EFFECTIVE COUNTING FOR DISCRETE LATTICE ORBITS IN THE PLANE VIA EISENSTEIN SERIES

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Abstract. In 1989 Veech showed that for the flat surface formed by gluing opposite sides of two regular n-gons, the set $Y \subset \mathbb{R}^2$ of saddle connection holonomy vectors satisfies a quadratic growth estimate $|\{y \in Y : \|y\| \leq R\}| \sim c_Y R^2$, and computed the constant c_Y. In 1992 he recorded an observation of Sarnak that gives an error estimate $|\{y \in Y : \|y\| \leq R\}| = c_Y R^2 + O\left(R^{\frac{4}{3}}\right)$ in the asymptotics. Both Veech's proof of quadratic growth, and Sarnak's error estimate, rely on the theory of Eisenstein series, and are valid in the wider context of counting points in discrete orbits for the linear action of a lattice in $\mathrm{SL}_2(\mathbb{R})$ on the plane. In this paper we expose this technique and use it to obtain the following results. For lattices Γ with trivial residual spectrum, we recover the error estimate $O\left(R^{\frac{4}{3}}\right)$, with a simpler proof. Extending this argument to more general shapes, and using twisted Eisenstein series, for sectors $\mathcal{S}_{\alpha,\beta} = \{re^{\mathbf{i}\theta} : r > 0, \alpha \leq \theta \leq \alpha + \beta\}$ we prove an error estimate

$$|\{y \in Y : y \in \mathcal{S}_{\alpha,\beta}, \|y\| \leq R\}| = c_Y \frac{\beta}{2\pi} R^2 + O_\varepsilon\left(R^{\frac{8}{5}}\right).$$

For dilations of smooth star bodies $R \cdot B_\psi = \{re^{\mathbf{i}\theta} : 0 \leq r \leq R\psi(\theta)\}$, where $R > 0$ and ψ is smooth, we prove an estimate

$$|\{y \in Y : y \in R \cdot B_\psi\}| = c_{Y,\psi} R^2 + O_{\psi,\varepsilon}\left(R^{\frac{12}{7}}\right).$$

Dedicated with admiration to the memory of Bill Veech

1. Introduction

We recall the Gauss circle problem, which aims to provide an estimate for the cardinality $\left|B \cap \mathbb{Z}^2\right|$ of the intersection of a large ball B in the plane with the integer lattice. The estimate

$$\left|B(0,R) \cap \mathbb{Z}^2\right| = \pi R^2 + O(R)$$

is easy to prove and is attributed to Gauss (here $B(x,r) \subset \mathbb{R}^2$ is the Euclidean ball of radius r around x). There have been several improvements to the error term and this is still the topic of intense investigation (see [IKKN06] for a recent survey). A more general problem in the same vein aims to replace the set $\mathbb{Z}^2$ with another discrete set Y, and replace large balls B with more general sets. For more general sets Y, the first step is establishing *quadratic growth*, i.e. showing $|B(0,R) \cap Y| = c_Y R^2 + o(R^2)$ for some $c_Y > 0$, and this can already be very challenging. In cases where quadratic growth has been established, the natural next questions are to evaluate the *quadratic growth constant* c_Y, and to obtain error estimates. A well-studied example is when Y is the set of primitive points in $\mathbb{Z}^2$, which is a discrete orbit under the group $\mathrm{SL}_2(\mathbb{Z})$. This paper is concerned with the case in which Y is a discrete orbit for a lattice in $G = \mathrm{SL}_2(\mathbb{R})$ acting on the plane. An important contribution to the study of these discrete orbits was made by Veech in a celebrated 1989 paper [Vee89], and in the subsequent papers [Vee92, Vee98]. We begin by recalling the context of Veech's work.

A translation surface is a compact oriented surface equipped with a translation structure. Since the main results of this paper will not involve translation surfaces, we omit the precise definitions, referring the interested reader to the surveys [Vor96, MT02, Z06]. For any translation surface M, the collection of *holonomy vectors of saddle connections* is a discrete set Y_M in $\mathbb{R}^2$, consisting of planar holonomies of certain straightline paths on M. The

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group G acts on a moduli space of translation surfaces, as well as on the plane by linear transformations, satisfying an equivariance property $Y_{gM} = gY_M$. For any M, its *stabilizer group* (or Veech group) is

$$\Gamma_M = \{g \in G : gM = M\}.$$

If Γ_M is a *lattice* in G, i.e. is discrete and of finite covolume, then M is called a *lattice surface* (or Veech surface). These lattices are non-uniform and thus have discrete orbits in the plane. Here is a summary of the results of [Vee89] which are relevant to this paper.

Theorem 1.1 (Veech 1989)**.**
(a) *The surfaces M_n obtained by gluing sides in two copies of a regular n-gon are lattice surfaces, and the corresponding lattice is non-arithmetic unless $n \in \{3, 4, 6\}$.*
(b) *For lattice surfaces, Y_M is a finite union of Γ_M-orbits.*
(c) *Discrete orbits of lattices in G acting on the plane, satisfy quadratic growth. In particular, the sets Y_M satisfy quadratic growth when M is a lattice surface.*
(d) *The quadratic growth constants for the surfaces M_n in (a) are computed.*

Veech proved statement (c) by reducing the problem to previous work in analytic number theory. We will review this below in §3. He also computed quadratic growth constants for the examples in statement (a), and in [Vee92], computed quadratic growth constants for more examples. Veech revisited statement (c) in [Vee98], where he introduced a number of techniques which make it possible to establish quadratic growth in more general situations, and compute quadratic growth constants. Among other things he also reproved (c) by ergodic methods, in particular using an ergodic-theoretic tool of Eskin and McMullen [EMc93]. Another ergodic-theoretic proof of (c) was given by Gutkin and Judge in [GJ00], also using ideas of [EMc93]. In subsequent work, Eskin and Masur [EM01] improved on Veech [Vee98] and proved that almost every translation M (with respect to the natural measures on the moduli spaces of translation surfaces) satisfies quadratic growth. Their arguments are also ergodic-theoretic and rely on an ergodic theorem appearing in [Ne17].

In the presence of some spectral estimates, it is possible to improve on quadratic growth by establishing *effective quadratic growth*, by which we mean proving an error term of the form

$$|B(0,R) \cap Y| = c_Y R^2 + O\left(R^{2-\delta}\right) \tag{1.1}$$

for some $\delta > 0$. In [NRW17], relying on spectral estimates established in [AGY06, AG13], such an error bound was given for almost every translation surface. In particular, the results of [NRW17] imply effective quadratic growth for Veech surfaces. However the constant δ appearing in [NRW17] is far from optimal, and a much better error estimate for the case of lattice surfaces has long been known to experts. In fact, already in [Vee92, Remark 1.12], Veech included the remark (which he attributed to Sarnak) that work of Selberg and Good can be used to prove to an estimate of the form

$$|B(0,R) \cap Y| = c_Y R^2 + O\left(R^{\frac{4}{3}}\right), \tag{1.2}$$

where Y is the orbit of tempered lattice in G (see §2). See also [RR09, Tr13] for related results.

An initial goal of this paper was to provide an exposition of the method sketched in [Vee92, Remark 1.12], specifically for the benefit of those who might be familiar with ergodic-theoretic counting techniques but not with the techniques used in analytic number theory. While studying this topic, the authors obtained several extensions and improvements. Thus the paper acquired an additional goal of proving these new results; however we believe that a survey on these matters has not lost its relevance, and we chose to write our paper on the level of a tutorial.

The structure of the paper is as follows. In §2 we define the objects which will be the focus of our discussion and state our results, comparing our new results with those which were obtained by previous authors (or could be easily deduced from their work). Specifically we define the class of *tempered* lattices and the larger class of *lattices with trivial residual spectrum*, which are the subgroups for which the relevant spectral estimates are as strong as one could hope for. As we will explain, our improvements concern counting points in more general shapes than Euclidean balls; e.g. sectors or dilates of star-shaped bodies. In §3 we define Eisenstein series and collect some results about them. We also explain how Veech obtained statement (c) of Theorem 1.1. In §4 we prove the bound (1.2) (see Theorem 2.4) for counting in balls, and for lattices with trivial residual spectrum. Our work bypasses difficult work of Good [Go83] by taking advantage of the fact that in our particular setting, counting can be achieved by making a contour shift of a truncated Eisenstein series to the critical line, for a general lattice. This strategy is classical in analytical number theory (see e.g. [Dav80]), and indeed goes back to the proof of the Prime Number Theorem, but in our situation requires an extra averaging argument, see Proposition 4.2. In §5 we use this idea to prove our improvements. Our analysis is further influenced by the work of Sarnak [Sa04]. Although §5 is the one containing the proof of the new results, the proofs use the ideas involved in proving earlier results, and so we do not recommend starting with §5. The results of §4 and §5 both rely on reducing counting problems to fundamental estimates about Eisenstein series, which are collected in §3, and whose proofs we do not explain.

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2. Definitions and statement of results

In this section we set our notation, recall certain preliminary results, and state our results.

2.1. **Some actions and subgroups of $G = \mathrm{SL}_2(\mathbb{R})$.** Recall that G acts on the left on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and on the plane $\mathbb{R}^2$ respectively, by the rules

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

The G-action on $\mathbb{H}$ preserves the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ and hence the hyperbolic area form $\frac{dxdy}{y^2}$. Let

$$K = \mathrm{SO}_2(\mathbb{R}) = \{r_\theta : \theta \in [0, 2\pi]\}, \text{ where } r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Let $\mathbf{i} = \sqrt{-1}$ so that K is the stabilizer of $\mathbf{i}$. Also let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^2$; it is also preserved by K. Let $\mathbf{e}_1 = (1, 0)$ so that the stabilizer of $\mathbf{e}_1$ is

$$N = \{u_s : s \in \mathbb{R}\}, \text{ where } u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Let $\Gamma \subset G$ be a discrete subgroup. It then acts properly discontinuously on $\mathbb{H}$. We call Γ a *lattice* if there is a finite G-invariant measure on G/Γ or equivalently, a fundamental domain for the Γ-action on $\mathbb{H}$ of finite hyperbolic area. If there is a compact fundamental domain then Γ is called *cocompact* or *uniform.* If Γ is a lattice we write $X_\Gamma = \mathbb{H}/\Gamma$, denote the G-invariant measure on X_Γ induced by the hyperbolic area form by μ_Γ and write $\mathrm{covol}(\Gamma) = \mu_\Gamma(X_\Gamma)$.

A subgroup of Γ is called *maximal unipotent* if it is conjugate (in G) to the group

$$N_0 = \{u_n : n \in \mathbb{Z}\}$$

and is not properly contained in a subgroup conjugate to N_0. For a lattice $\Gamma \subset G$, the quotient $\mathbb{H}/\Gamma$ has a finite number of topological ends called *cusps.* The number of cusps is zero if and only if Γ is cocompact, and in the non-uniform case, there is a bijection between cusps and conjugacy classes (in Γ) of maximal unipotent subgroups. For a lattice Γ and $v \in \mathbb{R}^2 \smallsetminus \{0\}$, the orbit Γv is discrete if and only if the stabilizer of v in Γ is a maximal unipotent group, and we refer to the conjugacy class of the stabilizer of v as the *cusp corresponding to* Γv. In particular a cocompact lattice has no discrete orbits in its action on $\mathbb{R}^2$. Clearly Γv is discrete if and only if $\Gamma(tv) = t\Gamma v$ is discrete for all $t \neq 0$, and hence the number of discrete orbits, considered up to dilation, is the same as the number of cusps. We are interested in counting points in discrete orbits for the Γ-action on $\mathbb{R}^2$.

Warning: In the literature, one often works with $\mathrm{PSL}_2(\mathbb{R}) = G/\{\pm\mathrm{Id}\}$, the group of orientation-preserving isometries of $\mathbb{H}$. Since we are interested in point sets in the plane, which need not be invariant under the action of $-\mathrm{Id}$, it will be more natural for us to work with G. This discrepancy may result in minor deviations with other texts; there will be no discrepancy whenever Γ contains $-\mathrm{Id}$, and for many counting problems we can reduce to this situation as follows:

Proposition 2.1. *Let $\pi : G \to \mathrm{PSL}_2(\mathbb{R})$ be the natural projection, and suppose Γ is a lattice in G, which does not contain $-\mathrm{Id}$. Let $\Gamma^{(\pm)} = \pi^{-1}(\pi(\Gamma))$, so that $\Gamma^{(\pm)}$ is a degree 2 central extension of Γ. Then either $\Gamma^{(\pm)}v = \Gamma v$ or $\Gamma^{(\pm)}v = \Gamma v \sqcup -\Gamma v$ (a disjoint union).*

Proof. Since $-\mathrm{Id}$ and Γ generate $\Gamma^{(\pm)}$, we have $\Gamma^{(\pm)}v = \Gamma v \cup -\Gamma v$. If this is not a disjoint union then there is $u \in \Gamma v$ for which $-u \in \Gamma v$, say $u = \gamma_1 v$ and $-u = \gamma_2 v$. Then

$$-v = -\gamma_1^{-1}(u) = \gamma_1^{-1}(-u) = \gamma_1^{-1}\gamma_2 v$$

so that $\Gamma v = -\Gamma v$. $\square$

Let Δ be the Laplace-Beltrami differential operator on $\mathbb{H}$, which is expressed in coordinates as

$$\Delta f(x + \mathbf{i}y) = -y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right). \tag{2.1}$$

It is not hard to check that Δ is G-invariant and hence descends to a well-defined differential operator on X_Γ which we continue to denote by Δ. The eigenvectors for Δ which belong to $L^2(X_\Gamma, \mu_\Gamma)$ are called *Maass forms.* The corresponding eigenvalues satisfy

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots,$$

and the *nontrivial small eigenvalues* are those satisfying $\lambda_i \in \left(0, \frac{1}{4}\right]$.

Definition 2.2. We say that Γ is *tempered* if it has no nontrivial small eigenvalues.

Examples of tempered subgroups are the Veech groups of the surfaces M_n of Theorem 1.1(1): Veech showed that they are $(2, n, \infty)$ Schwarz triangle groups, these triangle groups were shown to be tempered by Sarnak [Sa80, §3], and all triangle groups were shown to be tempered by Zograf [Z82]. All but finitely many non-uniform triangle groups arise as Veech groups of lattice surfaces, see [BM10, Hoo13, Wr13].

Suppose Γ is a non-uniform lattice with k cusps, and for $i = 1, \dots, k$ choose $\mathfrak{s}_i \in G$ so that $\Gamma'_i = \mathfrak{s}_i^{-1}\Gamma\mathfrak{s}_i$ contains N_0 as a maximal unipotent subgroup, where the groups $\mathfrak{s}_i N_0 \mathfrak{s}_i^{-1}$ are mutually nonconjugate maximal unipotent subgroups of Γ. Let

$$N'_0 = \begin{cases} \pm N_0 & \Gamma \text{ contains -Id} \\ N_0 & \text{otherwise} \end{cases},$$

and set

$$E_i(z, s) = \sum_{\gamma \in N'_0 \backslash \Gamma'_i} \mathrm{Im}(\gamma' z)^s, \quad \text{where } \gamma' = \gamma\mathfrak{s}_i^{-1}, \tag{2.2}$$

where $z \in \mathbb{H}$ and the sum ranges over any collection of coset representatives. Then E_i is the *Eisenstein series corresponding to the i-th cusp of* Γ. It will play a major role in our discussion and will be slowly introduced in §3. As we will see, for each fixed i and z, the sum (2.2), considered as a map $s \mapsto E_i(z, s)$, converges for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to the complex plane (for $\mathrm{Re}(s) \le 1$, the notation $E_i(z, s)$ refers to the analytic continuation). We use this fact for the following important definition:

Definition 2.3. Let Γ, k, i be as above. The *residual spectrum of $E_i(z, s)$* is the set of $s \in (1/2, 1)$ for which $s \mapsto E_i(z, s)$ has a pole at s. If there are no such poles we say that *the residual spectrum of $E_i(z, s)$ is trivial.*

We remark that the choice of z in the above definition is unimportant as all functions $E_i(z,\cdot)$ have poles at the same values of s, see [Iwa95, Thm. 6.10].

