Dynamics on Parameter Spaces: Submanifold and Fractal Subset Questions

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Abstract Our theme is the following: when it is known that a certain property holds for almost every point in a manifold, we want to know whether the property holds for almost every point in a submanifold or fractal subset. Such results were proved by Kleinbock and Margulis for Diophantine approximation via dynamics on homogeneous spaces, and by Masur and Veech for interval exchanges via dynamics on quadratic differential spaces. We survey some recent work along these lines, and also prove some new results, including a generalization of the convergence case of Khinchin's theorem to a class of fractals in $\mathbb{R}^d$.

1 Introduction

In recent years, through the work of H. Masur, W. Veech, A. Eskin, and others, a remarkable bridge has developed between work in two areas: dynamics of subgroup actions on homogeneous spaces, with applications to number theory, and dynamics of the $\mathrm{SL}(2,\mathbb{R})$ action on the moduli space of quadratic differentials on a surface, with applications to interval exchange transformations and rational polygonal billiards. Eskin's recent survey [3] describes work in this vein, and focuses on counting problems which have been tackled in both contexts by similar methods. In this survey we describe more work bringing out some of the interaction between these fields.

Our focus however will be on questions of a different type, which may all be seen as offsprings of a famous question of K. Mahler. Say that a vector $\mathbf{s} \in \mathbb{R}^d$ is *not very well approximable (NVWA)* if for all $\delta > 0$ there is $c > 0$ such that for all $\mathbf{p} \in \mathbb{Z}^d, q \in \mathbb{Z}$ we have $\|q\mathbf{s} - \mathbf{p}\| \geq c|q|^{-(1/d+\delta)}$. It is easily seen that almost every vector in $\mathbb{R}^d$ is NVWA. In 1932 Mahler asked whether almost every vector on the curve

$$\{(t, t^2, \ldots, t^d) : t \in \mathbb{R}\}$$

is NVWA (with respect to the natural measure on the curve). The conjecture, settled affirmatively by V. Sprindzhuk in the '60s, spawned interest in questions of the following general form. Let P be a property of points in $\mathbb{R}^d$, which holds for Lebesgue almost every point in $\mathbb{R}^d$, and let μ be a measure on $\mathbb{R}^d$. We say that μ *inherits* P if μ almost every point satisfies P. The general problem is:

Problem: *Suppose it is known that P holds for Lebesgue almost every point in $\mathbb{R}^d$. Describe the measures which inherit P.*

To make the discussion more concrete, let us mention three influential papers in which questions of this type were addressed. The work we will describe can be seen as an effort to clarify the relationship between these results.

Kleinbock and Margulis: Diophantine approximation on submanifolds. Let $V \subset \mathbb{R}^k$ be open, and let $\mathbf{f} : V \to \mathbb{R}^d$ be a C^m function, $m \geq 1$. We say that $\mathbf{f}$ is *nondegenerate* if for almost every $s \in V$, $\mathbb{R}^d$ is spanned by the partial derivatives of $\mathbf{f}$, up to some order, at s. In 1996, D. Kleinbock and G. Margulis applied the theory of flows on homogeneous spaces to prove that if $\mathbf{f}$ is nondegenerate and μ is the natural smooth measure on $\mathbf{f}(V)$ then μ-almost every $\mathbf{s}$ is NVWA. In other words, the smooth measure class on a nondegenerate submanifold in $\mathbb{R}^d$ inherits the property of being NVWA. These and stronger results are contained in [7]. For further developments, see [1].

Kerckhoff, Masur and Smillie: ergodicity of rational billiards. A foliation of a surface S is called *uniquely ergodic* if there exists a unique measure (up to scaling), on segments transverse to the foliation, which is invariant under holonomy. Let $\mathcal{Q}$ be the moduli space of unit area holomorphic quadratic differentials on S. This space, which will be described in more detail below, is an orbifold on which $\mathrm{SL}(2,\mathbb{R})$ acts in a natural way, and each $q \in \mathcal{Q}$ determines two transverse measured foliations of the complement of a finite set in S; these are called the horizontal and vertical foliations of q. It is known by work of Masur [12] that $\mathcal{Q}$ carries a natural finite smooth invariant measure, and that with respect to this measure, for almost every q, the vertical foliation of q is uniquely ergodic. Let $r_\theta \in \mathrm{SL}(2,\mathbb{R})$ be the rotation matrix corresponding to an angle θ. In 1980, as part of their pioneering work on ergodicity of rational billiards, S. Kerckhoff, H. Masur and J. Smillie [5] proved that for any $q \in \mathcal{Q}$, the smooth measure on $\{r_\theta q : \theta \in \mathbb{R}\}$ inherits the unique ergodicity of the vertical foliation.

