We take this opportunity to rectify an inaccuracy: in* [SmWe1, Prop. 4, case (2)] *instead of a torus we should have the quotient of the torus by a finite group. This does not affect the validity of other statements in* [SmWe1]*.*

4.2. **Vertical rel flow in twist coordinates.** Now we will specialize to the setting where $\mathbf{x} \in \mathcal{H}_{\mathrm{m}}$ has two singularities, which we distinguish as a black singularity and a white singularity. Figure 7 shows an example. We continue to suppose that the vertical direction is completely periodic, but now we will also assume that $\mathbf{x}$ admits no vertical saddle connections joining distinct singularities. That is, each boundary edge of each vertical cylinder contains only one of the two singularities. We will order the cylinders so that $C_1, \ldots, C_k$ have the white singularity on their left and the black singularity on their right, so that $C_{k+1}, \ldots, C_\ell$ have the black singularity on their left and the white singularity on their right, and $C_{\ell+1}, \ldots, C_m$ have the same singularity on both boundary components.

We observe that the vertical rel flow applied to a surface in the vertical twist space can be viewed as only changing the twist parameters. Concretely, $\mathrm{Rel}_r^{(v)}$ decreases the twist parameters of cylinders $C_1, \ldots, C_k$ by r, increases the twist parameters of $C_{k+1}, \ldots, C_\ell$ by r, and does not change the twist parameters of $C_{\ell+1}, \ldots, C_m$. Therefore we find:

Proposition 4.5. *The vertical twist space $\mathcal{VT}_x$ is invariant under* $\mathrm{Rel}^{(v)}$*. Define*

$$\vec{w} \in \mathbb{R}^m, \quad w_i = \begin{cases} -1 & \text{if } i \leqslant k \\ 1 & \text{if } k < i \leqslant \ell, \\ 0 & \text{if } i > \ell. \end{cases}$$

Then the straightline flow

$$F_{\vec{w}}^r : \vec{y} \mapsto \vec{y} + r\vec{w}$$

on $\mathbb{R}^m$ induces a well-defined straightline flow on $\mathbb{R}^m/\mathrm{M}(\mathbf{x})$ and Φ is a topological conjugacy from this flow to the restriction of $\mathrm{Rel}_r^{(v)}$ to $\mathcal{VT}_x$; that is $\Phi \circ F_{\vec{w}}^r = \mathrm{Rel}_r^{(v)} \circ \Phi$ for every r.

Proof. In the case $\mathrm{M}(\mathbf{x}) = \mathrm{M}_0(\mathbf{x})$, the vertical twist space $\mathcal{VT}_x$ is isomorphic to the torus $\mathbb{T}$ and it is straightforward to check that the effect of applying $\mathrm{Rel}_r^{(v)}$ on the twist coordinates is exactly $y_i \mapsto y_i + rw_i$, giving the required conjugacy. In the general case we need to show that the action of $F_{\vec{w}}^r$ and of the group Δ on $\mathbb{T}$ commute; indeed this will imply both that the action of $F_{\vec{w}}^r$ on $\mathbb{T}/\Delta \cong \mathcal{VT}_x$ is well-defined, and that Φ intertwines the straightline flow $F_{\vec{w}}^r$ on $\mathbb{R}^m/\mathrm{M}(\mathbf{x})$ with the vertical rel flow $\mathrm{Rel}_r^{(v)}$ on $\mathcal{VT}_x$. The definition of the w_i implies that $F_{\vec{w}}^r$ and Δ commute provided the permutation action of $\mathrm{M}(\mathbf{x})$ on the vertical cylinders preserves each of the three collections of cylinders $\{C_1, \ldots, C_k\}, \{C_{k+1}, \ldots, C_\ell\}, \{C_{\ell+1}, \ldots, C_m\}$; this in turn follows from our assumption that singularities are labeled, and the definition of $\mathrm{M}(\mathbf{x})$. □