If $E_i(z,\cdot)$ has a pole at $s \in (1/2, 1)$ then Δ has an eigenvalue $\lambda = s(1-s) \in (0, 1/4)$. Thus, if Γ is tempered then all of its cusps have trivial residual spectrum. With regard to the converse, consider for instance *principal congruence groups* (the principal congruence group of level n is the group of all matrices in $\mathrm{SL}_2(\mathbb{Z})$ congruent to $\mathrm{Id} \mod n$). In this case it is known that the Eisenstein series associated to any cusp for any congruence group has trivial residual spectrum (see [Iwa95, Thm. 11.3]), but the question of whether all of these group are tempered is a famous longstanding open question posed by Selberg.

With this terminology we will prove:

Theorem 2.4. *Suppose Γ is a nonuniform lattice in G, $Y = \Gamma v$ is a discrete orbit for which the corresponding Eisenstein series has trivial residual spectrum. Then there is $c_Y > 0$ such that*

$$|B(0,R) \cap Y| = c_Y R^2 + O\left(R^{\frac{4}{3}}\right). \tag{2.3}$$

Moreover, the asymptotic (2.3) *holds when one replaces $B(0,R)$ with the dilate $R \cdot \mathbf{E}$ of any centered ellipse $\mathbf{E}$ (with the constant c_Y and the implicit constant in the O-notation depending on $\mathbf{E}$).*

In this result one also obtains a precise formula for the quadratic growth constant c_Y, and an asymptotic expansion for the error in case the residual spectrum is not trivial, with one term for every pole at $s \in \left(\frac{1}{2}, 1\right)$. See Theorem 4.1. The error estimate in (2.3) is not new. In fact, as we saw in (1.2), Sarnak and Veech could prove it already in 1993. However the proof we will give below will be simpler than the proof outlined in [Vee92], which relies on difficult work of Good concerning counting results in both $\mathbb{R}^2$ and in $\mathbb{H}$. See §2.3 for a more detailed comparison with Good's work.

An interesting open question is whether the error term in Theorem 2.4 is optimal. In this regard, note that for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the error term in (1.2) can be improved to $o(R)$ (see [HN96]); the same is true for congruence groups. However we are not aware of any non-arithmetic non-uniform lattices for which a bound better than that of (1.2) is known.

2.2. **Counting in more general domains.** We turn to new results. In these results we strive to take sets more general than Euclidean balls in the counting problem, while still obtaining a good bound for the error. We will need a further definition.

For each $\gamma \in G$, we set $c_\gamma = c$, $d_\gamma = d$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that c_γ, d_γ only depend on the coset $N\gamma$. For each $n \in \mathbb{Z}$, and for $\Gamma, \Gamma_i, \mathfrak{s}_i, N_0'$ as in the discussion preceding (2.2), define a *twisted Eisenstein series*

$$E_i(z,s)_{2n} = \sum_{\gamma \in N_0' \backslash \Gamma_i'} \mathrm{Im}(\gamma' z)^s \left(\frac{c_{\gamma'} z + d_{\gamma'}}{|c_{\gamma'} z + d_{\gamma'}|}\right)^{2n} \quad \text{where } \gamma' = \gamma \mathfrak{s}_i^{-1}. \tag{2.4}$$

This is sometimes also referred to as the *weight* $2n$ *Eisenstein series.* Note that this definition makes sense for any m in place of $2n$ but we are only interested in the even values. Once again it is true that $s \mapsto E_i(z,s)_{2n}$ has a meromorphic continuation to the entire complex plane, whose poles do not depend on z, and we generalize Definition 2.3 as follows:

Definition 2.5. For i, n as above, the *residual spectrum* are those $s \in (1/2, 1)$ for which $E_i(z,\cdot)_{2n}$ has a pole at s. If there are no such s we say *the residual spectrum of $E_i(z,s)_{2n}$ is trivial.*

Once again it is true that $E_i(z,s)_{2n}$ has finitely many poles and a tempered group Γ has trivial residual spectrum for each i and n.

Let $S \subset \mathbb{R}^2$ be a bounded closed set. We say that S is *star shaped at* 0 if it can be written as

$$S = \{r(\cos\theta, \sin\theta) : \theta \in [0, 2\pi], r \in [0, \rho(\theta)]\}$$

for $\theta \mapsto \rho(\theta)$ a non-negative bounded 2π-periodic function of compact support.

We say that S is a *sector* if it is of the above form with the function ρ the indicator of a nondegenerate subinterval. We say S is a *smooth star shape* if it is of the above form and ρ is smooth and everywhere positive. We write $R \cdot S$ for the dilated set $\{Rx : x \in S\}$.

With these notations we have:

Theorem 2.6. *Suppose Γ is a non-uniform lattice in G containing $-\mathrm{Id}$, $Y = \Gamma v$ is a discrete orbit corresponding to the i-th cusp, and suppose that for each n, $E_i(z,s)_{2n}$ has trivial residual spectrum. Then:*

- *If S is a smooth star shape then there is $c_{Y,S} > 0$ such that for every $\varepsilon > 0$,*
$$|Y \cap R \cdot S| = c_{Y,S} R^2 + O\left(R^{\frac{12}{7}+\varepsilon}\right).$$
- *If S is a sector then there is $c_{Y,S} > 0$ such that*
$$|Y \cap R \cdot S| = c_{Y,S} R^2 + O\left(R^{\frac{8}{5}}\right); \tag{2.5}$$
moreover, the asymptotic (2.5) *is also valid if one replaces S with a sector in a centered ellipse (i.e. the image of a sector under an invertible linear map), with implicit constants depending also on the ellipse.*

In the above results, the quadratic growth constants can be written down explicitly and the implicit constants depend on the sets S.

The fact that the error terms in Theorem 2.6 are worse than those in Theorem 2.4 is an artifact of our method: when working in Iwasawa coordinates (see §2.3) the functions which arise when counting in balls have a much simpler form. In particular, their analysis does not require bounds on the twisted Eisenstein series. We do not know whether one should expect the true error asymptotics for balls to be significantly different from those of smooth star shaped domains. Regarding counting in sectors, as is often the case, we incur a price for approximating indicator functions by smooth functions. Thus it would not be surprising if the optimal error terms for sectors are worse than those for balls.

2.3. **Relation to the work of Good.** Let $G = KAN$ be the Iwasawa decomposition of G, that is K, A, N are respectively the subgroup of orthogonal, diagonal, and unipotent upper triangular matrices, and let Γ be a non-uniform lattice normalized so that it contains N_0 as a maximal unipotent subgroup. The counting problem considered in Theorem 2.4 can be thought of as a counting problem in the double coset space
$$\mathfrak{S} = (\Gamma \cap K)\backslash \Gamma / (\Gamma \cap N).$$
In fact, one can easily verify that
$$\begin{aligned} |\Gamma \mathbf{e}_1 \cap B(0,R)| &= |\{[\gamma] \in \mathfrak{S} : \|\gamma \mathbf{e}_1\| \le R\}| \\ &= \left|\left\{[\gamma] \in \mathfrak{S} : \mathrm{Im}(\gamma^{-1}\mathbf{i}) \ge R^{-2}\right\}\right| \\ &= \left|\{(\theta \mod 2\pi, y, x \mod 1) : [r_\theta a_y u_x] \in \mathfrak{S},\ y \le R\}\right|, \end{aligned}$$
where $a_y = \mathrm{diag}(y, y^{-1})$ and the last identification relies on the Iwasawa decomposition of G. Fix $y > 0$, and set
$$\mathcal{K}_y = \{(\theta \mod 2\pi, x \mod 1) : [r_\theta a_y n_x] \in \mathfrak{S}\}.$$
The character sums, for $m, n \in \mathbb{Z}, y > 0$,
$$\mathbf{S}(m,n,y) = \sum_{(\theta,x)\in\mathcal{K}_y} e^{\mathbf{i}m\theta} e^{2\pi \mathbf{i} n x}$$
indexed over the above Iwasawa double coset decomposition are a natural generalization of the classical Kloosterman sums from number theory (which appeared already in work of Poincaré about Fourier expansions of Eisenstein series, see [P11]). Note in particular that
$$|\Gamma \mathbf{e}_1 \cap B(0,R)| = \sum_{y \le R} \mathbf{S}(0,0,y).$$

In [Go83, Thm. 4], Good proved bounds on the asymptotic growth of sums of various generalizations of Kloosterman sums as above, meaning over various double coset decompositions of Γ in G. This corresponds to the problem we have discussed above (counting for Γ-orbits in the plane) as well as other counting problems such as Γ-orbits in $\mathbb{H}$ and in the space of geodesics. For the case of the linear action on the plane, which is the one of interest here, Good obtains

$$\sum_{y \leq R} \sum_{(\theta,x)\in K_y} e^{\mathbf{i}m\theta} e^{2\pi\mathbf{i}nx} = c_{m,n}R^2 + O_{m,n}\left(R^{\frac{4}{3}}\right),$$

where $c_{m,n} > 0$ if and only if $m = n = 0$. The dependence of implicit constants in the remainder term on m, n is however not worked out explicitly (and difficult to trace over the 100 pages of build-up Good relies on to prove this asymptotic). If it were, one would be able to deduce results similar to Theorem 2.6 from [Go83].

2.4. **Counting in still more general (well-rounded) domains.** A common assumption in the theory of counting lattice points in a family of domains in a Lie group is that of well-roundedness, introduced in [DRS93] and [EMc93]. In [GN12] this assumption was combined with an estimate of the spectral gap that arises in the automorphic representation of G on $L^2_0(G/\Gamma)$ to prove an effective estimate for the lattice point count. We will show that the problem we consider, namely counting points in discrete orbits for the linear action of a lattice on the plane, can be reduced to the lattice point counting problem for domains in $\mathrm{SL}_2(\mathbb{R})$. This will allow us to count orbit points in more general sets in the plane using just the existence of a spectral gap, but this additional generality compromises the error estimate, leading to bounds which are inferior to the ones stated above. Thus the techniques we describe in this subsection are applicable in more general situations, but lead to weaker bounds.

We will show:

Theorem 2.7. *Let Γ be any non-uniform lattice in $\mathrm{SL}_2(\mathbb{R})$, and $Y = \Gamma v$ any discrete orbit of Γ in $\mathbb{R}^2 \setminus \{0\}$. Let $S \subset \mathbb{R}^2$ be a star-shaped domain at 0 with $\rho(\theta)$ a non-negative piecewise Lipschitz function, and let $R \cdot S$ be the dilation of S by a factor of R. Then, for all $\varepsilon > 0$*

$$|Y \cap R \cdot S| = C_{Y,S}R^2 + O\left(R^{q_\Gamma+\varepsilon}\right),$$

where the implicit constants in the O-notation depend on Γ, Y, S and ε, and with q_Γ depending only on the spectral gap of the automorphic representation of G on $L^2_0(G/\Gamma)$. In particular, if the lattice Γ is tempered, then we can set $q_\Gamma = \frac{7}{4}$.

Note that Theorem 2.7 applies, in particular, to all convex sets with piecewise Lipchitz boundary (containing the origin in their interior), and in particular, to all convex polygons.

3. A bit of Eisenstein series

In this section we go into more details about our main actor, the Eisenstein series introduced in (2.2). We refer the reader to [Kub73, Hej83, Ter85, Sa81] for more information.

3.1. **Some sums and their relation to the counting function.** For a non-uniform lattice Γ, a discrete orbit $Y = \Gamma v$ in the plane, $g \in G$ and $s \in \mathbb{C}$, we set

$$E(g,s) = E^{(\Gamma,v)}(g,s) = \sum_{u \in g\Gamma v} \|u\|^{-2s}. \tag{3.1}$$

Note that in (2.2), (3.1) we introduce the notation E and $E^{(\Gamma,v)}$ to denote two different functions, one of which has an argument $z \in \mathbb{C}$ and the other, $g \in G$. This ambiguity is common in the literature and is explained below, see (3.6). Our first task will be to motivate this new definition, in the context of the counting problem for points in Γv. For the moment we consider (3.1) as a formal sum, postponing the discussion of convergence issues.

Warning (continued). In the literature, there are two conflicting conventions regarding the definition of $E(g,s)$. What is denoted $E(g,s)$ in [Vee89] is denoted $E(g^{-1},s)$ in [Kub73].

We will follow Veech's convention, and we say more about the source of this discrepancy below.

Let $\mathbf{N}(g,R) = |B(0,R) \cap gY|$. Considering the measure $\nu^{(g)} = \sum_{u \in g\Gamma v} \delta_u$, which is a Radon measure on $\mathbb{R}^2$ (since Γv is discrete), and considering the radial function $f^{(s)}(w) = \|w\|^{-2s}$, we have

$$E(g,s) = \int_{\mathbb{R}^2} f^{(s)} d\nu^{(g)} = \int_0^\infty \frac{d\mathbf{N}(g,R)}{R^{2s}},$$

where in the last equality we have written a Lebesgue-Stieltjes integral. Using integration by parts (and recalling that convergence issues will be addressed further below), we have

$$E(g,s) = 2s \int_0^\infty \frac{\mathbf{N}(g,R)}{R^2} R^{1-2s} dR. \tag{3.2}$$

We now recall the definition of the Mellin transform and Mellin inversion, which are multiplicative analogues of the Fourier transform and Fourier inversion. Recall that a Schwartz function is a function $\mathbb{R} \to \mathbb{R}$ which is infinitely differentiable and for which all derivatives decay to zero at infinity faster than any power. We will say that $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a Schwartz function on $\mathbb{R}_+ = (0,\infty)$ if $f(x) = \psi(e^x)$ is a Schwartz function. The Mellin transform of a Schwartz function $\psi : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$\mathcal{M}\psi : \mathbb{C} \to \mathbb{C}, \quad \mathcal{M}\psi(s) = \int_0^\infty \psi(y) y^{s-1} dy, \tag{3.3}$$

and Mellin inversion says that for $\sigma \in \mathbb{R}$ we have

$$\psi(y) = \frac{1}{2\pi \mathbf{i}} \int_{\mathrm{Re}(s)=\sigma} \mathcal{M}\psi(s)\, y^{-s} ds. \tag{3.4}$$

The above formulae follow immediately from the Fourier transform and Fourier inversion formula, from which they are obtained by a change of variables $y = e^x$. As we will explain below, under suitable conditions the formulae extend to functions which are not Schwartz functions. For the moment we proceed considering them as formal identities.

Comparing equations (3.2) and (3.3), and making a change of variables $y = R^{-1}$, we see that $\frac{E(g,s)}{2s}$ is the Mellin transform of the function $y \mapsto \mathbf{N}(g, y^{-1})$ evaluated at $2s$. Applying Mellin inversion we recover the counting function $\mathbf{N}$ as

$$\mathbf{N}(g,R) = \frac{1}{2\pi \mathbf{i}} \int_{\mathrm{Re}(s)=\sigma} \frac{E(g,s)}{s} R^{2s} ds \tag{3.5}$$

(this formal manipulation is given a precise meaning and justified in Corollary 3.3 below). The upshot of this discussion is that, at least formally, the counting function which we are interested in has an integral representation in terms of the function $E(g,s)$. Furthermore, if we know (as will turn out to be the case) that $s \mapsto E(g,s)$ is holomorphic or meromorphic, then the integral of (3.5) can be evaluated using standard tools of complex analysis like contour shifts, residue computations, etc. Using this, after justifying our manipulations we will indeed be able to obtain a detailed understanding of $R \mapsto \mathbf{N}(g,R)$ from an understanding of $E(g,s)$. Note also that up to this point no use has been made of the dependence of all quantities on the variable g. This dependence will not play much of a role in our discussion, but it is crucial when one wants to say something about E.