Veech: decaying measures and uniquely ergodic foliations. Let μ be a measure on $\mathbb{R}^d$, let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, and let $B(x,r)$ denote the Euclidean ball of radius r around $x \in \mathbb{R}$. We say that μ is *F-decaying* if for any $x \in \mathbb{R}$, every $0 < r < 1$ and every $0 < \varepsilon < 1$ we have:

$$\mu(B(x, \varepsilon r)) \leq F(\varepsilon)\mu(B(x,r))\,.$$

Note that Lebesgue measure is F-decaying for $F(x) = x$. In 1999 it was proved by Veech [21] that if μ is an F-decaying measure on $\mathbb{R}$, where F is any function satisfying $F(\varepsilon) \to_{\varepsilon \to 0} 0$, then for any $q \in \mathcal{Q}$, the pushforward of μ to $\mathcal{Q}$ via

$$\theta \mapsto r_\theta q$$

inherits unique ergodicity of the vertical foliation.

In this paper we describe some conjectures and results concerning Diophantine approximation on fractal sets and unique ergodicity of foliations. The paper is organized as follows. In Sect. 2 we state the results and conjectures. The proofs of the results are contained in several papers, some of which are complete and some of which are still under construction. In Sect. 3 we define, hopefully in a user-friendly manner, the dynamical systems with which we will work. In Sect. 4 we describe the interactions between work on homogeneous spaces and on quadratic differential spaces, emphasizing the similar dynamical strategies which are used to prove the results. We close in Sects. 5, 6 with some apparently new applications of the Borel–Cantelli lemma, including a Diophantine result which interpolates the convergence case of Khinchin's theorem on metric to measures on fractal sets, and a result about quadratic differentials which shows that the natural measure on any Teichmüller horocycle inherits the upper logarithm law for geodesics.

2 Conjectures and Results

Fix a permutation σ on $d+1$ symbols which is irreducible, i.e., for all $1 \leq k \leq d$,

$$\sigma(\{1, \ldots, k\}) \neq \{1, \ldots, k\}.$$

Let Δ_d be the open d-dimensional simplex, namely:

$$\Delta_d = \{(a_1, \ldots, a_d) : a_i > 0, \ \sum a_i < 1\}.$$

Any $\mathbf{a} = (a_1, \ldots, a_d) \in \Delta_d$ defines an interval exchange transformation $\mathcal{IE}_\sigma(\mathbf{a}) : [0,1) \to [0,1)$ by cutting $[0,1)$ into $d+1$ adjacent intervals whose lengths respectively are $a_1, a_2, \ldots, a_d, 1 - \sum a_i$ and rearranging them by orientation preserving isometries according to σ. In 1982 Masur [12] and Veech [19] settled a conjecture of M. Keane, proving that for Lebesgue almost every choice of $\mathbf{a} \in \Delta_d$, $\mathcal{IE}_\sigma(\mathbf{a})$ is *uniquely ergodic* (i.e., the Lebesgue measure is the only $\mathcal{IE}_\sigma(\mathbf{a})$-invariant measure on $[0,1)$). Masur's strategy — reducing the question to a question about quadratic differential dynamics — will be described in Sect. 4.

It is natural to ask which measures on Δ_d inherit unique ergodicity of $\mathcal{IE}_\sigma(\cdot)$. First let us state some conjectures regarding this question.

Conjecture 2.1 *Let $V \subset \mathbb{R}^k$ be open, let $f : I \to \Delta_d$ be nondegenerate and let $\sigma \in S_{d+1}$ be irreducible. Then the natural smooth measure on $f(V)$ inherits the unique ergodicity of $\mathcal{IE}_\sigma(\cdot)$.*

Recall that an interval exchange is called *minimal* if the orbit of every point is dense. Conditions for minimality are well understood (see e.g. [4]). In particular, the set

$$\{\mathbf{a} \in \Delta : \mathcal{IE}_\sigma(\mathbf{a}) \text{ is not minimal}\}$$

is a countable union of affine subspaces of Δ which may be explicitly described.

Conjecture 2.2 *For any affine subspace $\bar{\mathcal{L}} \subset \mathbb{R}^d$, let $\mathcal{L} = \bar{\mathcal{L}} \cap \Delta$. Then one of the following holds:*

1. *The natural measure on $\mathcal{L}$ inherits unique ergodicity of $\mathcal{IE}_\sigma(\cdot)$;*
2. *For every $\mathbf{a} \in \mathcal{L}$, $\mathcal{IE}_\sigma(\mathbf{a})$ is not minimal.*

In [23], we obtain, following Masur's strategy, partial results supporting these conjectures from corresponding results about measures on $\mathcal{Q}$ which inherit uniquely ergodic vertical foliations. The results supporting Conjecture 2.2 are applications of:

Theorem 2.3 ([17, Corollaries 2.8, 2.9]) *Let $q \in \mathcal{Q}$, and let μ be*

- *either the natural length measure on the Teichmüller horocycle $h_t q$, $t \in \mathbb{R}$; or,*
- *more generally, the pushforward, under the map $t \mapsto h_t q$, of a measure on $\mathbb{R}$ which is F-decaying for some F satisfying $F(\varepsilon) \to_{\varepsilon \to 0} 0$.*

Then μ inherits the unique ergodicity of the vertical foliation.

This result is analogous to those of Kerckhoff, Masur and Smillie (resp. Veech) described above, with trajectories of the Teichmüller horocyclic flow $\{h_t : t \in \mathbb{R}\}$ replacing trajectories under the circle group action $\{r_\theta : \theta \in \mathbb{R}\}$.