Given a real vector space V, a $\mathbb{Q}$-*structure* on V is a choice of a $\mathbb{Q}$-linear subspace V_0 such that $V = V_0 \otimes_{\mathbb{Q}} \mathbb{R}$ (i.e. there is a basis of V_0 as a vector space over $\mathbb{Q}$, which is a basis of V as a vector space over $\mathbb{R}$). The elements of V_0 are then called *rational* points of V. If V_1, V_2 are vector spaces with $\mathbb{Q}$-structures, then a linear transformation $T : V_1 \to V_2$ is said to be *defined over* $\mathbb{Q}$ if it maps rational points to rational points. There is a natural $\mathbb{Q}$-structure on $H^1(S, \Sigma; \mathbb{R}^2)$, namely $H^1(S, \Sigma; \mathbb{Q}^2)$. Since the action of $\mathrm{Mod}(S, \Sigma)$ preserves $H_1(S, \Sigma; \mathbb{Z})$ this induces a well-defined $\mathbb{Q}$-structure on $\mathcal{H}$. Moreover since $\mathcal{H}$ is an affine manifold locally modeled on $H^1(S, \Sigma; \mathbb{R}^2)$, the tangent space to $\mathcal{H}$ at any $x \in \mathcal{H}$ inherits a $\mathbb{Q}$-structure. With respect to this $\mathbb{Q}$-structure, we obtain:

Proposition 4.6. *Retaining the notation above, let*

$$\mathcal{O}(x) = \overline{\{\mathrm{Rel}_s^{(v)} x : s \in \mathbb{R}\}} \subset \mathcal{VT}_x. \tag{4.8}$$

Then $\mathcal{O}(x)$ is a d-dimensional affine sub-orbifold of $\mathcal{H}$, where d is the dimension of the $\mathbb{Q}$-vector space

$$\mathrm{span}_{\mathbb{Q}} \left\{ \frac{-1}{c_1}, \cdots, \frac{-1}{c_k}, \frac{1}{c_{k+1}}, \ldots, \frac{1}{c_\ell} \right\} \subset \mathbb{R}. \tag{4.9}$$

Moreover, for every x, the tangent space to $\mathcal{O}(x)$ is a $\mathbb{Q}$-subspace of $H^1(S, \Sigma; \mathbb{R}^2)$.

Proof. We will work with the standard m-torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, and define

$$\Psi : \mathbb{T}^m \to \mathcal{VT}_x; \quad (t_1, \dots, t_m) \mapsto \Phi(c_1 t_1, \dots, c_m t_m). \tag{4.10}$$

Then the conjugacy from Proposition 4.5 leads to a semi-conjugacy from the straight-line flow on $\mathbb{T}^m$ in direction

$$\vec{v} = \left(\frac{-1}{c_1}, \cdots, \frac{-1}{c_k}, \frac{1}{c_{k+1}}, \cdots, \frac{1}{c_\ell}, 0, \dots, 0 \right).$$

The orbit closure of the origin of this straight-line flow is a rational subtorus, and the tangent space to the origin is the smallest real subspace V of $\mathbb{R}^m$ defined over $\mathbb{Q}$ and containing the vector $\vec{v}$. Every rational relation among the coordinates of $\vec{v}$ gives a linear equation with $\mathbb{Q}$-coefficients satisfied by $\vec{v}$, and vice versa; this implies that the dimension of V is the same as the rational dimension of (4.9).

Similarly to (4.10), let $\widetilde{\Psi} : \mathbb{R}^m \to \mathcal{H}_{\mathrm{m}}$ be defined by $\widetilde{\Psi}(t_1, \dots, t_m) = \widetilde{\Phi}(c_1 t_1, \dots, c_m t_m)$. Then $\widetilde{\Psi}$ intertwines the action of $\mathbb{Z}^m$ on $\mathbb{R}^m$ by translations, with the action of $\mathrm{M}_0(\mathbf{x})$ on the image of $\widetilde{\Psi}$, and we have $\mathcal{O}(x) = \pi(\widetilde{\Psi}(V))$. Let $\vec{v}_1, \dots, \vec{v}_d$ be a basis of V contained in $\mathbb{Z}^m$, and let $\Gamma \subset \mathbb{Z}^m$ be the sub-lattice $\langle \vec{v}_1, \dots, \vec{v}_d \rangle$. The subspace V is fixed by the action $\Gamma \subset \mathbb{Z}^m$ by translations, and $\widetilde{\Psi}$ intertwines this translation action with a translation action of a subgroup of $\mathrm{M}_0(\mathbf{x})$. Thus in order to prove that the tangent space to $\mathcal{O}(x)$ is defined over $\mathbb{Q}$, it is enough to prove that any element of $\mathrm{M}_0(\mathbf{x})$ acts by translation by a rational vector.

