RIGIDITY OF GROUP ACTIONS ON HOMOGENEOUS SPACES, III

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Abstract.

Consider homogeneous G/H and G/F,
for an S-algebraic group G.
A lattice Γ acts on the left
strictly conservatively.

The following rigidity results are obtained:
morphisms, factors and joinings defined apriori
only in the measurable category
are in fact algebraically constrained.

Arguing in an elementary fashion
we manage to classify
all the measurable Φ
commuting with the Γ-action:

assuming ergodicity, we find
they are algebraically defined.

1. Introduction

Flows on homogeneous spaces provide a rich and fruitful source of examples of dynamical systems. Most of the literature on this subject concerns actions of subgroups $H < G$ on G/Γ, where G is a Lie group and $\Gamma < G$ is a lattice. In this paper we consider a situation where the roles of the subgroups $\Gamma, H < G$ are reversed, and study actions of a discrete subgroup $\Gamma < G$ on a homogeneous spaces G/H. We shall mostly focus on the situation where G is a Lie group, or a product of algebraic groups over local fields, $\Gamma < G$ is a lattice, $H < G$ is a closed (algebraic) subgroup, and G/H is equipped with the Haar measure class. Our aim is to study the classification problem for such objects as:

(1) measurable Γ-equivariant maps $G/H \to G/L$,
(2) relatively probability measure-preserving Γ-quotients of G/H,
(3) relatively p.m.p. joinings of Γ-actions on G/H with G/L,
(4) quasi-factors, i.e. Γ-equivariant maps $G/H \to \mathrm{Prob}(G/H)$,

and seek situations where these *measurable* Γ*-objects* (maps, quotients, joinings) are necessarily *algebraic*, and therefore can be explicitly described. These results expand the scope of problems previously studied in this context in [31] and [7] and generalize most of the results there using new methods. Before formulating the results, we define the precise framework of this study.

1.1. Γ-spaces.

Let G be a locally compact second countable group. A *Borel G-space* is a standard Borel space $(X, \mathcal{X})$ with a Borel action $G \curvearrowright X$, where the action map $G \times X \to X$ is Borel measurable. Two probability measures on X are called *equivalent* if they have the same nullsets. A *measurable G-space* is a Borel G-space X equipped with a probability measure μ, defined on the given Borel σ-algebra $\mathcal{X}$ on X, which is G-quasi-invariant, that is such that $g_*\mu$ is equivalent to μ for every $g \in G$. We shall often write $(X, [\mu])$ to emphasize that it is only the measure class $[\mu]$ of μ that is assumed to be G-invariant. If $(X, [\mu])$ is a measurable G-space and V is a Borel G-space, a map $f : X \to V$ is called *G-equivariant*, or just a *Borel G-map*, if it is Borel measurable and for every $g \in G$ for μ-a.e. $x \in X$,

$$f(g \cdot x) = g \cdot f(x). \tag{1.1}$$

Maps $f, f' : X \to V$ that agree μ-a.e. will be identified, and

$$\mathrm{Map}_G(X, V)$$

denotes the set of all equivalence class of G-maps. In light of [37, Appendix B], any map in $\mathrm{Map}_G(X, V)$ is equivalent to $f : X \to V$ such that on a conull subset $X_0 \subset X$, (1.1) holds for every $g \in G$ and every $x \in X_0$. If G_1 is a subgroup of G then any G-space is automatically a G_1-space and G-maps are G_1-maps, yielding an injective map $\mathrm{Map}_G(X, Y) \to \mathrm{Map}_{G_1}(X, Y)$. If this map is surjective we write $\mathrm{Map}_G(X, Y) = \mathrm{Map}_{G_1}(X, Y)$. With the usual abuse of notations $f \in \mathrm{Map}_G(X, V)$ will often mean that f is an actual G-map, rather than an equivalence class of such maps.

A measurable G-space $(X, [\mu])$ is *ergodic* if every G-invariant measurable subset is μ-null or μ-co-null. Let T be a standard Borel space with the trivial G-action. Then $(X, [\mu])$ is ergodic iff every G-map $X \to T$ is a constant map, that is $\mathrm{Map}_G(X, T) \cong T$.

1.2. Relatively probability measure-preserving factors.

Let $(X, [\mu])$, $(Y, [\nu])$ be measurable G-spaces and $p : X \to Y$ be a G-map so that $[p_*\mu] = [\nu]$. Such a map will be called *G-morphism*, *G-quotient*, or *G-factor (map)*. We will be particularly interested in relatively measure preserving morphisms, which we now define. Recall that if X is a Borel G-space then so is the space $\mathrm{Prob}(X)$ of Borel probability measures on X. Additionally recall that if X, Y are standard Borel spaces, μ is a Borel measure on X, $p : X \to Y$ is a Borel map, and ν is the pushforward $p_*\mu$, then there is a *disintegration of* μ, i.e. a measurable map $Y \to \mathrm{Prob}(X)$, $y \mapsto \mu_y$, such that

$$\mu = \int_Y \mu_y \, d\nu(y), \qquad \text{i.e.} \qquad \mu(E) = \int_Y \mu_y(E) \, d\nu(y) \tag{1.2}$$

for all measurable $E \subset X$, and such that

$$\mu_y(p^{-1}(\{y\})) = 1 \qquad \text{for} \quad \nu\text{−a.e. } y \in Y. \tag{1.3}$$

This map is unique in the sense that any two such maps differ on a nullset for ν. We now say that a G-morphism $p : (X, [\mu]) \to (Y, [\nu])$ is *relatively probability measure-preserving* (or relatively p.m.p.) if there is $\mu' \in [\mu]$ such that in the disintegration of μ' with respect to $p_*\mu'$, the corresponding map $Y \to \mathrm{Prob}(X)$ is G-equivariant, that is,

$$\mu_{g \cdot y} = g_*\mu_y \qquad (g \in G) \tag{1.4}$$

for ν-a.e. $y \in Y$. If the measure class $[\mu]$ contains a G-invariant probability measure μ_0, then $[\nu]$ contains a G-invariant probability measure $\nu_0 = p_*\mu_0$ and p is relatively p.m.p. If G is a locally compact second countable group and $H < G$ is a closed subgroup, then there is a unique invariant measure class on G/H induced by Haar measure on G, which we will call the *Haar measure class* on G/H; the invariant measure class may or may not contain an invariant probability measure. If $L < G$ is another closed subgroup containing H, then the G-equivariant map $G/H \to G/L$, $gH \mapsto gL$ is relatively p.m.p. iff L/H carries an L-invariant probability measure.

1.3. **Relatively p.m.p. joinings and quasi-factors.**

A *relatively p.m.p. joining* of two measurable Γ-spaces $(X_i, [\mu_i])$, $i = 1, 2$ is a G-quasi-invariant probability measure ν on $X_1 \times X_2$, so that both projections

$$p_i : (X_1 \times X_2, \nu) \to (X_i, \mu_i) \qquad (i = 1, 2)$$

are relatively p.m.p. maps. For instance, if $p : (X, \mu) \to (Y, \nu)$ is a relatively p.m.p. quotient map, then the pushforward of μ under the map $x \mapsto (x, p(x))$ has the required properties; this is the *relatively independent self-joining* of (X, μ) associated to $p : X \to Y$.

Given measurable G-spaces (X, μ) and (Y, ν), we say that (Y, ν) is a *quasi-factor* of (X, μ) if there is a G-map $\phi \in \mathrm{Map}_G(Y, \mathrm{Prob}(X))$, $y \mapsto \phi_y$, so that

$$\mu = \int_Y \phi_y \, d\nu(y).$$

Thus every relatively p.m.p. quotient map $p : (X, \mu) \to (Y, \nu)$ defines a quasi-factor by disintegration.

Furthermore, any relatively p.m.p. joining ν of (X_1, μ_1) and (X_2, μ_2) defines a pair of quasi-factor maps $X_1 \to \mathrm{Prob}(X_2)$, $X_2 \to \mathrm{Prob}(X_1)$, via the disintegration of the projections $p_i : \nu \to \mu_i$. Hence a classification of all quasi-factors may lead to a classification of relatively p.m.p. joinings; and relatively p.m.p. factors. For example, if $p : (X, \mu) \to (Y, \nu)$ is a relatively p.m.p. G-factor, and $y \mapsto \mu_y$ is the associated disintegration of μ with respect to ν, then

$$X \to \mathrm{Prob}(X), \qquad x \mapsto \mu_{p(x)}$$

is a G-quasi-factor.

1.4. **S-algebraic groups, subgroups, actions.**

Let S be a finite set, for each $v \in S$ let k_v be a non-discrete local field of zero characteristic (i.e., $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$ for some prime p), and $\mathbf{G}_v$ be an algebraic group defined over k_v. Let $G = \prod_{v \in S} \mathbf{G}_v(k_v)$ be the locally compact second countable group formed by the product of the k_v-points $\mathbf{G}_v(k_v)$ of $\mathbf{G}_v$. We shall use the term *S-algebraic group* to describe groups G that arise in this way. By an *S-algebraic subgroup* H of an S-algebraic group G, we mean any locally compact subgroup $H = \prod_{v \in S} \mathbf{H}_v(k_v)$ where $\mathbf{H}_v < \mathbf{G}_v$ is a k_v-algebraic subgroup for each $v \in S$. Similarly, V is an *S-variety* if $V = \prod_{v \in S} \mathbf{V}_v(k_v)$ is a product of k_v-points of k_v-varieties for $v \in S$. We shall say that V is an *S-algebraic G-space* if G is an S-algebraic group and V is an S-algebraic variety, equipped with a G-action $G \curvearrowright V$ associated to k_v-algebraic actions $\mathbf{G}_v \curvearrowright \mathbf{V}_v$ for each $v \in S$. Homogeneous S-algebraic spaces are $V = G/H$ where $H < G$ is an S-algebraic subgroup.

Any S-algebraic space V can be considered as a Borel G-space, where the Borel structure of V is induced by its locally compact topology. A homogeneous S-algebraic G-space, $V = G/H$, equipped with the unique G-invariant measure class $[\mu]_{G/H}$ is a measurable G-space. If $M < G$ is a closed (not necessarily S-algebraic) subgroup, any S-algebraic space V can be viewed as a Borel M-space, and any S-algebraic G-homogeneous space G/H is a measurable M-space.

Let G be a general locally compact second countable group. A discrete subgroup $\Gamma < G$ is a *lattice* if G/Γ has a G-invariant finite measure. One of the reasons to consider the framework of S-algebraic groups, rather than just algebraic groups over a single field, is that an S-algebraic group G may contain lattices which are not products of lattices in the factors $\mathbf{G}_v(k_v)$. For example $\Gamma = \mathrm{SL}_n\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)$ is a lattice in the S-algebraic group $G = \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_p)$. A lattice is called *irreducible* if it does not have a discrete image in any proper factor of G.

Recall that if H, M are closed subgroups in some locally compact group G, then $M \curvearrowright G/H$ is ergodic iff $H \curvearrowright G/M$ is ergodic, where both spaces are considered with the Haar measure classes. If $\mathbf{G}_v$ are semi-simple k_v-groups without compact factors, $\mathbf{H}_v(k_v)$ are non-compact for every $v \in S$, and $\Gamma < G$ is an irreducible lattice, then $H \curvearrowright G/\Gamma$ and $\Gamma \curvearrowright G/H$ are ergodic by Moore's ergodicity theorem.

1.5. **Statements of the main results.**

We shall now present the main results for actions of a lattice Γ in an S-algebraic group G on G/H where $H < G$ is an S-algebraic subgroup. We shall assume ergodicity of $\Gamma \curvearrowright G/H$ (equivalently $H \curvearrowright G/\Gamma$). Let us first state the results under the following simplifying assumption

(*) H has no non-trivial S-algebraic compact factor group.

Hereafter for any group action $G \curvearrowright S$ we denote by S^G the set of G-fixed points in S.

Theorem 1.1 (Equivariant Borel maps)**.**
Let G be an S-algebraic group, $H < G$ an S-algebraic subgroup, $\Gamma < G$ a lattice, and G/H an ergodic Γ-space. Assume (*). Let V be an S-algebraic G-space, viewed as a Borel Γ-space. Then

$$\mathrm{Map}_\Gamma(G/H, V) = \mathrm{Map}_G(G/H, V) \cong V^H$$

where $v \in V^H$ corresponds to the G-map $f_v(gH) = g \cdot v$.

As an immediate corollary we obtain measure-theoretic rigidity results. Given a measurable Γ-space $(X, [\mu])$ denote by $\mathrm{Aut}(X, [\mu])$ the group of measure-class automorphisms of X, up to null sets, and let $\mathrm{Aut}_\Gamma(X, [\mu])$ denote the subgroup of Γ-equivariant ones.

We denote conjugation by $h^g = g^{-1}hg$, $H^g = g^{-1}Hg$, and normalizers by

$$\mathcal{N}_G(H) = \{g \in G \mid H^g = H\}.$$

Recall that $\mathcal{N}_G(H)/H$ is $\mathrm{Aut}_G(G/H)$ – the group of G-equivariant bijections of G/H as a set. Here $nH \in \mathcal{N}_G(H)/H$ acts on G/H by $gH \mapsto gnH$.