A related result about logarithm laws for Teichmüller geodesics along Teichmüller horocycle paths is given in Sect. 6.

The following result concerns Diophantine approximation on fractal sets. We say that μ is α-decaying if it is cx^α-decaying for some c. In [21], Veech gave conditions guaranteeing that a measure supported on a compact subset of the real line is α-decaying for some $0 < \alpha < 1$. For instance it turns out that the natural coin tossing measure on Cantor's middle thirds set is α-decaying for $\alpha = \log(2)/\log(3)$ (the same as its Hausdorff dimension). Similarly, many measures arising naturally on dynamically defined fractal sets are α–decaying for some α.

In [22] we prove:

Theorem 2.4 *If μ is an α-decaying measure on $\mathbb{R}$, then μ inherits NVWA.*

The proof of this statement is very simple. An indication of the proof is given in Sect. 5, along with somewhat weaker results about measures on $\mathbb{R}^d$.

There is an intimate connection between Theorem 2.4 and the preceding theorems: it can be proved by specializing some of the arguments of [21] and [17] to the case in which S is the torus! In this case, $\mathcal{Q}$ coincides with $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ and thus has applications to Diophantine approximation on the real line. Of course, as this is a somewhat degenerate case, most of the difficulties disappear and one is left with a very easy argument.

Now suppose that μ is an α-decaying measure on $\mathbb{R}$, so that the decay of lengths of subintervals is controlled from above, and that in addition, the following condition (sometimes called the *Federer condition*) holds: there are positive α_1 and c_1 such that for every $x \in \mathrm{supp}\mu$, every $0 < r < 1$ and every $0 < \varepsilon < 1$,

$$\mu(B(x, \varepsilon r)) \geq c_1 \varepsilon^{\alpha_1} \mu(B(x, r)) \, .$$

In joint work with E. Lindenstrauss [11], the following was proved:

Theorem 2.5 *Suppose μ is a measure on $\mathbb{R}$ for which the conditions above hold, and let $\mathbf{f} : I \to \mathbb{R}^d$ be a nondegenerate curve. Then the pushforward of μ via the map $s \mapsto \mathbf{f}(s)$ inherits NVWA.*

The proof of this result combines the arguments of [7] with simple arguments as in [22].

An interesting and apparently difficult project is to unify the results of Kleinbock and Margulis with Theorems 2.4 and 2.5, by specifying a purely measure-theoretic condition on a measure μ on $\mathbb{R}^d$ (encompassing both nondegeneracy and decay) which would ensure that μ inherits NVWA.

3 A Gentle Reminder Regarding Dynamics on Homogeneous / Quadratic Differential Spaces

Homogeneous spaces. Let $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, and $\pi : G \to G/\Gamma$ be the projection map $\pi(g) = g\Gamma$. The *homogeneous space* G/Γ carries the following structure:

- It is a manifold parameterizing the set of cocompact discrete subgroups $\Lambda \subset \mathbb{R}^n$ such that the Lebesgue measure of a fundamental domain for Λ in $\mathbb{R}^n$ is 1;
- Any subgroup $H < G$ acts naturally by
$$h\pi(g) = \pi(hg) \, ;$$
- The Haar measure of a fundamental domain for Γ in G is finite. Since G is unimodular, the Haar measure defines a finite measure on G/Γ which is invariant under the action of G (and any of its subgroups);
- **Mahler's compactness criterion.** For $X \subset G$, $\overline{\pi(X)} \subset G/\Gamma$ is compact if and only if
$$\inf\{\|x \cdot \mathbf{v}\| : x \in X, \ \mathbf{v} \in \mathbb{Z}^n - \{0\}\} > 0 \, .$$

These features can of course be generalized to a more general context in which G is a Lie group and Γ is a nonuniform lattice. Besides its own intrinsic interest, the study of dynamics of the actions of various subgroups of G on G/Γ has been of fundamental importance in various applications. There are many excellent survey papers about these matters. Among them, we recommend the recent and very detailed survey [9].

Spaces of Quadratic differential. We now describe the space of quadratic differentials. To the best of our knowledge, all of the features above have their counterparts in this context. Even the notorious problem of the lack of a suitable survey was recently addressed by Masur and S. Tabachnikov [16], survey of which the author became aware while preparing this paper.

Let S be a closed surface of genus $g \geq 1$. Informally, the space of quadratic differentials describes the possible constructions of S out of flat pieces of paper, in which the paper is checkered with vertical and horizontal straight lines and the construction is required to respect this pattern and the orientation. By the Gauss–Bonnet's formula no such construction is possible for a surface of genus greater than 1, so we allow a finite set of points in which several pieces of paper are attached, giving a total angle which is more than 2π. Note that in general S may also be allowed to have finitely many punctures but this case introduces some technicalities and will be omitted from the discussion.