As we have seen (see (4.3)), the action of each τ_i on $H^1(S, \Sigma; \mathbb{R}^2)$ is induced by its action on $H_1(S, \Sigma; \mathbb{Z})$ via

$$\gamma \mapsto \gamma + (\gamma \cap C_i) C_i,$$

i.e. by translating by a vector in $H_1(S, \Sigma; \mathbb{Z})$. Now the assertion follows from the definition of the $\mathbb{Q}$-structure on $H^1(S, \Sigma; \mathbb{R}^2)$. □

4.3. **Vertical rel flow on deformations of the Arnoux-Yoccoz surface.** We now specialize further to $x_r = \mathrm{Rel}_r^{(h)} x_0$. Throughout this section we assume that $r > 0$ and r not an integral power of α. Then x_r admits a decomposition into four cylinders by Lemma 3.4. For such r, define $\mathcal{O}_r = \mathcal{O}(x_r)$ via (4.8), and denote the tangent space to $\mathcal{O}_r$ at x_r by T_r. Then we have:

Lemma 4.7. *With the notation above, T_r is a 3-dimensional $\mathbb{Q}$-subspace of $H^1(S, \Sigma; \mathbb{R}^2)$, and $\mathcal{O}_r$ is a three dimensional affine torus.*

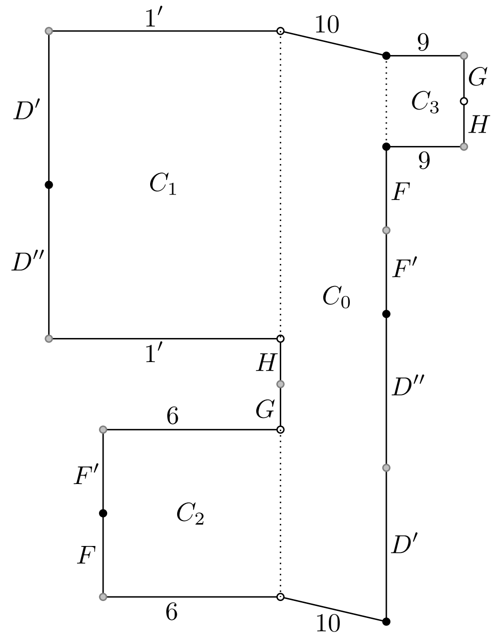


FIGURE 7. Cylinders on the surface x_r with $1 < r < \alpha^{-1}$.

Proof. Let $k \in \mathbb{Z}$ such that $\alpha^{-k} < r < \alpha^{-(k+1)}$. Lemma 3.4 tells us that x_r has a four cylinder decomposition with cylinders named C_k, C_{k+1}, C_{k+2} and C_{k+3}, and by Lemma 3.2, their circumferences are

$$c_i = 2\alpha^i(1 + \alpha^2). \tag{4.11}$$

Following Proposition 4.6 we consider the $\mathbb{Q}$-vector space

$$S = \operatorname{span}_{\mathbb{Q}} \left\{ \frac{-1}{c_k}, \frac{1}{c_{k+1}}, \frac{1}{c_{k+2}}, \frac{1}{c_{k+3}} \right\} \subset \mathbb{R}.$$

(The signs are irrelevant for our purposes, but can be determined by noting that there is a $k \in \mathbb{Z}$ so that $\widetilde{g}^{-k}x_r$ is one of the surfaces represented by Figure 7.) Scaling by c_{k+3} we have $c_{k+3}S = \operatorname{span}_{\mathbb{Q}}(1, \alpha, \alpha^2, \alpha^3) \cong \mathbb{Q}(\alpha)$. Since α is cubic, $\dim_{\mathbb{Q}} S = \dim_{\mathbb{Q}} \mathbb{Q}(\alpha) = 3$.

