DIVERGENT TRAJECTORIES ON NONCOMPACT PARAMETER SPACES

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1. Introduction

The pioneers of dynamical systems theory were interested primarily in physical systems, for example planetary motion, and began by studying the asymptotic behavior of *typical* trajectories. In recent developments, researchers motivated by classical problems in pure mathematics were led to consider dynamical systems on various paramterizing spaces, and to questions about *special* trajectories. Two outstanding developments with these features are the study of dynamics of Lie group actions on homogeneous spaces, with applications to classical questions in number theory, and the study of dynamics of the $\mathrm{SL}(2,\mathbb{R})$-action on the moduli space of quadratic differentials with applications to interval exchange transformations and polygonal billiards. We refer to [KlShSt] and [MasTa] for recent, detailed accounts. We also refer to [We2] for a survey highlighting the parallels between these theories.

The parameter spaces which are studied from this point of view, for example the space of lattices $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})$ or the space of unit area quadratic differentials over complex structures on a surface, are often noncompact orbifolds which carry a smooth finite invariant measure. Hence by Poincaré recurrence, a typical orbit returns along an unbounded sequence of times to any neighborhood of its starting point. At the opposite extreme are the *divergent* trajectories, that is, trajectories which eventually escape any compact subset of the space. Such atypical trajectories are very interesting for applications: for actions on homogeneous spaces, through work of Dani [Da], they are related to singular systems of linear forms which had been previously studied in the theory of diophantine approximation, and for actions on spaces of quadratic differentials, they are related, by work of Masur [Mas2], to the unique ergodicity of interval exchange transformations.

Our goal in this paper is a systematic study of divergent trajectories on noncompact parameter spaces. We begin with an abstract approach, which is well-adapted to study both homogeneous spaces and spaces of quadratic differentials, and to study actions of one-parameter groups as well as multidimensional groups and semigroups. We then specialize to specific spaces and specific actions. Let us first state the three problems which we address.

A. Existence. Do divergent trajectories exist?

B. Obvious vs. non-obvious reasons to escape. In the parameter spaces we consider, there are certain easily described divergent trajectories which we call 'obvious' (the terms 'degenerate' and 'spiraling' have also appeared in the literature). Consider for example the space of lattices $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ with the action of a one-parameter diagonalizable subgroup $\{a(t) : t \in \mathbb{R}\}$. It follows easily from Mahler's compactness criterion that the trajectory $\{a(t)\pi(x) : t \geq 0\}$ (where $\pi : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ is the quotient map) is divergent if there is $0 \neq v \in \mathbb{Z}^n$ such that

$$\|a(t)x \cdot v\| \longrightarrow_{t\to+\infty} 0.$$

Similarly a trajectory of a quadratic differential is divergent if some nontrivial element of the surface's fundamental group has a representative which is a vertical loop, that is, lies entirely in the leaves of the vertical folitation associated to the quadratic differential. Precise definitions will appear later; loosely speaking, a divergent trajectory is obvious if there is a finite set of explicit algebraic data which account for the divergence of the trajectory.

The non-obvious divergent trajectories are the interesting ones for the applications. In the homogeneous space setting they have been associated with irrational singular forms, and in quadratic differential spaces, with minimal but non-uniquely-ergodic interval exchanges. In each context, one would like to know whether non-obvious divergent trajectories exist, or whether all divergent trajectories are obvious.

C. Rates of divergence. Fixing a natural metric on the space, and a basepoint, it is natural to ask how quickly a divergent trajectory escapes, that is, what is the rate of growth of the distance from a point on the trajectory at time t to the basepoint, as a function of t. For the obvious divergent trajectories, it is simple to compute the rate of escape, and it remains to describe the possible rates for non-obvious trajectories. Specifically, it is interesting to know whether they may escape as quickly as the obvious ones do, and whether they may escape arbitrarily slowly.

In this paper, answers to the above questions are given in many specific contexts, for both homogeneous spaces and quadratic differentials. It turns out that for homogeneous spaces a greater variety of cases arise, and they occupy us for most of the paper. Let us informally describe our main results.

In §2 we expose a scheme for constructing non-obvious divergent trajectories. The construction is based on ideas of Khintchine [Kh], introduced in the context of diophantine approximation. These ideas were later developed by Cassels [Ca] and adapted by Dani [Da] for flows on homogeneous spaces. We abstract and refine the scheme further, obtaining results which are general enough to treat the problems described above.

In §§3–5 we study flows on homogeneous spaces. First, in section 3 we study problems A and B for one-parameter flows, improving some results of Dani (Proposition 3.5 and Theorem 3.9). In §4 we study problems A and B for actions of multi-dimensional groups and semigroups. The situation for cones in the maximal diagonalizable subgroups turns out to be quite interesting. We show (Theorem 4.5) that actions of many cones, including the Weyl chamber, admit non-obvious divergent trajectories, but other cones (Theorem 4.8) do not. See figure 1.

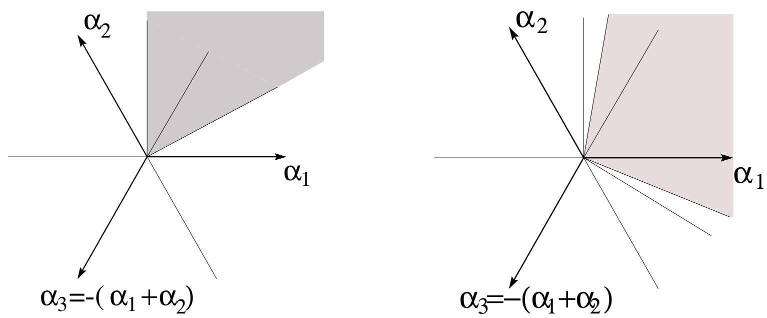

FIGURE 1. Although the semigroup on the left appears 'smaller', it does not admit non–obvious divergent trajectories, while the one on the right does.

It is known through previous work of G. Tomanov and the author [ToWe] that the action of the full $\mathbb{R}$-diagonalizable subgroup (e.g., the full diagonal group in $\mathrm{SL}(n,\mathbb{R})$) does not admit non-obvious divergent trajectories, and only admits divergent trajectories when $\mathrm{rank}_{\mathbb{Q}} G = \mathrm{rank}_{\mathbb{R}} G$. We formulate a general conjecture (Conjecture 4.11) describing what we think are the answers to problems A and B for all multidimensional $\mathbb{R}$-diagonalizable subgroups, and prove partial results (Proposition 4.12 and Corollary 4.14) supporting the conjecture.

In §5 we study rates of escape on homogeneous spaces. For the important special case $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ we give in Theorem 5.2 a complete description of the fastest possible rates of escape for non-obvious divergent trajectories. We then generalize the analysis to general one-parameter semigroups on a general homogeneous space (Theorem 5.9). The obvious divergent trajectories escape with linear speed and our analysis shows that non-obvious divergent trajectories may also escape with linear speed. On the other hand it is possible (Theorem 5.4) to construct non-obvious divergent trajectories which diverge arbitrarily slowly on an unbounded subsequence of times.

In §6 we discuss quadratic differential spaces. Divergent trajectories in this context have been studied quite extensively by many authors, especially Masur. However the question of rates of escape has not been studied. We give a complete description (Theorem 6.4) of the fastest possible rates non-obvious divergent trajectories. Then we describe a stronger type of non-obvious divergent trajectory, in which not one but many disjoint simple closed curves are being pinched. In Theorem 6.6 we show the existence of such trajectories and analyze their possible speed of escape.

To conclude this introduction we mention two additional general problems we have not addressed in this paper. Specific additional questions are raised throughout the paper.

The first problem is to work out the significance of our dynamical results, for both diophantine approximation and interval exchange transformations. This work is currently in progress. The second problem is the prevalence of divergent trajectories, e.g. to compute the Hausdorff dimension of points whose trajectory is divergent and to consider the intersection of the divergent set with various subsets of the space under consideration, such as curves, submanifolds, or fractal subsets. These topics have been extensively studied for quadratic differentials, see [Mas2] for a survey, and also [We2]. For homogeneous spaces, a solution in a specific case is contained in [Ch].

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2. A scheme of Khintchine, Cassels and Dani

We present a scheme for constructing divergent trajectories. As mentioned in the introduction, this scheme originated in the diophantine approximation literature, in a proof of Khintchine [Kh], which was later discussed in [Ca, Theorem 14]. Dani generalized the construction and adapted it to a dynamical framework in his proof of [Da, Theorem 7.3]. The version presented here is an abstraction of Dani's.

Let Y be a locally compact Hausdorff space on which a noncompact locally compact topological group or semigroup A acts.

A trajectory $Ay \subset Y$ is **divergent** if for any compact subset $K \subset Y$ there is a compact $C \subset A$ such that

$$a \in A \smallsetminus C \Longrightarrow ay \notin K$$

(equivalently, if the map $a \mapsto ay$ is proper).

Now suppose that the action lifts to a covering X of Y, i.e., there is an action of A on a locally compact Hausdorff space X and a surjective equivariant map $\pi : X \to Y$:

$$\begin{tikzcd} A \times X \arrow[r] \arrow[d, "\mathrm{Id}\times\pi"'] & X \arrow[d, "\pi"] \\ A \times Y \arrow[r] & Y \end{tikzcd}$$

Theorem 2.1. *Let $X_1, X_2, \ldots$ be a sequence of subsets of X such that $A\pi(x)$ is divergent for every $x \in \bigcup X_i$. Assume the following:*

1. **Density:** *For every j,*
$$X_j = \overline{X_j \cap \bigcup_{i \neq j} X_i}.$$
2. **Transversality:** *For every $i \neq j$, $\overline{X_i \smallsetminus X_j} = X_i$.*
3. **Local Uniformity:** *For any i, any $x \in X_i$, and any compact $K \subset Y$ there is a compact $C \subset A$ and a neighborhood $\mathcal{U}$ of x such that for every $a \in A \smallsetminus C$ and every $z \in \mathcal{U} \cap X_i$, we have $a\pi(z) \notin K$.*

Then there is $x_0 \in X \smallsetminus \bigcup_i X_i$ such that the trajectory $A\pi(x_0)$ is divergent.

Proof. The required point x_0 is obtained as follows. Let us fix an increasing sequence of compact sets $K_k \subset Y$, with $Y = \bigcup K_k$ and $K_k \subset \mathrm{int}(K_{k+1})$. We will construct a sequence of open sets with compact closure $\Omega_0, \Omega_1, \Omega_2, \ldots$ in X, an increasing sequence of indices $i_1, i_2, \ldots$, and an increasing collection of compact sets $C_0, C_1, C_2, \ldots \subset A$ such that the following hold for $k = 1, 2, \ldots$:

a. $\overline{\Omega_k} \subset \Omega_{k-1}$.
b. For every $j < i_k$, $X_j \cap \Omega_k = \varnothing$.
c. $X_{i_k} \cap \Omega_k$ is nonempty and for every $z \in X_{i_k} \cap \Omega_k$ and every $a \in A \smallsetminus C_k$ we have $a\pi(z) \notin K_k$.
d. For every $z \in \Omega_k$ and every $a \in C_k \smallsetminus \mathrm{int}(C_{k-1})$, $a\pi(z) \notin K_{k-1}$.

First let us show why such sequences suffice. The intersection $\bigcap_k \Omega_k$ is nonempty by condition a. For $x_0 \in \bigcap_k \Omega_k$, we have by condition b that $x_0 \notin \bigcup X_i$ and by condition d that $A\pi(x_0)$ is divergent.