For the benefit of readers not satisfied with this non-rigorous derivation of (3.5), we include another non-rigorous derivation. Let $u \in g\Gamma v$ and consider its contribution to both sides of (3.5). Assume for simplicity that $g\Gamma v$ does not contain vectors of length precisely R, and set $y = R/\|u\|$, so that u contributes 1 to $\mathbf{N}(g,R)$ when $y > 1$ and contributes 0 when $y < 1$. Recalling (3.1), and exchanging the order of summation and integration in the right hand side of (3.5), we see that each u contributes $\frac{1}{2\pi\mathbf{i}} \int_{\mathrm{Re}(s)=\sigma} \frac{y^{2s}}{s} ds$. This integral is the limit as $T \to \infty$ of line integrals along the vertical lines $L_{\sigma,T} = \{\sigma + \mathbf{i}t : -T \leq t \leq T\}$. For each fixed T we can evaluate this line integral by Cauchy's integral formula, replacing $L_{\sigma,T}$ with $L_{\zeta,T}$ (the total contribution along the horizontal lines $\mathrm{Im}(s) = \pm T$ becomes negligible as $T \to \infty$), where in case $y > 1$, we let $\zeta \to -\infty$, and get a contribution of 1 due to the pole at the origin, and in case $y < 1$ we let $\zeta \to +\infty$ and get a contribution of 0.

3.2. **Simple properties and the relation to Eisenstein series.** Having motivated our interest in the function defined by (3.1), we now make the link with the functions defined by (2.2). Let $\Gamma_v = \{\gamma \in \Gamma : \gamma v = v\}$. Note that if $g = r_\theta \operatorname{diag}\left(y, y^{-1}\right) u_s$, where $y > 0$ and $r_\theta \in K$, $u_s \in N$ (Iwasawa decomposition), then y can be detected in both the linear action as $\|g\mathbf{e}_1\|$, and in the action on the upper half-plane as $\operatorname{Im}(g^{-1}\mathbf{i})^{-\frac{1}{2}}$. Using this observation, the following properties follow readily from definition (3.1) and from the fact that the Euclidean norm is K-invariant.

Proposition 3.1. *(1) For $r_\theta \in K$ and $\gamma \in \Gamma_v$ we have $E(r_\theta g \gamma, s) = E(g, s)$.*

(2) For $g, \mathfrak{s} \in G$, if $\Gamma^{\mathfrak{s}} = \mathfrak{s}^{-1}\Gamma\mathfrak{s}$ then $E^{(\Gamma^{\mathfrak{s}}, \mathfrak{s}^{-1}v)}(g\mathfrak{s}, s) = E^{(\Gamma, v)}(g, s)$.

(3) Suppose Γ contains N_0 as a maximal unipotent subgroup. If Γ does not contain -Id *then $E^{(\Gamma,\mathbf{e}_1)}(g,s) = \sum_{\gamma \in N_0' \backslash \Gamma} \operatorname{Im}(\gamma g^{-1}\mathbf{i})^s$; if Γ contains* -Id *then $E^{(\Gamma,\mathbf{e}_1)}(g,s) = 2\sum_{\gamma \in N_0' \backslash \Gamma} \operatorname{Im}(\gamma g^{-1}\mathbf{i})^s$.*

(4) If $v_1, v_2 \in \mathbb{R}^2 \smallsetminus \{0\}$ satisfy $v_2 = tv_1$ for $t > 0$, then $E^{(\Gamma, v_2)}(g,s) = t^{-2s}E^{(\Gamma,v_1)}(g,s)$.

By property (1), the dependence of (3.1) on g is actually only a dependence on the coset Kg, and we can identify these cosets with $\mathbb{H}$ via $Kg \leftrightarrow z = g^{-1}\mathbf{i}$ to replace g with z. And with the normalization that the stabilizer of v is N_0, we see by rescaling and using properties (2) and (3) that

$$z = g^{-1}\mathbf{i} \quad \Longrightarrow \quad E_i(g,s) = \begin{cases} E_i(z,s) & \Gamma \text{ does not contain -Id} \\ 2E_i(z,s) & \text{otherwise} \end{cases}. \tag{3.6}$$

Thus for each non-uniform lattice Γ with k cusps, up to the trivial transformations recorded above, there are k essentially different functions of this form. They are normalized by conjugating so that $\Gamma_v = N_0$ and rescaling so that $v = \mathbf{e}_1$. It will become clearer later why this normalization is convenient. It will also develop that in order to understand these functions in detail, it is best not to focus on one of them, but to consider their properties as a vector valued function $(z, s) \mapsto (E_1(z,s), \ldots, E_k(z,s))$.

The discrepancy between the notation used in [Vee89] and that used in [Kub73] is related to the substitution $z = g^{-1}\mathbf{i}$ above. If one followed the convention of Kubota one would make the substitution $z = g\mathbf{i}$ instead. The convention of Veech, which we follow, gives simpler formulae involving discrete orbits in $\mathbb{R}^2$ and is consistent with working with the space of left cosets G/Γ. The convention of Kubota gives simpler formulae when discussing the action of G on $\mathbb{H}$ by Möbius transformations, and is consistent with working with right cosets $\Gamma\backslash G$. Thus the discrepancy between these notations is collateral damage in a larger battle.

We now explain our interest in the twisted Eisenstein series (2.4). Above we motivated Eisenstein series by explaining its relation to the counting problem in the plane, where each orbit point is assigned the same mass 1. In this application the counting function is K-invariant, and so we can equivalently view the first parameter of the Eisenstein series as ranging in $g \in G$ or in $z \in \mathbb{H}$ (as in the preceding paragraph). In more general situations it is desirable to assign different masses to different points, and in particular allow functions which depend on g rather than on the coset Kg. This will arise when we deal with more refined counting problems as in Theorem 2.6, and also arises in many other problems of geometric origin.

For a vector $u \in \mathbb{R}^2 \smallsetminus \{0\} \cong \mathbb{C} \smallsetminus \{0\}$ we define polar coordinates $u = \|u\|e^{\mathbf{i}\theta_u}$, where $\theta_u \in \mathbb{R}/2\pi\mathbb{Z}$. Let Γv be a discrete orbit corresponding to the i-th cusp Γ, normalized so that $v = \mathfrak{s}_i\mathbf{e}_1$, and set

$$E_i(g,s)_n = \sum_{u \in g\Gamma v} \|u\|^{-2s} e^{-\mathbf{i}n\theta_u}. \tag{3.7}$$

Note that these functions vanish for n odd when Γ contains $-\mathrm{Id}$. It is not hard to formulate an analogue of Proposition 3.1 and, by comparing (2.4) and (3.7), to verify that

$$E_i(z,s)_{2n} = E_i(g,s)_{2n} \quad \text{when } z = x + \mathbf{i}y \text{ and } g = \begin{pmatrix} y^{-1/2} & -xy^{-1/2} \\ 0 & y^{1/2} \end{pmatrix}. \tag{3.8}$$

Note that the choice of g in (3.8) ensures $z = g^{-1}\mathbf{i}$, and if we choose another g with this property, this will only affect $E_i(g,s)_{2n}$ by multiplication with a complex number of modulus 1.

Warning (continued): In (3.7), it would have been more natural, and consistent with the Veech convention mentioned after Proposition 3.1, to define the Eisenstein series using $e^{\mathbf{i}n\theta_u}$ instead of $e^{-\mathbf{i}n\theta_u}$. However this would have made it necessary to introduce a change of signs in (2.4) and would have caused a discrepancy between our notation and that of [Se89, Sa81].

Treating more general weights of points on the plane also leads to the Θ-transform which we will discuss in §3.6.

3.3. **Convergence properties.** We now begin our discussion of convergence properties of the various series introduced so far, and give a more rigorous justification of (3.5). Convergence rests on the following weak (and standard) counting estimate.

Proposition 3.2. *For each $g \in G$ we have $\mathbf{N}(g,R) = O(R^2)$. Moreover the implicit constant can be taken to be independent of g.*

Proof. We will give a simple proof in which the implicit constant will appear to depend on g. For a similar but more careful proof, which explains how to take the constant independent of g, see [Vee98, Lemma 16.10].

Make a change of variables so that $v = \mathbf{e}_1$ and $\Gamma_v = N_0$, and compare the actions on $\mathbb{R}^2$ and $\mathbb{H}$. Let $\gamma v \in \Gamma v$, and using Iwasawa decomposition, write

$$\gamma = r_\theta a_\gamma u_s.$$

Since $v = \mathbf{e}_1$ is fixed by u_s, the condition $\|\gamma v\| \le R$ is equivalent to $y_\gamma \le R$, where $a_\gamma = \mathrm{diag}\left(y_\gamma, y_\gamma^{-1}\right)$. Furthermore, we can choose $\gamma \mod N_0$ so that u_s is bounded. Now apply γ^{-1} to $\mathbf{i}$. Since r_θ preserves $\mathbf{i}$, $a_\gamma^{-1}\mathbf{i} = \frac{1}{y_\gamma^2}\mathbf{i}$, and u_s is bounded, we see that $\gamma^{-1}\mathbf{i}$ is contained in a set A_R which is an r-neighborhood of the ray $\left\{t\mathbf{i} : t \ge R^{-2}\right\}$, for some $r > 0$ independent of R. The hyperbolic area of A_R is $O(R^2)$. On the other hand, since $\Gamma\mathbf{i} \subset \mathbb{H}$ is discrete and since G acts on $\mathbb{H}$ by isometries, there is $r > 0$ small enough so that the balls of radius r around points of $\Gamma\mathbf{i}$ are disjoint. So the intersection of A_R with $\Gamma\mathbf{i}$ contains $O(R^2)$ points. ☐

Corollary 3.3. The quantities in (2.2), (3.1) and (3.2) converge absolutely on $\{\mathrm{Re}(s) > 1\}$ and converge uniformly on compact subsets of $\{\mathrm{Re}(s) > 1\}$. For any $\zeta > 1$, (2.2) and (3.1) are bounded on $\{\mathrm{Re}(s) \ge \zeta\}$ by a bound which can be taken to be uniform as z and g vary in a compact set. For fixed g, and for $\sigma > 1$,

$$\frac{\mathbf{N}(g,R^+) + \mathbf{N}(g,R^-)}{2} = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{E(g,\sigma+\mathbf{i}t)}{\sigma+\mathbf{i}t} R^{2(\sigma+\mathbf{i}t)}\, dt, \tag{3.9}$$

where $\mathbf{N}(g,R^+), \mathbf{N}(g,R^-)$ denote the one-sided limits of $\mathbf{N}(g,R)$ as $x \to R$.

Proof. The claim regarding (2.2) and (3.1) follows easily from Proposition 3.2. For instance, for (3.1), split the sum into sums over the 'rings'

$$\left\{w \in \Gamma v : \|w\| \in \left[2^n, 2^{n+1}\right]\right\}$$

for $n \in \mathbb{N}$. Also note that by discreteness, $\mathbf{N}(g,R)$ vanishes for all R close to 0, so the convergence of (3.2) is proved in the same way.

For (3.9), fix $g \in G$ and $\sigma_0 > 2$, and define the function $\psi_{\sigma_0}(\tau) = \mathbf{N}(g,e^\tau)e^{-\sigma_0\tau}$. Then $\psi_{\sigma_0}(\tau)$ has finitely many discontinuities on every bounded interval, with well-defined one-sided limits, and vanishes when we take $\tau \to -\infty$ (by discreteness of Y). Also, by

Proposition 3.2, we have $\psi_{\sigma_0}(\tau) = O(e^{(2-\sigma_0)\tau})$ as $\tau \to \infty$, and hence $\psi_{\sigma_0} \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Write

$$\widehat{\psi_{\sigma_0}}(u) = \int_{-\infty}^{\infty} \psi_{\sigma_0}(\tau) e^{-2\pi \mathbf{i} u \tau} d\tau$$

for the Fourier transform of ψ_{σ_0}. Using (3.2) and making changes of variables $R = e^\tau$, $2s = \sigma_0 + 2\pi \mathbf{i} u$ we have

$$\widehat{\psi_{\sigma_0}}(u) = \frac{E(g,s)}{2s}, \text{ where } \mathrm{Re}(s) > 1.$$

Then by Fourier inversion (see e.g. [Ter85, Ex. 1.2.7]), for all $\tau \in \mathbb{R}$ we have

$$\frac{\psi^+_{\sigma_0}(\tau) + \psi^-_{\sigma_0}(\tau)}{2} = \lim_{T\to\infty} \int_{-T}^{T} \widehat{\psi_{\sigma_0}}(u) e^{2\pi \mathbf{i} u \tau} du,$$

and hence (with the changes of variables $\sigma = \sigma_0/2$, $t = \pi u$, $R = e^\tau$, $s = \sigma + \mathbf{i} t$)

$$\frac{\mathbf{N}(g,R^+) + \mathbf{N}(g,R^-)}{2} = R^{\sigma_0} \frac{\psi^+_{\sigma_0}(\log R) + \psi^-_{\sigma_0}(\log R)}{2}$$
$$= \lim_{T\to\infty} \int_{-T}^{T} R^{\sigma_0} \frac{E(g, \sigma + \pi \mathbf{i} u)}{\sigma_0 + 2\pi \mathbf{i} u} R^{2\pi \mathbf{i} u} du = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{E(g, \sigma + \mathbf{i} t)}{\sigma + \mathbf{i} t} R^{2(\sigma + \mathbf{i} t)} dt.$$

□

3.4. **Selberg's results: meromorphic continuation and functional equation.** We now move beyond elementary results and come to much deeper results about Eisenstein series. Most of these results are due to celebrated work of Selberg, see [Se56, Se89, Hej83, Kub73] (the introduction to [Se89] contains some historical notes). The proofs exploit the dependence of $E(z,s)$ on the variable z, and we content ourselves with two comments, in order to clarify the connection with objects appearing in the preceding sections.

For $s \in \mathbb{C}$, the functions $f(x + \mathbf{i} y) = y^s$ clearly satisfy $\Delta f = s(1-s) f$, i.e. are eigenfunctions for the Laplace-Beltrami operator. Since Δ is G-invariant, formula (3.1) shows that for fixed s, the Eisenstein series also gives rise (at least formally) to a Laplace-Beltrami eigenvector $z \mapsto E_i(z,s)$, thus furnishing a connection between the Eisenstein series and the representation theory of G. Similarly, the functions $g \mapsto E_i(g,s)_n$ defined in (3.7) are eigenfunctions for the Casimir operator on G.

Also recall our normalization sending a cusp of Γ to ∞ so that the stabilizer group becomes N_0. If Γ has one cusp then this means that $z \mapsto E(z,s)$ has a periodicity property $E(z,s) = E(u_1 z, s) = E(z+1, s)$. We can exploit this periodicity by developing $E(z,s) = E(x + \mathbf{i} y, s)$ in a Fourier series $\sum_m a_m(y,s) e^{2\pi \mathbf{i} m x}$. Furthermore, if Γ has more than one cusp and i,j represent two of them, then $z \mapsto E_i(\mathfrak{s}_j z, s) = \sum_m a_{i,j,m}(y,s) e^{2\pi \mathbf{i} m x}$ is also 1-periodic, and this leads to interesting relations between the functions $a_{i,j,m}$.

We now turn to Selberg's results. By Corollary 3.3, as an absolutely convergent series of holomorphic functions, the functions $s \mapsto E_i(g,s)$ are holomorphic on $\{\mathrm{Re}(s) > 1\}$. A fundamental issue is to extend the functions to the entire plane, and here we have:

Theorem 3.4 (Selberg c. 1953)**.** *The functions $s \mapsto E_i(z,s)$ have a meromorphic continuation to the complex plane. There is a pole at $s = 1$ with residue $\frac{1}{\mathrm{covol}(\Gamma)}$, and all other poles with* $\mathrm{Re}(s) \geq \frac{1}{2}$ *are contained in* $\left(\frac{1}{2}, 1\right]$ *(in particular there are no poles at $s = 1/2$). All poles are simple.*

The second basic result is a functional equation according to which one may recover the values of $E_i(g,\cdot)$ at s from the values at $1-s$. To state this we use the notation introduced after Definition 2.2, and let $\mathbf{\Gamma}$ denote the classical Γ-function. For each $1 \leq i,j \leq k$ let

$$\varphi_{ij}(s) = \sqrt{\pi}\, \frac{\mathbf{\Gamma}\left(s - \frac{1}{2}\right)}{\mathbf{\Gamma}(s)} \sum |c|^{-2s}, \quad \Phi(s) = (\varphi_{ij}(s))_{i,j=1}^k \tag{3.10}$$

where the sum ranges over distinct representatives $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of double cosets in $N_0' \backslash \mathfrak{s}_i^{-1} \Gamma \mathfrak{s}_j / N_0'$ with $c \neq 0$. The function φ_{ij} has another definition in terms of the constant term in the

Fourier expansion of $z \mapsto E_i(\mathfrak{s}_j z, s)$, see [Kub73, §2.2]. The matrix $\Phi(s)$ is sometimes called the *constant term matrix* corresponding to Γ, and sometimes called the *scattering matrix.* The poles of $s \mapsto E_i(z, s)$ with $\mathrm{Re}(s) \geq 1/2$ are also poles of Φ.