This proves the first assertion. For the second one, we note by inspecting Figure 6 that x_r satisfies the conditions of Proposition 4.2 when $1 < r < \alpha^{-1}$. We can extend to all r as above by an appropriate action of a power of $\widetilde{g}$. This means that the map Φ given in (4.6) is injective, and the image is a 4-dimensional rational torus in $\mathcal{H}(2,2)$. By Proposition 4.6, $\mathcal{O}_r$ is an affine 3-dimensional sub-torus. $\square$

Define the *vertical twist cohomology subspace* P to be the real subspace of $H^1(S, \Sigma; \mathbb{R}^2)$ spanned by the classes of the form $(0, C^*)$, where C varies over the cylinders of x_r. As we have seen, P is the tangent space to $\mathcal{VT}_r$. In light of Proposition 3.1 and Lemma 3.4, P is independent of r, and by Proposition 4.6, contains the subspace T_r for all r as above.

The subspace P is related to horizontal holonomy in the surface x_r, as follows. If C_i are the vertical cylinders on x_r, ξ_i are their widths (that is ξ_i are the x_i of (4.1)), and $C_i^* \in H^1(S, \Sigma; \mathbb{Z})$ are their dual classes as in (4.4), then for each $\gamma \in H_1(S, \Sigma; \mathbb{Q})$,

$$\mathrm{hol}_{\mathrm{x}}(\gamma, x_r) = \sum_i \xi_i C_i^*(\gamma). \tag{4.12}$$

Therefore we see:

Corollary 4.8. *The action of φ^* on $H^1(S, \Sigma; \mathbb{R}^2)$ preserves the subspace P. The cohomology class $\big(0, \mathrm{hol}_{\mathrm{x}}(x_0)\big) \in H^1(S, \Sigma; \mathbb{R}^2)$ lies in P and is a dominant eigenvector for the action of φ^* on P. The corresponding eigenvalue is α^{-1}.*

Proof. Since $\tilde{g}x_r = x_{\alpha^{-1}r}$, φ maps the cohomology classes represented by core curves of vertical cylinders on x_r, to cohomology classes represented by core curves of vertical cylinders on $x_{\alpha^{-1}r}$. Since P is independent of r, we find that φ^* preserves P. By equation (4.12) and from the definition of P, we see that $\big(0, \mathrm{hol}_{\mathrm{x}}(x_r)\big)$ lies in P for all r. By projecting onto the second summand in $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ we can identify P with a subspace of $H^1(S, \Sigma; \mathbb{R})$, and we continue to denote this subspace by P, then we have shown that $\mathrm{hol}_{\mathrm{x}}(x_r) \in P$ for all r as above, so letting $r \to 0$ we find that $\mathrm{hol}_{\mathrm{x}}(x_0) \in P$. Since φ is a pseudo-Anosov on x_0 with derivative $\tilde{g}$, we know that

$$\varphi^* \mathrm{hol}_{\mathrm{x}}(x_0) = \alpha^{-1} \mathrm{hol}_{\mathrm{x}}(x_0).$$

Moreover it is known [R] that the action of a pseudo-Anosov map on $H^1(S, \Sigma; \mathbb{R})$ has a unique dominant eigenvector whose eigenvalue is the expansion coefficient of the pseudo-Anosov, in this case α^{-1}. Thus $\mathrm{hol}_{\mathrm{x}}(x_0)$ is an dominant eigenvector for the action of φ^* on $H^1(S, \Sigma; \mathbb{R})$ and hence also for the action on P. $\square$

Let $P_1 = \mathrm{span}\{\mathrm{hol}_{\mathrm{x}}(x_0)\} \subset P$. Since P_1 is generated by a dominant eigenvector, there is a φ^*-invariant complementary subspace P_2. We have:

Corollary 4.9. *For any r as above, the subspace $T_r \subset P$ is not contained in P_2.*

Proof. Suppose by way of contradiction that $T_r \subset P_2$. By counting dimensions, we obtain that $T_r = P_2$ and in particular that $\varphi^*(T_2) = T_2$.