More precisely, let $\mathbf{Q}$ be the set of all atlases of charts $\mathbf{q}$ of the following type. Away from a finite set $\Sigma \subset S$, every point on S has a neighborhood with a chart to $\mathbb{R}^2$, so that the transition maps are of the form $\mathbf{s} \mapsto \pm\mathbf{s} + c$. Thus the Euclidean metric, the Euclidean area form, and the set of lines of any given slope in $\mathbb{R}^2$ are preserved by the transitions, and make sense on $S - \Sigma$ as well. The preimages of the horizontal lines and the vertical lines are called the horizontal and vertical foliations of $\mathbf{q}$. We normalize our atlases by requiring that the total area of the surface with respect to the Euclidean area form is 1. In general the vertical and horizontal foliations do not admit an orientation. If they do, $\mathbf{q}$ is said to be orientable.

Around a singular point $x \in \Sigma$ there is a neighborhood U and a k-fold branched cover ($k = k(x) \geq 3$) from U to $\mathbb{R}^2/\pm$ which is compatible with the charts around nonsingular points. Thus both the horizontal and vertical foliations have a k-pronged singularity at x, and the metric in a neighborhood is inherited from k Euclidean halfplanes glued cyclically together along rays.

Let $\mathrm{Homeo}_+(S)$ denote the group of orientation-preserving homeomorphisms of S, $\mathrm{Homeo}^0_+(S)$ its identity component, and let

$$\mathrm{Mod}(S) = \mathrm{Homeo}_+(S)/\mathrm{Homeo}^0_+(S)$$

denote the mapping class group. The natural action of $\mathrm{Homeo}_+(S)$ by composition of each chart in an atlas, gives us quotients

$$\begin{aligned} \widetilde{\mathcal{Q}} &= \mathbf{Q}/\mathrm{Homeo}^0_+(S) \\ \mathcal{Q} &= \mathbf{Q}/\mathrm{Homeo}_+(S) = \widetilde{\mathcal{Q}}/\mathrm{Mod}(S)\,. \end{aligned}$$

The data consisting of the number of singularities, the set of singularity types $\{k(x) : x \in \Sigma\}$, and the orientability of the foliations, make sense on an equivalence class modulo $\mathrm{Homeo}^0_+(S)$. There are finitely many possible values for these data, and a level set for these data is called a *stratum*.

There is a standard description of $\widetilde{\mathcal{Q}}$ as a bundle of holomorphic tensors over the Teichmüller space, which may be used to equip $\widetilde{\mathcal{Q}}$ with a manifold topology, in which the strata are locally closed submanifolds of various dimensions. Alternatively, each stratum $\mathcal{M}$ can be given a manifold topology, as follows. Let $\mathbf{M}$ be the subset of $\mathbf{Q}$ projecting to $\mathcal{M}$, let $\mathbf{q} \in \mathbf{M}$, and assume first that we are in the orientable case. Any differentiable path α on S has a $\mathbf{q}$-length, defined by integrating the local projections of $d\alpha$ to the x and y axes in each chart. It can be checked that this length is constant on a suitable homology class in S rel Σ. This defines a map from $\mathbf{M}$ into a cohomology group (depending on $\mathcal{M}$), which is $\mathrm{Homeo}^0_+(S)$-invariant, hence descends to a well-defined map on $\mathcal{M}$, and which can be shown to be a local homeomorphism, (see [20] for details). In the non-orientable case, there is a double cover of S by a surface of higher genus $\hat{S}$, which is ramified precisely at $\{x \in \Sigma : k(x) \text{ is odd}\}$, and by pullback we get for each quadratic differential in $\mathcal{M}$ an orientable quadratic differential on $\hat{S}$. This produces a map of $\mathcal{M}$ into a stratum of orientable quadratic differentials over $\hat{S}$, which can be checked to be an immersion (note that for different strata one obtains different surfaces $\hat{S}$ – see [10] for more details, and for a description of the connected components of the orientable strata). Thus in both cases $\mathcal{M}$ is a manifold.

The action of $\mathrm{Mod}(S)$ on $\widetilde{\mathcal{Q}}$ and on each stratum is properly discontinuous, and this gives $\mathcal{Q} = \widetilde{\mathcal{Q}}/\mathrm{Mod}(S)$ the structure of an orbifold. Let $\pi : \widetilde{\mathcal{Q}} \to \mathcal{Q}$ be the quotient map. The space $\mathcal{Q}$ is stratified by *projections of strata*, that is, the immersed sub-orbifolds $\pi(\mathcal{M})$ for all strata $\mathcal{M} \subset \widetilde{\mathcal{Q}}$. There is an open and dense stratum in $\widetilde{\mathcal{Q}}$ corresponding to the maximal number of singularities. It is called the *principal stratum*.