Corollary 1.2.
Let G be an S-algebraic group, $H, H_1, H_2 < G$ be S-algebraic subgroups satisfying (*), $\Gamma < G$ a lattice whose action on G/H_i is ergodic. Then

(1) $\mathrm{Map}_\Gamma(G/H_1, G/H_2) = \mathrm{Map}_G(G/H_1, G/H_2)$ *and all such maps are given by* $gH_1 \mapsto gg_0H_2$, *where* $H_1^{g_0} < H_2$.
(2) *The spaces* G/H_1 *and* G/H_2 *are isomorphic as ergodic* Γ*-spaces iff they are algebraically isomorphic, i.e.,* $H_1^{g_0} = H_2$ *for some* $g_0 \in G$.
(3) $\mathrm{Aut}_\Gamma(G/H) = \mathrm{Aut}_G(G/H) \cong \mathcal{N}_G(H)/H$.

Given an ergodic Γ-space G/H as above, it is possible to classify all of its relatively p.m.p. Γ-factors.

Theorem 1.3 (Relatively p.m.p. factors)**.**
Let $\Gamma, H < G$ *be as in Theorem 1.1, and let* $p : G/H \to Y$ *be a relatively p.m.p.* Γ*-factor.*

Then (Y, ν) *is a relatively p.m.p.* G*-factor of* G/H*. More precisely, there is a closed subgroup* $H \lhd L < G$ *with* $K = L/H$ *compact, so that* $Y \cong G/L$ *and*

$$p : G/H \to Y \cong G/L \qquad \text{is given by} \qquad gH \mapsto gL.$$

The following result shows that under assumption (*), ergodic measurable Γ-spaces G/H_1 and G/H_2 have no non-trivial relatively p.m.p. joinings, unless $G/H_1 \cong G/H_2$.

Theorem 1.4 (Relatively p.m.p. joinings)**.**
Let $\Gamma, H_1, H_2 < G$ *be as in Corollary 1.2. Then the ergodic* Γ*-spaces* G/H_1 *and* G/H_2 *admit a relatively p.m.p. joining iff* H_1 *and* H_2 *are conjugate in* G.

Theorem 1.5 (Quasi-factors)**.**
Let $\Gamma, H < G$ *and* V *be as in Theorem 1.1, and let* $\phi : G/H \to \mathrm{Prob}(V)$ *be a* Γ*-quasi-factor, where* V *is equipped with the* Γ*-quasi-invariant measure*

$$\nu = \int_{G/H} \phi_{gH}\, d\mu_{G/H}.$$

Then $V^H \neq \emptyset$ *and* $\phi_{gH} = g \cdot \nu_0$ *for some fixed* $\nu_0 \in \mathrm{Prob}(V^H)$.

In short, the results above show that certain classes of measurable Γ-maps on a G-homogeneous space G/H are necessarily G-maps, and therefore can be explicitly described using transitivity of the G-action. These results depend on the assumption that the S-algebraic subgroup $H < G$ has no S-algebraic compact factors. In the case of a general S-algebraic subgroup $H < G$, measurable Γ-maps as above need not be G-maps. We shall show, however, that such maps are M-maps, where M is some closed cocompact subgroup of G containing Γ and acting transitively on G/H. Moreover, M is a *fat complement* of H in G as described in Definition 6.1.

Theorem 1.6 (Quasi-factors, general case)**.**
Let G *be an* S*-algebraic group,* $H < G$ *an* S*-algebraic subgroup,* $\Gamma < G$ *a lattice and assume that* G/H *is an ergodic* Γ*-space. Let* V *be an* S*-algebraic* G*-space with a* Γ*-quasi-factor* $\phi : G/H \to \mathrm{Prob}(V)$*, where* V *is equipped with the* Γ*-quasi-invariant measure*

$$\nu = \int_{G/H} \phi_{gH}\, d\mu_{G/H}.$$

Then there exists a closed cocompact subgroup $M < G$*, containing* Γ*, acting transitively on* G/H *(and being a fat complement of* H *as in Definition 6.1), so that*

$V^{M\cap H} \neq \emptyset$, and a probability measure ν_0 supported on $V^{M\cap H}$, such that for a.e. $m \in M$,

$$\phi_{mH} = m \cdot \nu_0.$$

Moreover, if G satisfies (*) then M has finite index in G.

When M acts transitively on G/H, for any $g \in G$ there is $m \in M$ such that $gH = mH$, and any two such elements of M differ by right-multiplication by an element of $M \cap H$. Therefore for $v \in V^{M\cap H}$, the map $f_v(gH) = m \cdot v$ is well-defined and is an M-map; and clearly all M-maps $G/H \to V$ arise in this way.

Corollary 1.7 (Equivariant maps, general case)**.**
Let $\Gamma, H < G$ and V be as above. Then there exists $M < G$ as in Theorem 1.6, so that

$$\mathrm{Map}_\Gamma(G/H, V) = \mathrm{Map}_M(G/H, V) \cong V^{M\cap H}.$$

Corollary 1.8 (Relatively p.m.p. factors, general case)**.**
Let $\Gamma, H < G$ be as above, and $p : G/H \to Y$ be a relatively p.m.p. Γ-factor. Then there exists $M < G$ as in Theorem 1.6, a closed subgroup $M \cap H \triangleleft L < M$ so that $L/(M \cap H)$ has a finite L-invariant measure, $Y \cong M/L$, and

$$p : G/H \to Y \cong M/L \qquad \text{is given by} \qquad mH \mapsto mL \qquad (m \in M).$$

1.6. **Previous results.** The general inspiration for the questions considered here is the pioneering work of Marina Ratner [23–25] where questions of measurable isomorphism, classification of factors and joinings where studied for actions of unipotent subgroup $H < G$ on G/Γ, where Γ is a lattice in a Lie group G (these and more general results can also be deduced from the general Ratner's theorem [26], see also [35]). The assumption that $H < G$ is unipotent is very important for these results, as they fail for the diagonal subgroup $A < \mathrm{SL}_2(\mathbb{R})$. It should also be emphasized that these are probability measure-preserving actions.

Questions of measure-theoretic rigidity for the Γ-action on infinite measure homogeneous spaces G/H were addressed by Shalom and Steger in [31]. In this (unpublished) work the authors obtained a number of rigidity results using representation theoretic techniques; including rigidity of lattices in $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{R}^2$, classification of relatively p.m.p. Γ-factors of $\mathbb{R}^n$ where $\Gamma < \mathrm{SL}_n(\mathbb{R})$ is a lattice, etc.

In [7] further results were obtained for Γ-actions on G/H using purely dynamical methods (*alignment* property). These results included classification of centralizers, relatively p.m.p. quotients, and joinings for a particular type of homogeneous space G/H. In particular, in [7] the group G was assumed to be semi-simple, $H < G$ to be "super-spherical", and G/H to carry an infinite G-invariant measure.

The present paper provides a more systematic study of Γ-actions on G/H, reproving most of the results of [7] and [31] and significantly expanding the scope of the spaces G/H and of the questions. In particular, we impose no restrictions on G and H except for being S-algebraic. For instance, even the very special case $G = \mathrm{SL}_2(\mathbb{R})$, $H_1 = H_2 = A$ (the group of diagonal matrices) of Corollary 1.2 is new. The new questions include equivariant maps to S-algebraic G-spaces (Theorems 1.1 and Corollary 1.7) and classification of quasi-factors (Theorems 1.5 and 1.6).

Our results bear some similarity to Margulis's factor theorem [16] that asserts that all measurable Γ-equivariant quotients of G/P are G-equivariant, and therefore have the form $G/P \to G/Q$, $gP \mapsto gQ$, where $P < Q < G$ are parabolic subgroups. However, the similarity is only superficial, as the context, phenomenology, and the idea of the proof are completely different. Margulis's factor theorem concerns *higher rank* semi-simple group G and a *parabolic* subgroup $P < G$, but the quotient maps are not assumed to be (and never are) relatively p.m.p. Similarly for related work on factor theorems by Zimmer [36], Nevo-Zimmer [19], and Bader-Shalom [1].

1.7. **Some ideas in the proofs.**

If H is a subgroup of G, one may *restrict* any G-space to obtain an H-space, and there is a complementary operation of *inducing* an H-space to obtain a G-space. These operations have been extensively studied in other contexts in representation theory and ergodic theory. In §2 we adapt them to our framework and establish an analogue of Frobenius reciprocity, which allows us to convert questions about Γ-maps, where Γ is a lattice, to questions about H-maps, where H is algebraic. Then, to study algebraic actions, we apply generalizations of the classical Borel density theorem [3], which says that if G is a semi-simple S-algebraic group with no compact factors and $\Gamma < G$ is a lattice, then Γ is Zariski dense in G. This theorem was generalized by a number of authors (see [5, 8, 30, 33]), and it was realized that this phenomenon is related to the classification of G-invariant probability measures on V, where V is an S-algebraic G-space, leading to a more abstract version which says that every such measure must be supported on the set of fixed points, i.e.

$$\mathrm{Prob}(V)^G = \mathrm{Prob}(V^G).$$

We obtain one such generalization (Theorem 3.1) as a straightforward corollary of previous work. Frobenius reciprocity and Borel density are already enough to imply our results in case H satisfies (*). To obtain our results in the general case, we need a more general version of Borel density (Theorem 6.2), which we call *relative Borel density.* Theorem 6.2 is the main technical innovation in our paper. Its proof relies on classical arguments, along with a detailed study of the Mautner envelope and a general version of ergodic decomposition.

1.8. **Some additional results.**

Our discussion so far has focused on actions of *lattices* $\Gamma < G$ on homogeneous spaces G/H where H is an S-algebraic subgroup of an S-algebraic group G. In fact, we have only used the fact that G/Γ carries a finite G-invariant measure and the results apply verbatim to actions of not necessarily discrete closed subgroups L of G provided G/L carries a *finite* G-invariant measure. Such subgroups $L < G$ are said to be of *finite covolume* in G.

However, the finite covolume assumption is not necessary for the results as above to hold; for it is primarily used in arguments involving Borel's density theorem, where existence of finite invariant measure provides recurrence via Poincaré recurrence theorem. In some situations one can exploit recurrence phenomena for actions of subgroups $\Gamma < G$ of *infinite covolume* on some G/H to prove results analogous to [7, Theorem B]. For an example of such a situation, let G be the group of orientation preserving isometries of a symmetric space of rank one $\mathbf{H}^n_K$, with $K = \mathbb{R}$, or $K = \mathbb{C}$. Such G acts transitively on the space of pairs of distinct points at the boundary $\partial\mathbf{H}^n_K$

$$\partial^2\mathbf{H}^n_K = \partial\mathbf{H}^n_K \times \partial\mathbf{H}^n_K \setminus \Delta.$$

Hence the latter space can be identified with G/A where A is the Cartan subgroup of G. We shall consider $\partial^2\mathbf{H}^n_K \cong G/A$ equipped with the G-invariant measure class.

Theorem 1.9. *Let $n \geq 2$ and $G = \mathrm{Isom}_+(\mathbf{H}^n_K)$, and $\Gamma < G$ be a discrete subgroup with an ergodic action on $G/A \cong \partial^2\mathbf{H}^n_K$. Then*

$$\mathrm{Map}_\Gamma(G/A, G/A) = \mathrm{Map}_G(G/A, G/A) = \mathrm{Aut}_G(G/A) \cong \mathcal{N}_G(A)/A = \{1, F\}$$

where F denotes the flip $F : (x, y) \mapsto (y, x)$ on $\partial^2\mathbf{H}^n_K$.

In the statement above the source G/A is considered as a measurable G-space (resp. Γ-space), and the target G/A can be viewed either as a measurable or merely as a Borel G-space (resp. Γ-space). The ergodicity assumption in the theorem is equivalent to the ergodicity of $A \curvearrowright G/\Gamma$. If $\Gamma < G$ is a lattice, ergodicity is guaranteed by Moore's theorem. Thus this case is covered by Theorem 1.1. On the other hand, there exist discrete subgroups $\Gamma < G$ with infinite covolume with $\Gamma \curvearrowright G/A$ ergodic. For instance, if $\Gamma \triangleleft \Lambda$ where $\Lambda < G$ is a cocompact lattice and $\Lambda/\Gamma \simeq \mathbb{Z}$ or $\mathbb{Z}^2$ then $A \curvearrowright G/\Gamma$ is ergodic ([28], [11]). Examples of such infinite covolume $\Gamma < G$ can be constructed in all groups $G = \mathrm{SO}(n, 1)$ (that is, $K = \mathbb{R}$) and $\mathrm{SU}(n, 1)$ ($K = \mathbb{C}$).

Remark 1.10. Theorem 1.9 is also valid in case $G = \mathrm{Sp}(n, 1)$ (that is, $K = \mathbb{H}$) or $G = \mathrm{F}_{4\,(-20)}$ (K the octonion numbers), but gives no examples which are not covered by Theorem 1.1. This is due to the fact that in these cases, G has property (T) and ergodicity of $A \curvearrowright G/\Gamma$ implies $\mathrm{vol}(G/\Gamma) < +\infty$.

Another possible modification of our basic setup concerns the assumptions on $H < G$, namely instead of assuming H to be an S-algebraic subgroup, one might consider a general closed subgroup, or a discrete subgroup as an extreme case. More specifically, consider the space of Γ-equivariant measurable maps

$$\mathrm{Map}_\Gamma(G/\Lambda, G/\Delta)$$

where $\Lambda, \Delta < G$ are discrete subgroups and Γ is a lattice in G. As before, the source space G/Λ is equipped with the Haar measure class and is viewed as an ergodic Γ-space, while the target space G/Δ may be viewed either as a measurable Γ-space, or as a Borel Γ-space.