Now let us construct the sequences inductively. Choose $C_0 = \varnothing$, $i_1 = 1$. Let $x \in X_1$ and let Ω_1 be a small enough open neighborhood of x, and $C_1 \subset A$ a large enough compact set so that for all $z \in X_1 \cap \Omega_1$ and all $a \in A \smallsetminus C_1$, we have $a\pi(z) \notin K_1$. This is possible by the local uniformity assumption. In addition let Ω_1 have compact closure. Now defining $K_0 = \varnothing$ and Ω_0 any open set with compact closure such that $\overline{\Omega_1} \subset \Omega_0$, we see that conditions a, b, and d are vacuous for $k = 1$ and condition c is satisfied by our choice of Ω_1 and C_1.

Suppose we have chosen i_s, Ω_s, C_s for $s = 1, \ldots, k$. By the density assumption there are $\ell \neq i_k$ such that

$$X_\ell \cap \Omega_k \cap X_{i_k} \neq \varnothing. \tag{1}$$

Choose i_{k+1} to be any such ℓ. Note that by condition b, $i_{k+1} > i_k$. Let $x \in X_{i_k} \cap \Omega_k \cap X_{i_{k+1}}$. By the local uniformity assumption, there is a small enough open neighborhood $\mathcal{U}$ around x and a large enough compact $C_{k+1} \subset A$ such that for all $z \in \mathcal{U} \cap X_{i_{k+1}}$ and all $a \in A \smallsetminus C_{k+1}$, we have $a\pi(z) \notin K_{k+1}$. In addition let $\mathcal{U}$ be small enough so that $\overline{\mathcal{U}} \subset \Omega_k$. Since $K_k \subset \mathrm{int}(K_{k+1})$ and $C_{k+1} \smallsetminus \mathrm{int}(C_k)$ is compact, by continuity there is a neighborhood $\widetilde{\Omega}$ of x, contained in $\mathcal{U}$, such that:

$$z \in \widetilde{\Omega},\ a \in C_{k+1} \smallsetminus C_k \Longrightarrow a\pi(z) \notin K_k. \tag{2}$$

Now let

$$\Omega_{k+1} = \widetilde{\Omega} \smallsetminus \bigcup_{j<i_{k+1}} X_j. \tag{3}$$

Let us verify that $i_{k+1}, \Omega_{k+1}, C_{k+1}$ satisfy the required conditions. Condition a is satisfied by our choice of $\mathcal{U}$. Condition b is satisfied by our definition of Ω_{k+1}. In condition c, $\Omega_{k+1} \cap X_{i_{k+1}} \neq \varnothing$ because $x \in \widetilde{\Omega} \cap X_{i_{k+1}}$, and because of the transversality assumption. The second assertion in condition c holds because of the choice of C_{k+1} and $\mathcal{U}$. Condition d holds because of the choice of $\widetilde{\Omega}$. □

Modifying the argument, we now obtain three refinements of the above result. The first two of these involve the rate of escape of a divergent trajectory. Since we will only be discussing rates for the action of one–dimensional groups and semigroups, let us assume that $A = \{a(t) : t \geq 0\}$. Let us first explain what we mean by a rate of escape.

Definition 2.2. *A* **rate of growth** *is a collection* $\{K(t) : t \geq 0\}$ *of subsets of* Y*, satisfying:*

- *Any compact subset of* Y *is contained in* $K(t)$ *for some* $t \geq 0$.
- *If* $t_1 < t_2$ *then* $K(t_1) \subset \mathrm{int}(K(t_2))$.
- **Continuity of** $\{K(t)\}$**:** *For any* $0 \leq a \leq b \leq \infty$, *the set* $\{(t, x) : x \in K(t),\ a \leq t \leq b\}$ *is closed in* $\mathbb{R} \times Y$.

Definition 2.3. *We say that a trajectory* $\{a(t)y : t \geq 0\}$ *is* **divergent with rate given by** $\{K(t)\}$ *if there is* t_0 *such that for every* $t \geq t_0$ *we have* $a(t)y \notin K(t)$.

Theorem 2.4. *Let a rate of growth* $\{K(t)\}$ *be given. Let* $X_i,\ i = 1, 2, \ldots$ *be a sequence of subsets of* X *such that every* $x \in \bigcup_i X_i$ *is divergent with rate given by* $\{K(t)\}$. *Assume also that the* X_i *satisfy the density and transversality hypotheses, and the following:*

Local uniformity w.r.t. $\{K(t)\}$: *for every* i *and every* $x \in X_i$ *there exists a neighborhood* $\mathcal{U}$ *of* x *and* t_0 *such that for every* $z \in \mathcal{U} \cap X_i$ *and every* $t > t_0$, $a(t)\pi(z) \notin K(t)$.

Then there exists $x_0 \in X \smallsetminus \bigcup_i X_i$ *such that* $A\pi(x_0)$ *is divergent with rate given by* $\{K(t)\}$.

Proof. We follow an inductive procedure similar to that of the preceding proof, constructing open sets with compact closure $\Omega_0, \Omega_1, \Omega_2, \ldots$ in X, an unbounded sequence $T_1 < T_2 < \cdots$ of positive numbers and an increasing sequence of indices $i_1, i_2, \ldots$ such that for $k = 1, 2, \ldots$, the following hold:

a, b. As in the proof of Theorem 2.1.

c'. $X_{i_k} \cap \Omega_k$ is nonempty and for every $z \in X_{i_k} \cap \Omega_k$ and every $t \geq T_k$ we have $a(t)\pi(z) \notin K(t)$.

d'. For $k = 2, 3, \dots$, for every $z \in \Omega_k$ and every $t \in [T_{k-1}, T_k]$ we have $a(t)\pi(z) \notin K(t)$.

The sequences are constructed inductively. We start with $i_1 = 1$ and choose, using the local uniformity assumption, Ω_1 with compact closure small enough and T_1 big enough so that $X_1 \cap \Omega_1 \neq \varnothing$ and for every $t \geq T_1$ and every $z \in \Omega_1 \cap X_1$, $a(t)\pi(z) \notin K(t)$. Then conditions a, b, and c' are satisfied for $k = 1$.

Now suppose we have constructed i_s, Ω_s, T_s for $s \leq k$. Again let i_{k+1} be any ℓ for which (1) holds, and let $x \in X_{i_k} \cap \Omega_k \cap X_{i_{k+1}}$. By the local uniformity with respect to $\{K(t)\}$ there is T_{k+1} and a neighborhood $\mathcal{U}$ of x such that $\overline{\mathcal{U}} \subset \Omega_k$ and for all $t \geq T_{k+1}$ and all $z \in \mathcal{U} \cap X_{i_{k+1}}$ we have $a(t)\pi(z) \notin K(t)$. Since $x \in X_{i_k}$, the subsets

$$\{(t, a(t)\pi(x)) : t \in [T_k, T_{k+1}]\}$$

and

$$\{(t, z) : z \in K(t),\ t \in [T_k, T_{k+1}]\}$$

of $\mathbb{R} \times Y$ are disjoint, and by the continuity of $\{K(t)\}$, they are closed. Hence by the continuity of the action and the compactness of $[T_k, T_{k+1}]$, a small enough neighborhood $\widetilde{\Omega}$ of x contained in $\mathcal{U}$ can be chosen so that

$$z \in \widetilde{\Omega},\ t \in [T_{k-1}, T_k] \Longrightarrow a(t)\pi(z) \notin K(t).$$

Now we can define Ω_{k+1} by (3) and check that a, b, c' and d' are satisfied for $k + 1$. This completes the construction.

Now for $x_0 \in \bigcap \Omega_i$ we will have $x_0 \notin \bigcup X_i$ and $a(t)\pi(x_0) \notin K(t)$ for all $t \geq T_1$. □

The second modification of the above scheme is useful for finding divergent trajectories which do not diverge too quickly, that is, for a given rate of growth $\{K(t) : t \geq 0\}$, we will construct divergent trajectories which do not diverge with rate given by $\{K(t)\}$. We need the following definition.

Let x, $x' \in X$ and let $\mathcal{U}$ be a connected open set containing x and x'. We say that x, $x' \in X$ are **connected by** $\{X_j\}$ **in** $\mathcal{U}$ if there is $r > 0$ and indices $j_1, \dots, j_r$ such that x and x' belong to the same connected component of

$$\mathcal{U} \cap (X_{j_1} \cup \cdots \cup X_{j_r}).$$

We denote this by $x \overset{\{X_j\},\mathcal{U}}{\longleftrightarrow} x'$.

Theorem 2.5. *Suppose $\{K(t)\}$ is a rate of growth. Suppose X_i, $i = 1, 2, \dots$ is a sequence of subsets of X such that for every $x \in \bigcup X_i$, $\{a(t)\pi(x)\}$ is divergent with rate given by $\{K(t)\}$. Assume that the X_i satisfy the assumptions of Theorem 2.4, and assume in addition:*

- *The $\{a(t)\}$ action on Y is* **topologically transitive,** *that is, for any pair of nonempty open sets $A, B \subset Y$ the set*

$$\{t \in \mathbb{R}_+ : a(t)A \cap B \neq \varnothing\}$$

is unbounded.

- **Density of connected components:** *For any i, any $x \in X_i$, and any neighborhood $\mathcal{U}$ of x, the set*
$$\overline{\{x' \in \mathcal{U} : x \overset{\{X_j\},\mathcal{U}}{\longleftrightarrow} x'\}}$$
contains a neighborhood of x.

Then there is $x_0 \in X \smallsetminus \bigcup_i X_i$ such that $\{a(t)\pi(x_0)\}$ is divergent but not divergent with rate given by $\{K(t)\}$.

Question 2.6. *We do not prove that $a(t)x \in K(t)$ for all $t \geq t_0$, only that there are arbitrarily large t for which $a(t)x \in K(t)$. Is there a trajectory $\{a(t)x\}$ for which the stronger statement holds? This seems considerably more difficult, and would be of interest number–theoretically, see the discussion in* [St, §30.2].

Proof. By definition of a rate of growth, for all large t we have $\mathrm{int}(K(t)) \neq \varnothing$. With no loss of generality let us assume this holds for all t. For all $t \geq 0$ let
$$K_1(t) = K(t/2).$$
Then $\{K_1(t)\}$ is a rate of growth, and $K_1(t) \subset \mathrm{int}(K(t))$ for all $t > 0$. We will find $x \in X$ such that $\{a(t)\pi(x)\}$ is divergent with rate given by $\{K_1(t)\}$ but not with rate given by $\{K(t)\}$.

We construct inductively open sets with compact closure $\Omega_0, \Omega_1, \Omega_2, \ldots$ in X, an unbounded sequence $T_1 < T_2 < \cdots$ of positive numbers and an increasing sequence of indices $i_1, i_2, \ldots$ satisfying:

a, b. As in the proof of Theorem 2.1.

c', d'. As in the proof of Theorem 2.4 with $\{K_1(t)\}$ in place of $\{K(t)\}$.

e. For $k = 2, 3, \ldots$, there is $s \in [T_{k-1}, T_k]$ such that for every $z \in \Omega_k$, $a(s)\pi(z) \in \mathrm{int}(K(s))$.

f. For $k = 1, 2, \ldots$, and any $z \in X_{i_k} \cap \Omega_k$,
$$\overline{\{z' : z \overset{\{X_j\},\Omega_k}{\longleftrightarrow} z'\}}$$
contains a neighborhood of z.

We choose Ω_1, i_1 and X_1 as in the proof of Theorem 2.4, additionally making sure that condition f holds for Ω_1 by applying the density of connected components hypothesis.