We have:

- $\mathcal{Q}$ is an orbifold parameterizing the (essential) ways in which S can be constructed out of pieces of paper as above, for which the total area of S is 1. The projection of each stratum in $\mathcal{Q}$ is an immersed sub-orbifold parameterizing the constructions above in which the number and type of singular points, and the orientability of the construction, is fixed;
- There is a natural action on $\widetilde{\mathcal{Q}}$ by $\mathrm{SL}(2,\mathbb{R})$: for any $\mathbf{q} \in \mathbf{Q}$ and $A \in \mathrm{SL}(2,\mathbb{R})$, replace each chart ϕ of $\mathbf{q}$ by $A \circ \phi$ where A acts linearly on $\mathbb{R}^2$. This preserves the compatibility condition and commutes with the action of $\mathrm{Homeo}_+(S)$, hence descends to a well-defined action on $\mathcal{Q}$. The action $\{h_s\}$ (resp. $\{g_t\}$) of the one-parameter group of upper triangular unipotent matrices (resp. positive diagonal matrices) in $\mathrm{SL}(2,\mathbb{R})$ defines the Teichmüller horocycle flow (resp., the Teichmüller geodesic flow);
- As shown by Masur [12] for the principal stratum and by Veech [20] in general, on the projection of each stratum there is a smooth finite measure which is $\mathrm{SL}(2,\mathbb{R})$-invariant;

- **Mumford's compactness criterion.** For $X \subset \widetilde{\mathcal{Q}}$, $\overline{\pi(X)} \subset \mathcal{Q}$ is compact if and only if
$$\inf\{\ell_{[\gamma]}(q) : [\gamma],\ q \in X\} > 0\,,$$
where $[\gamma]$ ranges over free homotopy classes on S and $\ell_{[\gamma]}(q)$ is the length of the shortest representative of $[\gamma]$ w.r.t. the Euclidean metric defined by q.

Fix a stratum $\mathcal{M}$. By passing to a finite cover of $\mathcal{M}$, we may assume that $\Sigma = \Sigma(q)$ is independent of $q \in \mathcal{M}$ – i.e. that the singularities are enumerated. For any pair of points $x_1, x_2 \in \Sigma$, the shortest path in the homotopy class in S fixing the endpoints x_1, x_2 is called a *saddle connection* (we allow $x_1 = x_2$ but do not allow homotopically trivial paths from x_1 to itself). It consists of finitely many straight segments $\alpha : [0,1] \to S$ such that $\{0,1\} = \alpha^{-1}(\Sigma)$. Every $q \in \mathcal{M}$ assigns to each saddle connection δ a length $\ell_\delta(q)$, and so we obtain a *length function* $\ell_\delta : \mathcal{M} \to \mathbb{R}_+$.

- **Compactness criterion on the projection of a stratum.** For $X \subset \mathcal{M}, \pi(\mathcal{M}) \cap \overline{\pi(X)}$ is a compact subset of $\mathcal{Q}$ if and only if
$$\inf\{\ell_\delta(q) : \delta \text{ is a saddle connection, } q \in X\} > 0\,.$$

4 Quantitative Nondivergence and Applications

In this section we explain what quantitative nondivergence results are and how they are used to prove results such as those mentioned above. We start with another list of features which are common to the papers described in Sect. 1.

Visits outside large compact sets. The first step in the argument of [7] is a reduction (developed first in work of S. G. Dani [2] and extended recently by Kleinbock and Margulis in [8]) in which Diophantine properties of vectors in $\mathbb{R}^{n-1}$ are related to dynamical properties of flows on G/Γ. First we define the compact sets
$$K_\varepsilon = \pi(\{g \in G : \text{for all nonzero } \mathbf{v} \in \mathbb{Z}^n, \|g \cdot \mathbf{v}\| \geq \varepsilon\})\,.$$
By Mahler's compactness criterion the sets $\{K_\varepsilon : \varepsilon > 0\}$ give an exhaustion of G/Γ.

We then have:

Proposition 4.1 ([7, §2]). *There is a one-parameter diagonalizable subgroup $\{\tilde{g}_t : t \in \mathbb{R}\} \subset G$, and a map $\tau : \mathbb{R}^{n-1} \to \mathrm{SL(n,\mathbb{R})}$, such that if for all $\delta > 0$, and for all large t we have*
$$\tilde{g}_t \pi(\tau(\mathbf{v})) \in K_{\exp(-\delta t)}\,, \tag{1}$$
then $\mathbf{v}$ is NVWA.

Thus to ensure NVWA one must rule out infinitely many visits outside a sequence of compacts growing at a certain rate.

Very similarly, for the space of quadratic differentials we define

$$K_\varepsilon = \pi(\{q \in \widetilde{\mathcal{Q}} : \text{for every saddle connection } \delta, \ell_\delta(q) \geq \varepsilon\})\,.$$

The sets $\{K_\varepsilon : \varepsilon > 0\}$ are again an exhaustion of $\mathcal{Q}$, this time by Mumford's criterion. Moreover their intersections with the projection of each stratum are an exhaustion of it.

In analogy with Proposition 4.1, we have the following reduction developed by Masur:

Proposition 4.1 ([12], [14], [21]) *Given a permutation $\sigma \in S_{d+1}$ there is a surface S, a stratum $\mathcal{M} \subset \mathcal{Q} = \mathcal{Q}(S)$ and a map $\tau : \Delta_d \to \mathcal{M}$, such that $\mathcal{IE}_\sigma(\mathbf{a})$ is uniquely ergodic if and only if the vertical foliation of $\tau(\mathbf{a})$ is uniquely ergodic.*

For any $q \in \mathcal{Q}$, if the vertical foliation of q is not uniquely ergodic then the trajectory $\{g_t q : t \geq 0\}$ is divergent in the projection of its stratum in $\mathcal{Q}$. That is, for any $\varepsilon > 0$ there is t_0 such that for all $t \geq t_0,\ g_t q \notin K_\varepsilon$.