Theorem 3.5 (Selberg c. 1953)**.** *The matrix valued function Φ satisfies*

$$\Phi(s)\Phi(1-s) = \mathrm{Id},$$

and the column vector $\mathcal{E}(z, s) = (E_1, \ldots, E_k)$ satisfies

$$\mathcal{E}(z, s) = \Phi(s)\mathcal{E}(z, 1-s).$$

3.5. **Main term asymptotics and quadratic constant.** As Veech noted, it is well-known to number-theorists that the existence of a meromorphic continuation with a simple pole at $s = 1$, already implies Theorem 1.1, part (c). To see this, recall the Wiener-Ikehara Tauberian theorem (see e.g. [W46, Theorem 17]), which was developed in order to simplify proofs of the prime number theorem, and states:

Suppose $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is monotone non-decreasing, $A \in \mathbb{R}$, and suppose the integral $\int_0^\infty e^{-st}\psi(t)dt$, where $s = \sigma + \mathbf{i}\tau$, converges for $\sigma > 1$ to a function $f(s)$ which satisfies that $\lim_{\sigma\to 1+}\left(f(s) - \frac{A}{s-1}\right)$ exists, converges uniformly, and defines a uniformly bounded function in every interval $\tau \in [-a, a]$, for all $a > 0$. Then $\frac{\psi(t)}{e^t} \to_{t\to\infty} A$.

To obtain part (c) of Theorem 1.1, suppose Γv is a discrete orbit for a nonuniform lattice Γ corresponding the i-th cusp of Γ, and apply the Wiener-Ikehara theorem with A the residue of $E_i(g, s)$ at $s = 1$, and $\psi(t) = \mathbf{N}(g, e^{t/2})$. The hypotheses of the Wiener Ikehara theorem are justified by (3.2), a change of variables $R = e^{t/2}$, and Theorem 3.4. Here

$$A = \begin{cases} \frac{1}{\mathrm{covol}(\Gamma)} & \Gamma \text{ does not contain -Id} \\ \frac{2}{\mathrm{covol}(\Gamma)} & \text{otherwise} \end{cases} \tag{3.11}$$

will be the quadratic growth constant $c_{\Gamma v}$ provided v satisfies $v = \mathfrak{s}_i \mathbf{e}_1$.

Warning (continued): In [Vee89] the quadratic growth constant is given as $\frac{1}{\mathrm{covol}(\Gamma)}$, but the groups he considers do contain -Id. The discrepancy is due to the fact that Veech only counts closed cylinders and saddle connections on surfaces, and each of these gives rise to two holonomy vectors, depending on orientation.

Veech was not content with deriving Theorem 1.1(c) from known results about Eisenstein series. In 1998 he reversed the logic, reproving the result using ergodic-theoretic ideas introduced in [EMc93], and using this, obtained a continuation result for $E(g, s)$. Namely he showed that the limit $\lim_{s\to 1}(s-1)E(g, s)$ exists along any sequence approaching $s = 1$ nontangentially from $\{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$, and used this to provide an alternative derivation of the formula (3.11) for the quadratic growth constant. See [Vee98, §16] for more details.

3.6. **Θ-transform.** Let Γ be a non-uniform lattice in G, and Γv a discrete orbit in the plane. We will assume throughout this section that v corresponds to the i-th cusp of Γ and is normalized so that $v = \mathfrak{s}_i \mathbf{e}_1$. Putting different weights on different points on the plane amounts to choosing $f : \mathbb{R}^2 \to \mathbb{C}$, and defining

$$\Theta_f : G/\Gamma \to \mathbb{C}, \quad \Theta_f(g\Gamma) = \sum_{u \in g\Gamma v} f(u).$$

We will refer to the map $f \mapsto \Theta_f$ as the Θ-transform. Note that this definition extends (3.1), in that $g \mapsto E^{(\Gamma,v)}(g, s) = \Theta_f(g)$ for $f(u) = \|u\|^{-2s}$. As before we need to worry about convergence issues, and we will assume for the moment that f has compact support contained in $\mathbb{R}^2 \smallsetminus \{0\}$. Note that this is not satisfied for (3.1) and it will make the Θ-transforms we consider easier to handle analytically. This will already be apparent in the following proposition, in which we discuss the Θ-transform of smooth functions which have a special form.

Write $h_R(x) = h\left(\frac{x}{R}\right)$. With this notation we have the following extension of (3.5):

Proposition 3.6. *Let $f : \mathbb{R}^2 \smallsetminus \{0\} \to \mathbb{C}$ be a smooth compactly supported function, let $\rho : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}_+$ be smooth, let $\psi : \mathbb{R}_+ \to \mathbb{C}$ be smooth and compactly supported, and let $\Psi = \mathcal{M}\psi$ be the Mellin transform of ψ as in (3.3). Let $\sigma > 2$ and denote by Θ the transform associated with the orbit Γv corresponding to the i-th cusp of Γ, normalized so that $v = \mathfrak{s}_i \mathbf{e}_1$. Then:*

(1) If $f(u) = \psi(\|u\|)$ is purely radial, then

$$\Theta_{f_R}(g\Gamma) = \frac{1}{2\pi\mathbf{i}} \int_{\mathrm{Re}(s)=\sigma} \Psi(s) E_i\left(g, \frac{s}{2}\right) R^s ds. \tag{3.12}$$

(2) Suppose $f\left(re^{\mathbf{i}\theta}\right) = \psi\left(\frac{r}{\rho(\theta)}\right)$, and $\rho(\theta)^s = \sum_{n\in\mathbb{Z}} \hat{\rho}_n(s) e^{\mathbf{i}n\theta}$ is the Fourier expansion of ρ^s. Then

$$\Theta_{f_R}(g\Gamma) = \frac{1}{2\pi\mathbf{i}} \sum_{n\in\mathbb{Z}} \int_{\mathrm{Re}(s)=\sigma} \Psi(s) \hat{\rho}_{-n}(s) E_i\left(g, \frac{s}{2}\right)_n R^s ds. \tag{3.13}$$

(3) Suppose f splits into angular and radial parts as $f\left(re^{\mathbf{i}\theta}\right) = \psi(r)\rho(\theta)$, and let $\rho(\theta) = \sum_n \hat{\rho}_n e^{\mathbf{i}n\theta}$ be the Fourier expansion of ρ. Then

$$\Theta_{f_R}(g\Gamma) = \frac{1}{2\pi\mathbf{i}} \sum_{n\in\mathbb{Z}} \hat{\rho}_{-n} \int_{\mathrm{Re}(s)=\sigma} \Psi(s) E_i\left(g, \frac{s}{2}\right)_n R^s ds. \tag{3.14}$$

Proof. There is no need to prove (1) since it is the special case of (2) with $\rho(\theta) \equiv 1$. We will write the Fourier expansion $\rho(\theta)^s = \sum_n \hat{\rho}_n(s) e^{\mathbf{i}n\theta}$ as $\sum_n \hat{\rho}_{-n}(s) e^{-\mathbf{i}n\theta}$. The Fourier series converges absolutely for each s since ρ is smooth, and the coefficients admit an upper bound

$$|\hat{\rho}_n(s)| \leq 2\pi \|\rho\|_\infty^\sigma, \quad \text{where } \sigma = \mathrm{Re}(s). \tag{3.15}$$

More generally, applying integration by parts twice, we see that

$$|\hat{\rho}_n(s)| \ll \frac{|s|^2}{n^2}, \tag{3.16}$$

where the implicit constant depends on σ, $\|\rho\|_\infty$, $\|\rho'\|_\infty$, and $\|\rho''\|_\infty$. The Mellin transform $\mathcal{M}\psi$ satisfies $(\mathcal{M}\psi_R)(s) = R^s(\mathcal{M}\psi)(s)$ and so by Mellin inversion

$$\psi_R\left(\frac{y}{\rho}\right) = \frac{1}{2\pi\mathbf{i}} \int_{\mathrm{Re}(s)=\sigma} \Psi(s) y^{-s} R^s \rho^s ds.$$

Plugging this into the definition of Θ_{f_R} and writing each u as $\|u\|e^{\mathbf{i}\theta_u}$ we obtain

$$\begin{aligned}
\Theta_{f_R}(g\Gamma) = \sum_{u\in g\Gamma v} \psi_R\left(\frac{\|u\|}{\rho(\theta_u)}\right) &= \frac{1}{2\pi\mathbf{i}} \int_{\mathrm{Re}(s)=\sigma} \Psi(s) R^s \left(\sum_{u\in g\Gamma v} \|u\|^{-s} \rho(\theta_u)^s\right) ds \\
&= \frac{1}{2\pi\mathbf{i}} \int_{\mathrm{Re}(s)=\sigma} \Psi(s) R^s \sum_{n\in\mathbb{Z}} \hat{\rho}_{-n}(s) \left(\sum_{u\in g\Gamma v} \|u\|^{-s} e^{-\mathbf{i}n\theta_u}\right) ds \\
&= \frac{1}{2\pi\mathbf{i}} \sum_{n\in\mathbb{Z}} \int_{\mathrm{Re}(s)=\sigma} \Psi(s) \hat{\rho}_{-n}(s) R^s \left(\sum_{u\in g\Gamma v} \|u\|^{-s} e^{-\mathbf{i}n\theta_u}\right) ds.
\end{aligned}$$

To justify switching order of integration and summation in the first line use the quadratic growth of the set $g\Gamma v$ (Proposition 3.2) and the assumption $\sigma > 2$. In the second line, use also (3.15), and in the third line use Proposition 3.2, (3.16), and the fact that Ψ decays faster than any polynomial along the line $\mathrm{Re}(s) = \sigma$. Formula (3.13) now follows by plugging in (3.7).

For (3), we have

$$\begin{aligned}\Theta_{f_R}(g\Gamma) &= \sum_{u\in g\Gamma v} \rho(\theta_u)\psi_R\left(\|u\|\right)\\ &= \sum_{u\in g\Gamma v}\left(\sum_{n\in\mathbb{Z}}\hat{\rho}_{-n}e^{-\mathbf{i}n\theta_u}\right)\frac{1}{2\pi\mathbf{i}}\int_{\mathrm{Re}(s)=\sigma}\Psi(s)R^s\|u\|^{-s}ds\\ &=\frac{1}{2\pi\mathbf{i}}\sum_{n\in\mathbb{Z}}\hat{\rho}_{-n}\int_{\mathrm{Re}(s)=\sigma}\Psi(s)R^s\left(\sum_{u\in g\Gamma v}\|u\|^{-s}e^{-\mathbf{i}n\theta_u}\right)ds,\end{aligned}$$

and again we plug in (3.7), leaving it to the reader to justify changing the order of sums and integrals. □

3.7. **Additional properties.** We will need the extensions of the results of §3.4 to twisted Eisenstein series, and also some further properties. For convenience we collect all the results we will need, including results already discussed above, in the following list.

Theorem 3.7. *Let Γ be a nonuniform lattice in G with k cusps. Let $\mathfrak{s}_i$ be the elements conjugating these cusps to ∞ as in the discussion preceding* (2.2)*. Let $E_i(z,s)$ (resp. $E_i(z,s)_{2n}$) denote the (twisted) Eisenstein series as in* (2.2) *(resp.* (2.4)*). Then there is a function $\omega:\mathbb{R}\to\mathbb{R}$ (see* (A.1) *for an explicit definition) such that the following hold:*

- **(AC)** *The functions $E_i(z,s)_{2n}$ are absolutely convergent for $\mathrm{Re}(s)>1$, and for any $\zeta>0$, they are uniformly bounded and uniformly convergent on sets of the form $\{\mathrm{Re}(s)\geq 1+\zeta\}$.*
- **(M)** *The functions $s\mapsto E_i(z,s)_{2n}$ have a meromorphic continuation to all of $\mathbb{C}$.*
- **(P)** *The poles of $s\mapsto E_i(z,s)_{2n}$ with $\mathrm{Re}(s)\geq 1/2$ are all simple, lie in $(1/2,1]$, and are contained in the set (s_ℓ) of poles of the constant term matrix Φ of* (3.10)*.*
- **(1)** *There is a pole at $s=1$ if and only if $n=0$, with residue $\mathrm{covol}(\Gamma)^{-1}$.*
- **(1/2)** *The functions $E_i(z,s)_{2n}$ have no poles on the line $\mathrm{Re}(s)=1/2$.*
- (ω) *For all $t\in\mathbb{R}$, $\omega(t)\geq 1$, $\omega(-t)=\omega(t)$ and for $T\geq 1$, $\int_{-T}^{T}\omega(t)dt\ll T^2$, where implicit constants depend on Γ.*
- **(G1)** *If $n\in\mathbb{Z}$, $\mathrm{Re}(s)\geq 1/2$ and $|t|\geq|n|+1$ then $E_i(z,s)_{2n}\ll|t|\sqrt{\omega(t)}$, where $t=\mathrm{Im}(s)$ (implicit constants depend on z and Γ but not on n or $\mathrm{Re}(s)$).*
- **(G1/2)** *For all n, $\int_{-T}^{T}\left|E_i\left(z,\frac{1}{2}+\mathbf{i}t\right)_{2n}\right|^2dt\ll(T+|n|)^2$ (implicit constants depend on Γ).*

Proof. For $n=0$, all items are given in [Se89], see also [Hej83, Kub73]. The extension of the first five properties to general n is given in [Kub73, Chapter 6] (see also [Sa81]). Property **(G1/2)** is extended to arbitrary n by Marklof and Strömbergsson in [MS03] (in [MS03] only the case of the integral over $[0,T]$ is discussed, but the proof extends verbatim to the interval $[-T,0]$). To the best of our knowledge, there is no presentation of property **(G1)** for general n in the literature. We fill this gap in the appendix to this paper, see Theorem A.1. □

4. A BOUND $O\left(R^{\frac{4}{3}}\right)$

The following is the main result of this section. It immediately implies Theorem 2.4.

Theorem 4.1. *Suppose Γ is a lattice in G, Γv is a discrete orbit corresponding to the i-th cusp of Γ, $E_i(g,s)$ is the corresponding Eisenstein series, and $s_0=1>s_1>\cdots>s_r>1/2$ are the poles of $E_i(g,\cdot)$. Then there are $c_0,\dots,c_r$ such that*

$$\mathbf{N}(g,R)=c_0R^2+\sum_{\ell=1}^{r}c_\ell R^{2s_\ell}+O\left(R^{\frac{4}{3}}\right).$$

Furthermore, if v is rescaled so that $\mathfrak{s}_i\mathbf{e}_1=v$, then the c_ℓ are the residues of $s\mapsto\frac{E_i(g,s)}{s}$ at the poles s_ℓ. In particular, the quadratic growth constant c_0 is given by formula (3.11)*.*

The basic idea for the proof of Theorem 4.1 is a 'contour shift' argument, as follows. We recall (3.5) and (3.9), which imply that for $\sigma > 1$, for a large parameter T, $\mathbf{N}(g,R) \approx \frac{1}{2\pi}\int_{-T}^{T} E(z,\sigma+\mathbf{i}t)\frac{R^{2(\sigma+\mathbf{i}t)}}{\sigma+\mathbf{i}t}dt$. This is a path integral over the line segment $L_{\sigma,T} = \{\sigma+\mathbf{i}t : t \in [-T,T]\}$ introduced in §3.1. Since $s \mapsto E(z,s)$ is meromorphic in all of $\mathbb{C}$, the Cauchy residue formula makes it possible to replace this path integral over $L_{\sigma,T}$ with a path integral over $L_{1/2,T}$ and the two horizontal segments $H^{\pm} = \{s \pm \mathbf{i}T : s \in [1/2,\sigma]\}$, taking into account the residues in the rectangle bounded by these segments. We need to show that the contribution of the integral over the segments $H^{\pm}$ is negligible, compute the contribution of the poles in the Cauchy formula, and evaluate the integral over $L_{1/2,T}$. Each of these steps presents difficulties as stated. To bypass them we recall that if $\chi = \chi_{[0,1]}$ denotes the indicator function of $[0,1]$ and $f(u) = \chi(\|u\|)$ then $\mathbf{N}(g,R) = \Theta_{f_R}(g\Gamma)$. We can justify the contour shift argument if f is replaced by a smooth compactly supported approximation $f^{(U)}$ (where U is an approximation parameter), and in this way, obtain bounds on the growth of $\Theta_{f_R^{(U)}}(g\Gamma)$ as $R \to \infty$. To make use of this we bound the difference $\left|\mathbf{N}(g,R) - \Theta_{f_R^{(U)}}(g\Gamma)\right|$ as well as the differences in the residues of the sums for f and $f^{(U)}$, and optimize the choice of U as a function of R to make the combined error as small as possible.