Suppose that for $s = 1, \ldots, k$ we have found Ω_s, i_s, T_s satisfying conditions a, b, c', d', e and f. Let $z \in X_{i_k} \cap \Omega_k$ and, using condition f, let $\widehat{\Omega}_k$ be an open neighborhood of z contained in $\overline{\{z' : z \overset{\{X_j\},\Omega_k}{\longleftrightarrow} z'\}}$. Using topological transitivity, we find $t_0 \geq T_k$ for which $a(t_0)\pi(\widehat{\Omega}_k) \cap \mathrm{int}(K(T_k)) \neq \varnothing$. Let $z' \in \widehat{\Omega}_k \cap \pi^{-1}(a(-t_0)(\mathrm{int} K(T_k)))$ such that $z \overset{\{X_j\},\Omega_k}{\longleftrightarrow} z'$. That is, there are r, $i_k = j_1, j_2, \ldots, j_r$ and z, $z' \in \Omega_k$ satisfying:

- $a(t_0)\pi(z') \in \mathrm{int}(K(T_k))$.

- $z, z' \in C$, where C is a connected component of $\Omega_k \cap (X_{j_1} \cup \cdots \cup X_{j_r})$.

By locally uniform escape with respect to $\{K_1(t)\}$, choose $\widetilde{T}$ such that for all $t \geq \widetilde{T}$ and all $z_1 \in \Omega_k \cap (X_{j_1} \cup \cdots \cup X_{j_r})$ we have $a(t)\pi(z_1) \notin K_1(t)$, and let

$$T_{k+1} = \max\{\widetilde{T}, t_0\}.$$

Let

$$C_1 = \{y \in C : \forall t \in [T_k, T_{k+1}],\, a(t)y \notin \mathrm{int}(K(t))\},$$

$$C_2 = \{y \in C : \exists t \in [T_k, T_{k+1}],\, a(t)y \in K_1(t)\}.$$

It is easy to check using the continuity of $\{K_1(t)\}$ that C_1 and C_2 are closed in C. They are nonempty since $z \in C_1$ by property c' and $z' \in C_2$. They are disjoint since $K_1(t) \subset \mathrm{int}(K(t))$ for all t. Since C is connected there is $y \in C \smallsetminus (C_1 \cup C_2)$.

That is, for all $t \in [T_k, T_{k+1}]$, $a(t)y \notin K_1(t)$ but for some $s \in [T_k, T_{k+1}]$, $a(s)y \in \mathrm{int}(K(s))$. By condition c' (for stage k) $y \notin X_{i_k}$. Since $y \in C$ we have $y \in X_{j_t}$ for some t, and we set $i_{k+1} = j_t > i_k$.

Now for a small enough neighborhood $\widetilde{\Omega}$ of y, conditions a, c', d' and e are satisfied. We guarantee condition b by defining

$$\widetilde{\Omega}_{k+1} = \widetilde{\Omega} \smallsetminus \bigcup_{j<i_{k+1}} X_j,$$

and we guarantee condition f by taking for Ω_{k+1} a connected component of $\widetilde{\Omega}_{k+1}$ which intersects $X_{i_{k+1}}$ and using the density of connected components.

This completes the inductive construction. A point in $\bigcap_k \Omega_k$ will satisfy the conclusion of the theorem. $\square$

Remark 2.7. *The proof of Theorem 2.5 yields a more precise result about the rate of escape of the trajectory which is constructed. Namely, for any rate of growth $\{K(t)\}$, and any unbounded increasing function $\phi(t)$ satisfying $\phi(t) < t$ for all t, let $K_1(t) = K(\phi(t))$, and suppose the hypotheses of the Theorem are satisfied for $\{K(t)\}$. Then $\{K_1(t)\}$ is a rate of growth, and the proof constructs a trajectory which is divergent with rate given by $\{K_1(t)\}$ but not with rate given by $\{K(t)\}$.*

We now turn to the third variant of Theorem 2.1. To motivate it, note that in the construction of Theorem 2.1, the sets $\{X_i\}$ play two roles: the contructed point x_0 is sufficiently close to some of the X_i's, causing its trajectory to diverge, and additionally, the $\{X_i\}$ are avoided, that is $x_0 \notin \bigcup X_i$. It is sometimes useful to retain one list of sets $\{X_i\}$ in order to make the trajectory $A\pi(x_0)$ divergent, and add an additional list of subsets $\{X'_j\}$ which we want to avoid.

Before stating the result we introduce some terminology. Given sequences $\{X_i\}_{i\in\mathbb{N}}$, $\{X'_j\}_{j\in\mathbb{N}}$ of subsets of X, a **level function for** $(\{X_i\}, \{X'_j\})$ is a function $L : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ such that:

- $X_i \not\subset X'_j$ if and only if $L(i,j) = \infty$.
- For each j there is $M = M(j) \in \mathbb{N}$ such that for all i, $L(i,j) \in \{1, \dots, M, \infty\}$.

Theorem 2.8. *Let $X_1, X_2, \dots$ be a list of subsets of X satisfying the density, transversality and local uniformity hypotheses. Let $X'_1, X'_2, \dots$ be another list of subsets of X and let L be a level function for $(\{X_i\}, \{X'_j\})$, and assume:*

- **Transversality relative to $\{X'_j\}$:** *For any i, j, if $X_i \not\subset X'_j$ then $X_i = \overline{X_i \smallsetminus X'_j}$.*
- **Density of level-increasing points:** *For every i, j for which $L(i,j) < \infty$,*
$$X_i = \overline{X_i \cap \bigcup \{X_k : L(k,j) > L(i,j)\}}.$$

Then there is $x_0 \in X \smallsetminus (\bigcup_i X_i \cup \bigcup_j X'_j)$ such that $A\pi(x_0)$ is divergent.

Proof. We construct sequences Ω_k, C_k, i_k as before, and an additional non-decreasing sequence of positive integers $j_1, j_2, \dots$ satisfying the following:

a, c, d. As in the proof of Theorem 2.1.
b'. For every $i < i_k$ and every $j < j_k$, $X_i \cap \Omega_k = X'_j \cap \Omega_k = \varnothing$.
e'. For each k, if $L(i_k, j_k) < \infty$ then $L(i_{k+1}, j_k) > L(i_k, j_k)$.

To construct the sequences, start with $j_1 = 1$ and C_1, Ω_1, i_1 as in the proof of Theorem 2.1. Supposing Ω_s, C_s, i_s, j_s have been constructed for $s = 1, \dots, k$, let i_{k+1} be an index ℓ for which (1) holds and in addition:
$$L(i_k, j_k) < \infty \Longrightarrow L(\ell, j_k) > L(i_k, j_k).$$
Such indices ℓ exist by the hypothesis on density of level-increasing points. Necessarily $i_{k+1} > i_k$. Now define C_{k+1}, $\widetilde{\Omega}$ as in the proof of Theorem 2.1, and, in case $L(i_{k+1}, j_k) < \infty$ define $j_{k+1} = j_k$ and Ω_{k+1} by (3). In case $L(i_{k+1}, j_k) = \infty$ define $j_{k+1} = j_k + 1$ and
$$\Omega_{k+1} = \widetilde{\Omega} \smallsetminus (X'_{j_k} \cup \bigcup_{j < i_{k+1}} X_j).$$

We verify that the required conditions hold for Ω_{k+1}, C_{k+1}, i_{k+1}, j_{k+1}. For c, in case $L(i_{k+1}, j_k) = \infty$ we have $X_{i_{k+1}} \not\subset X'_{j_k}$ and hence, by transversality and transverality relative to $\{X'_j\}$, we obtain that $\Omega_{k+1} \cap X_{i_{k+1}} \neq \varnothing$. The case $L(i_{k+1}, j_k) < \infty$, as well as the second assertion in c and conditions a and d, follow as in the proof of Theorem 2.1. b' (resp. e') follows from the definition of Ω_{k+1} (resp. j_{k+1}).

A construction satisfying these conditions suffices because by e', and the fact that L is a level function for $(\{X_i\}, \{X'_j\})$, we have $j_k \to \infty$ and hence, by b', a point $x_0 \in \bigcap_k \Omega_k$ will not be contained in $\bigcup_i X_i \cup \bigcup_j X'_j$. $\square$

Remark 2.9. *By combining the proofs of Theorems 2.4 and 2.8 in an obvious way, we see that if $X_1, X_2, \dots, X'_1, X'_2, \dots$ satisfy the hypotheses of Theorem 2.8 and Theorem 2.4 then there is $x \in G \smallsetminus (\bigcup X_i \cup \bigcup X'_j)$ satisfying the conclusion of Theorem 2.4.*

The details are left to the reader.

3. Divergence on Homogeneous Spaces

In this section we consider the space $Y = G/\Gamma$, where G is a semisimple real algebraic group and Γ is a non-uniform arithmetic lattice, and the flow is induced by a one-parameter subgroup. We define *obvious* divergent trajectories, which are an a-priori wider class than Dani's *degenerate* divergent trajectories. We apply the results of the previous section to prove the existence of non-obvious divergent trajectories. This strengthens [Da, Thm. 7.3]. We also show, generalizing [Da, Prop. 4.5] that obvious divergent trajectories exist on any noncompact homogeneous space if the acting semigroup is not quasi-unipotent.

3.1. **Terminology.** We will freely use terminology and standard results about the structure of real algebraic groups, homogeneous spaces, and lattices. We refer the reader to [Ra], [Bo2] or [St] for more details.

Let G denote a semisimple real algebraic group defined over $\mathbb{Q}$, let Γ be an arithmetic subgroup of G (that is, Γ is commensurable with $G(\mathbb{Z})$) and let $\pi : G \to G/\Gamma$ be the natural quotient map. G and any of its subgroups acts on G/Γ by the rule

$$g \cdot \pi(h) = \pi(gh).$$

Recall that Γ is a lattice in G, that is, the Haar measure on G descends to a finite G-invariant measure on G/Γ. Recall that G is said to be $\mathbb{Q}$-**simple** if it has no proper normal infinite $\mathbb{Q}$-algebraic subgroups. In this case Γ is irreducible.

Let D (resp. S) denote a maximal $\mathbb{R}$-split (resp. $\mathbb{Q}$-split) subtorus in G. Since there is a conjugate (in G) of D which contains S (see [BoTi1]) we may replace D with such a conjugate and assume $S \subset D$. The dimension of D (resp. S) is denoted by $\mathrm{rank}_{\mathbb{R}} G$ (resp. $\mathrm{rank}_{\mathbb{Q}} G$). We denote by $X(D)$ the group of $\mathbb{R}$-characters of D, and by $X(S)$ the group of $\mathbb{Q}$-characters of S (which coincides with the group of $\mathbb{R}$-characters on S). Characters are written additively and are identified with their derivatives, that is, we think of a character as a linear functional on $\mathrm{Lie}(D)$ or $\mathrm{Lie}(S)$. Given a representation $\varrho : G \to \mathrm{GL}(V)$, a nonzero ($\mathbb{Q}$-) character χ is a ($\mathbb{Q}$-) **weight for** ϱ or a **weight appearing in** ϱ if there is a nonzero vector $v \in V$ such that for all $d = \exp(X)$ in D (resp., in S) we have

$$\varrho(d)v = e^{\chi(X)}v. \tag{4}$$

The **weight–space** corresponding to χ is the subspace V_χ consisting of all vectors v for which (4) holds for all d. If χ is a $\mathbb{Q}$-root then V_χ is defined over $\mathbb{Q}$ and therefore is spanned by $V_\chi \cap V(\mathbb{Q})$. The set of all ($\mathbb{Q}$-) weights for ϱ is denoted by Λ_ϱ (resp. $\Lambda_{\mathbb{Q}}(\varrho)$). Given $a \in D$, we write

$$\Lambda_\varrho^-(a) = \{\lambda \in \Lambda_\varrho : \lambda(a) < 0\}$$

and

$$V_\varrho^-(a) = \bigoplus_{\lambda \in \Lambda_\varrho^-(a)} V_\lambda.$$

If ϱ is clear from context we omit it from the notation. Also, if $\mathbf{a} \in \mathrm{Lie}(D)$ we write $V^-(\mathbf{a})$ (resp. $\Lambda^-(\mathbf{a})$) for $V^-(\exp(\mathbf{a}))$ (resp. $\Lambda^-(\exp(\mathbf{a}))$).