Thus to ensure unique ergodicity one must rule out trajectories which eventually leave every large compact set in the projection of a stratum.

Quantitative nondivergence. Let X be a space and let $\mathbf{f} : \mathbb{R}^k \to X$ be a map. Let B be the ball in $\mathbb{R}^k$. For $K \subset X$ we set

$$\mathrm{Avg}_{B,\mathbf{f}}(K) = \frac{|\{t \in B : \mathbf{f}(t) \in K\}|}{|B|}\,,$$

where $|\cdot|$ is the Lebesgue measure. By a quantitative nondivergence result we mean a result which gives a lower bound on $\mathrm{Avg}_{B,\mathbf{f}}(K_\varepsilon)$ in terms of an expression which depends on ε, is independent of B and depends only weakly on $\mathbf{f}$. Such a result is the main tool of [7], sharpening earlier results of Dani, Margulis and N. A. Shah [18] on the nondivergence of trajectories under the action of a unipotent subgroup. With the work of Kleinbock and Margulis as a model, we prove in [17] similar results for quadratic differentials.

For example we have:

Theorem 4.2 ([17, Theorem 6.3]) *For any $q \in \mathcal{Q}$, let $\mathbf{f}_q(s) = h_s q$. There are positive constants C, α, ϱ_0 (depending only on S) such that if $q \in \mathcal{Q}$, an interval $B \subset \mathbb{R}$, and $0 < \varrho < \varrho_0$ have the property that:*

$$\text{for any saddle connection } \delta \text{ there is } s \in B \text{ such that } \ell_\delta(h_s q) \geq \varrho\,, \tag{2}$$

then for any $0 < \varepsilon < \varrho$ we have:

$$\mathrm{Avg}_{B,\mathbf{f}_q}(K_\varepsilon) \geq 1 - C\left(\frac{\varepsilon}{\varrho}\right)^\alpha.$$

Using nondivergence results. We illustrate the use of quantitative nondivergence results by explaining how the first part of Theorem 2.3 is deduced from Proposition 4.1 and Theorem 4.2. By Proposition 4.1, we must rule out the possibility that the set

$$\{s \in \mathbb{R} : \{g_t h_s q : t \geq 0\} \text{ is divergent in the projection of its stratum}\}$$

has positive measure. If it did, there would be an interval $B \subset \mathbb{R}$ and $\theta > 0$ such that for all $\varepsilon > 0$,

$$|\{s \in B : g_t h_s q \notin K_\varepsilon \text{ for all large enough } t\}| \geq \theta \, .$$

By a simple calculation [17, Claim 7.5], we find $\varrho > 0$ such that for all $t \geq 0$, and any saddle connection δ, there is $s \in B$ such that $\ell_\delta(g_t h_s q) \geq \varrho$. This guarantees that (2) holds uniformly for all $t \geq 0$. Now taking ε small enough so that $C(\varepsilon/\varrho)^\alpha < \theta/2$, and t large enough so that

$$|\{s \in B : g_t h_s q \notin K_\varepsilon\}| \geq \theta/2 \, ,$$

we obtain a contradiction with Theorem 4.2.

5 Khinchin's Convergence Case for Fractals

In this section we present a simple result along the lines of Theorem 2.4.

Let $\psi : \mathbb{Z}_+ \to \mathbb{R}_+$ be a decreasing function. A vector $\mathbf{s} \in \mathbb{R}^d$ is said to be ψ*–approximable* if there are infinitely many $\mathbf{p} \in \mathbb{Z}^d$ and $q \in \mathbb{Z}_+$ for which

$$|q\mathbf{s} - \mathbf{p}| < \psi(q) \, . \tag{3}$$

Note that $s \in \mathbb{R}$ is NVWA iff it is not $q^{-(1/d+\delta)}$–approximable for some $\delta > 0$. Khinchin [6] proved the famous result that the Lebesgue measure of the (complement of the) ψ-approximable vectors is zero if

$$\sum_q \psi(q)^d < \infty \tag{4}$$

(resp., if the series diverges). In this section we show that an analogue of the convergence case of Khinchin's theorem holds for certain measures satisfying a decay condition.

Let μ be a measure on $\mathbb{R}^d$. Extending the previous definition, we say that μ is α*-decaying* if there exists C such that for every $\mathbf{s} \in \mathbb{R}^d$, every $0 < r < 1$ and every $0 < \varepsilon < 1$ we have

$$\mu(B(\mathbf{s}, \varepsilon r)) \leq C\varepsilon^\alpha \mu(B(\mathbf{s}, r)) \, .$$

Note that Lebesgue measure is d-decaying.

We prove:

Theorem 5.1 *If μ is α-decaying and $\sum_q \psi(q)^\alpha < \infty$ then μ-almost every* $\mathbf{s}$ *is not ψ-approximable.*

Notation: In the sequel, the notation $x \prec y$ means that x and y are quantities depending on some parameters, and there is some $C > 0$, independent of these parameters, such that $x \leq Cy$.

