ALMOST NO POINTS ON A CANTOR SET ARE VERY WELL APPROXIMABLE

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Let $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the positive integers and positive reals, respectively, and let $\psi : \mathbb{Z}_+ \to \mathbb{R}_+$ be a decreasing function. A real number s is said to be ψ–**approximable** if there are infinitely many $p \in \mathbb{Z}, q \in \mathbb{Z}_+$ such that

$$|qs - p| < \psi(q), \tag{1}$$

and is said to be **very well approximable (VWA)** if it is $q^{-(1+\delta)}$–approximable for some $\delta > 0$. It is well–known that the set of ψ–approximable numbers has zero Lebesgue measure provided

$$\sum_{q=1}^{\infty} \psi(q) < \infty. \tag{2}$$

This forms the easy direction of Khinchin's theorem [Kh, Theorem 30]. In particular the set of VWA numbers has zero Lebesgue measure. In this note we point out that a variant of Khinchin's statement above also holds if instead of Lebesgue measure we consider certain measures supported on fractal sets. This implies for example that the set of VWA numbers has zero measure with respect to the standard measure supported on the Cantor middle–thirds set, given by assigning the digits $0, 1, 2$ in the ternary expansion probabilities $1/2, 0, 1/2$ respectively (on the other hand the set of VWA numbers has full Hausdorff dimension [Ja]).

Our condition on the measure is that there are $C > 0$ and $0 < \alpha \leq 1$ such that for every $x \in \mathbb{R}$, $r > 0$ and $0 < \epsilon \leq 1$ we have:

$$\mu(B(x, \epsilon r)) \leq C\epsilon^{\alpha} \mu(B(x, r)), \tag{3}$$

where $B(x, r)$ is the interval $(x - r, x + r)$.

For Lebesgue measure on the real line one has, for $C = \alpha = 1$, equality in (3). For some other examples of measures satisfying this condition, see [Ve][section 2]. In particular Veech shows that the standard measure on the Cantor middle–thirds set satisfies the condition. In fact for the Cantor middle–thirds set α can be taken to be $\frac{\log 2}{\log 3}$ — the same as its Hausdorff dimension.

Theorem. *Suppose μ is a measure on $\mathbb{R}$ satisfying (3) for some C, α as above. Let $\psi : \mathbb{Z}_+ \to \mathbb{R}_+$ satisfy*

$$\sum_{q=1}^{\infty} q^{\alpha-1}\psi(q)^{\alpha} < \infty. \tag{4}$$

Then μ-almost every $s \in \mathbb{R}$ is not ψ–approximable.

In particular $\mu(\mathbf{VWA}\text{ numbers}) = 0$.

Proof: It is easy to see that if (1) is satisfied for $p = dp', q = dq'$, for some p', q' and infinitely many d, then s is rational. Since the set of rationals is countable we may assume in (1) that $(p,q) = 1$. Since a countable union of sets of zero measure has zero measure, it suffices to show that for any interval I, the set of $s \in I$ for which there are infinitely many coprime $p \in \mathbb{Z}, q \in \mathbb{Z}_+$ such that (1) is satisfied has zero measure with respect to μ. Since (1) is equivalent to $s \in B(p/q, \psi(q)/q)$, by (the easy part of) the Borel–Cantelli lemma, it suffices to show that

$$\sum_{\substack{p\in\mathbb{Z}, q\in\mathbb{Z}_+ \\ (p,q)=1, p/q\in I}} \mu(B(p/q, \psi(q)/q)) < \infty. \tag{5}$$

Notice that for every $Q \geq 0$, every $q,\ q_0 \in \{2^Q, \ldots, 2^{Q+1}-1\}$ and every $p, p_0 \in \mathbb{Z}$ such that $\frac{p}{q} \neq \frac{p_0}{q_0}$, we have

$$B(p/q, 2^{-2(Q+2)}) \cap B(p_0/q_0, 2^{-2(Q+2)}) = \emptyset.$$

Indeed, if s were in the intersection we would have:

$$\begin{aligned} 2^{-2(Q+1)} &\leq \frac{1}{qq_0} \\ &< \left|\frac{pq_0 - p_0 q}{qq_0}\right| \\ &= \left|\frac{p}{q} - \frac{p_0}{q_0}\right| \\ &\leq \left|\frac{p}{q} - s\right| + \left|\frac{p_0}{q_0} - s\right| \\ &< 2 \cdot 2^{-2(Q+2)}, \end{aligned}$$

a contradiction.

The intervals $B(p/q, 2^{-2(Q+2)})$, where $p/q \in I$, $(p,q) = 1$ and $2^Q \leq q < 2^{Q+1}$, are disjoint and their union is contained in a bounded neighborhood of I (which is independent of Q). This implies that there is

C' such that for all Q,

$$\sum_{\substack{p/q\in I\\(p,q)=1\\2^Q\le q<2^{Q+1}}} \mu(B(p/q,2^{-2(Q+2)})) < C'.$$

From (4) and the monotonocity of ψ it follows easily that for all but finitely many Q, for $2^Q \le q < 2^{Q+1}$ we have $\frac{\psi(q)/q}{2^{-2(Q+2)}} \le 1$. With no loss of generality we may assume that this holds for all Q.

We obtain:

$$\begin{aligned}\sum_{\substack{p,q\in\mathbb{Z}\\p/q\in I\\(p,q)=1}} \mu(B(p/q,\psi(q)/q)) &= \sum_{Q=0}^{\infty}\sum_{\substack{p/q\in I\\(p,q)=1\\2^Q\le q<2^{Q+1}}} \mu(B(p/q,\psi(q)/q))\\ &\le \sum_{Q=0}^{\infty}\sum_{\substack{p/q\in I\\(p,q)=1\\2^Q\le q<2^{Q+1}}} C(\frac{\psi(q)/q}{2^{-2(Q+2)}})^{\alpha}\mu(B(p/q,2^{-2(Q+2)}))\end{aligned}$$

by (3. Hence on rearranging

$$\begin{aligned}\sum_{\substack{p,q\in\mathbb{Z}\\p/q\in I\\(p,q)=1}} \mu(B(p/q,\psi(q)/q)) &\le C\sum_{Q=0}^{\infty}\sum_{\substack{p/q\in I\\(p,q)=1\\2^Q\le q<2^{Q+1}}} 2^{2\alpha(Q+2)}\psi(q)^{\alpha}q^{-\alpha}\mu(B(p/q,2^{-2(Q+2)}))\\ &\le C_1\sum_{Q=0}^{\infty}2^{\alpha(2Q-Q)}\psi(2^Q)^{\alpha}\sum_{\substack{p/q\in I\\(p,q)=1\\2^Q\le q<2^{Q+1}}} \mu(B(p/q,2^{-2(Q+2)}))\\ &\le C_2\sum_{Q=0}^{\infty}2^Q\cdot(2^Q)^{\alpha-1}\psi(2^Q)^{\alpha}.\end{aligned}$$

The monotonicity of ψ implies that of $q^{\alpha-1}\psi(q)^{\alpha}$. In view of (4), we obtain (5). □

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