Consider the φ^*-equivariant projection $\mathrm{Res} : H^1(S, \Sigma; \mathbb{R}) \to H^1(S; \mathbb{R})$ as in (2.4) (where we take coefficients in $\mathbb{R}$), and set

$$\bar{T}_r = \mathrm{Res}(T_r) \quad \text{and } \bar{P} = \mathrm{Res}(P).$$

By definition of the respective $\mathbb{Q}$-structures, Res is defined over $\mathbb{Q}$, so by Proposition 4.6, $\bar{T}_r$ is a $\mathbb{Q}$-subspace of $\bar{P}$, which is invariant under φ^*. Since T_r contains the tangent direction to $\mathrm{Rel}_r^{(v)}$, which is contained in the kernel of Res, we find that $\dim \bar{T}_r = 2$. Thus we have found a two-dimensional φ^*-invariant $\mathbb{Q}$-subspace of $\bar{P}$. But one of the eigenvalues of φ^* on $\bar{P}$ is the cubic number α^{-1}. This is a contradiction. □

Theorem 4.10. *For any r as above, let $r_n = \alpha^n r \to 0$. Then*

$$Vx_0 \subset \overline{\bigcup_n \mathcal{O}_{r_n}}.$$

Proof. In the affine orbifold structure on $\mathcal{H}$, the orbit Vx is a line, and the sets $\mathcal{O}_{r_n}$ are linear submanifolds. Since $x_{r_n} \to x_0$ it suffices to show that the set of accumulation points of the tangent space T_{r_n} contains the tangent direction to V. By definition of the V-action, the derivative of the vertical horocycle flow $\frac{d}{ds}[v_s x_0]$ (as an element of the tangent space to $\mathcal{H}$ at x_0, identified with $H^1(S, \Sigma; \mathbb{R}^2)$) is precisely $\big(0, \mathrm{hol}_{\mathrm{x}}(x_0)\big)$. By Corollary 4.8, $\big(0, \mathrm{hol}_{\mathrm{x}}(x_0)\big)$ is the dominant eigenvector for the action of φ^* on P. So it is enough to show that T_r contains a vector which projects non-trivially onto P_1, with respect to the decomposition $P = P_1 \oplus P_2$. But this is immediate from Corollary 4.9. □

5. The closure of a leaf

Proof of Theorem 1.3. Let $\mathcal{H}$ denote the set of surfaces in $\mathcal{H}^{\mathrm{odd}}(2,2)$ with the same area as x_0, and let $\Omega \subset \mathcal{H}$ denote the closure of the rel leaf of x_0. For any $r > 0$ and r not an integral power of α, Ω contains $\mathcal{O}_r = \mathcal{O}(x_r)$ (as in(4.8)). Hence by Theorem 4.10, Ω contains the orbit Vx_0. Now using Corollary 2.8, Ω contains the hyperelliptic locus $\mathcal{L}$.

Since Ω is saturated with respect to the Rel foliation, for any $z \in \mathcal{L}$ and any $u \in \mathfrak{R}$ for which $\mathrm{Rel}^u(z)$ is defined, Ω contains $\mathrm{Rel}^u(z)$. Given $z \in \mathcal{L}$ and $g \in G$, $gz \in \mathcal{L}$ since $\mathcal{L}$ is G-invariant. Moreover, if $\mathrm{Rel}^u(z)$ is defined, by Proposition 2.3, $\mathrm{Rel}^{gu}(gz) = g\mathrm{Rel}^u(z)$ is also defined and contained in Ω. The set

$$\{\mathrm{Rel}^u(z) : z \in \mathcal{L},\ u \in \mathfrak{R}, \mathrm{Rel}^u(z) \text{ is defined}\}$$

contains an open subset $\mathcal{U}$ of $\mathcal{H}(2,2)$ by Proposition 2.4. Since $\mathcal{U}$ has positive measure, with respect to the natural flat measure on $\mathcal{H}$, we can apply ergodicity of the G-action, to find that $\mathcal{U}$ contains a point z_0 for which $\overline{Gz_0} = \mathcal{H}$. Since $\overline{Gz_0} \subset \Omega$ we find that $\Omega = \mathcal{H}$. □

Appendix A. The vertical rel trajectory

In this appendix, we show that vertical rel deformations of x_0 are completely periodic in the horizontal direction. In fact we will prove:

Theorem A.1. *If $r \neq 0$, then $\mathrm{Rel}_r^{(v)} x_0$ has a horizontal cylinder decomposition. This cylinder decomposition has 3 cylinders if and only if $\frac{r}{1+\alpha^2}$ is an integral power of α. If $\frac{r}{1+\alpha^2}$ is not an integral power of α, then the cylinder decomposition has 4 cylinders.*