For a vector $\mathbf{w} \in \mathbb{Q}^d$, the notation $\mathbf{w} = \mathbf{p}/q$ means that $\mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{Z}^d$, $q \in \mathbb{Z}_+$, and $\gcd(q, p_1, \ldots, p_d) = 1$.

Proposition 5.2 *Let $J \subset \mathbb{R}^d$ be a bounded open set and let $B_1, B_2, \ldots$ be disjoint finite subsets of $\mathbb{Q}^d$ such that*

$$\mathbb{Q}^d \cap J = \bigcup_{r=1}^{\infty} B_r \,.$$

For each r, let

$$L_r = \min\{q : \mathbf{p}/q \in B_r\}$$

and let

$$D_r = \min\{\mathrm{dist}(s, B_r - \{s\}) : s \in B_r\} \,.$$

Suppose that μ is α-decaying and that

$$\sum_r \left(\frac{\psi(L_r)}{L_r}\right)^{\alpha} D_r^{-\alpha} < \infty \,. \tag{5}$$

Then μ almost every $\mathbf{s} \in J$ is not ψ-approximable.

Proof. Since the general term in (5) tends to zero, we have for large r that $\psi(q)/(qD_r) \leq 1/2$, whenever $\mathbf{p}/q \in B_r$. With no loss of generality we may assume that this holds for all r. This implies, by our assumption that μ is α-decaying, that for every $q \in \mathbb{Z}_+$ and every $\mathbf{p} \in \mathbb{Z}^d$, we have:

$$\mu(B(\mathbf{p}/q, \psi(q)/q)) \prec \left(\frac{\psi(q)/q}{D_r/2}\right)^{\alpha} \mu(B(\mathbf{p}/q, D_r/2)) \,.$$

The definition of D_r implies that the $B(\mathbf{p}/q, D_r/2)$, $\mathbf{p}/q \in B_r$ are pairwise disjoint, and are all contained in a fixed neighborhood of J. Therefore the quantity

$$\sum_{\mathbf{p}/q \in B_r} \mu(B(\mathbf{p}/q, D_r/2))$$

is bounded uniformly in r.

It is enough to prove the theorem for any open set J' compactly contained in J, in which case, for all large enough q, if (3) holds for $\mathbf{s} \in J'$, $\mathbf{p} \in \mathbb{Z}^d$ and q, then $\mathbf{p}/q \in \bigcup B_r$. Therefore we may assume with no loss of generality that whenever (3) holds, $\mathbf{p}/q \in \bigcup B_r$.

We obtain

$$\begin{aligned}
&\sum_{q=1}^{\infty} \mu\{\mathbf{s} \in J : \exists \mathbf{p} \in \mathbb{Z}^d \text{ s.t. } \mathbf{p}, q \text{ satisfy (3)}\} \\
&= \sum_{q=1}^{\infty} \mu\{\mathbf{s} \in J : \exists \mathbf{p} \in \mathbb{Z}^d,\ \mathbf{s} \in B(\mathbf{p}/q, \psi(q)/q)\} \\
&\leq \sum_{r=1}^{\infty} \sum_{\mathbf{p}/q \in B_r} \mu(B(\mathbf{p}/q, \psi(q)/q)) \\
&\prec \sum_{r=1}^{\infty} \left(\frac{\psi(q)/q}{D_r/2}\right)^{\alpha} \sum_{\mathbf{p}/q \in B_r} \mu(B(\mathbf{p}/q, D_r)) \\
&\prec \sum_{r=1}^{\infty} \left(\frac{\psi(L_r)}{L_r}\right)^{\alpha} D_r^{-\alpha} < \infty\,.
\end{aligned}$$

In view of Borel–Cantelli, this proves the theorem. □

Theorem 5.1 follows, taking

$$B_q = \{\mathbf{p}/q : \mathbf{p} \in \mathbb{Z}^d\} \cap J$$

for any ball $J \subset \mathbb{R}^d$.

Note that there are some fractal sets with Hausdorff dimension α, which support a natural α-decaying measure. For such measures, the expression in Theorem 5.1 naturally interpolates (4) for non-integer dimensions.

Taking

$$B_r = \{p/q : (p, q) = 1,\ q \in \{2^r, 2^{r+1}, \ldots, 2^{r+1} - 1\}\}\,,$$

we obtain the following statement, of which Theorem 2.4 is a special case.

Theorem 5.3 ([22]) *If μ is an α-decaying measure on $\mathbb{R}$ and ψ satisfies*

$$\sum_{q=1}^{\infty} q^{\alpha-1}\psi(q)^{\alpha} < \infty\,, \tag{6}$$

then μ-almost every $s \in \mathbb{R}$ is not ψ–approximable.