Theorem 1.11. *Let G be a connected non-compact simple real Lie group, $\Gamma < G$ a lattice, $\Lambda, \Delta < G$ discrete subgroups, with Λ Zariski dense in G. In addition assume one of the following, either:*

- (RS) *Λ is a lattice in a subgroup of G generated by unipotent elements, or*
- (BQ) *Δ is a contained in a lattice $\Delta_0 < G$.*

Then any element of $\mathrm{Map}_\Gamma(G/\Lambda, G/\Delta)$ is either:

1. *a G-map $g\Lambda \mapsto ga\Delta$, where $a \in G$ is such that $a^{-1}\Lambda a < \Delta$, or*
2. *a constant map $g\Lambda \mapsto a\Delta$, where $a \in G$ is such that $a^{-1}\Gamma a < \Delta$.*

1.9. **Organization of the paper.** In Section 2 we introduce the category of measurable G-spaces and establish a version of Frobenius reciprocity that will play a crucial role in the proof. Then in Section 3 we discuss the Borel density theorem and in Section 4, give a self-contained proof of our main results under the assumption that H has no compact algebraic quotients. Sections 2-4 exhibit our main ideas, and quickly prove a substantial part of our results, avoiding technicalities.

The proof in full generality requires further preparation. In Section 5 we introduce the notions of the Mautner envelope and the f.i.-algebraic kernel, and establish a general version of the Mautner property (Theorem 5.6), which is of independent interest. In Section 6 we discuss fat complements and deduce the relative Borel density theorem. The proofs of the remaining results are given in Section 7. As we show they follow easily from the relative Borel density theorem. In Section 8, we give several examples to demonstrate that the assumptions in the main theorems are essential. Finally, in an appendix to the paper, we summarize well-known results on the space of ergodic components and deduce some corollaries.

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2. Frobenius reciprocity

Let G be a locally compact second countable group. Denote by $\mathcal{B}_G$ the category of all Borel G-spaces. The objects of this category are Borel actions on standard Borel spaces $G \curvearrowright X$, and morphisms are Borel G-maps.

We shall denote by $\mathcal{MCP}_G$ the category of measurable G-spaces. The objects of this category are measure class preserving actions of G on standard probability spaces $G \curvearrowright (X, [\mu])$, and the morphisms are G-morphisms $p : X \to Y$ for which $[\nu] = [p_*\mu]$. We identify any two G-morphisms that agree μ-a.e. and write $\mathrm{Mor}_G(X, Y)$ for the set of equivalence classes. We shall distinguish a subset of morphisms:

$$\mathrm{Mor}^1_G(X, Y) = \{p \in \mathrm{Mor}_G(X, Y) : p \text{ relatively p.m.p.}\}.$$

Recall that given $X \in \mathcal{MCP}_G$ and $Z \in \mathcal{B}_G$ we denote by

$$\mathrm{Map}_G(X, Z)$$

the set of equivalence classes of Borel G-equivariant maps $\Phi : X \to Z$.

Let H be a closed subgroup of G. Then every Borel G-space is a Borel H-space, and every Borel G-map is a Borel H-map. Similarly for the measurable category. Hence, we have the *restriction functor*

$$\mathrm{Res}^G_H : \mathcal{B}_G \to \mathcal{B}_H, \qquad \mathrm{Res}^G_H : \mathcal{MCP}_G \to \mathcal{MCP}_H.$$

There is also a natural construction of the *induction functor* (cf. [37, §4.2])

$$\mathrm{Ind}_H^G : \mathcal{B}_H \to \mathcal{B}_G, \qquad \mathrm{Ind}_H^G : \mathcal{MCP}_H \to \mathcal{MCP}_G$$

defined as follows. Given a Borel H-space X, consider the factor map $\pi : G\times X \to (G\times X)/\sim$ where $\sim$ is the equivalence relation $(g,x)\sim(gh,h^{-1}x)$ for $g\in G$, $h\in H$, and $x\in X$. As a measure space, $(G\times X)/\sim$ is isomorphic to $G/H\times X$ so is a standard Borel space. The G-action on $(G\times X)/\sim$ is given by $g[g_1,x]=[gg_1,x]$ for $g,g_1\in G$ and $x\in X$. This defines the induced G-space $\mathrm{Ind}_H^G(X)$. Given an H-map $p : X\to Y$ between Borel H-spaces, define the induced map

$$\mathrm{Ind}_H^G(p) : \mathrm{Ind}_H^G(X) \to \mathrm{Ind}_H^G(Y)$$

by $[g,x]\mapsto[g,p(x)]$ for $g\in G$ and $x\in X$. In the measurable category one applies the same constructions to the underlying Borel spaces. As for the measures, let λ be a probability measure on G equivalent to Haar measure on G. We equip $(G\times X)/\sim$ with the probability measure obtained by pushing forward $\lambda\times\mu$ under the quotient map $G\times X\to(G\times X)/\sim$. The G-action preserves the induced measure class on $\mathrm{Ind}_H^G(X)$. We leave the proof of the following straightforward statement to the reader.

Lemma 2.1. *Let $X,Y\in\mathcal{MCP}_H$. Then*

(1) *If $p\in\mathrm{Mor}_H(X,Y)$ then $\mathrm{Ind}_H^G(p)\in\mathrm{Mor}_G(\mathrm{Ind}_H^G(X),\mathrm{Ind}_H^G(Y))$,*
(2) *If $p\in\mathrm{Mor}_H^1(X,Y)$ then $\mathrm{Ind}_H^G(p)\in\mathrm{Mor}_G^1(\mathrm{Ind}_H^G(X),\mathrm{Ind}_H^G(Y))$.*

□

Next we show that the functors Res_H^G and Ind_H^G are formally adjoint.

Proposition 2.2 (Frobenius reciprocity)**.**
Let $X\in\mathcal{MCP}_H$, $Y\in\mathcal{MCP}_G$ and $Z\in\mathcal{B}_G$. Then there are natural bijections

$$\begin{aligned}\mathrm{Mor}_H(X,\mathrm{Res}_H^G(Y)) &\simeq \mathrm{Mor}_G(\mathrm{Ind}_H^G(X),Y),\\ \mathrm{Mor}_H^1(X,\mathrm{Res}_H^G(Y)) &\simeq \mathrm{Mor}_G^1(\mathrm{Ind}_H^G(X),Y),\\ \mathrm{Map}_H(X,\mathrm{Res}_H^G(Z)) &\simeq \mathrm{Map}_G(\mathrm{Ind}_H^G(X),Z),\end{aligned}$$

in all three cases given by

$$\Phi \longmapsto ([g,x]\mapsto g\Phi(x)). \tag{2.1}$$

Proof. Let $\Phi : X\to Y$ be an H-map, where Y is a G-space. It follows from [37, B.5] that after modifying Φ on a null set, there exists an H-invariant Borel subset $X_0\subset X$ of full measure such that $\Phi|_{X_0}$ is H-equivariant. This shows that formula (2.1) gives a well-defined map from $\mathrm{Mor}_H(X,\mathrm{Res}_H^G(Y))$ to $\mathrm{Mor}_G(\mathrm{Ind}_H^G(X),Y)$.

Now we construct its inverse. Every element in $\mathrm{Mor}_G(\mathrm{Ind}_H^G(X),Y)$ can be lifted to a map $\Psi : G\times X\to Y$ such that

$$\Psi(gg',x)=g\Psi(g',x) \quad\text{and}\quad \Psi(g'h^{-1},h\cdot x)=\Psi(g',x) \tag{2.2}$$

for every $g\in G$, $h\in H$, and almost every $(g',x)\in G\times X$. Moreover, this correspondence respects the measure-class-preserving property. Applying [37, B.5] to the action of $G\times H$ on $G\times X$ given by

$$(g,h).(g',x)=(gg'h^{-1},h\cdot x),$$

we conclude that after modifying Ψ on a set of measure zero, we may assume that there exists an H-invariant Borel subset X_0 of X with full measure such that (2.2) holds for all $(g',x) \in G \times X_0$. Then $\Phi(x) = \Psi(e,x)$, $x \in X$, is an H-map and it is clear that it defines the inverse to the map (2.1). $\square$

The induced space $\mathrm{Ind}_H^G(X)$ is easy to describe for G-spaces.

Proposition 2.3. *For every $X \in \mathcal{MCP}_G$,*

$$\mathrm{Ind}_H^G(\mathrm{Res}_H^G(X)) \simeq G/H \times X$$

where the latter space is equipped with the product measure class and the diagonal action.

Proof. The map $G \times X \to G \times X : (g,x) \mapsto (g, g\cdot x)$ induces an isomorphism of the spaces in question. $\square$

Now we establish several corollaries that give correspondences among sets of maps in different categories.

Corollary 2.4. *Let $X, Y \in \mathcal{MCP}_G$, $Z \in \mathcal{B}_G$, and $H < G$ a closed subgroup. Then under the bijections given by Propositions 2.2 and 2.3,*

$$\begin{aligned} \mathrm{Mor}_H(X,Y) &\simeq \mathrm{Mor}_G(G/H \times X, Y), \\ \mathrm{Mor}^1_H(X,Y) &\simeq \mathrm{Mor}^1_G(G/H \times X, Y), \\ \mathrm{Map}_H(X,Z) &\simeq \mathrm{Map}_G(G/H \times X, Z). \end{aligned} \tag{2.3}$$

Proof. The claim follows from Propositions 2.2 and 2.3. $\square$

Corollary 2.5. *Let $H, L < G$ be closed subgroups, $Y \in \mathcal{MCP}_G$, $Z \in \mathcal{B}_G$. Then there are natural bijections*

$$\begin{aligned} \mathrm{Mor}_L(G/H, Y) &\simeq \mathrm{Mor}_H(G/L, Y), \\ \mathrm{Mor}^1_L(G/H, Y) &\simeq \mathrm{Mor}^1_H(G/L, Y), \\ \mathrm{Map}_L(G/H, Z) &\simeq \mathrm{Map}_H(G/L, Z). \end{aligned} \tag{2.4}$$

Proof. Indeed $\mathrm{Map}_L(G/H,Z) \simeq \mathrm{Map}_G(G/L \times G/H, Z) \simeq \mathrm{Map}_H(G/L,Z)$ using Corollary 2.4. Similarly for the other maps. $\square$

Remark 2.6. Let us record the correspondence $\mathrm{Map}_L(G/H,Z) \simeq \mathrm{Map}_H(G/L,Z)$, explicitly. Denote $Y = G/L$ and $\sigma : Y \to G$ a Borel cross-section of the projection $g \mapsto gL$. Then the map $\ell : G \times Y \to G$ defined by

$$\ell(g,y) = \sigma(g.y)^{-1} g \sigma(y)$$

takes values in L, and forms a Borel cocycle. Given an L-map $f : G/H \to Z$ we define $F : G/L \to Z$ by

$$F(y) = \sigma(y).f(\sigma(y)^{-1}H)$$

and observe that for $h \in H$ one has using L-equivariance of f:

$$\begin{aligned} F(h.y) &= \sigma(h.y).f(\sigma(h.y)^{-1}H) \\ &= \sigma(h.y).f(\ell(h,y)\sigma(y)^{-1}H) = \sigma(h.y)\ell(h,y).f(\sigma(y)^{-1}H) \\ &= \sigma(h.y)\ell(h,y)\sigma(y)^{-1}.F(y) = h.F(y). \end{aligned}$$

The correspondence $f \mapsto F$ is an explicit identification of $\mathrm{Map}_L(G/H,Z)$ with $\mathrm{Map}_H(G/L,Z)$, depending on $\sigma : G/L \to G$.

For our purposes it will suffice to consider maps from measurable spaces to Borel spaces. Hence we state the following Corollary for this case only, others being analogous.

Corollary 2.7. *Let H, L, M be closed subgroups of G with $L < M$, and let $Z \in \mathcal{B}_G$. Then the natural inclusion map*

$$\mathrm{Map}_M(G/H, Z) \hookrightarrow \mathrm{Map}_L(G/H, Z)$$

is a bijection if and only if the natural map

$$\mathrm{Map}_H(G/M, Z) \to \mathrm{Map}_H(G/L, Z),$$

induced by $G/L \to G/M$, is a bijection.

Proof. It is straightforward to check that one has the commutative diagram

$$\begin{tikzcd} \mathrm{Map}_M(G/H,Z) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_L(G/H,Z) \arrow[d, "\simeq"] \\ \mathrm{Map}_H(G/M,Z) \arrow[r] & \mathrm{Map}_H(G/L,Z) \end{tikzcd}$$

where the vertical arrows are the bijections given by Corollary 2.5. The claim follows. □

3. Borel density theorem

Fix a finite set S and a collection of non-discrete local fields k_v, $v \in S$, of zero-characteristic. Let $V = \prod_{v\in S} \mathbf{V}_v(k_v)$ be an S-variety. It carries the *Zariski topology*, by which we mean the product of the Zariski topologies of the factors $\mathbf{V}_v$, $v \in S$, and the *analytic topology* coming from the analytic structures of the local fields. An S-algebraic quotient $H \to K$ of an S-algebraic group H is called *compact* if K is compact in the analytic topology.

For an S-algebraic variety V, we set

$$\mathrm{Prob}^0(V) = V, \qquad \mathrm{Prob}^{n+1}(V) = \mathrm{Prob}(\mathrm{Prob}^n(V)) \qquad (n \geq 0).$$

We shall need the following version of the Borel density theorem.