Proof. Because of the action of $\widetilde{g}$ and the hyperelliptic involution of x_0 on the vertical rel leaf, it suffices to prove the theorem for a fundamental domain for the action of $\widetilde{g}$ on $r > 0$. We will analyze the fundamental domain

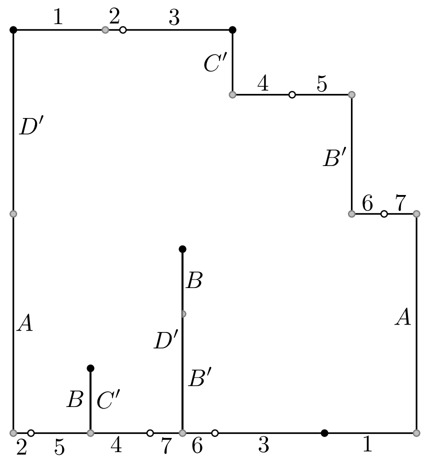
$$\{r \in \mathbb{R} : \ 1 + \alpha^2 \leqslant r < \alpha^{-1}(1 + \alpha^2) = 1 + 2\alpha + \alpha^2\}.$$

So, our first task is to rel flow the Arnoux-Yoccoz surface until we obtain $\mathrm{Rel}^{(v)}_{1+\alpha^2} x_0$.

Throughout this section, we make use of the following idea. The action of $\mathrm{Rel}_r^{(v)}$ is to move white singularities upward by r (relative to the black singularity), or equivalently to move the black singularity downward by r. We will take the latter point of view. Suppose y is in the stratum $\mathcal{H}(2,2)$ like x_0. There are three downward directions leaving the black singularity of y. To form $\mathrm{Rel}_r^{(v)} y$, we slit the surface along downward segments leaving the black singularity of length r. We will define a new translation surface by using the same polygonal presentation, keeping all gluing maps the same, except for the gluing along the slits. Along the slits we reglue so that the former location of the singularity is replaced by three regular points. This determines the gluings of the slits, and the bottom endpoints of the slit are forced to be identified into a 6π cone singularity. It is not hard to see by examining the effect on holonomies of paths, that this surface is obtained by following the imaginary rel vector field, and thus is equal to $\mathrm{Rel}_r^{(v)} y$.

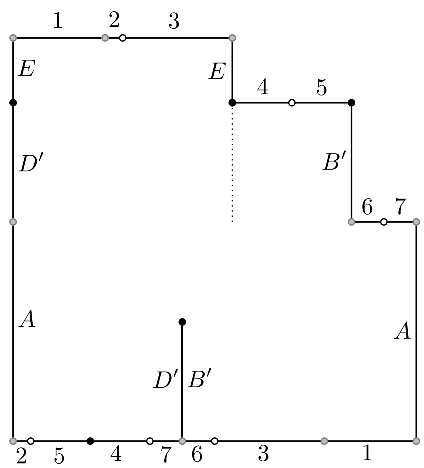
We apply this idea to the Arnoux-Yoccoz surface x_0 depicted in Figure 1 with $r = \alpha^2 + \alpha^4$. We slit open the edges labeled C and D, and the top subinterval of B of length $\alpha^2 + \alpha^4$. Regluing as described above has the effect of collapsing the boundary edges labeled C and D in the figure, and shortens the edge labeled B by $\alpha^2 + \alpha^4$. The resulting surface, $\mathrm{Rel}^{(v)}_{\alpha^2+\alpha^4} x_0$, is depicted in Figure 8.

Now we will vertically rel flow by an addition $2\alpha^3$, which is the common length of the edges labeled B and C' in Figure 8. We cut downward slits of lengths $2\alpha^3$, and re-identify as above. This has the effect of collapsing edges B and C', and also shortens the edge labeled



Label	Edge length
1	$1-\alpha$
2	$2\alpha-1$
3	α
4, 5	α^2
6, 7	α^3
A	2α
B, C'	$2\alpha^3$
B'	$2\alpha^2$
D'	$2\alpha^2+2\alpha^3$

FIGURE 8. The surface $\mathrm{Rel}^{(v)}_{\alpha^2+\alpha^4}x_0$.