A one–parameter subsemigroup $\{a(t) : t \geq 0\}$ of G is called **non-quasi-unipotent**, if at least one of the eigenvalues of $\mathrm{Ad}(a(1))$ is not on the unit circle.

3.2. **Remarks about the hypotheses.** Note that the questions we consider make sense in the more general setup in which G is a Lie group, Γ a closed subgroup, and the acting semigroup is an arbitrary subsemigroup of G. Using some standard reductions, most (but not all) questions about divergent trajectories in the general setup can be reduced to the setup considered here. In particular, it should be noted that:

(i) We have assumed that G is semisimple. Our questions are only of interest in case G/Γ is not compact, but typical orbits are not divergent. This is the case when G/Γ is non-compact and has finite volume, i.e. Γ is a **non-uniform lattice** in G. The case in which G is a general Lie group and Γ a non-uniform lattice in G reduces to the case in which G is semisimple, as follows (we are grateful to the referee for indicating the reduction). Let F be the maximal connected normal amenable subgroup of G. Then $G' = G/F$ is semisimple, and it is known that $F/F \cap \Gamma$ is compact. This implies that G/Γ is isomorphic to a fiber bundle, with compact fiber, over G'/Γ', where Γ' is the image of Γ in G'.

(ii) The assumption that Γ is arithmetic does not entail substantial loss of generality since, by a result of Dani [Da, Theorem 6.1] most of the questions we will consider are only interesting for groups of real rank at least two, hence the Margulis arithmeticity theorem can be used.

(iii) By Margulis' nondivergence lemma [Mar] there are no divergent trajectories for unipotent (and also quasi-unipotent) subsemigroups. Thus we may safely assume that A is non-quasi-unipotent.

(iv) In this section we will restrict our attention to the case that $\dim A = 1$, that is, A is either a line or a half-line. The higher–dimensional case presents new phenomena and will be considered in §4.

3.3. **The obvious divergent trajectories.** Let us describe some obvious reasons to escape to infinity in G/Γ. Let $A = \{a(t) : t \geq 0\}$. Suppose $\varrho : G \to \mathrm{GL}(V)$ is a representation defined over $\mathbb{Q}$, and suppose $0 \neq v \in V(\mathbb{Q})$. Fix some realization of G as a group of matrices. Since $\Gamma = G(\mathbb{Z})$, there is a uniform bound on the denominators of all matrices in $\varrho(\Gamma)$, and since

$v \in V(\mathbb{Q})$, it follows from this that $\varrho(\Gamma)v$ is a discrete subset of V. In particular, for any compact subset $K \subset G$ the set $\varrho(K\Gamma)v$ is closed and does not contain 0.

Now suppose for some $x \in G$ that

$$\varrho(a(t)x)v \to_{t\to+\infty} 0,$$

and suppose if possible that there is a compact subset $K' \subset G/\Gamma$ and an infinite unbounded subsequence $\{t_n\}$ such that for all n, $a(t_n)\pi(x) \in K'$. Then there is a compact subset $K \subset G$ such that $a(t_n)x \in K\Gamma$ for all n, and hence

$$\varrho(a(t_n)x)v \in \varrho(K\Gamma)v,$$

a contradiction. We have proved:

Proposition 3.1. *Let $x \in G$. If there is a $\mathbb{Q}$-representation $\varrho : G \to \mathrm{GL}(V)$, and a nonzero $v \in V(\mathbb{Q})$ such that $\varrho(a(t)x)v \to_{t\to+\infty} 0$ then $A\pi(x)$ is divergent.*

Definition 3.2. *We say that the trajectory $A\pi(x)$ is an* **obvious divergent trajectory** *if the hypotheses of Proposition 3.1 hold.*

From the point of view of reduction theory, it is natural to consider a more restricted class of representations. This results in an a-priori smaller class of divergent trajectories. We make the following definition (cf. [Da, Def. 5.5]):

Definition 3.3. *A trajectory $A\pi(x)$ is a* **degenerate divergent trajectory** *if there is a $\mathbb{Q}$-representation $\varrho : G \to \mathrm{GL}(V)$ and a nonzero $v \in V(\mathbb{Q})$ such that:*

- $\varrho(a(t)x) \longrightarrow_{t\to+\infty} 0$.
- $G_{[v]} = \{g \in G : \varrho(g)v$ is a scalar multiple of $v\}$ *is a parabolic subgroup of G.*

Question 3.4. *Are there obvious divergent trajectories which are not degenerate?*

3.4. **Existence of divergent trajectories.** It was proved by Margulis [Mar] that a unipotent subgroup has no divergent trajectories on G/Γ, and his argument also shows that a quasi-unipotent subsemigroup has no divergent trajectories. Recall also that G/Γ is non-compact if and only if $\mathrm{rank}_{\mathbb{Q}} G \geq 1$. The proposition below shows that these are the only obstructions to the existence of divergent trajectories.

Proposition 3.5. *Let G be a semisimple $\mathbb{Q}$-algebraic group and let $\Gamma = G(\mathbb{Z})$. Let $A = \{h(t) : t \in \mathbb{R}\}$ be a one-parameter subgroup, and suppose there is a $\mathbb{Q}$-simple factor G_1 of G such that* $\mathrm{rank}_{\mathbb{Q}} G_1 \geq 1$ *and the projection of A onto G_1 is not quasi-unipotent. Then there is $x \in G$ such that $A\pi(x)$ is divergent.*

Remark 3.6. *This answers a question of Starkov* [St, §25.1]. *Dani proved the result in* [Da, Prop. 4.5] *under the additional hypothesis that either A is diagonalizable over $\mathbb{C}$ or $\mathrm{rank}_{\mathbb{Q}}G = \mathrm{rank}_{\mathbb{R}}G$.*

We collect some facts about one-parameter subgroups and parabolic subgroups of algebraic groups.

Proposition 3.7 (Jordan decomposition over $\mathbb{R}$). *Given a one parameter subgroup $\{h(t)\}$ of an algebraic group G, there are one-parameter subgroups $\{k(t)\}$, $\{a(t)\}$, $\{u(t)\}$ satisfying:*

- *For all t, $u(t)$ is unipotent, $k(t)a(t)$ is diagonalizable over $\mathbb{C}$, $a(t)$ is diagonalizable over $\mathbb{R}$, and $h(t) = k(t)a(t)u(t)$.*
- *$\{k(t) : t \in \mathbb{R}\}$ is bounded in G.*
- *For all t, $k(t)$, $a(t)$ and $u(t)$ commute.*

Proof. This may be deduced from [Bo2, Thm. 4.4 and Prop. 8.15]. □

Proposition 3.8. *Let G be an algebraic group defined over a field k of characteristic zero which is almost k-simple (has no proper normal k-subgroups of positive dimension), let B be a minimal k-parabolic subgroup, and let P_1, P_2 be two proper k-parabolic subgroups containing B, with unipotent radicals U_1, U_2. Then* $\dim U_1 \cap U_2 \geq 1$.

Proof. Let D be a maximal k-split torus contained in B, and choose an order on $X_k(D)$ corresponding to the minimal parabolic subgroup B. Since G is almost k-simple the k-root system is irreducible [Bo2, Thm. 22.10]. Let λ be a dominant root with respect to this order. It follows from [Hu, §10.4, Lemma A] and the description of standard parabolic subgroups [BoTi2, §5] that $\mathcal{G}_\lambda \subset \mathrm{Lie}(U_1 \cap U_2)$. □

Proof of Proposition 3.5. We first reduce the problem to the case that $G = G_1$ is $\mathbb{Q}$-simple. Let $\pi_1 : G \to G_1$ be the quotient map. After replacing Γ with a commensurable lattice, we have $G = G_1G_2$, $\Gamma = \Gamma_1\Gamma_2$, where the G_i are commuting semisimple $\mathbb{Q}$-algebraic subgroups, $\Gamma_i = G_i(\mathbb{Z})$, and Γ_1 is commensurable with $\pi_1(\Gamma)$. Define $A_1 = \pi_1(A)$ and $\overline{\pi}_1(g\Gamma) = \pi_1(g)\Gamma_1$. The map $\overline{\pi}_1$ is well-defined and intertwines the action of A on G/Γ with the action of A_1 on G_1/Γ_1. By assumption $\mathrm{rank}_{\mathbb{Q}}G_1 \geq 1$ and A_1 is not quasi-unipotent. If we can find $x_1 \in G_1/\Gamma_1$ for which A_1x_1 is divergent then Ax is divergent for any $x \in \overline{\pi}_1^{-1}(x_1)$.

Since $\mathrm{rank}_{\mathbb{Q}}G \geq 1$ there is a proper $\mathbb{Q}$-parabolic subgroup P with unipotent radical U. We claim that U intersects any almost $\mathbb{R}$-simple factor of G nontrivially. Since U is normalized by a maximal $\mathbb{R}$-split torus, its Lie algebra is a sum of root spaces and in particular its intersection with any almost $\mathbb{R}$-simple factor coincides with its projection on that factor. Thus it suffices to show that U_1 projects nontrivially on any almost $\mathbb{R}$-simple factor of G. Let G_0 be the smallest normal subgroup of G containing U. Since U is defined over $\mathbb{Q}$, so is G_0. Since G is almost $\mathbb{Q}$-simple, $G = G_0$. On the other hand G_0 is the product of the almost $\mathbb{R}$-simple factors of G onto which U projects nontrivially. The claim follows.

Let $d = \dim U$ and let $\mathbf{p}_U \in V = \bigwedge^d \mathcal{G}$ be a corresponding vector. Let $\varrho : G \to \mathrm{GL}(V)$ be the d-th exterior power of the adjoint representation. This is a representation defined over $\mathbb{Q}$, and $\mathbf{p}_U$ can be chosen in $V(\mathbb{Q})$.

For $\chi \in \Lambda_\varrho$ we have

$$V_\chi = \mathrm{span}\{v_1 \wedge \cdots \wedge v_d : v_i \in \mathcal{G}_{\alpha_i},\, \alpha_1 + \cdots + \alpha_d = \chi.\} \tag{5}$$

Let $\{k(t)\}$, $\{a(t)\}$, $\{u(t)\}$ be as in Prop. 3.7. Let

$$P^- = \{g \in G : \{a(t)ga(-t) : t \geq 0\} \text{ is bounded in } G\}. \tag{6}$$

The unipotent radical of P^- is

$$U^- = \{g \in G : a(t)ga(-t) \to_{t\to+\infty} e\},$$

the contracting horospherical subgroup of $\{a(t)\}$.

We have

$$\mathrm{Lie}(P^-) = \bigoplus_{\chi\in\Phi,\, \chi(a(1))\leq 0} \mathcal{G}_\chi \quad \text{and} \quad \mathrm{Lie}(U^-) = \bigoplus_{\chi\in\Phi,\ \chi(a(1))<0} \mathcal{G}_\chi. \tag{7}$$

Since $\{h(t)\}$ is not quasi-unipotent, $\{a(t)\}$ is nontrivial, and hence P^- is a proper $\mathbb{R}$-parabolic subgroup of G, and U^- is nontrivial. Let G_1 be an almost $\mathbb{R}$-simple factor of G such that $P^- \cap G_1$ is a proper $\mathbb{R}$-parabolic subgroup of G_1. Let $U_1 = G_1 \cap U$, which is nontrivial in view of the above claim.