Theorem 3.1 (Borel Density)**.** *Let H be an S-algebraic group with no compact S-algebraic factors, and let V be an S-algebraic H-space. Then for all $n \geq 0$,*

$$\mathrm{Prob}^n(V)^H = \mathrm{Prob}^n\left(V^H\right).$$

Note that existence of H-invariant measures implies existence of H-fixed points. The set of H-fixed points V^H is the product of k_v-points of fixed point varieties

$$V^H = \prod_{v\in S} \mathbf{V}_v^{\mathbf{H}_v}(k_v).$$

Proof. The inclusion $\mathrm{Prob}^n(V)^H \supset \mathrm{Prob}^n\left(V^H\right)$ is trivial. We shall prove the other inclusion by induction on n, starting with $n = 1$. When $V = \mathbf{V}_v(k_v)$ is an algebraic variety defined over a single local field k_v, the claim is well-known (see, for instance, [30, Theorem 3.9]). To handle the general S-algebraic case, consider the projections $p_v : V \to \mathbf{V}_v(k_v)$, $v \in S$. For $\mu \in \mathrm{Prob}(V)^H$, the push-forward measures satisfy

$$p_{v*}(\mu) \in \mathrm{Prob}(\mathbf{V}_v(k_v))^{\mathbf{H}_v(k_v)} = \mathrm{Prob}(\mathbf{V}_v(k_v)^{\mathbf{H}_v(k_v)}),$$

where $H_v = \mathbf{H}_v(k_v)$. This implies that μ is supported on $V^H = \prod_{v\in S} V_v(k_v)^{H_v}$, as required.

Assume validity of the Theorem for $n \geq 1$. The barycenter map (see [20]) gives an H-equivariant map

$$\mathrm{bar} : \mathrm{Prob}(\mathrm{Prob}^n(V)) \to \mathrm{Prob}^n(V), \qquad \nu \mapsto \int_{\mathrm{Prob}^n(V)} \mu \, d\nu(\mu).$$

By equivariance, invariant measures are mapped to fixed points. Thus, by induction,

$$\mathrm{bar}(\mathrm{Prob}^{n+1}(V)^H) \subset \mathrm{Prob}^n(V)^H = \mathrm{Prob}^n(V^H). \tag{3.1}$$

If, for $\nu \in \mathrm{Prob}^{n+1}(V)^H$, we have

$$\nu(\{\mu : \mu(\mathrm{Prob}^{n-1}(V^H)) < 1\}) > 0,$$

then

$$\mathrm{bar}(\nu)(\mathrm{Prob}^{n-1}(V^H)) = \int_{\mathrm{Prob}^n(V)} \mu(\mathrm{Prob}^{n-1}(V^H)) \, d\nu(\mu) < 1,$$

which contradicts (3.1). It follows that ν-a.e measure in $\mathrm{Prob}^n(V)$ is supported on $\mathrm{Prob}^{n-1}(V^H)$. Therefore, $\nu \in \mathrm{Prob}^{n+1}(V^H)$. This completes the proof. □

We recall the notion of discompact radical introduced by Shalom in [30, Proposition 1.4]. The *discompact radical* $\mathrm{R}_{\mathrm{dc}}(H)$ of an S-algebraic group H is the maximal S-algebraic subgroup of H which does not have any nontrivial compact S-algebraic quotients. We note that $\mathrm{R}_{\mathrm{dc}}(H)$ is normal in H and $H/\mathrm{R}_{\mathrm{dc}}(H)$ is compact (see [30]).

We give two corollaries of the Borel Density Theorem.

Corollary 3.2. *Let H be an S-algebraic group and V an S-algebraic H-space. Let (X,ξ) be an H-space with an invariant probability measure. Then for any H-map $\Phi : X \to \mathrm{Prob}^n(V)$,*

$$\Phi(x) \in \mathrm{Prob}^n\left(V^{\mathrm{R}_{\mathrm{dc}}(H)}\right) \quad \text{for } \xi \text{ a.e. } x.$$

In particular, if the H-action on (X,ξ) is ergodic, then for some $\eta_0 \in \mathrm{Prob}^n(V)$

$$\Phi(x) \in H.\eta_0$$

for ξ-a.e. $x \in X$.

Proof. The first assertion follows from Theorem 3.1 applied to the measure $\Phi_*(\xi)$ and the group $\mathrm{R}_{\mathrm{dc}}(H)$. To prove the second assertion, observe that $\Phi_*(\xi)$ is ergodic with respect to the action of the compact group $H/\mathrm{R}_{\mathrm{dc}}(H)$. Hence it has to be supported on a single orbit. □

The following corollary explains the terminology "Borel density theorem".

Corollary 3.3. *Let H be an S-algebraic group, and $M < H$ a closed subgroup, so that H/M has a finite H-invariant measure. Then $\mathrm{R}_{\mathrm{dc}}(H)$ is contained in the Zariski closure of M.*

Proof. Let $\overline{M}$ be the Zariski closure of M, and $V = H/\overline{M}$. Applying Corollary 3.2 to the H-map $H/M \to V$, we deduce that $\mathrm{R}_{\mathrm{dc}}(H)$ acts trivially on $H/\overline{M}$, hence $\mathrm{R}_{\mathrm{dc}}(H) < \overline{M}$. □

4. Proofs of the basic results

Following these preparations we are in position to prove Theorems 1.1 and 1.5. They correspond to the special cases $n = 0, 1$ of the following result.

Theorem 4.1. *Let G be an S-algebraic group, H an S-algebraic subgroup with no compact S-algebraic factors, let $\Gamma < G$ be a lattice so that $\Gamma \curvearrowright G/H$ is ergodic, and let V be an S-algebraic G-space.*
Then for every $n = 0, 1, \ldots$, viewing $\mathrm{Prob}^n(V)$ as a Borel Γ-space, and G/H as an ergodic Γ-space, there is a natural isomorphism

$$\mathrm{Map}_\Gamma(G/H, \mathrm{Prob}^n(V)) = \mathrm{Map}_G(G/H, \mathrm{Prob}^n(V)) \cong \mathrm{Prob}^n(V^H)$$

where $w \in \mathrm{Prob}^n(V^H)$ corresponds to the map $\Phi_w(gH) = g.w$.

Proof. Denote $Y = \mathrm{Prob}^n(V)$. By Corollary 3.2, we have the bijections

$$\mathrm{Map}_H(G/\Gamma, Y) \simeq Y^H \simeq \mathrm{Map}_H(G/G, Y).$$

Hence, it follows from Corollary 2.7 that the inclusion

$$\mathrm{Map}_G(G/H, Y) \hookrightarrow \mathrm{Map}_\Gamma(G/H, Y)$$

is a bijection as well. This means that every Γ-map $G/H \to Y$ agrees almost everywhere with a G-map. Finally, it is easy to see that every G-map is of the given form. □

Proof of Theorem 1.3. Let μ_y be the measures on G/H coming from the disintegration of μ. We may assume that $y \mapsto \mu_y$ is a Γ-map. Applying Theorem 1.5 to the Γ-map $\phi : G/H \to \mathrm{Prob}(G/H)$ defined by $\phi(x) = \mu_{p(x)}$, we see that there is $\nu_0 \in \mathrm{Prob}(G/H)$, supported on $(G/H)^H \cong \mathcal{N}_G(H)/H$, such that $\phi(gH) = g\nu_0$. In particular ϕ is a G-map. Let L denote the stabilizer of ν_0. Since $y \mapsto \mu_y$ is injective, Y is identified with its image under $y \mapsto \mu_y = \mu_{p(gH)} = g\nu_0$; that is Y is isomorphic to G/L, and under this identification, p is given by $gH \mapsto gL$. Since $L < \mathcal{N}_G(H)$, we have that $H \lhd L$, and since the disintegration of $\mu_{G/H}$ has L-invariant probability measures as fiber measures, there is a finite invariant measure on L/H. This implies that L/H is compact. □

Proof of Theorem 1.4. As explained in §1.3, a relatively p.m.p. joining of G/H_1 and G/H_2 gives rise to a Γ-map $G/H_1 \to \mathrm{Prob}(G/H_2)$. By Theorem 1.5, $(G/H_2)^{H_1}$ is nonempty, i.e. H_1 is contained in a conjugate of H_2. Reversing the roles of H_1 and H_2 we see that H_1 contains a conjugate of H_2. Since the H_i are S-algebraic groups, this implies that H_1 and H_2 are conjugate. □

5. The Mautner property

Let G be a locally compact second countable group and H and L be closed subgroups. We say that (H, L, G) has the *Mautner property* if for every continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, a vector which is fixed by H is already fixed by L, i.e.

$$\mathcal{H}^{\pi(H)} \subset \mathcal{H}^{\pi(L)}.$$

When G is clear, we will say that the pair (H, L) has the Mautner property. In fact (fixing G and H) there is a maximal L such that (H, L, G) has the Mautner property: take the intersection of all fixators of all spaces of the form $\mathcal{H}^{\pi(H)}$ over all unitary G-representation π. We call this maximal L *the Mautner envelope* of H in G, and denote it by $\overline{H}^{MT}$. The alert reader will note that our notation suppresses the dependence on G. Thus $(H, \overline{H}^{MT}, G)$ has the Mautner property, and (H, L, G) has the Mautner property if and only if $L < \overline{H}^{MT}$. In this chapter we analyze Mautner envelopes in S-algebraic groups. A closed subgroup of an S-algebraic group is called *f.i.-algebraic* if it is a finite-index subgroup of an S-algebraic subgroup $L \subset G$. While the Mautner envelope does not have to be normal or f.i.-algebraic in general, we shall show (see Theorem 5.6 below) that it always contains a cocompact subgroup satisfying both of these properties. Before doing so, let us collect some trivial observations.

Lemma 5.1. *Let H be a closed subgroup in a locally compact second countable group G.*

(1) $\overline{\{e\}}^{MT} = \{e\}$.
(2) *If H is compact, then $H = \overline{H}^{MT}$.*
(3) *If $\overline{H}^{MT}$ contains a closed subgroup N which is normal in G, then*

$$\rho\left(\overline{H}^{MT}\right) = \overline{\rho(H)}^{MT}$$

where $\rho : G \to G/N$ is the factor map.

(4) *If N is a closed normal subgroup of G containing H then $\overline{H}^{MT} < N$.*

Proof.
(2) Let $\pi : G \to H\backslash G$ be the quotient map. Given $g \notin H$, let f be a continuous compactly supported function on $H\backslash G$ which vanishes on $\pi(g)$ but not on $\pi(e)$. Then $f \circ \pi \in L^2(G)$ is H-invariant but not g-invariant. (1) is a special case of (2).

(3) The inclusion $\subset$ follows from the fact that any G/N-representation is also a G-representation. For the other inclusion, let $M = \rho^{-1}\left(\overline{\rho(H)}^{MT}\right)$. We need to show that (H, M) has the Mautner property. Since $N < H$, given any unitary G-representation $\pi : G \to \mathcal{U}(\mathcal{H})$, one has

$$\mathcal{H}^{\pi(H)} \subset \mathcal{H}^{\pi(N)},$$

and since N is normal in G, $\mathcal{H}^{\pi(N)}$ is $\pi(G)$-invariant, so induces a unitary G/N-representation on $\mathcal{H}^{\pi(N)}$. In this representation $\overline{\rho(H)}^{MT}$ acts trivially, so M is trivial on $\mathcal{H}^{\pi(N)}$. In particular M fixes $\mathcal{H}^{\pi(H)}$.

(4) Can be seen by considering the representation $L^2(G/N)$. □

Let us review some basic properties of S-algebraic groups that follow from well-known properties of algebraic groups over local fields of characteristic zero (see, for instance, [21]).

Proposition 5.2 (cf. Proposition 3.3 and Theorem 6.14 in [21])**.**
Let $\mathbf{X} = \prod_{v \in S} \mathbf{X}_v$ be a product of algebraic varieties over local fields with a transitive action of $\mathbf{G} = \prod_{v \in S} \mathbf{G}_v$. Then the set X of S-points of $\mathbf{X}$ is a union of finitely many G-orbits. Each of these orbits is open and closed in X.

The following is a special case of Proposition 5.2:

Proposition 5.3.
Let $\rho : G \to H$ be an algebraic homomorphism of S-algebraic groups. Then the image $\rho(G)$ is f.i.-algebraic in H.

Lemma 5.4. *Let $\mathbf{G}$ be an S-algebraic group and G the corresponding locally compact group. For any closed subgroup $M < G$ the collection $\mathcal{N}$ of all subgroups $N < M$ which are normal in G and correspond to Zariski connected, f.i.-algebraic subgroups in G, contains a unique maximal element.*

Definition 5.5. The unique maximal subgroup N of M as above is called the *f.i.-algebraic kernel* of M, and is denoted by $\mathrm{N_a}(M)$.

To justify the term, we note that the f.i.-algebraic kernel $\mathrm{N_a}(M)$ of M is the maximal f.i.-algebraic, Zariski connected subgroup of G that acts trivially on G/M.

Proof of Lemma 5.4.
First let us show that a maximal element of $\mathcal{N}$ (with respect to inclusion) must be unique. Let $U, V \in \mathcal{N}$, and denote by $\mathbf{U}$, $\mathbf{V}$ their Zariski closures in $\mathbf{G}$, and by $\overline{U}$, $\overline{V}$ the corresponding locally compact subgroups in G. Then

$$[\overline{U} : U] < \infty, \qquad [\overline{V} : V] < \infty.$$

Applying Proposition 5.2 to the action of $\overline{U} \times \overline{V}$ on $\overline{U \cdot V}$, we see that $\overline{U} \cdot \overline{V}$ is of finite index in $\overline{U \cdot V}$. Similarly UV is of finite index in $\overline{U} \cdot \overline{V}$. Clearly UV is normal and contained in M, therefore $UV \in \mathcal{N}$.