In order to justify the contour shift we will need the following:

Proposition 4.2. *Let E be a meromorphic function on $\mathbb{C}$, let $a < b$, and let Ψ be a holomorphic function defined in a neighborhood of $\{s \in \mathbb{C} : a \le \mathrm{Re}(s) \le b\}$, such that:*

(i) E has finitely many poles (s_ℓ) with $\mathrm{Re}(s) \in [a,b]$. They are all simple poles, all on the real line, and there are no poles at $s = a$ and $s = b$.
(ii) There is a function $\omega : \mathbb{R} \to \mathbb{R}$ satisfying the conclusions of Theorem 3.7 item (ω), and such that for all $|t| \ge 1$, $|E(s)| \ll \sqrt{\omega(t)}\,|t|$ (where $t = \mathrm{Im}(s)$).
(iii) For any $k > 0$ there is C' such that for all $\sigma \in [a,b]$,

$$|\Psi(\sigma+\mathbf{i}t)| \le \frac{C'}{t^k}.$$

(iv) For $\sigma = a$ and $\sigma = b$, the integrals

$$\int_{-\infty}^{\infty} E(\sigma+\mathbf{i}t)\,\Psi(\sigma+\mathbf{i}t)\,dt \tag{4.1}$$

converge absolutely.

Then

$$\frac{1}{2\pi\mathbf{i}}\int_{\mathrm{Re}(s)=b} E(s)\Psi(s)ds = \frac{1}{2\pi\mathbf{i}}\int_{\mathrm{Re}(s)=a} E(s)\Psi(s)ds + \sum_{\ell}\Psi(s_\ell)\mathrm{Res}|_{s=s_\ell}(E) \tag{4.2}$$

(where $\mathrm{Res}|_{s=s_0}h(s) = \lim_{s\to s_0}(s-s_0)h(s)$).

Proof. Let $\tau > 1$, and consider the integral of $E(s)\,\Psi(s)$ on the rectangle

$$R_\tau = \{s \in \mathbb{C} : \mathrm{Re}(s) \in [a,b], \mathrm{Im}(s) \in [\tau,\tau+1]\}.$$

We have

$$\begin{aligned}\int_\tau^{\tau+1}\left|\int_a^b E(\sigma+\mathbf{i}t)\Psi(\sigma+\mathbf{i}t)\,d\sigma\right|dt &\overset{(iii)}{\ll} \frac{1}{\tau^3}\int_\tau^{\tau+1}\int_a^b |E(\sigma+\mathbf{i}y)|\,dy\,d\sigma \\ &\overset{(ii)}{\ll} \frac{1}{\tau^3}\int_{-(\tau+1)}^{\tau+1}\sqrt{\omega(y)}\;y\,dy \\ &\overset{\text{Cauchy-Schwarz}}{\ll} \frac{1}{\tau^3}\left(\int_{-(\tau+1)}^{\tau+1}\omega(y)dy\right)^{1/2}\tau^{3/2} \\ &\overset{(\omega)}{\ll} \tau^{-1/2} \longrightarrow_{\tau\to\infty} 0.\end{aligned}$$

Hence for each $n \in \mathbb{N}$ there is $\tau_n \in [n, n+1]$ such that

$$\left| \int_a^b E(\sigma + \mathbf{i}\tau_n)\Psi(\sigma + \mathbf{i}\tau_n)\, d\sigma \right| \to_{n\to\infty} 0. \tag{4.3}$$

By the same argument, there are $\tau_{-n} \in [-(n+1), -n]$ such that (4.3) also holds with τ_{-n} instead of τ_n.

Since Ψ is holomorphic, and E holomorphic outside a set of finitely many poles, we can now apply the Cauchy residue formula for the contour integral of $E(s)\Psi(s)$ over the boundary of the rectangle

$$\{s \in \mathbb{C} : \mathrm{Re}(s) \in [a, b], \mathrm{Im}(s) \in [\tau_{-n}, \tau_n]\}.$$

The intergrals along the horizontal boundaries $[a, b] \times \{\tau_{\pm n}\}$ go to 0 as $n \to \infty$, and the integrals along the vertical boundaries $\{a, b\} \times [\tau_{-n}, \tau_n]$ tend to the integrals in (4.1). The result follows.

$\square$

Proof of Theorem 4.1. Let $\beta : \mathbb{R} \to [0, 1]$ be a smooth function satisfying

$$\beta(x) = \begin{cases} 0 & \text{for } x \le 0.1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

and for a parameter $U > 2$ let

$$\psi^{-(U)} = \begin{cases} \beta(Ux) & x \le 1/2 \\ \beta(U(1-x)) & x \ge 1/2 \end{cases} \quad \text{and} \quad \psi^{+(U)} = \begin{cases} \beta(Ux) & x \le 1/2 \\ \beta(1 + U(1-x)) & x \ge 1/2 \end{cases}.$$

Let χ denote the indicator function of $[0, 1]$. Our choices imply that $\psi^{\pm(U)}$ are supported on a compact subset of the positive real line and satisfy

$$\begin{aligned} x \ge 0 \quad &\Longrightarrow \quad \psi^{-(U)}(x) \le \chi(x), \\ x \ge \frac{1}{U} \quad &\Longrightarrow \quad \chi(x) \le \psi^{+(U)}(x), \\ x \notin \left[0, \frac{1}{U}\right] \cup \left[1 - \frac{1}{U}, 1 + \frac{1}{U}\right] \quad &\Longrightarrow \quad \psi^{-(U)}(x) = \chi(x) = \psi^{+(U)}(x), \\ \sup_{x \in \mathbb{R}} \frac{d^\ell \psi^{\pm(U)}}{dx^\ell} \quad &= \quad O\left(U^\ell\right). \end{aligned} \tag{4.4}$$

From (4.4) we have

$$\psi^{-(U)}\left(\frac{\|u\|}{R}\right) \le \chi\left(\frac{\|u\|}{R}\right) \le \chi\left(\frac{\|u\|U}{R}\right) + \psi^{+(U)}\left(\frac{\|u\|}{R}\right).$$

Thus if we define $f^{\pm}_{R,U}(u) = \psi^{\pm(U)}\left(\frac{\|u\|}{R}\right)$, since the Θ-transform is order-preserving, we have

$$\Theta_{f^-_{R,U}}(g\Gamma) \le \mathbf{N}(g, R) \le \Theta_{f^+_{R,U}}(g\Gamma) + \mathbf{N}\left(g, \frac{R}{U}\right). \tag{4.5}$$

We will obtain bounds for $\Theta_{f^\pm_{R,U}}(g\Gamma)$ using (3.12) and a contour shift argument, and then combine this with (4.5) and optimize the choice of $U = U(R)$ to obtain good bounds for $\mathbf{N}(g, R)$. To simplify notation we omit the superscript $\pm$ from now, that is $f_{R,U}$ stands for any one of $f^\pm_{R,U}$ and $\Psi^{(U)}$ stands for the Mellin transform of any one of the $\psi^{\pm(U)}$.

Step 1. Dependence of the residues on the approximation parameter. Let c'_ℓ be the residue of $s \mapsto E_i(g, s)$ at $s = s_\ell$, let c_ℓ be the residue of $s \mapsto \frac{E_i(g,s)}{s}$ at $s = s_\ell$ and let $c_\ell(U)$ be the residue of $s \mapsto \Psi^{(U)}(s) E_i\left(g, \frac{s}{2}\right)$ at $s = 2s_\ell$. The c'_ℓ are nonzero by **(P)** and the s_ℓ satisfy $s_\ell > 1/2$, and we have $c_\ell = \frac{c'_\ell}{s_\ell}$ and $c_\ell(U) = 2c'_\ell\, \Psi^{(U)}(2s_\ell)$. By (4.4), we have

$$\begin{aligned}\left|\frac{1}{2s_\ell} - \Psi^{(U)}(2s_\ell)\right| &\le \left|\int_0^1 y^{2s_\ell - 1} dy - \int_0^\infty \psi^{(U)}(y) y^{2s_\ell-1} dy\right| \\ &\le \int_0^\infty \left|\psi^{(U)}(y) - \chi(y)\right| y^{2s_\ell-1} dy \\ &\overset{(4.4)}{\le} 2\left(\int_0^{1/U} y^{2s_\ell-1} dy + \int_{1-1/U}^{1+1/U} y^{2s_\ell-1} dy\right) = O\left(\frac{1}{U}\right),\end{aligned}$$

and thus

$$c_\ell - c_\ell(U) = 2c'_\ell\left(\frac{1}{2s_\ell} - \Psi^{(U)}(2s_\ell)\right) = O\left(\frac{1}{U}\right). \tag{4.6}$$

Step 2. Bounding the integral over $\mathrm{Re}(s) = 1$. We will bound $\int_{\mathrm{Re}(s)=1} \Psi^{(U)}(s) E_i\left(g, \frac{s}{2}\right) R^s ds$ in terms of R and U, and to this end we will bound

$$I = \int_1^\infty \Psi^{(U)}(1 + \mathbf{i}t) E_i\left(g, \frac{1+\mathbf{i}t}{2}\right) R^{1+\mathbf{i}t} dt.$$

We first prove that for all $U \ge 2$,

$$|t| \ge U \implies \Psi^{(U)}(1 + \mathbf{i}t) = O\left(\frac{U}{|t|^2}\right) \tag{4.7}$$

and

$$1 \le |t| \le U \implies \Psi^{(U)}(1 + \mathbf{i}t) = O\left(\frac{1}{|t|}\right). \tag{4.8}$$

Moreover we will establish such a bound for $\sigma + \mathbf{i}t$ in place of $1 + \mathbf{i}t$, where the implicit constant is uniform as long as σ varies in a closed interval of positive reals. To see (4.7), apply integration by parts twice, and use that $\psi^{(U)}$ and all its derivatives vanish for $y \notin \left(0, \frac{1}{U}\right] \cup \left[1 - \frac{1}{U}, 1 + \frac{1}{U}\right]$, to obtain for $s = \sigma + \mathbf{i}t$:

$$\begin{aligned}\left|\Psi^{(U)}(\sigma + \mathbf{i}t)\right| &= \left|\int_0^\infty \left[\psi^{(U)}\right]'(y) \frac{y^s}{s} dy\right| = \left|\int_0^\infty \left[\psi^{(U)}\right]''(y) \frac{y^{s+1}}{s(s+1)} dy\right| \\ &= \left|\int_0^{1/U} \left[\psi^{(U)}\right]''(y) \frac{y^{s+1}}{s(s+1)} dy + \int_{1-1/U}^{1+1/U} \left[\psi^{(U)}\right]''(y) \frac{y^{s+1}}{s(s+1)} dy\right| \\ &= 2^{\sigma+1}\, O\left(U^2\right)\, O\left(\frac{1}{U|t(t+1)|}\right) = O\left(\frac{U}{|t|^2}\right),\end{aligned}$$

proving (4.7).

Now if $1 \le t \le U$ and $U \ge 2$ then

$$\begin{aligned}\left|\Psi^{(U)}(\sigma + \mathbf{i}t)\right| &= \left|\int_0^\infty \left[\psi^{(U)}\right]'(y) \frac{y^s}{s} dy\right| \\ &= \left|\int_0^{1/U} \left[\psi^{(U)}\right]'(y) \frac{y^s}{s} dy + \int_{1-1/U}^{1+1/U} \left[\psi^{(U)}\right]'(y) \frac{y^s}{s} dy\right| \\ &= 2^{\sigma+1}\, O(U)\, O\left(\frac{1}{U}\right) O\left(\frac{1}{\sqrt{\sigma^2 + t^2}}\right) = O\left(\frac{1}{t}\right),\end{aligned} \tag{4.9}$$

proving (4.8).

Writing $E(t)$ for $E_i\left(g, \frac{1+\mathbf{i}t}{2}\right)$ and $\Psi(t)$ for $\Psi^{(U)}(1 + \mathbf{i}t)$, this leads to

$$|I| \le \int_1^\infty \left|\Psi(t) E(t) R^{1+\mathbf{i}t}\right| dt \ll R\left[\int_1^U |\Psi(t)E(t)|\, dt + \int_U^\infty |\Psi(t)E(t)|\, dt\right] \tag{4.10}$$

and we bound each of these integrals separately. By assumption **(G1/2)** and Cauchy-Schwarz, for $1 \leq A \leq B$,

$$\int_A^B |E(t)| dt \leq \sqrt{\int_A^B |E(t)|^2 dt} \sqrt{\int_A^B 1\, dt} \ll B^{\frac{3}{2}}. \tag{4.11}$$

Hence, by a dyadic decomposition,

$$\begin{aligned}
\int_1^U |E(t)|\ |\Psi(t)|\, dt &\overset{(4.8)}{\ll} \int_1^U \frac{|E(t)|}{t} dt \\
&\leq \sum_{0 \leq k \ll \log U} \frac{1}{U/2^{k+1}} \int_{U/2^{k+1}}^{U/2^k} |E(t)| dt \\
&\overset{(4.11)}{\ll} \sum_{0 \leq k \ll \log U} \frac{1}{U/2^k} \left(\frac{U}{2^k}\right)^{\frac{3}{2}} \\
&\ll U^{\frac{1}{2}} \sum_{k \geq 0} 2^{-\frac{k}{2}} \ll U^{\frac{1}{2}}.
\end{aligned}$$

Similarly, in the range $t \geq U$ we have

$$\begin{aligned}
\int_U^\infty |E(t)|\ |\Psi(t)|\, dt &\overset{(4.7)}{\ll} \int_U^\infty |E(t)| \frac{U}{t^2} dt \\
&\ll U \sum_{k \geq 0} \frac{1}{(2^k U)^2} \int_{2^k U}^{2^{k+1} U} |E(t)| dt \\
&\overset{(4.11)}{\ll} U \sum_{k \geq 0} \frac{1}{(2^k U)^2} \left(2^{k+1} U\right)^{\frac{3}{2}} \\
&\ll U^{\frac{1}{2}} \sum_{k \geq 0} 2^{-\frac{k}{2}} \ll U^{\frac{1}{2}}.
\end{aligned}$$

Putting these estimates together we obtain

$$I = O\left(R U^{\frac{1}{2}}\right).$$

The bound on the ray $\{1 + \mathbf{i}t : t \leq -1\}$ is similar, and on the finite interval $\{1 + \mathbf{i}t : -1 \leq t \leq 1\}$ the functions E_i and $\Psi^{(U)}$ are bounded independently of U. For the last claim, note that the calculation in (4.9) holds also for $0 \leq t \leq 1$, and the second to last equality there implies boundedness. In total we find

$$\int_{\mathrm{Re}(s)=1} \Psi^{(U)}(s) E_i\left(g, \frac{s}{2}\right) R^s ds = O\left(R U^{\frac{1}{2}}\right). \tag{4.12}$$

Step 3. Justifying the contour shift. We want to show that for any R, U,

(4.13)

$$\int_{\mathrm{Re}(s)=\sigma>2} \Psi^{(U)}(s) E_i\left(g, \frac{s}{2}\right) R^s ds = \sum_\ell c_\ell(U) R^{2s_\ell} + \int_{\mathrm{Re}(s)=1} \Psi^{(U)}(s) E_i\left(g, \frac{s}{2}\right) R^s ds,$$

where $\Psi^{(U)}$ is the Mellin transform of $\psi^{(U)}$, the sum ranges over the poles (s_ℓ) of $E_i(g, \cdot)$, and $c_\ell(U)$ are the corresponding residues. This follows from Proposition 4.2, with $a = 1, b = \sigma > 2$, $E(s) = E_i\left(g, \frac{s}{2}\right)$, and $\Psi(s) = R^s \Psi^{(U)}(s)$. Note that by (3.8), for upper bounds as needed for Proposition 4.2, it makes no difference if one works with the twisted Eisenstein series in (2.4) or in (3.7). Hypotheses (i) and (ii) of Proposition 4.2 hold by Theorem 3.7, (iii) follows by repeated integration by parts as in the proof of (4.7), and so we need to show (iv). The case $b = \sigma$ is trivial because $t \mapsto E(b + \mathbf{i}t)$ is bounded, and the case $\sigma = 1$ was proved in Step 2. Thus (iv) holds.