Let B^- be a minimal $\mathbb{R}$-parabolic subgroup of G which is contained in P^- and let B be a minimal $\mathbb{R}$-parabolic subgroup of P containing U. Since all minimal $\mathbb{R}$-parabolic subgroups are conjugate in G [Bo2, Thm. 20.9], there is $g_0 \in G$ such that $B^- = g_0 B g_0^{-1}$. We then have

$$g_0 U g_0^{-1} \subset P^- \tag{8}$$

and, applying Proposition 3.8 to $G_1 \cap P^-$ and $G_1 \cap g_0 P g_0^{-1}$,

$$\dim U^- \cap g_0 U g_0^{-1} \geq 1. \tag{9}$$

From this it follows, using (5),(7), (8) and (9), that

$$\varrho(g_0)\mathbf{p}_U \in V^-(a(1)). \tag{10}$$

In particular, $t \mapsto \|\varrho(a(t)g_0)\mathbf{p}_U\|$ decreases exponentially. Since the norm of $k(t)$ is uniformly bounded and the norm of $u(t)$ increases polynomially in t, we obtain that

$$\varrho(h(t)g_0)\mathbf{p}_U \to_{t\to+\infty} 0. \tag{11}$$

Repeating the same argument with $a(-t)$ in place of $a(t)$ we obtain $g' \in G$ such that

$$\varrho(g')\mathbf{p}_U \in V^-(a(-1))$$

and hence

$$\varrho(h(t)g')\mathbf{p}_U \to_{t\to-\infty} 0.$$

Let Q^- (resp. Q^+) denote the largest subgroup of G leaving $V^-(a(1))$ (resp. $V^-(a(-1))$) invariant. Note that Q^- and Q^+ contain opposite Borel subgroups, and hence Q^+Q^- contains an open (and in fact dense) subset of G. Therefore there is $g \in G(\mathbb{Q})$ such that $g'gg_0^{-1} = (q^+)^{-1}q^- \in Q^+Q^-$. Now letting $x = q^+g'g = q^-g_0$ and $\mathbf{p} = \varrho(g^{-1})\mathbf{p}_U \in V(\mathbb{Q})$ we obtain:

$$\varrho(h(t)x) \cdot \mathbf{p}_U \to_{t \to +\infty} 0 \quad \text{and} \quad \varrho(h(t)x)\mathbf{p} \to_{t \to -\infty} 0.$$

Hence, by Proposition 3.1, the orbit $A\pi(x)$ is divergent. □

3.5. **Existence of non-obvious divergent trajectories.** In this subsection we apply the results of §2 to prove the following:

Theorem 3.9. *Let G be a semisimple $\mathbb{Q}$-algebraic group and let Γ be an arithmetic subgroup. Let $\{h(t) : t \in \mathbb{R}\}$ be a one-parameter subgroup of G and let $A = \{h(t) : t \geq 0\}$. Let G_1 be the product of all the almost $\mathbb{Q}$-simple factors G_0 of G such that the projection of $h(1)$ onto G_0 is non-quasi-unipotent, and suppose* $\mathrm{rank}_{\mathbb{Q}} G_1 \geq 2$.

Then there are non-obvious divergent trajectories for A.

Remark 3.10.
1. *This improves* [Da, Thm. 7.3]*, where the existence of* nondegenerate *divergent trajectories is proved, under the additional hypotheses that* $\mathrm{rank}_{\mathbb{Q}} G = \mathrm{rank}_{\mathbb{R}} G$ *and G is $\mathbb{Q}$-simple.*
2. *Using arguments as in the proof of Prop. 3.5 one can obtain the same result for A a one-dimensional subgroup (rather than subsemigroup) of D.*
3. *The condition* $\mathrm{rank}_{\mathbb{Q}} G_1 \geq 2$ *is a necessary one, since by* [Da, §6]*, any divergent trajectory is obvious (even degenerate) when* $\mathrm{rank}_{\mathbb{Q}} G_1 = 1$.

Example. Let $G_1 = G_2 = \mathrm{SL}(2, \mathbb{R})$, $\Gamma_1 = \Gamma_2 = \mathrm{SL}(2, \mathbb{Z})$, $\pi_i : G_i \to G_i/\Gamma_i$, $G = G_1 \times G_2$, $\Gamma = \Gamma_1 \times \Gamma_2$, $\pi(x, y) = (\pi_1(x), \pi_2(y))$, $g_t = \mathrm{diag}(e^t, e^{-t}) \in G_i$, $h(t) = (g_t, g_t)$. It can be easily shown that a divergent trajectory $\{h(t)\pi(x_1, x_2) : t \geq 0\}$ is obvious in this case if and only if there is $i \in \{1, 2\}$ such that $\{g_t\pi_i(x_i) : t \geq 0\}$ is divergent in $\mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$. Our theorem applies to show the existence of non-obvious divergent trajectories. In recent work, Y. Cheung [Ch] explicitly describes these trajectories in terms of continued fraction expansions, and computes their Hausdorff dimension.

We list some facts we will need for the proof.

Proposition 3.11. *If P is a $\mathbb{Q}$-parabolic subgroup of a semisimple real algebraic $\mathbb{Q}$-group G then $P = \overline{P(\mathbb{Q})}$, that is, the $\mathbb{Q}$-points are dense in the $\mathbb{R}$-points, w.r.t. the topology of G as a Lie group.*

Proof. See e.g. [PlRa, Chapter 6]. □

Let $h(t) = k(t)a(t)u(t)$ where $\{k(t)\}$, $\{a(t)\}$, $\{u(t)\}$ are as in Proposition 3.7. We fix a maximal $\mathbb{R}$-diagonalizable torus D containing $\{a(t)\}$, and applying a conjugation if necessary, assume D is defined over $\mathbb{Q}$.

Given a $\mathbb{Q}$-irreducible $\mathbb{Q}$-representation $\varrho : G \to \mathrm{GL}(V)$ and $v \in V(\mathbb{Q})$, define

$$X_{\varrho,v} = \{g \in G : \varrho(h(t)g)v \to_{t\to+\infty} 0\} \tag{12}$$

Note that the set of obvious divergent trajectories is $A\pi(x)$ for all $x \in \bigcup_{\varrho,v} X_{\varrho,v}$.

Proposition 3.12. *With the above notations, we have*

$$X_{\varrho,v} = \{g \in G : \varrho(g)v \in V^-(a(1))\}.$$

Proof. Denote the set on the right hand side by $\hat{X}$. To see that $\hat{X} \subset X_{\varrho,v}$, repeat the argument for obtaining (11) from (10). Now let $x \in X_{\varrho,v}$. The values

$$\{\chi(a(1)) : \chi \in \Lambda(\varrho),\ \varrho(x)v \text{ projects nontrivially onto } V_\chi\}$$

are all real numbers. If at least one of them is positive then $t \mapsto \|\varrho(a(t)x)v\|$ increases exponentially in t and hence $x \notin X_{\varrho,v}$, a contradiction. If at least one of them is equal to 0, consider the projection v' of $\varrho(x)v$ onto the corresponding eigenspace. We have $\varrho(a(t))v' = v'$ for all t and hence

$$\varrho(k(t)u(t))v' = \varrho(h(t))v' \to_{t\to+\infty} 0.$$

Since the $\{k(t)\}$ are in a bounded subset of G we must have $\varrho(u(t))v' \to 0$ but in view of [Bo2, Prop. 4.10], this implies that $v' = 0$, a contradiction. So all the eigenvalues are less than 0, and $x \in \hat{X}$. $\square$

Proof of Theorem 3.9. Arguing as in the first paragraph of the proof of Proposition 3.5, we may assume that $G = G_1$, that is assume that the projection of $h(1)$ onto any $\mathbb{Q}$-simple factor of G is non-quasi-unipotent.

Let B^- be a minimal $\mathbb{R}$-parabolic subgroup containing D, such that $B^- \subset P^-$, where P^- is defined by (6). Let P_1, P_2 be two distinct maximal $\mathbb{Q}$-parabolic subgroups containing B^-. Two such parabolics exist because $\mathrm{rank}_{\mathbb{Q}} G \geq 2$. We have

$$\mathrm{Rad}_u(P_i) \subset B^- \subset P^-, \quad i = 1, 2. \tag{13}$$

By the reduction above to the case $G = G_1$, for every almost $\mathbb{Q}$-simple factor G_0 of G we have that $G_0 \cap P^-$ is a proper parabolic subgroup of G_0. There are noncompact almost $\mathbb{Q}$-simple factors G_i of G such that the projection of P_i onto G_i is a proper $\mathbb{Q}$-parabolic subgroup of G_i. Arguing as in the proof of Proposition 3.5 we obtain that the projection of $\mathrm{Rad}_u(P_i)$ onto any almost $\mathbb{R}$-simple factor of G_i is nontrivial. Using Proposition 3.8,

$$\dim \mathrm{Rad}_u(P_i) \cap \mathrm{Rad}_u(P^-) \geq 1. \tag{14}$$

It follows using (5), (7), (13), and (14) that for all $g \in P_i$,

$$a(t)g \cdot \mathbf{p}_i \to_{t\to+\infty} 0,$$

where $d_i = \dim \mathrm{Rad}_u(P_i)$ and $\mathbf{p}_i \in V = \bigwedge^{d_i} \mathcal{G}$ is a $\mathbb{Q}$-vector representing $\mathrm{Rad}_u(P_i)$, and where $g \cdot v$ denotes the natural action by the d_i-th exterior power of the adjoint representation.

Let $\{X_1, X_2, \cdots\}$ be an enumeration of the distinct elements of

$$\{P_i g : i = 1, 2,\ g \in G(\mathbb{Q})\}.$$

Let us verify that the X_i satisfy the hypotheses of density, transverality, and locally uniformity as in §2. Below we let $X_i = Q_i g_i$, where $Q_i \in \{P_1, P_2\}$ and $g_i \in G(\mathbb{Q})$.

- **Density.** By Proposition 3.11, $X_i = \overline{X_i \cap G(\mathbb{Q})}$ for each i. Since each $X_i = Q_i g_i$ is a coset, for each $g_0 \in X_i \cap G(\mathbb{Q})$ we have $X_i = Q_i g_0$. Take $Q \in \{P_1, P_2\}$, $Q \neq Q_i$ and let $X_j = Q g_0$. Then $X_j \neq X_i$, and we have shown that
$$X_i \cap G(\mathbb{Q}) \subset \bigcup_{i \neq j} X_j.$$
- **Transversality.** Suppose that for some i, j, $X_i \cap X_j$ contains a relatively open subset of X_i. Since both X_i and X_j are connected algebraic varieties, we must have $X_i \subset X_j$. Since X_i, X_j are cosets for Q_i, Q_j respectively, we must have $Q_i \subset Q_j$, and since P_1 and P_2 are maximal $\mathbb{Q}$-parabolics, $Q_i = Q_j$. This implies that $X_i = X_j$, hence $i = j$.
- **Local uniformity.** Given a compact $K \subset Y$ and $x \in X_i = Q_i g_i$ let $\mathbf{p} = g_i^{-1} \cdot \mathbf{p}_j$, with $j \in \{1, 2\}$ such that $Q_i = P_j$. By an argument as in the proof of Proposition 3.1, there is $\varepsilon > 0$ such that if $\|g \cdot \mathbf{p}\| < \varepsilon$ then $g \notin \pi^{-1}(K)$. For all $g \in X_i$ we have $a(t)g \cdot \mathbf{p} \to_{t \to +\infty} 0$, that is
$$g \cdot \mathbf{p} \in V^-(a(1)).$$
By continuity of the G-action there is a small enough neighborhood $\mathcal{U}$ of x such that for all $z \in \mathcal{U} \cap X_i$, $z \cdot \mathbf{p} \in V^-$ and $\|z \cdot \mathbf{p}\| < 2\|x \cdot \mathbf{p}\|$. Let $\|a(t)\|$ denote the operator norm of $a(t)$. Then
$$\|a(t)\| \leq c e^{\alpha t}, \quad \text{where} \quad \alpha = \max_{\chi \in \Lambda^-} \chi(a(1)) < 0$$
and c is a constant. Let $t_0 > \frac{1}{\alpha} \log \frac{\varepsilon}{2c\|x \cdot \mathbf{p}\|}$, then for $t \geq t_0$ and $z \in \mathcal{U} \cap X_i$ we have
$$\|a(t) z \cdot \mathbf{p}\| \leq \|a(t)\| \|z \cdot \mathbf{p}\| < \varepsilon.$$
This proves the required statement (taking $C = \{a(t) : t \in [0, t_0]\}$).