We now show that a maximal element of $\mathcal{N}$ exists. If $U_1 < U_2 < \cdots$ is an ascending chain in $\mathcal{N}$, then the corresponding chain of Zariski closures stabilizes after finitely many steps because the $\overline{U}_i$ are connected. Since U_i has finite index in $\overline{U}_i$, this implies that the original chain stabilizes as well. □

Below we use results from [18, 34] in order to prove the following theorem, which is the main result of this section. We remind our reader that in our setting all fields k_v are of characteristic zero.

Theorem 5.6 (Mautner envelope)**.**
Let G be an S-algebraic group and H an S-algebraic subgroup of G. Then

(1) $\overline{H}^{MT} / \mathrm{N_a}\left(\overline{H}^{MT}\right)$ *is a compact group.*
(2) $\overline{H}^{MT} / \mathrm{N_a}\left(\overline{H}^{MT}\right)$ *is a finite group, provided H has no non-trivial compact S-algebraic quotients.*

Let $\mathbf{H}$ be an algebraic group over a local field k, and $\mathbf{A} < \mathbf{H}$ be a one-dimensional connected algebraic subgroup, isomorphic over k either to the additive group $\mathbf{G}_a$ or to the multiplicative group $\mathbf{G}_m$. We shall say that $A < H$ is a *one-dimensional k-split* subgroup of H. Given an S-algebraic group $\mathbf{H}$, let $H^\vee$ be the smallest closed subgroup of the locally compact group $H = \prod_{\nu \in S} \mathbf{H}_\nu(k_\nu)$ containing all the one-dimensional connected split subgroups over the relevant local fields. Given an S-algebraic group G, containing H, let $H^\wedge$ be the closed normal subgroup of G generated by $H^\vee$ (the dependence on G is hidden in this notation).

Theorem 5.6 will be deduced from the following three propositions that we prove below.

Proposition 5.7. *Let G be an S-algebraic group and H an S-algebraic subgroup of G. Then $H^\wedge$ is the Mautner envelope of $H^\vee$ in G.*

Proposition 5.8. *Let G be an S-algebraic group. Then $G^\vee$ is a Zariski-dense subgroup of finite index in $\mathrm{R}_{\mathrm{dc}}(G)$.*

Proposition 5.9. *Let G be an S-algebraic group and H an S-algebraic subgroup of G. Then $H^\wedge$ is f.i.-algebraic and Zariski connected.*

Proof of Theorem 5.6 (assuming Propositions 5.7, 5.8, 5.9).
Let $N = H^\wedge$. By Proposition 5.7, N is contained in $\overline{H}^{MT}$. It is clearly normal, and also f.i.-algebraic and Zariski connected by Proposition 5.9. It follows that N is contained in the f.i.-algebraic kernel $\mathrm{N_a}\left(\overline{H}^{MT}\right)$ of $\overline{H}^{MT}$. Hence, in order to establish assertion (1) it suffices to show that $\overline{H}^{MT}/N$ is compact.

Denote by $\rho : G \to G/N$ the factor map. By Lemma 5.1(3), $\rho\left(\overline{H}^{MT}\right) = \overline{\rho(H)}^{MT}$, thus we need to establish the compactness of $\overline{\rho(H)}^{MT}$. Since $H^\vee < \ker(\rho) \cap H$ and $H/H^\vee$ is compact by Proposition 5.8, it follows that $\rho(H)$ is compact. Now Lemma 5.1(2) implies that $\overline{\rho(H)}^{MT}$ is compact, proving (1).

We now assume that H has no compact S-algebraic quotients. By Proposition 5.8, $H^\vee$ is Zariski dense in H. Then the Zariski closure $\overline{N}$ of N contains H. By Proposition 5.9, N is Zariski connected and of finite index in $\overline{N}$. Clearly $\overline{N}$ is normal in G, so by Lemma 5.1(4), $\overline{H}^{MT}$ is contained in $\overline{N}$. Hence, we have the inclusions

$$N < \mathrm{N_a}\left(\overline{H}^{MT}\right) < \overline{H}^{MT} < \overline{N}$$

that imply claim (2). $\square$

Proof of Proposition 5.7. Let $A = \mathbf{A}(k)$ be a one-dimensional k-split algebraic subgroup of G. We first show that $(A, A^\wedge)$ has the Mautner property. Note that we may assume without loss of generality that G is Zariski connected. We need to consider the cases when $\mathbf{A}$ is isomorphic over k to the additive group $\mathbf{G}_a$ and the multiplicative group $\mathbf{G}_m$.

When $\mathbf{A} \simeq \mathbf{G}_a$, it was proved in [18, Proposition 2.1] that there exists a closed normal subgroup N containing A such that (A, N) has Mautner property. This implies the claim in this case.

When $\mathbf{A} \simeq \mathbf{G}_m$, we use the results of S. Wang from [33]. Let g be an element of infinite order in A. We define the following subgroups of G:

$$U^+ = \{x \in G : \, g^n x g^{-n} \to e \text{ as } n \to \infty\},$$
$$U^- = \{x \in G : \, g^{-n} x g^{n} \to e \text{ as } n \to \infty\},$$
$$M = \{x \in G : \, g^{-n} x g^{n} \text{ is bounded for } n \in \mathbb{Z}\}.$$

It was shown in [33, Section 2] that:

(i) U^+, U^- and M are the k-points of Zariski connected k-algebraic subgroups.

(ii) U^+, U^- and M generate G.
(iii) the closed subgroup $W = \overline{\langle U^+, U^- \rangle}$ is normal in G.
(iv) The pair (C, W) has the Mautner property, where $C = \overline{\langle g \rangle}$.

We claim in addition, that

(iv) M commutes with A.

For this we recall the construction of M from [33]. Let $\mathbf{S}$ be the maximal k-split torus in $\mathbf{G}$ containing $\mathbf{A}$, and let Δ be a set of simple roots on $\mathbf{S}$ such that $|\alpha(g)| \le 1$ for $\alpha \in \Delta$, where $|\cdot|$ denotes the absolute value of k. Let $\Theta = \{\alpha \in \Delta : |\alpha(g)| = 1\}$ and let $\mathbf{S}_\Theta$ be the connected component of the identity in $\bigcap_{\alpha \in \Theta} \ker(\alpha)$. Then the subgroup $\mathbf{M}$ is precisely the centralizer of $\mathbf{S}_\Theta$ in G. Every $\alpha \in \Theta$ defines a k-character of $\mathbf{A}$ such that α is bounded on $A = \mathbf{A}(k)$. Hence, it follows that $\alpha(A) = 1$ for every $\alpha \in \Theta$, and $A \subset \mathbf{S}_\Theta(k)$. Therefore M commutes with A, as claimed.

Let $N = \overline{AW}$, where W is as in (iii). Since M commutes with A, it follows from (ii) that N is normal in G. By (iv), (A, N) has the Mautner property. Hence, $(A, A^\wedge)$ has the Mautner property as well. This proves our claim for the case $\mathbf{A} \simeq \mathbf{G}_m$.

To finish the proof of the Proposition, we recall that $H^\vee$ is the closed subgroup generated by the one dimensional split subgroups A of H, and observe that $H^\wedge$ is the closed subgroup generated by the corresponding groups $A^\wedge$. Since the pairs $(A, A^\wedge)$ have the Mautner property, it follows that the pair $(H^\vee, H^\wedge)$ has the Mautner property as well. As $H^\wedge$ is normal in G, we conclude that it is the Mautner envelope of $H^\vee$ by Lemma 5.1(4). $\square$

Given an S-algebraic group G, we introduce its *Lie algebra* $\mathrm{Lie}(G)$. It is defined to be $\prod_{v \in S} \mathrm{Lie}(\mathbf{G}_v)$ where $\mathrm{Lie}(\mathbf{G}_v)$ is the Lie algebra of the vth local factor. Since each group $\mathbf{G}_v$ is defined over k_v, its Lie algebras has a k_v-structure, and we consider $\mathrm{Lie}(\mathbf{G}_v)$ as a Lie algebra over k_v. The notion of the Lie algebra can be defined for any closed subgroup H of $G = \prod_{v \in S} \mathbf{G}_v(k_v)$. For this purpose we may assume without loss of generality that all the local fields in S are incompatible (i.e., have different characteristics of the residue fields). Then there exists an open normal subgroup of H that splits as a product of local factors (see e.g. [25, Proposition 1.5]). We define the Lie algebra of H as the product of the Lie algebras of local factors. The *exponential map* $\exp : \mathrm{Lie}(H) \to H$ is the product of exponential maps of the local factors, and it defines a diffeomorphism in a neighborhood of zero.

Proof of Proposition 5.8. It suffices to prove the claim in case $\mathbf{G}$ is a connected algebraic group defined over a local field k. Let $R = G^\vee$. Consider the factor map $\pi : \mathbf{G} \to \mathbf{G}/\mathrm{R}_{\mathrm{dc}}(\mathbf{G})$. By Proposition 5.3, for a one-dimensional split group A, the group $\pi(A)$ is f.i.-algebraic and hence compact. Then $\ker(\pi) \cap A$ is infinite, and since A is one-dimensional and connected, $\ker(\pi) \cap A$ is Zariski dense in A. This implies that $\pi(A) = 1$. Hence, $R \subset \mathrm{R}_{\mathrm{dc}}(G)$.

Since a discompact group has no finite index algebraic subgroups, it suffices to show that R has finite index in $\mathrm{R}_{\mathrm{dc}}(G)$. We may assume, without loss of generality, that $G = \mathrm{R}_{\mathrm{dc}}(G)$. Since the field k has characteristic zero, the group $\mathbf{G}$ has Levi decomposition

$$\mathbf{G} = \mathbf{STU}$$

where $\mathbf{S}$ is a connected semisimple subgroup defined over k, $\mathbf{T}$ is an algebraic torus defined over k that commutes with $\mathbf{S}$, and $\mathbf{U}$ is a normal unipotent subgroup defined over k. Since $G = \mathrm{R}_{\mathrm{dc}}(G)$, it follows from [21, Theorem 3.1] that the reductive group $\mathbf{ST}$ has no nontrivial k-anisotropic quotients.

In particular, this implies that the torus $\mathbf{T}$ is k-split. Then $\mathbf{T}$ is isomorphic over k to $\mathbf{G}_m^d$ and $T \subset R$. Similarly, $U \subset R$; indeed, since U is unipotent, the exponential map is a polynomial isomorphism $\mathrm{Lie}(U) \to U$. It follows that every element of U is contained in one-dimensional split unipotent subgroup of the form $\exp(tx)$, $x \in \mathrm{Lie}(U)$.

Let S^+ denote the closed subgroup of S generated by unipotent split subgroups of S. Since S does not have any k-anisotropic quotients, the subgroup S^+ has finite index in S by [21, §7.2]. It follows that $R \cap STU$ has finite index in STU. Since G is a homogeneous space with respect to the action of $S \times T \times U$ given by $(s,t,u) \cdot g = stgu^{-1}$, it follows from Proposition 5.2 that G is a finite union of double cosets of (ST, U). Using that U is normal, we conclude that STU is a finite index subgroup of G. This implies the claim. □

Proposition 5.9 will be deduced from the following more general theorem.

Theorem 5.10. *Let $\mathbf{G}$ be an algebraic group defined over a local field k and $\mathbf{A}_i$, $i \in I$, a family of connected k-subgroups closed under conjugation by elements of G. Then the closed subgroup N of G generated by A_i, $i \in I$, is f.i.-algebraic.*

Proof of Proposition 5.9 (assuming Theorem 5.10). Note that $H^\wedge$ is generated by all the conjugates of the one-dimensional split S-algebraic subgroups of H. By Theorem 5.10, it is f.i.-algebraic, and it is Zariski connected since it is generated by Zariski connected subgroups. □

The rest of this section will be devoted to the proof of Theorem 5.10. Our first step toward the proof of Theorem 5.10 is the case of a solvable group.

Lemma 5.11. *Let $\mathbf{G}$ be a solvable algebraic group defined over a local field k and $\mathbf{A}_i$, $i \in I$, a family of connected k-algebraic subgroups. Suppose there is no proper normal algebraic subgroup of $\mathbf{G}$ containing all of the $\mathbf{A}_i$. For each i let H_i be a finite index subgroup of A_i. Let N be a closed subgroup of G which is normal in a finite index subgroup of G and which contains all of the H_i. Then N is of finite index in G.*

Proof. The group $\mathbf{G}$ must be connected, as it is generated by the connected groups $\mathbf{A}_i$ and their conjugates. The group N is Zariski dense in $\mathbf{G}$ since it is normalized by a Zariski dense subgroup and its Zariski closure contains all of the $\mathbf{A}_i$. By dimension considerations we may assume that the collection I is finite. In case $\mathbf{G}$ is abelian the lemma follows from Proposition 5.3, by considering the homomorphism $\prod_I \mathbf{A}_i \to \mathbf{G}$.

For the general case, we will proceed by induction on $\dim \mathbf{G}$. We will show that $\mathbf{G}$ contains a normal algebraic subgroup $\mathbf{M}$ such that $M \cap N$ is of finite index in M; from this the statement will follow by applying the induction hypothesis to $\mathbf{G}/\mathbf{M}$. We will use the notation of [4], and let $\mathcal{D}^i(\mathbf{G})$ denote the subgroups in the derived series of $\mathbf{G}$. We will focus on the last nontrivial group $\mathbf{D}$ in this series, and distinguish two cases, according as $\mathbf{D}$ is or is not central in $\mathbf{G}$.