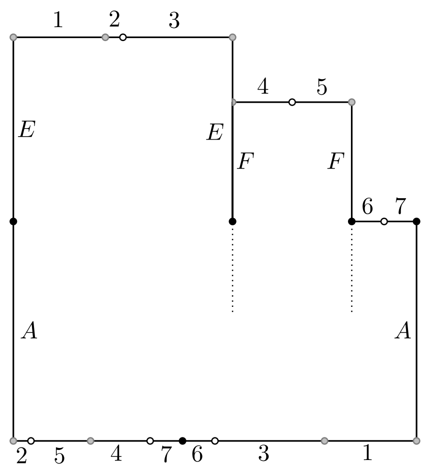


Label	Edge length
1	$1-\alpha$
2	$2\alpha-1$
3	α
4, 5	α^2
6, 7	α^3
A	2α
B', D'	$2\alpha^2$
E	$2\alpha^3$

FIGURE 9. The surface $\mathrm{Rel}^{(v)}_{\alpha+\alpha^3}x_0$.

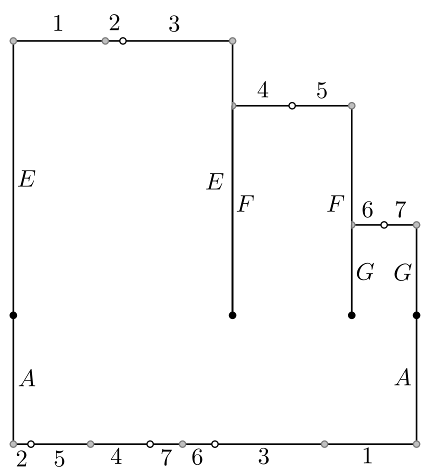
D by $2\alpha^3$. In addition, a new vertical edge (to be labeled E) is formed from the identified pieces of D' and C' in the top part of the picture. The resulting surface, $\mathrm{Rel}^{(v)}_{\alpha+\alpha^3}x_0$, is shown in Figure 9.

Next, we flow by an additional $2\alpha^2$, the length of B' and D' in Figure 9. The three slits cut in this step consist of the edges labeled B' and D', and an additional segment drawn as a dotted line in Figure 9. The edges B' and D' collapse, the edges labeled E grow by $2\alpha^2$, and there is a new boundary identification F formed by identifying the right half of the dotted line with the left side of B'. The resulting surface, $\mathrm{Rel}^{(v)}_{1+\alpha^2}x_0$, is depicted in Figure 10. This is the first surface in our



Label	Edge length
1	$1-\alpha$
2	$2\alpha-1$
3	α
4, 5	α^2
6, 7	α^3
A	2α
E	$2\alpha^2+2\alpha^3$
F	$2\alpha^2$

FIGURE 10. The surface $\mathrm{Rel}^{(v)}_{1+\alpha^2}x_0$.



Label	Edge length
1	$1-\alpha$
2	$2\alpha-1$
3	α
4, 5	α^2
6, 7	α^3
A	$2\alpha-t$
E	$2\alpha^2+2\alpha^3+t$
F	$2\alpha^2+t$
G	t

FIGURE 11. The surface $\mathrm{Rel}^{(v)}_{1+\alpha^2+t}x_0$ for $0<t<2\alpha$.

fundamental domain, and we see that the surface does indeed have a horizontal cylinder decomposition with three cylinders.

We parameterize the remainder of our interval, $[1+\alpha^2, 1+2\alpha+\alpha^2)$, as $1+\alpha^2+t$ with $0<t<2\alpha$. In order to find $\mathrm{Rel}^{(v)}_{1+\alpha^2+t}x_0$, we slit the surface of Figure 10 by vertical segments of length t. After re-identifying the slits as described above, we see that the edges labeled E and F have grown by t, edges labeled A have shrunk by t, and a new horizontal cylinder is formed of height t passing through new edges labeled G in Figure 11. By inspection, we observe that $\mathrm{Rel}^{(v)}_{1+\alpha^2+t}x_0$ has a horizontal cylinder decomposition consisting of four cylinders, as claimed in the theorem above. The combinatorics of the picture is the same for $0 < t < 2\alpha$. As noted in the first paragraph, this completes the proof of the theorem. □

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City College of New York, 160 Convent Ave, New York, NY, USA 10463 whooper@ccny.cuny.edu

Tel Aviv University barakw@post.tau.ac.il