We now let $X_1', X_2', \ldots$ be an enumeration of the sets $X_{\varrho, v}$ defined by (12), for all $\mathbb{Q}$-representations $\varrho : G \to \mathrm{GL}(V)$ which are irreducible over $\mathbb{Q}$ and all $v \in V(\mathbb{Q})$. For each $w \in V$ write $w = \sum_{\lambda \in \Lambda_\varrho} w_\lambda$, where $w_\lambda \in V_\lambda$ for all λ, and let

$$\Pi(w) = \{\lambda \in \Lambda_\varrho : w_\lambda \neq 0\}, \quad \Pi(z) = \Pi(\varrho(z)v). \tag{15}$$

By Proposition 3.12 we have

$$\lambda \in \Pi(z),\ z \in X_j' \implies \lambda(a(1)) < 0. \tag{16}$$

Now define $L(i,j)$ as follows. If $X_i \not\subset X'_j$ then $L(i,j) = \infty$, and if $X_i \subset X'_j = X_{\varrho,v}$ then

$$L(i,j) = \max_{z\in X_i} \#\Pi(z).$$

Taking $M(j) = \dim V$ we see that L is a level function for $(\{X_i\},\{X'_j\})$. We will complete the proof of the theorem by showing that the conditions of Theorem 2.8 are satisfied. The hypothesis of transversality relative to $\{X'_j\}$ follows automatically from the fact that both the X_i and the X'_j are real algebraic varieties, with X_i connected.

We now verify the density of level-increasing points. Assume (exchanging P_1 and P_2 if necessary) that $X_i = P_1 g_i \subset X'_j = X_{\varrho,v}$, where $g_i \in G(\mathbb{Q})$, and let

$$\widetilde{X}_i = \{g \in X_i \cap G(\mathbb{Q}) : L(i,j) = \#\Pi(g)\}.$$

We first claim that $\widetilde{X}_i$ is dense in X_i. For each $\lambda \in \Lambda_\varrho$, the set

$$Z_\lambda = \{g \in G : \lambda \in \Pi(g)\} = \{g \in G : \varrho(g)v \text{ has nonzero } V_\lambda\text{−component}\}$$

is Zariski open in G and hence its intersection with X_i is Zariski open in X_i. In particular, setting

$$\Pi_0 = \{\lambda \in \Lambda_\varrho : X_i \cap Z_\lambda \neq \varnothing\} \quad \text{and} \quad Z(\Pi_0) = \bigcap_{\lambda\in\Pi_0} Z_\lambda, \tag{17}$$

we have that $L(i,j) = \#\Pi_0$, and, since P_1, hence X_i, is connected, that $Z(\Pi_0)$ is Zariski open and dense in X_i. Also

$$\widetilde{X}_i = Z(\Pi_0) \cap G(\mathbb{Q}),$$

so by Proposition 3.11, $\widetilde{X}_i$ is dense in X_i.

For each $z \in \widetilde{X}_i$, $P_2 z$ is one of the X_ℓ's, and is different from $X_i = P_1 z$. Thus the density of level-increasing points, and hence the Theorem, follow from the following:

Claim 3.1. *For each $z \in \widetilde{X}_i$, there is $p \in P_2$ such that $\Pi_0 = \Pi(z) \subsetneqq \Pi(pz)$.*

Proof of Claim 3.1. By a Zariski density argument similar to the one above (replacing X_i with $P_2 z$), for all p in a Zariski dense subset of P_2 we have $\Pi_0 \subset \Pi(pz)$. Thus if the claim does not hold then for any p in a Zariski dense subset of P_2, $\Pi(pz) = \Pi_0$, and hence

$$p \in P_2 \implies \Pi(pz) \ \subset \Pi_0.$$

Also, by (17),

$$p \in P_1 \implies \Pi(pz) \ \subset \Pi_0.$$

Write

$$\mathrm{Lie}(P_i) = \mathrm{Lie}(Z_G(D)) \oplus \bigoplus_{\chi\in\Psi_i\subset\Phi} \mathcal{G}_\chi, \qquad i = 1,2.$$

We now show

$$\alpha \in \Psi_1 \cup \Psi_2,\ \chi \in \Pi_0 \implies \chi + \alpha \in \Pi_0. \tag{18}$$

Write $d\varrho : \mathrm{Lie}(G) \to \mathrm{End}(\mathcal{G})$ for the derivative of ϱ. Clearly, for all $\beta \in \Lambda_\varrho$ and $k \geq 0$, $d\varrho^k(\mathcal{G}_\alpha)V_\beta \subset V_{\beta+k\alpha}$. Moreover (this follows from the standard fact that for a $\mathbb{C}$-root α and a $\mathbb{C}$-weight β we have $d\varrho(\mathcal{G}_\alpha)(V_\beta) = V_{\alpha+\beta}$), for any nonzero $v \in V_\chi$, if $\alpha \in \Phi$ and $\chi + \alpha \in \Lambda_\varrho$ there is $\mathbf{a} \in \mathcal{G}_\alpha$ such that $d\varrho(\mathbf{a})(v) \neq 0$. Thus, writing

$$\varrho(z)v = \sum_{\lambda \in \Pi_0} w_\lambda, \quad \forall \lambda \in \Pi_0, \ w_\lambda \neq 0$$

there is $\mathbf{a} \in \mathcal{G}_\alpha$ and nonzero $w'_{\chi+\alpha} \in V_{\chi+\alpha}$ such that $d\varrho(\mathbf{a})w_\chi = w'_{\chi+\alpha}$.

Writing $\mathbf{a}_0 = d\varrho(\mathbf{a}) \in \mathrm{End}(V)$ and using the fact that $\varrho(\exp) = \exp(d\varrho)$, we obtain:

$$\begin{aligned}\varrho(\exp(t\mathbf{a})z)v &= \exp(t\mathbf{a}_0) \sum_{\lambda \in \Pi_0} w_\lambda \\ &= \sum_{k \geq 0} \frac{t^k \mathbf{a}_0^k}{k!} \sum_{\lambda \in \Pi_0} w_\lambda \\ &= \sum_{\lambda \in \Pi_0} \sum_{k \geq 0} \frac{t^k \mathbf{a}_0^k (w_\lambda)}{k!}.\end{aligned}$$

The $\chi + \alpha$ component in this sum is

$$\sum_{k \geq 0} \frac{t^k}{k!} \mathbf{a}_0^k (w_{\chi+(1-k)\alpha}) = t w'_{\chi+\alpha} + \sum_{k \geq 0,\, k \neq 1} \frac{t^k}{k!} \mathbf{a}_0^k (w_{\chi+(1-k)\alpha}).$$

Since $V_{\chi+(1-k)\alpha} = \{0\}$ for all large k, this is a finite sum, which defines a polynomial in t. Since $w'_{\chi+\alpha} \neq 0$, it is nonconstant, so vanishes for only finitely many t. For all other t we have $\chi + \alpha \in \Pi(\exp(t\mathbf{a})z) \subset \Pi_0$, and (18) follows.

From (18) we obtain that the subspace

$$V^0 = \bigoplus_{\lambda \in \Pi_0} V_\lambda$$

is $d\varrho(\mathrm{Lie}(P_i))$-invariant, and hence $\varrho(P_i)$-invariant, for $i = 1, 2$. Since P_1, P_2 generate G, V^0 is $\varrho(G)$-invariant, and since ϱ is irreducible over $\mathbb{R}$, we obtain that $V^0 = V$. In particular, $\Pi_0 = \Lambda_\varrho$, so by (16), for every $\lambda \in \Lambda_\varrho$, $\lambda(a(1)) < 0$. However, since G is semisimple $\Lambda_\varrho = -\Lambda_\varrho$, a contradiction. This proves the claim and completes the proof of the theorem. $\square$

4. Higher-dimensional semigroups

As a special case of Theorem 3.9 we obtain that if $\mathrm{rank}_\mathbb{Q} G \geq 2$, Γ is irreducible and A is a one-parameter subsemigroup of D, then there are non-obvious divergent trajectories for the action of A. On the other hand, in [ToWe, Thm. 1.1] it is proved that if $\mathrm{rank}_\mathbb{Q} G < \mathrm{rank}_\mathbb{R} G$ then there are no divergent trajectories for the action of D on G/Γ, and if $\mathrm{rank}_\mathbb{Q} G = \mathrm{rank}_\mathbb{R} G$

then the divergent trajectories of D on G/Γ admit a simple algebraic description. This leaves open the question of describing divergent trajectories for intermediate subgroups or subsemigroups of D.

In this section we define obvious divergent trajectories for an action of a subsemigroup A of D on a homogeneous space. The definition coincides with Definition 3.2 for one-parameter semigroups and is satisfied for the algebraic construction of divergent trajectories described in [ToWe]. We obtain two main results. First we apply the Khintchine-Cassels-Dani scheme to obtain non-obvious divergent trajectories for some semigroups. As a consequence we show that non-obvious divergent trajectories do exist when the semigroup is a Weyl chamber. Then, in the special case $G/\Gamma = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ we apply an elementary geometric argument to show that for certain semigroups, the only divergent trajectories are obvious. We conclude the section with some results about subgroups of intermediate dimension, that is $A \subset D$ with $1 < \dim A < \dim D$.