If $\mathbf{D}$ is not central in $\mathbf{G}$, since N is Zariski dense in $\mathbf{G}$, there is $n \in N$ which does not centralize $\mathbf{D}$. Consider the map $\varphi = \varphi_n : \mathbf{D} \to \mathbf{D}$ defined by $x \mapsto [n, x] =$

$nxn^{-1}x^{-1}$. Using the fact that $\mathbf{D}$ is abelian we obtain the formula

$$[n, xy] = [n, x][n, y]^x = [n, x][n, y], \tag{5.1}$$

i.e., φ is a k-algebraic homomorphism. Denote the Zariski closure of the image by $\mathbf{L}$. Applying Proposition 5.3 we find that $\varphi(D)$ is of finite index in L. On the other hand, whenever $x \in N_G(N)$, $\varphi(x) \in N$. Since $N_G(N)$ is of finite index in G, we also have that $N_D(N)$ is of finite index in D. This implies that $L \cap N$ is of finite index in L, and hence for any $n \in N$, the conjugate L^n also satisfies that $L^n \cap N$ is of finite index in L^n. Let $\mathbf{M}$ be the smallest k-algebraic group containing all the conjugates $\mathbf{L}^n$, for $n \in N$. Each L^n is contained in $\mathbf{D}$ since $\mathbf{D}$ is normal in $\mathbf{G}$. Thus $\mathbf{M}$ is contained in $\mathbf{D}$, hence abelian. Since N is Zariski dense, $\mathbf{M}$ is a normal subgroup of $\mathbf{G}$. Since the theorem is already proved for abelian groups, we now apply it to the groups $\mathbf{M}$ (in place of $\mathbf{G}$) and L^n (in place of H_i) to obtain that $M \cap N$ is of finite index in M, completing the proof in this case.

If $\mathbf{D}$ is central in $\mathbf{G}$, we let $\mathbf{E}$ be the preceding term in the derived series, i.e. $\mathbf{D} = (\mathbf{E}, \mathbf{E})$. For each $n \in N$ we can define φ_n as in the previous case. Using [4, Prop. 2.2], an induction shows that for each i, $N \cap \mathcal{D}^i(\mathbf{G})$ is Zariski-dense in $\mathcal{D}^i(\mathbf{G})$. In particular we can find $n \in N \cap E$ such that $\varphi_n(E)$ is non-trivial. We consider $\varphi = \varphi_n$ as a k-algebraic map $\mathbf{E} \to \mathbf{D}$. Since $\mathbf{D}$ is central in $\mathbf{G}$, (5.1) shows that φ is a homomorphism in this case as well. Now the same argument can be repeated. $\square$

Lemma 5.12. *Let $\mathbf{G}$ be a a connected k-algebraic group with no solvable non-trivial quotients. Let N be an open normal subgroup of G. Then N is of finite index in G.*

Proof. We denote the unipotent radical of $\mathbf{G}$ by $\mathbf{U}$, and let $\mathbf{G} = \mathbf{S} \ltimes \mathbf{U}$ be a Levi decomposition. The subgroup $\mathbf{S}$ is semi-simple (otherwise there is an abelian quotient). By passing to a covering group, we may and will assume that $\mathbf{S}$ is simply connected. Since N is open in G, it is enough to prove that N is cocompact in G. By [17, Corollary 2.3.2(a)], N contains the group S' where $\mathbf{S}' < \mathbf{S}$ is the subgroup consisting of the product of all k-isotropic factors. Since $S'U$ is cocompact in G, there is no loss of generality in assuming that $\mathbf{S}' = \mathbf{S}$ and hence $S < N$. In case $\mathbf{U}$ is trivial the proof is now complete. For the general case, we proceed by induction on the dimension of $\mathbf{U}$.

Since $\mathbf{S}'$ is generated by unipotent subgroups and is not normal in $\mathbf{G}$ (otherwise there will be a solvable quotient), there exists a connected unipotent subgroup $\mathbf{V} < \mathbf{S}$ which is not normalized by $\mathbf{U}$. Since U is Zariski dense in $\mathbf{U}$, we can find $u \in U$ which does not normalize $\mathbf{V}$. Consider the group $\mathbf{W}$ generated by $\mathbf{V}$ and $\mathbf{V}^u$. It is a subgroup of $\mathbf{VU}$, hence unipotent. V and V^u are contained in N, hence by Lemma 5.11, N contains a finite index subgroup of W. Denote by $\mathbf{H}_1$ the connected component of the identity of $\mathbf{W} \cap \mathbf{U}$. This is a connected unipotent subgroup of positive dimension in $\mathbf{U}$. N contains a finite index subgroup of H_1. Consider all the conjugations of H_1 by elements of N. These conjugates generate a subgroup $\mathbf{H}$ which is normal in $\mathbf{G}$. N contains a finite index subgroup of $\mathbf{H}$ by another application of Lemma 5.11. The proof now follows by applying the induction hypothesis to the group $\mathbf{G}/\mathbf{H}$. $\square$

Proof of Theorem 5.10. Replacing $\mathbf{G}$ by the algebraic subgroups generated by the groups $\mathbf{A}_i$, we need to prove that N is of finite index in G. The group N is clearly normal in G and Zariski dense. The group $\mathbf{G}$ must be Zariski connected, as it is generated by the connected groups $\mathbf{A}_i$. By dimension considerations we can replace I by a finite subset, so that $\mathbf{G}$ is still generated by the $\mathbf{A}_i$. By [4, Proposition (2.2)] and [17, Theorem (2.5.3)(2)], $N < G$ is open. Let $\mathbf{G}_0$ be the smallest normal subgroup of $\mathbf{G}$ containing all the semisimple subgroups of $\mathbf{G}$, and let $\mathbf{G}_1 = \mathbf{G}/\mathbf{G}_0$. Then $N_0 = N \cap G_0$ is open in G_0, so of finite index by Lemma 5.12. By Lemma 5.11, the image of N in G_1 is of finite index. This implies that N is of finite index in G. ☐

6. The relative Borel density theorem

6.1. **Relative Borel density.** In this section we will state and prove the relative Borel density theorem (Theorem 6.2).

Definition 6.1. Let G be an S-algebraic group and let L be a closed subgroup. A *fat complement for L in G* is a subgroup $M < G$ satisfying the following three conditions:

(F1) $L \cdot M = G$,
(F2) the Zariski closure of M contains the discompact radical $\mathrm{R}_{\mathrm{dc}}(G)$,
(F3) $\mathrm{N}_{\mathrm{a}}(M) \cap L$ is cocompact in L.

Theorem 6.2 (Relative Borel density)**.**
Let G be an S-algebraic group and H an S-algebraic subgroup of G. Let (X, ξ) be a G-space with an invariant probability measure which is ergodic for the action of H. Then there exists a closed subgroup M of G which is cocompact, has finite covolume, and is a fat complement for H in G, and a measure-preserving G-map $\pi : X \to G/M$, such that the following holds:

For every S-algebraic G-space V, for every $n \geq 0$, and for every H-map $i : X \to \mathrm{Prob}^n(V)$, there exists an H-map $j : G/M \to V$ such that $i = j \circ \pi$ almost everywhere. That is, we have the following diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\pi"'] & \mathrm{Prob}^n(V) \\ G/M \arrow[ur, dotted, "j"] & \end{tikzcd}$$

Note that by property (F3), if H has no compact algebraic factors, then i is constant a.e. We will see below (Corollary 6.3) that if G has no compact S-algebraic factors then i has an essentially finite image.

Proof. Let $R = \mathrm{R}_{\mathrm{dc}}(H)$. By Corollary 3.2, the image of i is contained $\mathrm{Prob}^n\left(V^R\right)$. Let $X /\!\!/ R$ denote the space of ergodic components for the action of R on (X, ξ), as described in Appendix A. Applying Proposition A.2(1), we find that i factors through $X /\!\!/ R$. Since ξ is G-invariant and finite, Corollary A.3 implies that $X /\!\!/ R$ is canonically identified with $X /\!\!/ \overline{R}^{MT}$. Let $N = \mathrm{N}_{\mathrm{a}}\left(\overline{R}^{MT}\right)$ be the f.i.-algebraic kernel of the Mautner envelope of R, see Definition 5.5. It follows that i factors through $X /\!\!/ N$ which, by Proposition A.2(2), is a G-space on which N acts trivially. Moreover the factor map $P_N : X \to X /\!\!/ N$ is a G-map.

Since H acts ergodically on X, $H/H \cap N$ acts ergodically on $X /\!\!/ N$. By Theorem 5.6(2), N is of finite index in the Mautner envelope of R. It follows that $N \cap R$ is of finite index in R. Since H/R is compact, $H/(H \cap N)$ is compact as well. Because of ergodicity, the measure $(P_N)_*\xi$ is supported on a single orbit of H. In particular, it follows that $X/\!\!/N$ is isomorphic as a G-space to G/M, where M is a closed cocompact subgroup of G of finite covolume, with $N < M$.

We denote by $\pi : X \to G/M$ the map corresponding to $P_N : X \to X/\!\!/N$. Then the existence of a Borel map $j : G/M \to Y$ such that $i = j \circ \pi$ follows from Proposition A.2(1). Since i and π are H-maps, j is an H-map as well.

It remains to show that M is a fat complement for H in G. Since $H/(H \cap N)$ is compact and acts ergodically on G/M, we conclude that H acts transitively on G/M and (F1) follows. To prove (F2), we observe that G/M supports a G-invariant measure $\pi(\xi)$. Hence, it follows from Corollary 3.3 that $\mathrm{R}_{\mathrm{dc}}(G)$ is contained in the Zariski closure of M. Since N acts trivially on G/M, we have $N < \mathrm{N}_{\mathrm{a}}(M)$. This implies (F3), because $H \cap N$ is cocompact in H. $\square$

6.2. **Fat complements.** In this subsection we discuss fat complements. Our goal is the following proposition, which will provide more information on the conclusion of Theorem 6.2.

Proposition 6.3. *Let G, H, and i be as in Theorem 6.2. If G has no compact S-algebraic factors then i has finite image.*

Proposition 6.3 is an immediate consequence of the following.

Theorem 6.4. *Let G be an S-algebraic group with no compact algebraic quotients. Let H be an S-algebraic subgroup, and let M be a fat complement of H in G. Then M is a finite index subgroup of G.*

For a Lie algebra $\mathfrak{g}$, we denote by $\mathfrak{g}'$ its commutator subalgebra. Let $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$ denote the adjoint representation. We recall that the differential of Ad is given by $\mathrm{ad}(x) = [x, \cdot]$, $x \in \mathrm{Lie}(G)$, and there exists a neighborhood $\mathcal{O}$ of the origin in $\mathrm{Lie}(G)$ such that $\exp : \mathcal{O} \to G$ is well-defined and

$$\mathrm{Ad}(\exp(x)) = \exp(\mathrm{ad}(x)) \quad \text{for all } x \in \mathcal{O}. \tag{6.1}$$

For the proof of Theorem 6.4 we will need the following version of the Malcev lemma, which follows e.g. from Corollary 7.9 in [4].

Lemma 6.5. *Let G be an S-algebraic group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{m}$ a Lie subalgebra of $\mathfrak{g}$. Suppose the subgroup M of G generated by $\exp(\mathfrak{m})$ is Zariski dense in G. Then $\mathfrak{g}' = \mathfrak{m}'$.*

Proof of Theorem 6.4. Let $N = \mathrm{N}_{\mathrm{a}}(M)$. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}$ and $\mathfrak{m}$ denote the Lie algebras corresponding to G, H, N and M. The hypothesis that G has no compact factors, along with (F2), imply that M is Zariski dense, hence

$$\mathrm{Ad}(G)\mathfrak{m} \subset \mathfrak{m}. \tag{6.2}$$

We choose open subgroups M°, N°, and G° of M, N, and G respectively such that

(i) G° normalizes M° and N°, and N° normalizes M°.
(ii) the group $M^\circ N^\circ$ is generated by open neighborhoods which are in the image of a neighborhood satisfying (6.1).

We observe that one can choose these subgroups as products of local factors:

$$M^\circ = \prod_v M_v^\circ, \qquad N^\circ = \prod_v N_v^\circ, \qquad G^\circ = \prod_v G_v^\circ$$

(see [27, Proposition 1.5]). For Archimedean v, (ii) holds provided that M_v° and N_v° are connected. For non-Archimedean v, the groups have bases of neighborhoods of identity consisting of Lie subgroups, so that M_v° and N_v° can be taken to be sufficiently small to satisfy (ii). Property (i) can be satisfied because of (6.2).

Since G is Zariski connected, the subgroup G° is Zariski dense in G. Let $\bar{M}$ be the group of S-points of the Zariski closure of M°. Then $\bar{M}$ is normal in G. It follows from (F1) and the Baire category theorem that the set HM° is open in G. In particular, it is Zariski dense in G. Since the subgroup $H\bar{M}$ is both f.i.-algebraic (by Proposition 5.3) and open in G, we conclude that $H\bar{M}$ has finite index in G.