Questions and Remarks:

1. Different choices of $B_1, B_2, \ldots$ give different conditions on functions ψ.

2. The fact that Theorem 5.1 is not strong enough to show that an α-decaying measure, with $\alpha < d$, inherits NVWA is not surprising; indeed, if $M \subset \mathbb{R}^d$ is a submanifold which is degenerate (e.g. M is a rational linear subspace) then the natural measure on M does not in general inherit NVWA. Nevertheless, it is possible to specify a stronger decay condition, under which stronger conclusions may be obtained. Such questions are addressed in [11].
3. It would be interesting to know whether the condition (6) is the best possible for an α-decaying measure. This does not seem to be known even for the coin tossing measure on Cantor's ternary set.

6 Logarithm Laws on a Teichmüller Horocycle

As before, let $\{h_s\}$ (respectively $\{g_t\}$) denote the Teichmüller horocycle (respectively geodesic) flow. We say that $q \in \mathcal{Q}$ *satisfies the upper logarithm law* if for any $\delta > 0$, for all sufficiently large t we have:

$$g_t q \in K_{t^{-(1/2+\delta)}} \,. \tag{7}$$

Note that this contains finer information than (1): in terms of distance from some fixed point in the space, it can be computed (cf. [8], [13]) that (1) describes compacts growing at a linear rate whereas in (7) the rate is logarithmic.

It was proved in [13] that almost every $q \in \mathcal{Q}$ satisfies the upper logarithm law, and in fact the following stronger statement holds:

Theorem 6.1 (Masur) *For any $q \in \mathcal{Q}$, let μ be the natural measure on $\{r_\theta q : \theta \in \mathbb{R}\}$. Then μ inherits the upper logarithm law.*

In this section we show that Masur's argument can be modified to yield:

Theorem 6.2 *For any $q \in \mathcal{Q}$, let μ be the natural measure on $\{h_s q : s \in \mathbb{R}\}$. Then μ inherits the upper logarithm law.*

Proof. Fix $q \in \mathcal{Q}$ and let $|\cdot|$ denote Lebesgue measure on $\mathbb{R}$. We need to show that $|L| = 0$, where

$$L = \{s \in [-1,1] : \exists t_n \to \infty,\ g_{t_n} h_s q \notin K_{t_n^{-(1/2+\delta)}}\} \,.$$

Any saddle connection η and any $q \in \mathcal{M}$ determine a vector $(x(q), y(q)) = (x_\eta(q), y_\eta(q)) \in \mathbb{R}^2$ by integrating the projection of $d\eta$ to the horizontal and vertical coordinates determined by q, in each coordinate patch. The coordinates corresponding to all saddle connections form a discrete subset of $\mathbb{R}^2$. It is easily checked that

$$\begin{pmatrix} x(g_t h_s q) \\ y(g_t h_s q) \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(q) \\ y(q) \end{pmatrix} = \begin{pmatrix} e^t(x(q) + s y(q)) \\ e^{-t} y(q) \end{pmatrix} \,. \tag{8}$$

Thus if (7) does not hold, for $h_s q$ instead of q and for some t, then for some saddle connection η, setting $(x, y) = (x_\eta(q), y_\eta(q))$ we have:

$$|x + sy| \le \frac{e^{-t}}{t^{1/2+\delta}}\,, \tag{9a}$$

$$|y| \le \frac{e^{t}}{t^{1/2+\delta}}\,. \tag{9b}$$

For each $r \in \mathbb{N}$, let

$$M_r = \{s \in [-1,1] : \text{for some } \eta \text{ and } t, \text{(9a) and (9b) hold and} |y| \in [e^r, e^{r+1}]\}$$

and let M_∞ consist of those s which belong to infinitely many of the M_r. Since for every saddle connection there is at most one s for which $|x+sy| = 0$, it follows using (9a) that $L - M_\infty$ is countable. Thus it suffices to show that $|M_\infty| = 0$.

Since $s \in [-1,1]$, we have from (9a) that $|x| \le 1+|y|$. We use the following estimate, which follows immediately from Masur's estimate [15]:

- **The number of saddle connections is quadratic**: For any $q \in \mathcal{Q}$ and any $r > 0$,
$$\#\{\eta : |y| \le e^r,\ |x| \le 1 + |y|\} \prec e^{2r}\,.$$

For each η, choose $t(y)$ for which there is equality in (9b). We have $t(y) \ge \ln |y|$. If (9a) and (9b) hold for some t then they also hold for $t(y)$. Multiplying (9a) and (9b) we get

$$|xy + y^2 s| \le \frac{1}{t(y)^{1+2\delta}}\,.$$

The left hand side of this inequality is a linear function in s with slope $|y^2|$ and thus for a fixed η,

$$\big|\{s \in [-1,1] : (9a), (9b) \text{ hold for some } t\}\big| \prec \frac{1}{|y|^2 t(y)^{1+2\delta}} \le \frac{1}{|y^2|(\ln|y|)^{1+2\delta}}\,.$$

Summing on all possible η and using Masur's estimate we obtain:

$$\begin{aligned} |M_r| &\prec \sum_{\eta,\ |y| \in [e^r, e^{r+1}]} \frac{1}{|y|^2 (\ln|y|)^{1+2\delta}} \\ &\prec \frac{1}{r^{1+2\delta}}\,. \end{aligned}$$

Therefore $\sum |M_r| < \infty$ and by Borel–Cantelli, $|M_\infty| = 0$. ∎

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