4.1. **Obvious divergent trajectories (multidimensional case).** We preserve the notation of §3. Let A be a subsemigroup of D. We say that A is a **closed affine cone** if there is a connected subgroup D_0 of D, finitely many linear functionals $\lambda_1, \dots, \lambda_r \in \mathrm{Lie}(D)^*$ and non-negative $m_1, \dots, m_r$ such that

$$A = \{\exp(\mathbf{a}) : \mathbf{a} \in \mathrm{Lie}(D_0),\ \forall i,\ \lambda_i(\mathbf{a}) \geq m_i\}.$$

Definition 4.1. *We say that a trajectory $A\pi(x) \subset G/\Gamma$ is an* **obvious divergent trajectory** *if for any unbounded sequence $\{a_n\} \subset A$ there is a subsequence $\{a'_n\} \subset \{a_n\}$, a $\mathbb{Q}$-representation $\varrho : G \to \mathrm{GL}(V)$, and a nonzero $v \in V(\mathbb{Q})$ such that $\varrho(a'_n x)v \to_{n\to\infty} 0$.*

It is clear (see the proof of Proposition 3.1) that an obvious divergent trajectory is divergent. This definition may involve infinitely many representations. However, if A is a closed affine cone then only finitely many are needed:

Proposition 4.2. *Suppose A is a closed affine cone, and $x \in G$. Then $A\pi(x)$ is an obvious divergent trajectory if and only if for $i = 1, \dots, \ell$ there is a $\mathbb{Q}$-representation $\varrho_i : G \to \mathrm{GL}(V_i)$, $0 \neq v_i \in V_i(\mathbb{Q})$, and subsemigroup A_i of A such that:*

(i) $A \subset \bigcup A_i$.
(ii) *For $i = 1, \dots, \ell$, and for any divergent (in G) sequence $\{a_n\} \subset A_i$, $\varrho_i(a_n x)v_i \to_{n\to\infty} 0$.*

Remark 4.3. *Proposition 4.2 shows that Definitions 3.2 and 4.1 coincide in case A is a one-dimensional closed affine cone.*

Proof. It is clear that if (i) and (ii) hold then $A\pi(x)$ is an obvious divergent trajectory. Conversely, suppose $A\pi(x)$ is an obvious divergent trajectory. Let $\lambda_1, \dots, \lambda_r$ be the functionals as in the definition of a closed linear cone, let $\|\cdot\|$ be a norm on $\mathrm{Lie}(D_0)$, and let

$$B = \{\mathbf{d} \in \mathrm{Lie}(D_0) : \|\mathbf{d}\| = 1, \, \forall i, \, \lambda_i(\mathbf{d}) \geq 0\}. \tag{19}$$

Fix some $\mathbf{a} \in \mathrm{Lie}(D_0)$ with $\exp(\mathbf{a}) \in A$ and $\mathbf{d} \in B$. Then

$$a_n = \exp(\mathbf{a} + n\mathbf{d}) \in A$$

for all $n \in \mathbb{N}$ so, by the definition of an obvious divergent trajectory, there is a $\mathbb{Q}$-representation $\varrho = \varrho_{\mathbf{d}} : G \to \mathrm{GL}(V_{\mathbf{d}})$, $v_{\mathbf{d}} \in V_{\mathbf{d}}(\mathbb{Q})$ and indices $n_k \to \infty$ such that $\varrho(a_{n_k}x)v_{\mathbf{d}} \to_{k\to\infty} 0$. This implies that

$$\varrho_{\mathbf{d}}(x)v_{\mathbf{d}} \in V^-_{\varrho_{\mathbf{d}}}(\mathbf{d}).$$

The set

$$B(\mathbf{d}) = \{\mathbf{d}' \in \mathrm{Lie}(D_0) : \forall \chi \in \Lambda^-_{\varrho_{\mathbf{d}}}(\mathbf{d}), \, \chi(\mathbf{d}') < \chi(\mathbf{d})/2\}$$

is open and contains $\mathbf{d}$, so by compactness of B there are $\mathbf{d}_1, \dots, \mathbf{d}_\ell$ for which $B \subset \bigcup_{i=1}^{\ell} B(\mathbf{d}_i)$. It is simple to verify that (i) and (ii) are satisfied with $\varrho_i = \varrho_{\mathbf{d}_i}$ and with

$$A_i = \{\exp(t\mathbf{d}') : \mathbf{d}' \in B(\mathbf{d}_i), \, t \geq 0\}.$$

□

Example. Let G, D, Γ be as above and suppose $\mathrm{rank}_{\mathbb{R}}G = \mathrm{rank}_{\mathbb{Q}}G$. We claim that for any $g \in G(\mathbb{Q})$, $D\pi(g)$ is an obvious divergent trajectory. Note first that for any $\mathbb{Q}$-representation $\varrho : G \to \mathrm{GL}(V)$, the action of $\varrho(g^{-1})$ preserves $V(\mathbb{Q})$. Using this we may assume that $g = e$. Now let $\varrho : G \to \mathrm{GL}(V)$ be any $\mathbb{Q}$-representation such that $v \in V(\mathbb{Q})$ is a weight-vector for $\chi \in X(D)$ (for example we could take $\varrho = \mathrm{Ad}$ and for v and $\mathbb{Q}$-vector in $\mathcal{G}_\alpha$ for any $\alpha \in \Phi$). Let

$$A^- = \{d \in D : \chi(d) < 0\}.$$

Let $w_1, \dots, w_r \in G(\mathbb{Q})$ be representatives of the Weyl group $N_G(D)/C_G(D)$ (such representatives exist because $\mathrm{rank}_{\mathbb{Q}}G = \mathrm{rank}_{\mathbb{R}}G$, see e.g. [BoTi1, §5]). For $i = 1, \dots, r$ let $\varrho_i = \varrho$, $v_i = \varrho(w_i)v$, $A_i = w_i^{-1}A^-w_i$. It is now easy to obtain (i) and (ii) of the Proposition using the fact that the Weyl group acts transitively on the Weyl chambers of D.

Note that these obvious divergent trajectories $D\pi(g)$, $g \in G(\mathbb{Q})$ are the only divergent trajectories for the action of D, by [ToWe, Thm. 1.1].

Question 4.4. *In analogy with Definition 3.3 one could also define degenerate divergent trajectories for actions of closed affine cones, and ask whether an obvious divergent trajectory is necessarily degenerate.*

4.2. **Existence results for cones.**

Theorem 4.5. *Suppose G is a semisimple $\mathbb{Q}$-algebraic group, $\Gamma = G(\mathbb{Z})$, and $A \subset G$ is a closed affine cone. Suppose that for $\ell = 1, 2$ there are $\mathbb{Q}$-representations $\varrho_\ell : G \to \mathrm{GL}(V_\ell)$ and $v_\ell \in V_\ell(\mathbb{Q})$ such that the following hold:*

1. *For any divergent (in G) sequence $\{a_n\} \subset A$ we have $\varrho_\ell(a_n)v_\ell \to_{n\to\infty} 0$ for both ℓ.*
2. *The groups $P_\ell = \{g \in G : \varrho_\ell(g)v_\ell \in \mathbb{R}v_\ell\}$, $\ell = 1, 2$ are $\mathbb{Q}$-parabolic subgroups of G and P_1, P_2 generate G.*

Then there is $x \in G$ such that $A\pi(x)$ is divergent, but for any one-parameter semigroup $\{a(t) = \exp(t\mathbf{a}) : t \geq 0\} \subset A$, any $\mathbb{Q}$-representation $\varrho : G \to \mathrm{GL}(V)$ and any $v \in V(\mathbb{Q})$ we have

$$\varrho(a(t)x)v \not\to_{t\to+\infty} 0.$$

In particular there are non-obvious divergent trajectories for A.

Proof. Let B be as in (19), let $\mathbf{d}_1, \mathbf{d}_2, \dots$, such that $\{\mathbf{d}_k : k \geq 1\}$ is dense in B, and let $a_k(t) = \exp(t\mathbf{d}_k)$. We claim that it is enough to find $x \in G$ such that $A\pi(x)$ is divergent, but for any $k \geq 1$, any $\mathbb{Q}$-representation $\varrho : G \to \mathrm{GL}(V)$, and any $v \in V(\mathbb{Q})$ we have

$$\varrho(a_k(t)x)v \not\to_{t\to+\infty} 0.$$

Indeed, suppose we have found such an x and suppose by contradiction that for some one-parameter subgroup $\{a(t) = \exp(t\mathbf{a}) : t \geq 0\} \subset A$, some $\mathbb{Q}$-representation $\varrho : G \to \mathrm{GL}(V)$ and some $v \in V(\mathbb{Q})$ we have

$$\varrho(a(t)x)v \to_{t\to+\infty} 0.$$

Normalize $\mathbf{a}$ so that $\|\mathbf{a}\| = 1$, i.e., $\mathbf{a} \in B$. Then we have

$$\varrho(x)v \in V_{\mathbf{a}}^-.$$

For $\mathbf{d}_k$ sufficiently close to $\mathbf{a}$ we have

$$\lambda \in \Lambda_\varrho,\ \lambda(\mathbf{a}) < 0 \implies \lambda(\mathbf{d}_k) < 0.$$

This implies

$$\varrho(a_k(t)x)v \to_{t\to+\infty} 0,$$

a contradiction.

Let $X_1, X_2, \dots$ be an enumeration of the sets $\{P_i g : i = 1, 2,\ g \in G(\mathbb{Q})\}$. Repeating the arguments of the proof of Theorem 3.9, one obtains that the conditions of density and transversality hold for the X_i. To verify local uniformity, let B be as in (19) and let $\Pi(w)$ be defined as in (15). It follows from the hypothesis that for $\ell = 1, 2$,

$$v_\ell \in V^-(A) = \bigcap_{\mathbf{d}\in B} V^-(\mathbf{d})$$

and moreover, using compactness of B, there is $c > 0$ such that for $\ell = 1, 2$,

$$\mathbf{d} \in B, \, \chi \in \Pi(v_\ell) \quad \Longrightarrow \quad \chi(\mathbf{d}) < -c. \tag{20}$$

Exchanging P_1 and P_2 if necessary, suppose $x \in X_i = P_1 g, \; g \in G(\mathbb{Q})$, and let $\widetilde{v} = \varrho_1(g^{-1})v_1 \in V_1(\mathbb{Q})$. Given a compact $K \subset G/\Gamma$, from the proof of Proposition 3.1 there is $\varepsilon > 0$ small enough so that if $\|\varrho_1(z)\widetilde{v}\| < \varepsilon$ then $\pi(z) \notin K$. Now using (20) and the fact that $\varrho(x)\widetilde{v}$ is a scalar multiple of v_1 we find t_0 and a neighborhood $\mathcal{U}$ of x such that for all $t \geq t_0$, all $\mathbf{d} \in B$, and all $z \in \mathcal{U} \cap X_i$, $\varrho(\exp(t\mathbf{d})z)\widetilde{v} < \varepsilon$. Thus

$$C = A \cap \{\exp(t\mathbf{d}) : \mathbf{d} \in B, \, t \in [0, t_0]\}$$

satisfies the local uniformity requirement.

Now, for any $k \geq 1$, any $\mathbb{Q}$-representation $\varrho : G \to \mathrm{GL}(V)$ and any $v \in V(\mathbb{Q})$, let

$$X_{\varrho,v,k} = \{g \in G : \varrho(a_k(t)g)v \to_{t \to +\infty} 0\}, \tag{21}$$

and let $X'_1, X'_2, \ldots$ be an enumeration of all the sets $\{X_{\varrho,v,k} : \varrho, v, k\}$. With these choices we apply Theorem 2.8; verifying the hypotheses of this theorem is done exactly as in the proof of Theorem 3.9, and is omitted. $\square$

4.3. **The Weyl chamber and some other cones.** We illustrate the use of Theorem 4.5 by exhibiting some closed affine cones which admit non-obvious divergent trajectories. Preserve the notation of the previous sections. In particular, D is a maximal $\mathbb{R}$-split torus, $\Phi \subset X(D)$ is the set of $\mathbb{R}$-roots, and $\Delta \subset \Phi$ a set of positive simple roots. The **$\mathbb{R}$-Weyl chamber** determined by Δ is by definition

$$\{\exp(\mathbf{d}) \in D : \forall \lambda \in \Delta, \, \lambda(\mathbf{d}) \geq 0\}.$$

Corollary 4.6. *Let G be an almost $\mathbb{Q}$-simple semisimple algebraic group, with* $\mathrm{rank}_{\mathbb{Q}} G \geq 2$*, and let A be an $\mathbb{R}$-Weyl chamber in G. Then there are non-obvious divergent trajectories for A.*

Proof. Replacing Δ with $-\Delta$, suppose that

$$A = \{\exp(\mathbf{d}) \in D : \forall \lambda \in \Delta, \, \lambda(\mathbf{d}) \leq 0\}.$$

Let P_0 be a minimal $\mathbb{R}$-parabolic subgroup of G corresponding to the choice of Δ, let P_1 and P_2 be two maximal $\mathbb{Q}$-parabolic subgroups of G containing P_0, let U_1, U_2 be the respective unipotent radicals, let $\Psi_i \subset \Phi$ be the roots appearing in $\mathrm{Lie}(U_i)$, let ϱ_i be the $\dim U_i$-th exterior power of the adjoint representation of G on $V_i = \bigwedge^{\dim U_i} \mathcal{G}$, and let $0 \neq v_i \in V_i(\mathbb{Q})$ represent U_i. Such a vector exists because U_i is defined over $\mathbb{Q}$. It is clear that condition (2) of Theorem 4.5 is satisfied, and to conclude the proof we verify condition (1).