The group $N = \mathrm{N}_{\mathrm{a}}(M)$ is f.i.-algebraic and Zariski connected. Since N° is open in N, the subgroup N° is Zariski dense in $\bar{N}$. Let L be the Zariski closure of the subgroup $N^\circ M^\circ$. Clearly, L is normal and $L \supset \bar{N}\bar{M}$. Hence, HL is of finite index in G. Since M is a fat complement of H, $H/(H\cap N)$ is compact. Hence, G/L is compact as well. By our assumption on G, the group $N^\circ M^\circ$ is Zariski dense in G. Hence, we may apply Lemma 6.5 with the subgroup $N^\circ M^\circ$ to conclude that

$$\mathfrak{g}' \subset \mathfrak{n} + \mathfrak{m} \subset \mathfrak{m}. \tag{6.3}$$

Let $\mathbf{G}'$ denote the (algebraic) commutator subgroup of $\mathbf{G}$ and $\pi : \mathbf{G} \to \mathbf{G}/(\mathbf{G}'\bar{\mathbf{N}})$ the corresponding factor map. Since the algebraic group $\mathbf{G}/(\mathbf{G}'\bar{\mathbf{N}})$ is abelian, it splits as an almost direct product of anisotropic and split subgroups. Moreover, by our assumption on G, the anisotropic component is trivial. On the other hand, since $H/(H\cap N)$ is compact, $\pi(H)$ is a compact f.i.-algebraic subgroup. Hence, $H \subset G'\bar{N}$. Now using (F1) and (6.3), we obtain

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \subset \mathfrak{g}' + \mathfrak{n} + \mathfrak{m} = \mathfrak{m}.$$

This shows that the group M is open in G. On the other hand, the homogeneous space G/M is compact. Hence, G/M has to be finite. This completes the proof. □

7. Completion of the proofs

We will first state and prove a useful corollary of the results of the previous section.

Corollary 7.1.
Let G be an S-algebraic group, H an S-algebraic subgroup of G, and Γ be a lattice, such that $H \curvearrowright G/\Gamma$ is ergodic.

Then there exists a closed subgroup $\Gamma < M < G$, where M is cocompact, of finite covolume in G, and a fat complement of H in G, such that for every S-algebraic G-space V, the inclusion map

$$\mathrm{Map}_M(G/H, \mathrm{Prob}^n(V)) \hookrightarrow \mathrm{Map}_\Gamma(G/H, \mathrm{Prob}^n(V))$$

and the map

$$\mathrm{Map}_H(G/M, \mathrm{Prob}^n(V)) \to \mathrm{Map}_H(G/\Gamma, \mathrm{Prob}^n(V)),$$

obtained by precomposing maps from $\mathrm{Map}_H(G/M, \mathrm{Prob}^n(V))$ with the projection $G/\Gamma \to G/M$, are bijections. When G has no nontrivial compact S-algebraic factors, M is of finite index in G.

Proof. Applying Theorem 6.2 to $X = G/\Gamma$, we obtain a group M and a G-map $G/\Gamma \to G/M$. Replacing M with a conjugate we may assume that $\Gamma < M$. The second bijection is a direct corollary of Theorem 6.2 and the first follows formally from the second using Corollary 2.7 using $L = \Gamma$. The last assertion follows from Proposition 6.3. □

7.1. **Proof of Theorem 1.6 and Corollaries 1.7, 1.8.**

Proof. Theorem 1.6 is just the first assertion of Corollary 7.1, in the case $n = 1$ (recall the bijection $X^{M\cap H} \cong \mathrm{Map}_H(G/M, X)$ described in the introduction). Corollary 1.7 is the case $n = 0$, and the deduction of Corollary 1.8 from Theorem 1.6 follows the same steps as the deduction of Theorem 1.4 from Theorem 1.5. □

7.2. **Proof of Theorem 1.11.**

Proof. Fix a map $f \in \mathrm{Map}_\Gamma(G/\Lambda, G/\Delta)$ where the target G/Δ is viewed as a Borel Γ-space. By Corollary 2.5 we have the duality

$$\mathrm{Map}_\Gamma(G/\Lambda, G/\Delta) \cong \mathrm{Map}_\Lambda(G/\Gamma, G/\Delta).$$

Given $f \in \mathrm{Map}_\Gamma(G/\Lambda, G/\Delta)$ let $F : G/\Gamma \to G/\Delta$ be the corresponding measurable Λ-equivariant map. As in Remark 2.6 we fix a Borel cross-section $\sigma : Y = G/\Gamma \to G$ of the projection $G \to G/\Gamma$, and take

$$F(y) = \sigma(y).f\left(\sigma(y)^{-1}\Lambda\right). \tag{7.1}$$

Let μ denote the probability measure on the space

$$X = Y \times Z, \qquad \text{where} \qquad Y = G/\Gamma, \quad Z = G/\Delta$$

obtained by pushing the Haar measure $m_{G/\Gamma}$ to the graph of F. Since F is a Λ-map, μ is invariant under the action of $\mathrm{Diag}(\Lambda) = \{(\lambda, \lambda) \in G \times G : \lambda \in \Lambda\}$ on X. Since Λ is a Zariski dense subgroup of G, it acts ergodically on G/Γ. Hence so is the action $\mathrm{Diag}(\Lambda) \curvearrowright (X, \mu)$. Next we want to use one of the assumptions (RS) or (BQ).

Assuming (RS), Λ is a lattice in a subgroup of G which is generated by unipotent elements. Note that we do not assume that this subgroup is connected; i.e. Λ itself may be generated by unipotent elements. In this situation one can apply the results of Ratner [26], and their extension by Shah [29] and Witte [35], to deduce that μ is L-homogeneous, where L is a closed subgroup of $G \times G$ containing $\mathrm{Diag}(\Lambda)$. This means that that there is $x_0 = (g_1\Gamma, g_2\Delta) \in X$ so that denoting

$$\Sigma = L \cap (g_1\Gamma g_1^{-1} \times g_2\Delta g_2^{-1})$$

is a lattice in L, and μ is the push-forward of the Haar measure $m_{L/\Sigma}$ to $L.x_0 \subset X$. Since μ projects onto $m_{G/\Gamma}$, it follows that L projects onto G. Consider

$$H = \{g \in G : (e, g) \in L\}.$$

This is a closed subgroup in G. Recall that μ is supported on the graph of $F : G/\Gamma \to G/\Delta$. Hence for $m_{G/\Gamma}$-a.e. $y \in G/\Gamma$ and $h \in H$ we have $h.F(y) = F(y)$. Note that given $z = g\Delta \in G/\Delta$ the group $\Delta_z = g\Delta g^{-1}$ is defined unambiguously. We have

$$H \subset \Delta_{F(y)}$$

for $m_{G/\Gamma}$-a.e. $y \in G/\Gamma$. Consider the measure $\eta = F_* m_{G/\Gamma}$ on $Z = G/\Delta$. Since H and Δ are closed sets, there is a conull (w.r.t. η) $Z_0 \subset Z$ such that

$$H < \bigcap_{z \in Z_0} \Delta_z. \tag{7.2}$$

Let M denote the projection of $L < G\times G$ to the second factor. Then $\Lambda < M$ and $\eta\in\mathrm{Prob}(Z)$ is M-homogeneous. The connected component M^0 of the identity in M is normal in M, and therefore normalized by Λ, which is Zariski dense in G. Hence M^0 is normal in G. Since G is simple and connected, we have

(1) either $M^0=\{e\}$ and M is discrete,
(2) or $M^0=M=G$.

Case (1). As M is discrete, η is atomic, and since it is a Λ-invariant and ergodic probability measure, it is supported on a finite Λ-orbit. That is, F has a finite image. Let Λ_0 be a finite index normal subgroup of Λ acting trivially on the image of F. By the Howe-Moore theorem, Λ_0 also acts ergodically on G/Γ, which implies that F is essentially constant. That is, $\eta=\delta_{z_0}$ for some Λ-fixed point $z_0=a\Delta\in G/\Delta$. We deduce $\Lambda\subset\Delta_{z_0}=a\Delta a^{-1}$ from (7.2), and have $F(g\Gamma)=y_0$ a.e. It follows from (7.1) that

$$f(g\Lambda)=ga\Delta.$$

Case (2): $M=G$ and $\eta=m_{G/\Delta}$. Since Z_0 in (7.2) is conull, it follows that $H=\{e\}$. Recalling the definition of H, we deduce that

$$L=\{(g,\rho(g)) : g\in G\}$$

for some continuous homomorphism $\rho:G\to G$, which is therefore algebraic. As $\mathrm{Diag}(\Lambda)<L$ we get $\rho(\lambda)=\lambda$ for $\lambda\in\Lambda$; and Zariski density of Λ in G implies $\rho(g)=g$ for all $g\in G$. Therefore $F(gg_1\Gamma)=gg_2\Delta$, or

$$F(g\Gamma)=gg_0\Delta$$

with $a=g_1^{-1}g_2$; in particular $\Gamma<\Delta_a$. From (7.1) we deduce, using a Borel cross-section $s:G/\Lambda\to G$, that

$$f(g\Lambda)=s(g\Lambda)F\left(s(g\Lambda)^{-1}\Gamma\right)=s(g\Lambda)s(g\Lambda)^{-1}a\Delta=a\Delta,$$

which is an a.e. constant function. This completes the proof of Theorem 1.11 under assumption (RS). Let us proceed to the proof under assumption (BQ).

Assuming (BQ), we have $\Delta<\Delta_0$ where Δ_0 is a lattice in G. Denote by $\pi:G/\Delta\to G/\Delta_0$ the natural projection, and let

$$F_0: G/\Gamma\xrightarrow{F}G/\Delta\xrightarrow{\pi}G/\Delta_0,\qquad \eta_0=\pi_*\eta\in\mathrm{Prob}(G/\Delta_0).$$

Then η_0 is a Λ-invariant and ergodic probability measure on G/Δ_0. Since Λ is assumed to be Zariski dense in G, we can apply the recent result of Benoist-Quint [2] to the action $\Lambda\curvearrowright G/\Delta_0$ to deduce the dichotomy:

(1) either η_0 is atomic, equidistributed on a finite Λ-orbit $\Lambda g_0\Delta_0\subset G/\Delta_0$,
(2) or $\eta_0=m_{G/\Delta_0}$ is the Haar measure on G/Δ_0.

In case (1), η is also atomic, and we conclude the proof as in the previous case. We are left with case (2), where the probability measure η on G/Δ projects onto the normalized Haar measure m_{G/Δ_0} on G/Δ_0. We claim that this is possible, only if Δ has finite index in Δ_0, and η is the normalized Haar measure $m_{G/\Delta}$.

First let us identify the G-action on G/Δ with the skew-product G-action on $G/\Delta_0\times\Delta_0/\Delta$ given by

$$g_1:(g\Delta_0,a\Delta)\mapsto(g_1g\Delta_0,c(g_1,g\Delta_0)a\Delta),$$

where $c:G\times G/\Delta_0\to\Delta_0$ is the cocycle

$$c(g_1,g\Delta_0)=\sigma(g_1g\Delta_0)^{-1}g_1\sigma(g\Delta_0)$$

associated to a choice of a Borel section $\sigma : G/\Delta_0 \to G$ for the projection $G \to G/\Delta_0$. That is we have a Borel isomorphism of the G-actions on

$$G/\Delta \quad \cong \quad G/\Delta_0 \times \Delta_0/\Delta.$$

Consider the restriction to the action of $\Lambda < G$, and view the Λ-invariant and ergodic probability measure η realized on $G/\Delta_0 \times \Delta_0/\Delta$. Since η projects to the Haar measure m_{G/Δ_0}, the disintegration of η with respect to m_{G/Δ_0} has the form

$$\eta = \int_{G/\Delta_0} \eta_x \, dm_{G/\Delta_0}(x), \qquad \eta_x \in \mathrm{Prob}(\Delta_0/\Delta).$$

Moreover, for $\lambda \in \Lambda$ and m_{G/Δ_0}-a.e. x one has

$$\eta_{\lambda.x} = c(\lambda, x)_* \eta_x \tag{7.3}$$

because η is Λ-invariant. The set $J = \Delta_0/\Delta$ is at most countable, so each probability measure η_x on J is atomic and therefore has a well-defined maximal 'weight'. That is, for $x \in G/\Delta_0$ we define

$$p(x) = \max_{j \in J} \eta_x(\{j\}), \qquad A(x) = \{i \in J : \eta_x(\{i\}) = p(x)\}.$$

It follows from (7.3) that

$$A(\lambda.x) = c(\lambda, x)^{-1} A(x) \qquad (x \in G/\Delta_0,\ \lambda \in \Lambda)$$

where $A(x) \subset J = \Delta_0/\Delta$ are finite sets; the cardinality $|A(x)|$ being a.e. constant by ergodicity of $\Lambda \curvearrowright (G/\Delta_0, m_{G/\Delta_0})$.

We claim that J is finite. Assume otherwise, and consider the probability space

$$(Y, \nu) = (Y_0, \nu_0)^J$$

where (Y_0, ν_0) is a non-trivial probability space, say $\{0, 1\}$ with $(1/2, 1/2)$-weights. Since $\Delta_0 \curvearrowright J = \Delta_0/\Delta$ is a transitive action on an infinite index set, the corresponding Δ_0-action on (Y, ν) is an ergodic measure-preserving action. Consider the induced G-action

$$G \curvearrowright (G/\Delta_0 \times Y, m_{G/\Delta_0} \times \nu), \qquad g : (x, y) \mapsto (g.x, c(g, x).y).$$

Actions induced to G from ergodic actions of a lattice $\Delta_0 < G$ are ergodic. So $G \curvearrowright G/\Delta_0 \times Y$ is ergodic. By Moore's theorem, the restriction to any unbounded subgroup, in particular to Λ, remains ergodic. Recall the sets $A(x) \subset J$ and the fact that $Y = \{0, 1\}^J$. Consider the set

$$Z = \{(x, y) \in G/\Delta_0 \times \{0, 1\}^J : \forall j \in A(x),\ y(j) = 0\}.$$

Observe that $m_{G/\Delta_0} \times \nu(Z) = 2^{-k} > 0$ where $k = |A(x)|$, and that Z is invariant under the Λ-action. This contradicts ergodicity of the Λ-action, showing that the assumption that J is infinite was wrong.