Step 4. Combining bounds. Using (4.13) with a main term $\sum_\ell c_\ell R^{2s_\ell}$, and matching the errors incurred in (4.6) and (4.12) gives an error estimate

$$\max_\ell \frac{R^{2s_\ell}}{U} = \frac{R^2}{U} = RU^{\frac{1}{2}}.$$

This leads to a choice $U = R^{\frac{2}{3}}$ and the combined error becomes $O\left(R^{\frac{4}{3}}\right)$. This error is valid when using either one of $f^-_{R,U}$ and $f^+_{R,U}$. Proposition 3.2 implies that

$$\mathbf{N}\left(g, \frac{R}{U}\right) = O\left(\frac{R^2}{U^2}\right) = R^{\frac{2}{3}}.$$

Thus appealing to (3.12) and (4.5) completes the proof. □

Remark 4.3. We are grateful to Ze'ev Rudnick for explaining to us how to replace our earlier result $O\left(R^{\frac{4}{3}+\varepsilon}\right)$ with $O\left(R^{\frac{4}{3}}\right)$. Specifically, Rudnick suggested the use of dyadic decomposition in Step 2.

Remark 4.4. Any improvement in the bound **(G1/2)** gives a corresponding improvement in the error term. In fact, for the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, or its principal congruence subgroups, and $n = 0$, one can replace the term T^2 appearing in the right hand side of **(G1/2)** by T. Using this, and modifying (4.7) to a bound $O\left(\frac{1}{|t|^{1+\varepsilon}}\right)$ in Step 2, yields an error term $O\left(R^{1+\varepsilon}\right)$ for any $\varepsilon > 0$, in place of $O\left(R^{\frac{4}{3}}\right)$. The recent papers [HX16] and [Nor19] contain sup norm bounds for Eisenstein series for some arithmetic groups Γ which are not principal congruence subgroups. These bounds lead to improvements for **(G1/2)**, and using them, one obtains a better estimate than $R^{\frac{4}{3}}$ in (1.2) for the discrete orbits arising in these cases.

5. More general shapes

5.1. **Counting in smooth star shaped domains.** We first state a more detailed version of the first part of Theorem 2.6, for counting in a smooth star shape.

Theorem 5.1. *Suppose Γ is a non-uniform lattice in G containing $-\mathrm{Id}$, $Y = \Gamma v$ is a discrete orbit corresponding to the i-th cusp, and suppose that for each n, $E_i(z,s)_n$ has trivial residual spectrum. Let $\rho : \mathbb{R} \to \mathbb{R}_+$ be a smooth 2π-periodic function, $S = \{re^{\mathbf{i}\theta} : 0 \le r \le \rho(\theta)\}$, and let $c_{Y,S} = \frac{\mathrm{vol}(S)}{\pi\,\mathrm{covol}(\Gamma)}$. Then for every $\varepsilon > 0$,*

$$|Y \cap RS| = c_{Y,S}R^2 + O\left(R^{\frac{12}{7}+\varepsilon}\right),$$

where the implicit constant depends on ε and ρ.

Proof. The proof follows the same steps as in the proof of Theorem 4.1. We define the same approximants $\psi^{\pm(U)}$ of the indicator function of the unit interval, so that $x \mapsto \psi^{\pm(U)}\left(\frac{x}{\rho(\theta)}\right)$ are approximations of the indicator function of the interval $[0, \rho(\theta)]$, in the sense of (4.4). Then we set

$$f^{\pm}_{R,U}(re^{\mathbf{i}\theta}) = \psi^{\pm(U)}\left(\frac{r}{\rho(\theta)\,R}\right),$$

so that in analogy with (4.5), we have

(5.1)
$$\Theta_{f^-_{R,U}}(g\Gamma) \le |Y \cap RS| \le \Theta_{f^+_{R,U}}(g\Gamma) + N\left(g, R\,\frac{\max_\theta \rho(\theta)}{U}\right) \le \Theta_{f^+_{R,U}}(g\Gamma) + O\left(\frac{R^2}{U^2}\right).$$

As before we continue with $f_{R,U}$ standing for one of the $f^{\pm}_{R,U}$ and $\Psi^{(U)}$ standing for the Mellin transform of one of the $\psi^{\pm(U)}$. Using (3.13) we have

$$\Theta_{f_{R,U}}(g\Gamma) = \frac{1}{2\pi\mathbf{i}} \sum_{n\in\mathbb{Z}} \int_{\mathrm{Re}(s)=\sigma} \Psi^{(U)}(s)\hat{\rho}_{-n}(s)E_i\left(g, \frac{s}{2}\right)_n R^s\, ds \quad (\text{where } \sigma > 2). \tag{5.2}$$

Since we have assumed that $-\mathrm{Id} \in \Gamma$, the terms corresponding to odd n all vanish. For each $n \neq 0$, the functions $s \mapsto \Psi^{(U)}(s)\hat{\rho}_{-n}(s)E_i\left(g, \frac{s}{2}\right)_{2n}$ are holomorphic on $\{s \in \mathbb{C} : \mathrm{Re}(s) \geq 1\}$ by our assumption that all of the $E_i(g,s)_{2n}$ have trivial residual spectrum and by **(P)** and **(1)**. For $n = 0$, the function $s \mapsto \Psi^{(U)}(s)E_i\left(g, \frac{s}{2}\right)$ has a simple pole at $s = 2$, and by a computation as in the proof of (4.6), the residue $c_0(U)$ satisfies

$$c_0 - c_0(U) = O\left(\frac{1}{U}\right). \tag{5.3}$$

Here $c_0 = \hat{\rho}_0(2)\,\mathrm{covol}(\Gamma)^{-1}$, and since $\hat{\rho}_0(2) = \frac{1}{2\pi}\int_0^{2\pi} \rho(\theta)^2\, d\theta$, computing the area of S in polar coordinates we obtain $c_0 = \frac{\mathrm{Vol}(S)}{\pi\,\mathrm{covol}(\Gamma)}$.

We now bound the integral of

$$s \mapsto h^{(U)}(s)E_i\left(g, \frac{s}{2}\right)_n, \quad \text{where } h^{(U)}(s) = \Psi^{(U)}(s)\hat{\rho}_{-n}(s),$$

along the critical line $\{\mathrm{Re}(s) = 1\}$, by a bound depending on both U and n. Thus from now on implicit constants may depend on ρ but not on n and U. We will use parameters k, λ, ε which we will optimize further below.

For each $k \geq 0$ we have

$$\Psi^{(U)}(1 + \mathbf{i}t) \ll \frac{U^k}{|t|^{k+1}}, \tag{5.4}$$

and for each $\lambda \geq 0$ and $n \neq 0$, we have

$$|\hat{\rho}_n(s)| \ll \frac{|t|^\lambda}{|n|^\lambda}, \quad \text{where } s = 1 + \mathbf{i}t. \tag{5.5}$$

Indeed, we get (5.4) for $|t| \geq U$ by performing integration by parts $\lfloor k \rfloor + 1$ times (see (4.7)), and for $|t| \leq U$ by applying integration by parts $\lfloor k \rfloor$ times. The proof of (5.5) is similar. Using this and recalling that for $|t| \leq 1$ the integral is bounded (see the discussion preceding (4.12)), we have

$$\begin{aligned}
&\int_{\mathrm{Re}(s)=1} h^{(U)}(s)E_i\left(g, \tfrac{s}{2}\right)_n ds \\
&\ll \int_1^\infty \frac{U^k}{t^{k+1}}\frac{t^\lambda}{|n|^\lambda}E_i\left(g, \tfrac{1+\mathbf{i}t}{2}\right)_n dt \\
&\ll \frac{U^k}{|n|^\lambda}\left(\int_1^\infty \left(t^{1+\varepsilon/2}\frac{t^\lambda}{t^{k+1}}\right)^2 dt\right)^{1/2}\left(\int_1^\infty \frac{|E_i(g, \frac{1+\mathbf{i}t}{2})_n|^2}{t^{2+\varepsilon}} dt\right)^{1/2}.
\end{aligned} \tag{5.6}$$

To ensure finiteness of the first integral we will assume that

$$2k > 2\lambda + 1 + \varepsilon. \tag{5.7}$$

For the second integral, we define $H(T) = \int_1^T \left|E_i\left(g, \frac{1+\mathbf{i}t}{2}\right)_n\right|^2 dt$, so that **(G1/2)** gives $H(T) \ll (T+n)^2$. Then integration by parts gives

$$\int_1^\infty \frac{|E\left(g, \frac{1+\mathbf{i}t}{2}\right)_n|^2}{t^{2+\varepsilon}}\, dt \ll |n|^2.$$

Using these estimates in (5.6) gives

$$\int_{\mathrm{Re}(s)=1} \Psi^{(U)}(s)E_i\left(g, \frac{s}{2}\right)_n R^s\hat{\rho}_{-n}(s)\, ds \ll |n|^{1-\lambda}RU^k, \tag{5.8}$$

and the implicit constant depends on ε.

For each fixed n, the contour shift replacing the integral along $\mathrm{Re}(s) = \sigma$ with the integral along the line $\mathrm{Re}(s) = 1$ is justified by Proposition 4.2 (note that in condition (ii) of the

Proposition, the implicit constants are allowed to depend on E and thus on n). We only pick up one residue, corresponding to $n=0$ and $s=2$. Thus collecting estimates we get

$$\begin{aligned}\Theta_{f_{R,U}}(g\Gamma) =& c_0R^2 + O\left(\frac{R^2}{U}\right) + O\left(RU^{1/2}\right) + \left(\sum_{n\in\mathbb{Z}\smallsetminus\{0\}} |n|^{1-\lambda}\right) O\left(RU^k\right) \\ =& c_0R^2 + O\left(\frac{R^2}{U}\right) + O\left(RU^{1/2}\right) + O\left(RU^k\right),\end{aligned} \tag{5.9}$$

where we have used the bound (4.12) for $n=0$, and where we set $\lambda = 2+\varepsilon$ to ensure convergence of the infinite series. Setting $k=\frac{5}{2}+2\varepsilon$ ensures (5.7), and setting the two error terms equal to each other gives $U = R^{\frac{1}{k+1}}$, which also ensures that that last term in (5.1) is negligible. Thus (5.9) becomes

$$\Theta_{f_{R,U}}(g\Gamma) = c_0R^2 + O\left(R^{2-1/(\frac{7}{2}+2\varepsilon)}\right),$$

completing the proof. □

Remark 5.2. 1. As before, any improvement in the dependence on n, of the bound **(G1/2)**, would lead to a corresponding improvement in the error estimate. For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and principal congruence subgroups, this improvement leads to an error estimate $O\left(R^{\frac{5}{3}+\varepsilon}\right)$.

2. We do not prove a version of Theorem 5.1 for lattices for which the twisted Eisenstein series has nontrivial poles. If such poles s_ℓ existed, in performing the contour shift argument, one would need to analyze the sum $\sum_{n\in\mathbb{Z}} \mathrm{Res}|_{s=s_\ell} E_i\left(g,\frac{s}{2}\right)_{2n}$. As far as we are aware, this series is only known to be summable in the sense of distributions, and thus analyzing it leads to technical issues we prefer not to enter into.

3. The assumption $-\mathrm{Id}\in\Gamma$ ensured that we only need **G(1/2)** for n even, which is the context in which it was proved in [MS03]. We are not aware of a proof of **(G1/2)** in the literature for n odd.

For the proof of Theorem 2.6 we will need another construction which interestingly is also due to Selberg, see [M94, Chap. 1, §2].

Proposition 5.3. *For each interval $J\subset\mathbb{R}$ and each $V\in\mathbb{N}$ there are trigonometric polynomials $P^\pm = P^\pm_{J,V}$ such that*

(1) For all $x\in\mathbb{R}$,
$$P^-(x)\le\chi_J(x)\le P^+(x)$$
(where χ_J is the indicator function of J);

(2) The degree of $P^\pm$ is at most V;

(3) for each $0<|k|\le V$, the k-th Fourier coefficient satisfies $|\widehat{P^\pm}_k|\ll\frac{1}{k}$; and

(4) $\left|\left(\widehat{P^\pm}\right)_0 - (\widehat{\chi_J})_0\right| \ll \frac{1}{V}$.

We now state a more detailed version of the second part of Theorem 2.6, for counting in a sector.

Theorem 5.4. *Suppose Γ is a non-uniform lattice in G, $Y=\Gamma v$ is a discrete orbit corresponding to the i-th cusp, and suppose that for each n, $E_i(z,s)_n$ has trivial residual spectrum. Suppose also that $-\mathrm{Id}\in\Gamma$. Let $J\subset\mathbb{R}$ be an interval of length $|J|\le 2\pi$, let $S=\{re^{\mathrm{i}\theta}: 0\le r\le 1,\ \theta\in J\}$, and let $c_{Y,S}=\frac{|J|}{2\pi\,\mathrm{covol}(\Gamma)}$. Then*

$$|Y\cap RS| = c_{Y,S}R^2 + O\left(R^{\frac{8}{5}}\right), \tag{5.10}$$

where the implicit constant depends on J.

Proof. We follow the same steps with the same notations, but now we introduce an additional approximation parameter V, and let $\rho^{\pm(V)}(\theta)$ be approximations of the indicator function χ_J of J, namely they will satisfy

$$\forall s\in\mathbb{R}/2\pi\mathbb{Z},\quad \rho^{-(V)}(\theta)\le\chi_J(\theta)\le\rho^{+(V)}(\theta), \tag{5.11}$$

so that $re^{\mathbf{i}\theta} \mapsto \rho^{\pm(V)}(\theta)\psi^{\pm(U)}(r)$ are approximations of the indicator function of S. Then we set

$$f^{\pm}_{R,U,V}(re^{\mathbf{i}\theta}) = \rho^{\pm(V)}(\theta)\,\psi^{\pm(U)}\left(\frac{r}{R}\right),$$

so that

$$\Theta_{f^-_{R,U,V}}(g\Gamma) \le |Y \cap RS| \le \Theta_{f^+_{R,U,V}}(g\Gamma) + O\left(\frac{R^2}{U^2}\right). \tag{5.12}$$

As before to lighten notation we omit the superscripts for upper and lower bounds. Using (3.14) we have

$$\Theta_{f_{R,U,V}}(g\Gamma) = \frac{1}{2\pi\mathbf{i}}\sum_{n\in\mathbb{Z}} \hat{\rho}_{-n,V}\int_{\mathrm{Re}(s)=\sigma}\Psi^{(U)}(s)E_i\left(g,\frac{s}{2}\right)_n R^s\,ds,$$

where $\hat{\rho}_{n,V}$ is the nth Fourier coefficient of $\rho^{(V)}$ and $\sigma > 2$.