We have proved that $[\Delta_0 : \Delta] < +\infty$; in particular $\Delta < G$ is a lattice. Hence [2] implies that the Λ-invariant and ergodic probability measure η on G/Δ is the normalized Haar measure $\eta = m_{G/\Delta}$ (because it cannot be atomic under our assumption). Now consider the pushforward measure m of $m_{G/\Gamma}$ to the graph of $F : G/\Gamma \to G/\Delta$. This is a probability measure on

$$G \times G/\Gamma \times \Delta$$

invariant under the diagonal action of Λ. By [2] such a measure should be homogeneous for a subgroup $L < G \times G$ containing Λ. The argument can now be completed as in the (RS) case, by ruling out the possibility that $L = G \times G$ (because m cannot be a product measure $m_{G/\Gamma} \times m_{G/\Delta}$), and deducing that the Λ-equivariant map $F : G/\Gamma \to G/\Delta$ is actually G-equivariant. This, in turn, implies that the original Γ-equivariant map $f : G/\Lambda \to G/\Delta$ is a constant map. $\square$

In order to prove Theorem 1.9 we use recurrence instead of the existence of a finite invariant measure. An action of a group A on an A-space Z is called **strictly conservative** if for any non-null Borel $B \subset Z$ and any compact $C \subset A$, there is $a \in A \setminus C$ such that $aB \cap B$ is non-null.

Lemma 7.2. *Let A be an lcsc group and Y a second countable topological space equipped with a Borel probability measure ν on which A acts strictly conservatively. Then for almost every $y \in Y$, there is a sequence $a_n \in A$ such that $a_n \to \infty$ and $a_n y \to y$.*

Proof. For an open $U \subset Y$ of positive measure and compact $C \subset A$, let

$$Y_{C,U} = \{z \in U : \forall a \in A \setminus C,\, az \notin U\}.$$

This is a nullset by strict conservativity. Taking an exhaustion of A by countably many compacts $\{C_i\}$ and a countable basis U_j of open sets of Z, we have that $Y_\infty = \bigcup_{i,j:\nu(U_j)>0} Y_{C_i,U_j}$ is also a nullset, and any point in $\operatorname{supp}\nu \setminus Y_\infty$ has the required properties. $\square$

Proof of Theorem 1.9. We identify the group G of orientation preserving isometries of $\mathbb{H}$ with $\mathrm{PSL}_2(\mathbb{R})$. Without loss of generality we may assume that $\Gamma \subset G$. The space $\partial\mathbb{H} \times \partial\mathbb{H}$ is isomorphic as a G-space to the factor space G/A where A is the diagonal subgroup of G. By Corollary 2.5,

$$\mathrm{Map}_\Gamma(G/A, G/A) \simeq \mathrm{Map}_A(G/\Gamma, G/A). \tag{7.4}$$

The action of A on G/Γ is precisely the geodesic flow on the unit tangent bundle of the hyperbolic surface $\mathbb{H}/\Gamma$. According to [12], the geodesic flow is either ergodic and conservative, or totally dissipative. Since Γ acts ergodically on G/A, we are dealing with the former case.

Let $\Phi : G/\Gamma \to G/A$ be an A-map. Let μ be a Haar measure on G/Γ and $\nu = \Phi_*\mu$. Then ν is ergodic and conservative with respect to the action of A. From this we deduce that for ν-almost every $x \in G/A$ there exists a sequence $a_i \to \infty$ in A such that $a_i x \to x$ (Lemma 7.2). On the other hand, it is easy to check that the sequence $a_i x$ may only accumulate on the set $(G/A)^A$ of fixed points of A. This shows that for μ-almost every $x \in G/\Gamma$, we have $\Phi(x) \in (G/A)^A$. Note that $(G/A)^A \simeq \mathcal{N}_G(A)/A$ where $N_G(A)$ denotes the normalizer of A in G. Now it follows from ergodicity that the set $\mathrm{Map}_A(G/\Gamma, G/A)$ consists of exactly two elements indexed by $N_G(A)/A$. Under identification (7.4), these two elements correspond to the maps

$$G/A \to G/A, \qquad gA \mapsto gnA, \qquad (n \in \mathcal{N}_G(A)/A),$$

which are the identity map and the flip. $\square$

8. Examples

In this section we give several elementary examples which illustrate that the assumptions imposed in our main results are essential.

Example 8.1. This example demonstrates that a fat complement M may be of infinite index, and that the subgroup M in Theorem 1.7 may in general be of infinite index. Consider an S-algebraic group $\mathbf{G} = \mathbf{H} \times \mathbf{L}$ such that H is compact and set $\Gamma = L$, $Y = G$. Then

$$\Phi : G/H \to Y, \quad lH \mapsto (e, l), \quad l \in L,$$

defines a Γ-map. The largest group which could play the role of M in the statement is $M = L$.

Example 8.2. This example shows that the group M in Theorem 1.6 does not have to be f.i.-algebraic. Let $\mathbf{G} = \mathrm{SO}_\mathbf{n} \times \mathrm{G}_\mathbf{a}$ where $\mathrm{SO}_\mathbf{n}$ denotes the orthogonal group and $\mathrm{G}_\mathbf{a}$ denotes the additive group. We fix a compact one-parameter subgroup $\{k(t)\}_{t\in\mathbb{R}}$ of $\mathrm{SO}_n(\mathbb{R})$ and set $\Gamma = \{(k(t), t)\}_{t\in\mathbb{R}}$. We also set $H = \mathrm{SO}_n$. Then the map

$$\Phi : G/H \to G, \quad (e, t)H \mapsto (k(t), t), \quad t \in \mathbb{R},$$

is Γ-equivariant, but not equivariant almost everywhere with respect to any larger subgroup.

Example 8.3. This example shows that even when G in Theorem 1.6 has no nontrivial compact S-algebraic quotients, the subgroup M could be proper (cf. Theorem 6.4). Let $G = \mathrm{G}_m$, $H = \langle \pm 1\rangle$, $\Gamma = \mathbb{R}^\times_+$, and

$$\Phi : \mathbb{R}^\times / \langle \pm 1\rangle \to \mathbb{R}^\times, \quad x \langle \pm 1\rangle \mapsto x, \quad x \in \mathbb{R}^\times_+,$$

be the factor map. Then Φ is $\mathbb{R}^\times_+$–equivariant but not $\mathbb{R}^\times$-equivariant.

Example 8.4. This example shows that the assumption in Corollary 1.8 that the factor is measure-preserving is essential. Let $\mathbf{G} = \mathrm{PSL}_2$, $k = \mathbb{R}$, and let H be the diagonal subgroup of G. Let Γ be a cocompact lattice in G and $X = G/H$ equipped with the Haar measure class μ. Then Γ acts ergodically on X, and Theorem 1.6 implies that every measure-preserving Γ-factor of X is of the form G/Q where Q is a closed subgroup of G such that H is a normal cocompact subgroup of Q. Hence, there are only two measure-preserving Γ-factors of X corresponding to $Q = H$ and $Q = \mathcal{N}_G(H)$.

On the other hand, there exists an infinite normal subgroup Λ of Γ such that Λ does not act ergodically on X. For instance, one can take Λ such that $\Gamma/\Lambda \simeq \mathbb{Z}^k$ with $k \geq 3$ (see [28]). Then there is a nontrivial measure-class-preserving Γ-factor $Y = (X /\!\!/ \Lambda, P_\Lambda(\mu))$ (see Appendix A). Since Λ acts trivially on Y and G is simple, the space Y cannot be isomorphic to a G-space.

Appendix A. The space of ergodic components

Let G be an lcsc group. Given a G-space (X, μ), we denote by $\mathcal{B}(X)$ the Boolean σ-algebra consisting of measurable sets modulo the ideal of null sets. Note that every measure-class-preserving G-map $\Phi : (X, \mu) \to (Y, \nu)$ between G-spaces induces a G-equivariant Boolean-σ-algebra homomorphism $\Phi^* : \mathcal{B}(Y) \to \mathcal{B}(X)$ defined by $\Phi^*([A]) = [\Phi^{-1}(A)]$ for a Borel subset $A \subset X$.

For a G-space (X, μ), we introduce the space of ergodic components $X /\!\!/ G$. This space can be characterized by the universal property — Proposition A.2(1) below.

Proposition A.1 (Ergodic decomposition)**.** *Given a G-space (X, μ), there exist a standard Borel space $X /\!\!/ G$ and a Borel map $P_G : X \to X /\!\!/ G$ such that*

- *$P_G(gx) = P_G(x)$ for all $g \in G$ and almost every $x \in X$,*
- *$P_G^*(\mathcal{B}(X/\!\!/G))$ is the algebra of G-invariant elements in $\mathcal{B}(X)$.*

Proposition A.1 is a part of folklore in ergodic theory, see, for instance, [10, Theorem 5.2] for a proof.

We shall use the following functorial properties of the space $X/\!\!/G$:

Proposition A.2. (1) *Let (X, μ) be a G-space, Y a standard Borel space, and $\Phi : X \to Y$ a Borel map such that $\Phi(gx) = \Phi(x)$ for all $g \in G$ and almost every $x \in X$. Then there exists a Borel map $\bar{\Phi} : X/\!\!/G \to Y$ such that $\Phi = \bar{\Phi} \circ P_G$ on a conull set.*

(2) *If H is a closed normal subgroup of an lcsc group G and (X, μ) is a G-space, then $X/\!\!/H$ is equipped with a structure of a G-space so that $P_H : X \to X/\!\!/H$ is a G-map.*

Proof. In the proof we shall use the well-known correspondence between standard Borel spaces and Boolean algebras (see [15, 22]). Specifically, we shall use the following:

(i) ([22, Theorem 2.1]) If $\Phi, \Psi : (X, \mu) \to (Y, \nu)$ are G-maps such that $\Phi^* = \Psi^*$, then $\Phi = \Psi$ on a conull set.

(ii) ([15, Theorem 3], [22, Theorem 3.6]) If (X, μ) and (Y, ν) are G-spaces and $\phi : \mathcal{B}(Y) \to \mathcal{B}(X)$ is a G-equivariant homomorphism, then there exists a measure-class-preserving G-map such that $\Phi^* = \phi$.

(iii) ([15, Theorem 1], [22, Theorem 3.3]) Every Boolean space equipped with a G-action is G-equivariantly isomorphic to a Boolean measure space associated with a G-space.

Now we start with the proof of (1). By [37, B.5] there exists a measurable map $\Psi : X \to Y$ such that $\Psi = \Phi$ on a conull set and Ψ is G-equivariant on a G-invariant Borel sets of full measure. This implies that the Boolean σ-algebra $\Phi^*(\mathcal{B}(Y))$ consists of G-invariant elements, i.e.,

$$\Phi^*(\mathcal{B}(Y)) \subset P_G^*(\mathcal{B}(X/\!\!/G)).$$

Let $\bar{\phi} : \mathcal{B}(Y) \to \mathcal{B}(X/\!\!/G)$ be the corresponding embedding. By (ii) there exists a G-map $\bar{\Phi} : X/\!\!/G \to Y$ such that $\bar{\Phi}^* = \phi$. Then $P_G^* \circ \bar{\Phi}^* = (\bar{\Phi} \circ P_G)^* = \Phi^*$, and it follows from (i) that $\bar{\Phi} \circ P_G = \Phi$ on a conull set, as required.

Let H be a closed normal subgroup of G. Since $P_H^*(\mathcal{B}(X/\!\!/H))$ is the algebra of H-invariant elements, it invariant under G. This defines an action of G on $\mathcal{B}(X/\!\!/H)$ such that the map $P_H^* : \mathcal{B}(X/\!\!/H) \to \mathcal{B}(X)$ is G-equivariant. By (iii), the G-action on $\mathcal{B}(X/\!\!/H)$ comes from a structure of a G-space on $X/\!\!/H$. Since for every $g \in G$, we have $g^* \circ P_H^* = P_H^* \circ g^*$, it follows from (i) that $P_G \circ g = g \circ P_G$ on a conull set. This completes the proof. □

Corollary A.3. *Let G be an lcsc group, let H be its closed subgroup, and let $\overline{H}^{MT}$ be the Mautner envelope of H as in §5. Let (X, ξ) be a G-space with a G-invariant probability measure. Then the natural map $X/\!\!/H \to X/\!\!/\overline{H}^{MT}$ is a Borel isomorphism.*

Proof. To prove the corollary we need to show that every Borel subset B of X such that $\mu(B \triangle hB) = 0$ for $h \in H$, we have $\mu(B \triangle hB) = 0$ for $h \in \overline{H}^{MT}$. Indeed, the characteristic function of B is an H-invariant vector in $L^2(X, \mu)$, hence it is also invariant by $\overline{H}^{MT}$. □

Proof. Since $N < \overline{H}^{MT}$, the natural map $X /\!\!/ \overline{H}^{MT} \to (X /\!\!/ N) /\!\!/ \overline{H}^{MT}$ is an isomorphism. Since by Corollary A.3, $X /\!\!/ H \simeq X /\!\!/ \overline{H}^{MT}$ and $(X /\!\!/ N) /\!\!/ H \simeq (X /\!\!/ N) /\!\!/ \overline{H}^{MT}$, this implies the claim. $\square$

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