For each n we perform a contour shift to shift the integral to the line $\{\mathrm{Re}(s) = 1\}$, justifying it with Proposition 4.2. By our assumption on Γ, the only residue $c_0(U,V)$ that we need to take into account occurs for $n = 0$ at $s = 2$. Setting $c_0 = \frac{|J|}{2\pi\mathrm{covol}(\Gamma)}$, the residue satisfies

$$c_0 - c_0(U,V) = O\left(\frac{1}{U} + \left|\hat{\rho}_{0,V} - \frac{|J|}{2\pi}\right|\right). \tag{5.13}$$

Motivated by this, for the functions $\rho^{\pm(V)}$ we use the polynomials $P^{\pm}_{J,V}$ of Proposition 5.3 with $U = V$. With this choice, using item (4) of Proposition 5.3, (5.13) becomes

$$c_0 - c_0(U,V) = O\left(\frac{1}{U}\right),$$

and we get a bound

$$\sum_{n\in\mathbb{Z}}|n||\hat{\rho}_{n,V}| \ll \sum_{0<|n|\le U}|n|\left|\frac{1}{n}\right| \ll U.$$

We now repeat the arguments in Step 2 of the proof of Theorem 4.1 to obtain

$$\int_{\mathrm{Re}(s)=1}\Psi^{(U)}(s)E_i\left(g,\frac{s}{2}\right)_n R^s\,ds \ll |n|RU^{\frac{1}{2}}.$$

Note the explicit dependence on n which arises by using **(G1/2)** in (4.11).

Collecting estimates we get

$$\Theta_{f_{R,U}}(g\Gamma) = c_0R^2 + O\left(\frac{R^2}{U}\right) + O\left(RU^{\frac{3}{2}}\right),$$

and equating the two error terms and plugging into (5.12) leads easily to (5.10). □

5.2. **Counting in well-rounded sets.** In the present section we will prove Theorem 2.7, using an argument based on a general lattice point counting result. In order to state it, we recall the following:

Definition 5.5. ([GN12]) Let G be a connected Lie group with Haar measure m_G. Assume $\{G_t\} \subset G$ is a family of bounded Borel sets of positive measure such that $m_G(G_t) \to \infty$ as $t \to \infty$. Let $\mathcal{O}_\eta \subset G$ be the image of a ball of radius η (with respect to the Cartan-Killing norm) in the Lie algebra under the exponential map. Denote

$$G_t^+(\eta) = \mathcal{O}_\eta G_t \mathcal{O}_\eta = \bigcup_{u,v\in\mathcal{O}_\eta} u\,G_t\,v, \quad G_t^-(\eta) = \bigcap_{u,v\in\mathcal{O}_\eta} u\,G_t\,v.$$

The family $\{G_t\}$ is *Lipschitz well-rounded* if there exist positive c, η_0, t_0 such that for every $0 < \eta \le \eta_0$ and $t \ge t_0$,

$$m_G\left(G_t^+(\eta)\right) \le (1 + c\eta)\,m_G\left(G_t^-(\eta)\right).$$

Theorem 5.6. [GN12] *Let G be any connected almost simple non-compact Lie group (e.g. $\mathrm{SL}_2(\mathbb{R})$), and let $\{G_t\}$ be a Lipschitz well-rounded family of subsets of G. Let Γ be a lattice in G, and let m_G be Haar measure on G, normalized so that* $\mathrm{covol}(\Gamma) = 1$. *Define the corresponding averaging operators*

$$\beta_t(f)(x) = \frac{1}{m_G(G_t)} \int_{G_t} f(g^{-1}x)dm_G(g), \quad f \in L^2_0(G/\Gamma).$$

Suppose the β_t satisfy the following (operator-norm) bound:

$$\|\beta_t\|_{L^2_0(G/\Gamma)} \le C m_G(G_t)^{-\kappa}.$$

Then the lattice point counting problem in G_t has the effective solution

$$\frac{|\Gamma \cap G_t|}{m_G(G_t)} = 1 + O\left(m_G(G_t)^{-\frac{\kappa}{\dim G+1}}\right).$$

The proof of Theorem 2.7 proceeds by reducing the problem of counting points in the orbit Γv lying in bounded subsets of the plane, to counting lattice points in suitable bounded domains in the group $G = \mathrm{SL}_2(\mathbb{R})$. The domains constructed in $\mathrm{SL}_2(\mathbb{R})$ bijectively cover the domains in the plane, under the orbit map $g \mapsto gv$, and will depend non-trivially on the orbit under consideration, and not just on Γ.

In this section we write

$$a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Let $v \in \mathbb{R}^2 \smallsetminus \{0\}$ and use polar coordinates in the plane to write $g = r_{\theta_v} a_{t_v}$ where $v = g\mathbf{e}_1 = r_{\theta_v} a_{t_v} \mathbf{e}_1 = e^{t_v/2} r_{\theta_v} \mathbf{e}_1$. The stability group of v is $N^g = gNg^{-1}$, and for any t, x and θ,

$$r_\theta(g a_t g^{-1})(g n_x g^{-1})(v) = r_\theta g a_t \mathbf{e}_1 = e^{t/2} r_\theta v = e^{\frac{1}{2}(t+t_v)} r_{\theta+\theta_v} \mathbf{e}_1. \tag{5.14}$$

This gives a bijective parameterization of $\mathbb{R}^2 \smallsetminus \{0\}$ by $\mathbb{R} \times [0, 2\pi)$, with each pair $(t,\theta) \in \mathbb{R} \times [0, 2\pi)$ determining a unique vector $r_\theta g a_t \mathbf{e}_1 = e^{t/2} r_\theta v$ in $\mathbb{R}^2 \smallsetminus \{0\}$. Since $\mathbb{R}^2 \smallsetminus \{0\} = G/N^g$, we conclude that $G = KA^gN^g$, and this decomposition gives unique coordinates to each point in G. Note however that this is *not* an Iwasawa decomposition, the latter being given by $G = K^gA^gN^g$.

Let us denote

$$A_{t_1,t_2} = \{a_t : t_1 \le t \le t_2\}, \; N_{x_1,x_2} = \{n_x : x_1 \le x \le x_2\} \quad \text{and } K_{\theta_1,\theta_2} = \{r_\theta : \theta_1 \le \theta \le \theta_2\}.$$

Let $D \subset \mathbb{R}^2$ be a compact set in the Euclidean plane given in polar coordinates by

$$D = \{\rho(\cos\theta, \sin\theta) \,:\, \theta \in I \;,\; 0 \le \rho \le \rho(\theta)\}$$

where $I = [\theta_1, \theta_2] \subset [0, 2\pi]$ is an interval of angles contained in the unit circle, and $\rho(\theta)$ is a positive Lipschitz continuous function on the interval I. The set D can also be written in the form

$$D = \{r_\theta a_t \mathbf{e}_1 \,:\, \theta \in I \;,\; t \le 2\log\rho(\theta)\} \cup \{0\}.$$

Let $b' > 0$ be such that Γv contains no points of norm less than b'. For any $T \ge 1$ consider as before the dilated set $T \cdot D = \{Tx : x \in D\}$, and also the set

$$D_T = \{r_\theta a_t \mathbf{e}_1 \,:\, \theta \in I \;,\; 2\log b' \le t \le 2\log(T\rho(\theta))\}.$$

Then $D_T = T \cdot D \smallsetminus B(0, b')$, and hence $|\Gamma v \cap D_T| = |\Gamma v \cap T \cdot D|$.

We will now define bounded domains $\widetilde{D}_T \subset G$ which bijectively cover D_T. Fix a positive number $x_0 = x_0(g)$ so that the set $N^g(x_0) = \left\{g n_x g^{-1} : 0 \le x < x_0\right\}$ is a fundamental domain for the subgroup $\Gamma \cap N^g \cong \mathbb{Z}$ in the group $N^g \cong \mathbb{R}$. For each $\theta \in I$ define

$$J(T,\theta) = \left\{t \in \mathbb{R} \,:\, 2\log\frac{b'}{\|v\|} \le t \le 2\log\frac{T\rho(\theta)}{\|v\|}\right\} = [t_1, t_2(T,\theta)]$$

and with respect to the decomposition $G = KA^gN^g$, define

$$\widetilde{D}_T = \left\{r_{\theta-\theta_v} g a_t g^{-1} \,:\, \theta \in I, t \in J(T,\theta)\right\} \cdot N^g(x_0).$$

Then, as the reader may verify using (5.14), the orbit map $G \to \mathbb{R}^2, g \mapsto gv$, restricted to $\Gamma \cap \widetilde{D}_T$, is a bijection with its image $\Gamma v \cap D_T$, and as a consequence we obtain:

Lemma 5.7. $|\Gamma v \cap T \cdot D| = |\Gamma \cap \widetilde{D}_T|$.

To complete the proof of Theorem 2.7, it remains to prove that the family $\{\widetilde{D}_T\}$ is Lipschitz well-rounded. It will be convenient to use the following two facts.

We will need the following result:

Lemma 5.8. *(see* [GN12]*) If $\{G_t\}$ is a Lipschitz well-rounded family of subsets of G, then for each $g, h \in G$, so are the families $\{gG_t\}$, $\{G_t g\}$ and $\{gG_t h\}$. Furthermore the corresponding constants c, t_0, η_0 are bounded above and away from zero, as g varies on a compact set Q in G.*

It therefore suffices to prove that the sets

$$\widetilde{D}_T g = \{r_{\theta-\theta_v} g a_t n_x \,:\, \theta \in I, t \in J(T,\theta), 0 \le x < x_0\}$$

are Lipschitz well-rounded. Recalling that $g = r_{\theta_v} a_{t_v}$ and setting $T = e^\tau$, we have

$$\widetilde{D}_T g = \mathcal{C}_\tau = \{r_\theta a_{t+t_v} n_x \,:\, \theta \in I, t \in J(e^\tau,\theta), 0 \le x < x_0\}\,.$$

The Iwasawa coordinates $K \times A \times N \to G$ given by $(k,a,n) \mapsto kan$ satisfy the following Lipschitz property, established in [HoN17, Prop. 4.4]. For every fixed $S_0 \in \mathbb{R}$, there exist $C_1 = C_1(S_0) > 0$ and $\eta_1 = \eta_1(S_0) > 0$, such that for all θ, all x with $0 \le x \le x_0$, and all $t \ge S_0$, $0 < \eta < \eta_1$:

$$\mathcal{O}_\eta r_\theta a_t n_x \mathcal{O}_\eta \subset K_{\theta - C_1\eta, \theta + C_1\eta} A_{t - C_1\eta, t + C_1\eta} N_{x - C_1\eta, x + C_1\eta}\,. \tag{5.15}$$

Let $S_0 = 2\log b'$, let $C_1 = C_1(S_0)$, and let $C = C_1 L$, where L is Lipschitz constant of the function ρ. Finally for

$$\eta < \min\left\{\frac{\eta_1}{4C}, \frac{\theta_2 - \theta_1}{4C}, \frac{x_0}{4C}\right\},$$

let

$$W^-(\tau,\eta) = \left\{kan : k \in K_{\theta_1 + C\eta, \theta_2 - C\eta},\ a \in A_{t_1 + C\eta, t_2(e^\tau,\theta) - C\eta},\ n \in N_{C\eta, x_0 - C\eta}\right\}.$$

Applying (5.15) to $g \in W^-(\tau,\eta)$ it follows readily that $W^-(\tau,\eta) \subset C_\tau^-(\eta)$. A straightforward verification, using the explicit form of Haar measure in Iwasawa coordinates and the fact that ρ is Lipschitz, shows that $m_G(W^-(\tau,\eta)) \ge (1 - c_1\eta) \cdot m_G(\mathcal{C}_\tau)$, for a suitable $c_1 > 0$. In the other direction, note that $C_t^+(\eta) = \mathcal{O}_\eta \mathcal{C}_t \mathcal{O}_\eta$ is contained in

$$W^+(\tau,\eta) = \left\{kan : k \in K_{\theta_1 - C\eta, \theta_2 + C\eta},\ a \in A_{t_1 - C\eta, t_2(e^\tau,\theta) + C\eta},\ n \in N_{-C\eta, x_0 + C\eta}\right\},$$

where for $\theta \in [\theta_1 - C\eta, \theta_1] \cup [\theta_2, \theta_2 + C\eta]$ we define $t_2(e^\tau,\theta) = \max(t_2(e^t,\theta_1), t_2(e^t,\theta_2))$. Again a similar direct verification shows that $m_G(W^+(\tau,\eta)) \le (1 + c_2\eta) \cdot m_G(\mathcal{C}_\tau)$. The Lipschitz well-roundedness of the family $\mathcal{C}_\tau$ follows, and this completes the proof of Theorem 2.7. $\square$

Remark 5.9.
(1) Let us note that an error estimate established for the count in the dilates $R \cdot S$ of any given figure (in the plane, say), immediately implies an error estimate for the count in *shells* of shrinking width, namely with the sets $R \cdot S \smallsetminus (R - R^{-\alpha}) \cdot S$ for a suitable range of positive parameters α. Similarly it is also possible to intersect shells with sectors of shrinking angle, namely with sets $\left\{r(\cos\theta, \sin\theta) \,:\, \theta \in [\theta_0 - R^{-\beta}, \theta_0], r \in \mathbb{R}_+\right\}$, for a suitable range of positive parameters β, and obtain an effective estimate. This follows from the fact that Theorem 5.6 allows counting in a variable family of domains, provided that their Lipschitz well-roundedness parameters are controlled.
(2) A straightforward modification of the proof of Theorem 2.7 applies to counting in discrete orbits of non-uniform lattices in $\mathrm{SL}_2(\mathbb{C})$ acting linearly on $\mathbb{C}^2$, as well as discrete orbits of non-uniform lattices in $\mathrm{SO}^0_d(n,1)$ acting linearly on $\mathbb{R}^{n+1}$. This is based on Theorem 5.6 and the Lipchitz property of the Iwasawa decomposition, established in [HoN17] for any non-exceptional group of real rank one.

Appendix A. The growth estimate (G1) for general n, and the function $\omega(t)$

An important input to our argument is the estimate **(G1)** which is used to bound the average growth of the Eisenstein series along vertical lines $\mathrm{Re}(s) = \sigma$, for $\sigma \in [1/2, 1]$. This is a crucial input to our method, see condition (ii) of Proposition 4.2. This growth estimate was proved by Selberg for $n = 0$ but as far as we are aware, does not appear in the literature for the twisted Eisenstein series for general n. In this appendix we close this gap in the literature, and also provide estimates of the dependence of the implicit constant on n. Many of our arguments are based on ideas in [CS80, Sa81, Iwa95, MS03].

We first introduce standard notation. Let $\Phi(s) = (\varphi_{ij}(s))_{ij}$ be the constant term matrix as in (3.10), let (s_ℓ) denote all the poles of the functions φ_{ij} in the interval $[1/2, 1]$, let $q = q_\Gamma > 0$ be a real number specified by Selberg (see [Se89, p. 655]) and set

$$\text{(A.1)}\quad \Psi_0(s) = \det \Phi(s), \quad \Psi^*(s) = q^{2s-1} \prod_\ell \frac{s - s_\ell}{s - 1 + s_\ell} \Psi_0(s), \quad \omega(t) = 1 - \frac{\Psi^{*'}}{\Psi^*}\left(\frac{1}{2} + \mathbf{i}t\right).$$

The function $\omega : \mathbb{R} \to \mathbb{R}$ thus defined is the function appearing in Theorem 3.7. It satisfies $\omega(t) > 1$ for all t (see [Se89, p. 656]).

Theorem A.1. *For any non-uniform lattice Γ in G, the twisted Eisenstein series $E_i(\cdot)_n$ corresponding to the i-th cusp as in* (3.7) *satisfies*

$$\text{(A.2)}\qquad \mathrm{Re}s \geq \frac{1}{2}, |t| \geq |n| + 1 \quad \Longrightarrow \quad E_i(z, s)_{2n} \ll |t| \sqrt{\omega(t)} \quad (\textit{where } t = \mathrm{Im}(s)),$$

where implicit constants depend on Γ and z (but not on n).

Proof. We will divide the proof into a series of steps. Throughout the proof we will write

$$\sigma = \mathrm{Re}(s),\ t = \mathrm{Im}(s), \quad \text{and} \quad \varepsilon = \sigma - \frac{1}{2},$$

and will assume $\sigma \in \left[\frac{1}{2}, \frac{3}{2}\right]$, which entails no loss of generality as $E_i(z, \cdot)_{2n}$ is uniformly bounded on $\left\{s : \mathrm{Re}(s) \geq \frac{3}{2}\right\}$. Implicit constants in the $\ll$ and O notation depend on Γ and z (and not on n or σ). Since some of our arguments will depend on the dependence of the Eisenstein series on the variable z, from now on we will write z_0 instead of z and consider it as a fixed element of $\mathbb{H}$.

Step 1. Bounding Φ and the $\mathbf{\Gamma}$-factor. Let $\Phi(s) = (\varphi_{ij}(s))_{i,j}$ be the constant term matrix as in (3.10). By [Se89, p. 655], Φ is uniformly bounded as long as $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$ and $|t| \geq 1$. Following [Kub73, Chapter 6], define

$$\Phi_{2n}(s) = (\varphi_{ij}(s)_{2n})_{i,j} \quad \text{by} \quad \Phi_{2n}(s) = (-1)^n B_n(s)\Phi(s),$$

where

$$B_n(s) = \frac{\mathbf{\Gamma}(s)^2}{\mathbf{\Gamma}(s-n)\mathbf{\Gamma}(s+n)}.$$

Then

$$\text{(A.3)}\qquad |t| \geq 1,\ \sigma \leq 1 \implies |B_n(s)| \leq 1,$$

with equality for $\sigma = \frac{1}{2}$, and

$$\text{(A.4)}\qquad |t| \geq |n| \quad \Longrightarrow \quad 1 - |B_n(s)|^2 \ll \varepsilon.$$

To see this, since $B_n(s) = B_{-n}(s)$ we can assume that $n > 0$. Using the recurrence formula $\mathbf{\Gamma}(z + 1) = z\mathbf{\Gamma}(z)$ one obtains

$$\text{(A.5)}\qquad B_n(s) = \frac{\mathbf{\Gamma}(s-n)^2\left((s-1)\dots(s-n)\right)^2}{(s-n)\dots(s+n-1)\mathbf{\Gamma}(s-n)^2} = \prod_{k=1}^{n} \frac{s-k}{s+k-1}.$$

Since $\sigma \geq \frac{1}{2}$, $|s - k| \leq |s + k - 1|$ with equality when $\sigma = \frac{1}{2}$. This implies (A.3).

For (A.4), set $z_{s,k} = s + k - 1 = k - \frac{1}{2} + \varepsilon + \mathbf{i}t$. Plugging into (A.5) gives

$$|B_n(s)|^2 = \prod_{k=1}^{n} \left| \frac{-\overline{z_{s,k}} + 2\varepsilon}{z_{s,k}} \right|^2 = \prod_{k=1}^{n} \left(\frac{|z_{s,k}|^2 - 4\varepsilon \mathrm{Re}(z_{s,k}) + 4\varepsilon^2}{|z_{s,k}|^2} \right)$$
$$= \prod_{k=1}^{n} \left(1 - 4\varepsilon \frac{\mathrm{Re}(z_{s,k}) - \varepsilon}{|z_{s,k}|^2} \right) = \prod_{k=1}^{\mathbf{N}} \left(1 - 2\varepsilon \frac{2k-1}{|z_{s,k}|^2} \right).$$

Write

$$f(k) = \frac{2k-1}{|z_{s,k}|^2}, \quad \text{so that} \quad F(n) = \sum_{k=1}^{n} \log(1 - 2\varepsilon f(k)) = \log\left(|B_n(s)|^2\right).$$

Let $k \leq n$ and $|t| \geq n$, then $f(k) \leq \frac{k}{n^2}$. Taking a second order Taylor approximation for $x \mapsto \log(1-x)$ we have

$$-\log(1 - 2\varepsilon f(k)) = 2\varepsilon f(k) + O\left(\varepsilon^2 f(k)^2\right),$$

and hence

$$-F(n) \ll \sum_{k=1}^{n} \varepsilon f(k) + \varepsilon^2 f(k)^2 \ll \varepsilon \sum_{k=1}^{n} \frac{k}{n^2} \ll \varepsilon.$$

Now by second order Taylor approximation for $x \mapsto 1 - e^x$ we get

$$1 - |B_n(s)|^2 = 1 - e^{F(n)} \ll -F(n) \ll \varepsilon,$$

and we have shown (A.4).

Step 2. Regularized Eisenstein Series. We choose a parameter Y depending on z_0 by

$$Y = 1 + \max_j \mathrm{Im}(\mathfrak{s}_j^{-1} z_0), \tag{A.6}$$

and define a regularized Eisenstein series

$$E_i^Y(z,s)_{2n} = \begin{cases} E_i(z,s)_{2n} - \delta_{ij} y_j^s - \varphi_{ij}(s)_{2n}\, y_j^{1-s} & \text{if } y_j = \mathrm{Im}(\mathfrak{s}_j^{-1} z) \geq Y \text{ for some } j \\ E_i(z,s)_{2n} & \text{otherwise} \end{cases} \tag{A.7}$$

(note that the condition $y_j \geq Y$ can occur for at most one index j). Let $\mathcal{E}^Y(z,s)_{2n} = \left(E_1^Y(z,s)_{2n}, \ldots, E_k^Y(z,s)_{2n}\right)^{\mathrm{tr}}$. Then (see e.g. [Sa81, p. 727]) for $\sigma > \frac{1}{2}$, $t \neq 0$ we have the following inner product formula, which is known as the Maass-Selberg relation:

$$\int_{X_\Gamma} \mathcal{E}^Y(z,s)_{2n} \overline{\mathcal{E}^Y(z,s)_{2n}}^{\mathrm{tr}} \, d\mu_\Gamma(z)$$
$$= \frac{1}{2\varepsilon} \left(Y^{2\varepsilon} \, \mathrm{Id}_{k\times k} - \Phi_{2n}(s) \overline{\Phi_{2n}(s)}^{\mathrm{tr}} Y^{-2\varepsilon} \right) + \frac{\overline{\Phi_{2n}(s)}^{\mathrm{tr}} Y^{2\mathbf{i}t} - \Phi_{2n}(s) Y^{-2\mathbf{i}t}}{2\mathbf{i}t}.$$

Step 3. Trace of Maass-Selberg Relations. Let $\left\| E_j^Y(\cdot,s)_{2n} \right\|_2^2 = \int_{X_\Gamma} \left| E_j^Y(z,s)_{2n} \right|^2 d\mu_\Gamma(z)$, $\|\Phi_{2n}(s)\|^2 = \sum_{ij} |\varphi_{ij}(s)_{2n}|^2$, and set $\Psi_{2n}(s) = \det \Phi_{2n}(s)$. Then for $\sigma > \frac{1}{2}$, $t \neq 0$ we have

$$\sum_{j=1}^{k} \left\| E_j^Y(\cdot,s)_{2n} \right\|_2^2$$
$$= \frac{1}{2\varepsilon} \left(k Y^{2\varepsilon} - \|\Phi_{2n}(s)\|^2 Y^{-2\varepsilon} \right) + \frac{1}{2\mathbf{i}t} \left(Y^{2\mathbf{i}t} \sum_{i=1}^{k} \overline{\varphi_{ii}}(s)_{2n} - Y^{-2\mathbf{i}t} \sum_{i=1}^{k} \varphi_{ii}(s)_{2n} \right). \tag{A.8}$$

Indeed, this is a matrix computation that involves taking the trace of the inner product formula. See [Iwa95, p. 140] for the computation in case $n = 0$.

Step 4. Bounds on traces and norms. For $n = 0$ and $|t| \geq 1$ we have

$$k - \|\Phi(s)\|^2 \ll \varepsilon\, \omega(t), \tag{A.9}$$

and for $n \in \mathbb{N}$, $\sigma \in \left[\frac{1}{2}, \frac{3}{2}\right]$ and $|t| \geq n$ we also have

$$k - \|\Phi_{2n}(s)\|^2 \ll \varepsilon\, \omega(t). \tag{A.10}$$

Indeed, (A.9) is proved in [Se89, p. 657], and since

$$\|\Phi_{2n}(s)\|^2 = |B_n(s)|^2\|\Phi(s)\|^2,$$

(A.10) follows from (A.9), the boundedness of Φ, and (A.4).

As to L^2 bounds, for $n \in \mathbb{Z}$ and $\sigma \in \left[\frac{1}{2}, \frac{3}{2}\right]$ we have

$$\left\|E_j^Y(\cdot, s)_{2n}\right\|_2^2 \ll \omega(t), \tag{A.11}$$

where the implicit constant depends also on Y, and hence on z_0. . For this we use $Y^{\pm 2\varepsilon} = 1 + O_Y(\varepsilon)$. Using (A.3) we have that $\Phi_{2n}(s)$ is uniformly bounded for $\sigma \in \left[\frac{1}{2}, \frac{3}{2}\right]$, and hence the right hand summand in (A.8) is bounded. For the left hand summand, we have by (A.10)

$$\frac{1}{2\varepsilon}\left(kY^{2\varepsilon} - Y^{-2\varepsilon}\|\Phi_{2n}(s)\|^2\right) = \frac{1}{2\varepsilon}\left(k - \|\Phi_{2n}(s)\|^2 + O_Y(\varepsilon)\right) = O_Y(\omega(t) + 1).$$

Combining bounds and recalling $\omega(t) > 1$ gives (A.11) for $\sigma > \frac{1}{2}$. Since the implicit constant in (A.11) is independent of σ we can take a limit and get the same bound for $\sigma = \frac{1}{2}$.

Step 5. Convolution and point-pair invariant. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $n \in \mathbb{Z}_{\geq 0}$ and $z \in \mathbb{H}$, let

$$\varepsilon_\gamma(z)^{2n} = \frac{(cz+d)^{2n}}{|cz+d|^{2n}}.$$

Say a function f on $\mathbb{H}$ is *of weight* $2n$ if it transforms like $f(\gamma z) = \varepsilon_\gamma(z)^{2n} f(z)$, and denote the automorphic functions, whose restriction to a Dirichlet fundamental domain for Γ on $\mathbb{H}$ is in $L^2(X_\Gamma, \mu_\Gamma)$, by $L^2(\Gamma, 2n)$. For $\delta \in (0,1)$, let χ_δ be the indicator function of $[0, \delta]$, and for $z, w \in \mathbb{H}$ define

$$\begin{aligned} u(z,w) =& \frac{|z-w|^2}{4\,\mathrm{Im}z\,\mathrm{Im}w} \\ H(z,w) =& (-1)^n \frac{(w-\overline{z})^{2n}}{|w-\overline{z}|^{2n}} \\ k(z,w) = k_\delta(z,w) =& H(z,w)\chi_\delta(u(z,w)) \\ K(z,w) = K_\delta(z,w) =& \sum_{\gamma \in \Gamma} k_\delta(z, \gamma w)\varepsilon_\gamma(w)^{2n}. \end{aligned}$$

Functions such as k are called *point pair invariants of weight* $2n$. They satisfy (see [Hej83, Vol. 1, Prop. 2.11, p. 359]) the following transformation rules:

$$\begin{aligned} H(\gamma z, \gamma w) &= \varepsilon_\gamma(z)^{2n} H(z,w)\varepsilon_\gamma(w)^{-2n} \\ k(\gamma z, \gamma w) &= \varepsilon_\gamma(z)^{2n} k(z,w)\varepsilon_\gamma(w)^{-2n}. \end{aligned}$$

The operator L_k defined by

$$L_k f(z) = \int_{\mathbb{H}} k(z,w) f(w)dw = \int_{\Gamma\backslash\mathbb{H}} K(z,w)f(w)dw, \tag{A.12}$$

is a bounded self-adjoint operator on $L^2(\Gamma, 2n)$, see [Hej83, Vol. 1, Prop 2.13, p. 363]. Let Δ be as in (2.1), and let

$$\Delta_n u(z) = \Delta u(z) + \mathbf{i} n y \frac{\partial u}{\partial x}(z) \quad (\text{where } z = x + \mathbf{i}y)$$

be the *weighted Laplacian.* Then the Eisenstein series $z \mapsto E_i(z,s)_{2n}$ is a Δ_{2n}-eigenfunction and therefore (see [Hej83, Vol. 1, Prop. 2.14, p. 364]) is an eigenfunction for L_k, that is there is $h_{i,n,\delta}(s)$ such that for all $z \in \mathbb{H}$, $L_k E_i(z,s)_{2n} = h_{i,n,\delta}(s) E_i(z,s)_{2n}$.

Step 6. Bounding the eigenvalue. The eigenvalue $h_{i,n,\delta}(s)$ satisfies a bound

$$\sigma \in \left[\frac{1}{2}, \frac{3}{2}\right], \ |t| \geq |n| + 1, \ \delta = \frac{1}{100|t|^2} \implies |h_{i,n,\delta}(s)| \gg \frac{1}{|t|^2}. \tag{A.13}$$

Indeed, the bound (A.13) is proved in [MS03, Lemma 2.1] for $\sigma = \frac{1}{2}$, and the proof goes through for general $\sigma \in \left[\frac{1}{2}, \frac{3}{2}\right]$.

Step 7. Pointwise bounds. We now note that our choice (A.6) implies that

$$L_k E_i^Y(z_0, s)_{2n} = L_k E_i(z_0, s)_{2n}. \tag{A.14}$$

Indeed, considering (A.7) and the definitions of δ and u we see that $E_i^Y(w, s)_{2n}$ and $E_i(w, s)_{2n}$ coincide for all w in the neighborhood of z_0 consisting of the points for which the integrand in (A.12) is nonzero.

We now claim

$$\int_{\Gamma\backslash\mathbb{H}} |K_\delta(z_0, w)|^2 \, dw \ll \delta. \tag{A.15}$$

Indeed, by [Hej83, Vol. 1, Prop. 2.12b, p. 360] we have $K_\delta(z_0, w) = \overline{K_\delta(w, z_0)}$. Hence

(A.16)
$$\begin{aligned}\int_{\Gamma\backslash\mathbb{H}} |K_\delta(z_0, w)|^2 \, dw &= \int_{\Gamma\backslash\mathbb{H}} K_\delta(z_0, w) K_\delta(w, z_0) dw \overset{(A.12)}{=} \int_{\mathbb{H}} k_\delta(z_0, w) K_\delta(w, z_0) dw \\ &= \sum_\gamma \int_{\mathbb{H}} k_\delta(z_0, w) k_\delta(w, \gamma z_0) \varepsilon_\gamma(z_0)^{2n} dw \leq \sum_\gamma \int_{\mathbb{H}} \chi_\delta(u(z_0, w)) \chi_\delta(u(w, \gamma z_0)) dw.\end{aligned}$$

To bound the sum on the right-hand side of (A.16), we note from [Iwa95, p. 100] that points z_0 which satisfy both $u(w, \gamma z_0) < \delta$ and $u(z_0, w) < \delta$ for some γ, w, also satisfy $u(\gamma z_0, z_0) < 4\delta(\delta + 1)$. By discreteness (see [Iwa95, Cor. 2.12]), for fixed z_0 and small enough δ, the number of γ for which this happens is bounded. So the right-hand side of (A.16) is $\ll \int_{\mathbb{H}} \chi_\delta(u(z_0, w)) dw = \int_B \chi_\delta(u(z_0, w)) dw$, where B is a hyperbolic ball of area $\ll \delta$, as required.

To conclude the proof of (A.2), we apply Cauchy-Schwartz to find

$$\begin{aligned}|E_i(z_0, s)_{2n}| &= \frac{1}{|h_{i,n,\delta}(s)|} |L_k E_i(z_0, s)_{2n}| \overset{(A.14)}{=} \frac{1}{|h_{i,n,\delta}(s)|} \left| L_k E_i^Y(z_0, s)_{2n} \right| \\ &\overset{(A.13)}{\ll} |t|^2 \left| \int_{\Gamma\backslash\mathbb{H}} K_\delta(z_0, w) E_i^Y(w, s)_{2n} \, dw \right| \\ &\overset{(A.11)}{\ll} |t|^2 \sqrt{\omega(t)} \sqrt{\int_{\Gamma\backslash\mathbb{H}} |K_\delta(z_0, w)|^2 \, dw} \overset{(A.15)}{\ll} |t|^2 \sqrt{\omega(t)} \sqrt{\delta} \overset{(A.13)}{\ll} |t| \sqrt{\omega(t)}.\end{aligned}$$

□

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