Let G_0 be any $\mathbb{R}$-almost simple factor of G, with roots Φ_0 and simple roots Δ_0. Arguing as in the proof of Proposition 3.5, we see that $\dim G_0 \cap U_i \geq 1$.

Moreover, by the proof of Proposition 3.8, if $\alpha_{\max} \in \Phi_0$ is a dominant root then

$$\mathcal{G}_{\alpha_{\max}} \subset \mathrm{Lie}(U_i \cap G_0).$$

Also, by [Hu, §10.4, Lemma A],

$$\alpha_{\max} = \sum_{\lambda \in \Delta_0} a_\lambda \lambda, \quad \text{where} \quad \forall \lambda \in \Lambda_0,\ a_\lambda > 0. \tag{22}$$

Now let $\chi_i \in \Lambda_{\varrho_i}$ be the weight associated to v_i. Calculating using (5) and (22) we obtain that

$$\chi_i = \sum_{\lambda \in \Delta} b_\lambda \lambda, \quad \text{where} \ \ \forall \lambda \in \Lambda,\ b_\lambda > 0.$$

For any unbounded sequence $\{a_n\} \subset A$, we have $\lambda(a_n) \le 0$ for all $\lambda \in \Delta$, and, passing to a subsequence, there is at least one $\lambda \in \Delta$ such that $\lambda(a_n) \to_{n\to\infty} -\infty$. Hence for any divergent (in G) sequence $\{\mathbf{a}_n\} \subset \mathrm{Lie}(A)$ we have

$$\chi_i(\mathbf{a}_n) \to -\infty, \quad i = 1, 2.$$

Therefore

$$\begin{aligned} \varrho(a_n) v_i &= \varrho(\exp(\mathbf{a}_n)) v_i \\ &= e^{\chi_i(\mathbf{a}_n)} v_i \to_{n\to\infty} 0, \end{aligned}$$

proving (1). □

Corollary 4.7. *Let* $G = \mathrm{SL}(3,\mathbb{R})$, $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ *and let*

$$A = \{\mathrm{diag}(e^a, e^b, e^c) : a + b + c = 0,\ \varepsilon a + c \le 0,\ a + \varepsilon c \ge 0,\ a \le 0\},$$

where $\varepsilon > 0$ *(see the right hand side of Figure 1). Then there are non-obvious divergent trajectories for* A*.*

Proof. Let $V_1 = \mathbb{R}^3$, let $v_1 = \mathbf{e}_1$ be the first vector in the standard basis of $\mathbb{R}^3$, and let $\varrho_1 : G \to \mathrm{GL}(V_1)$ be the standard action (i.e. the given representation $G = \mathrm{SL}(3,\mathbb{R})$). Let $V_2 = \bigwedge^2 \mathbb{R}^3$, let $\varrho_2 = \bigwedge^2 \varrho_1$, and let $v_2 = \mathbf{e}_1 \wedge \mathbf{e}_2$. Then it is easy to verify that all conditions of Theorem 4.5 hold. □

4.4. **A cone admitting obvious divergence only.**

Theorem 4.8. *Let* $G = \mathrm{SL}(3,\mathbb{R})$, $\Gamma = \mathrm{SL}(3,\mathbb{Z})$*, and let*

$$A^+ = \{\mathrm{diag}(e^{d_1}, e^{d_2}, e^{d_3}) : \sum d_i = 0,\ d_2, d_3 \ge 0\}$$

(see the left hand side of Figure 1).

Then there are no non-obvious divergent trajectories for the action of A^+ *on* G/Γ*.*

Remark 4.9. *Let $A_1, \dots, A_6$ be the 6 closed affine cones obtained by rotating A^+ by multiples of $\pi/3$. Then the group of automorphisms of the root system, which is naturally isomorphic to $\{X \in \mathrm{Aut}(G) : X(D) = D\}$ acts transitively on the A_i (but the Weyl group doesn't!), and hence the theorem holds for any of the A_i in place of A^+.*

Question 4.10. *Our argument essentially uses the two-dimensionality of D. It would be interesting to see if this result could be extended to groups of* $\mathrm{rank}_{\mathbb{Q}} G \geq 3$.

Proof. We first note that for a sequence $\{\exp(\mathbf{d}_n)\} \subset A^+$,

$$\mathbf{d}_n = \mathrm{diag}(d_1^n, d_2^n, d_3^n) \text{ is divergent in } G \iff d_1^n \to -\infty. \tag{23}$$

This follows from the fact that $d_1^n = -(d_2^n + d_3^n),\ d_2^n, d_3^n \geq 0$ for all n. Also, if $\{\exp(\mathbf{d}_n)\} \subset A^+$ is divergent (in G) then, along a subsequence, either $d_2^n \to +\infty$ or $d_3^n \to +\infty$.

Suppose $A^+\pi(g)$ is divergent. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the standard basis of $\mathbb{R}^3$, and let

$$\mathbf{e}_{ij} = \mathbf{e}_i \wedge \mathbf{e}_j, \quad 1 \leq i < j \leq 3.$$

Let $\varrho_1(g)v$ denote the standard (given) representation of G on $\mathbb{R}^3$, and let $\varrho_2 = \varrho_1 \wedge \varrho_1$ be the representation of G on $\bigwedge^2 \mathbb{R}^3$. Equip $\mathbb{R}^3$ and $\bigwedge^2 \mathbb{R}^3$ with the sup-norms

$$\| \sum_1^3 a_i \mathbf{e}_i \| = \max |a_i|, \quad \| \sum_{1 \leq i < j \leq 3} b_{ij} \mathbf{e}_{ij} \| = \max |b_{ij}|.$$

Let $\widehat{\mathbb{Z}^3}$ denote the primitive vectors in $\mathbb{Z}^3$; that is, the nonzero vectors in $\mathbb{Z}^3$ which are not a multiple of a shorter vector in $\mathbb{Z}^3$. Let

$$\mathcal{L}_1 = \varrho_1(g)\widehat{\mathbb{Z}^3}, \quad \mathcal{L}_2 = \{v_1 \wedge v_2 : v_i \in \mathcal{L}_1,\ v_1 \wedge v_2 \neq 0\}.$$

For $v = \sum a_i \mathbf{e}_i \in \mathcal{L}_1$ (resp. $v = \sum b_{ij} \mathbf{e}_{ij} \in \mathcal{L}_2$), and $\varepsilon > 0$, let

$$Z(v) = \{j : a_j = 0\}, \quad (\text{resp. } Z(v) = \{i, j : 1 \leq i < j \leq 3,\ b_{ij} = 0\})$$

and

$$D_{\varepsilon,v} = \{d \in D : \|\varrho_i(d)v\| < \varepsilon\}, \quad \mathfrak{d}_{\varepsilon,v} = \log(D_{\varepsilon,v}).$$

In case $v = \sum a_i \mathbf{e}_i \in \mathcal{L}_1$ we have

$$\begin{aligned} \mathrm{diag}(d_1, d_2, d_3) \in \mathfrak{d}_{\varepsilon,v} &\iff |a_i| e^{d_i} < \varepsilon,\ i = 1, 2, 3 \\ &\iff d_j < \log \frac{\varepsilon}{|a_j|} \text{ whenever } j \notin Z(v). \end{aligned} \tag{24}$$

Similarly, in case $v = \sum b_{ij} \mathbf{e}_{ij} \in \mathcal{L}_2$, writing $k = k(ij)$ such that $\{i, j, k\} = \{1, 2, 3\}$, we have:

$$\mathrm{diag}(d_1, d_2, d_3) \in \mathfrak{d}_{\varepsilon,v} \iff d_k > \log \frac{\varepsilon}{|b_{ij}|} \text{ whenever } i, j \notin Z(v). \tag{25}$$

Possible shapes of $\mathfrak{d}_{\varepsilon,v}$, for various values of $\#Z(v)$ are shown in figure. Note that the shapes are of importance for the argument.

We now claim that there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the following hold:

a) for all $v \in \mathcal{L}_1 \cup \mathcal{L}_2$, $0 \notin \mathfrak{d}_{\varepsilon,v}$.
b) if for $v_1, v_2, v_3 \in \mathcal{L}_1$ and $d \in A^+$ we have $d \in \bigcap D_{\varepsilon,v_i}$, then the v_i-s are linearly dependent.
c) if for $v_1, v_2, v_3, v_4 \in \mathcal{L}_1$ and $d \in A^+$ we have $d \in \bigcap D_{\varepsilon,v_i}$ and $v_1 \wedge v_2 \neq 0 \neq v_3 \wedge v_4$ then $v_1 \wedge v_2 = \pm v_3 \wedge v_4$.

Indeed, property a) follows from the discreteness of the $\mathcal{L}_i$. Property b) holds since $\det(d) = 1$ and hence, for a constant C depending only on our choice of norms,

$$\mathbb{Z} \ni \|v_1 \wedge v_2 \wedge v_3\| = \|\varrho_1(d)v_1 \wedge \varrho_1(d)v_2 \wedge \varrho_1(d)v_3\| \leq C \prod_1^3 \|\varrho_1(d)v_i\| < C_2\varepsilon_0^3.$$

So for small enough ε_0 we have $v_1 \wedge v_2 \wedge v_3 = 0$. Property c) follows immediately from b) and the fact that elements of $\mathcal{L}_1$ are primitive.

Now fix a norm $\|\cdot\|$ on $\mathrm{Lie}(D)$, denote

$$\begin{aligned} \mathfrak{a}^+ &= \log(A^+), \\ \mathfrak{a}^+(r) &= \{\mathbf{a} \in \mathfrak{a}^+ : \|\mathbf{a}\| = r\}, \\ \mathfrak{a}^+(r_1, r_2) &= \{\mathbf{a} \in \mathfrak{a}^+ : r_1 \leq \|\mathbf{a}\| \leq r_2\}, \end{aligned}$$

and let $0 < \varepsilon < \varepsilon_0$. Since $A^+\pi(g)$ is divergent, by Mahler's compactness criterion [Ra, Chap. X] there is $r > 0$ such that for all $a \in A^+$ with $\|a\| \geq r$ there is $w \in \widehat{\mathbb{Z}^3}$ such that $\|\varrho_1(ag)w\| < \varepsilon$. That is, for all $R > r$,

$$\mathfrak{a}^+(r, R) \subset \bigcup_{v \in \mathcal{L}_1} \mathfrak{d}_{\varepsilon,v}.$$

We now claim that at least one of the regions in this covering is unbounded, that is

$$\exists v \in \mathcal{L}_1,\ \mathfrak{d}_{\varepsilon,v} \cap \mathfrak{a}^+ \neq \varnothing \ \text{ and } \ Z(v) \neq \varnothing. \tag{26}$$

Suppose otherwise; then, by (24), $D_{\varepsilon,v}$ is bounded for every $v \in \mathcal{L}_1$ for which $\mathfrak{d}_{\varepsilon,v} \cap \mathfrak{a}^+ \neq \varnothing$. By compactness of $\mathfrak{a}^+(r)$, and discreteness of $\mathcal{L}_1$, the set

$$S = \{v \in \mathcal{L}_1 : \mathfrak{a}^+(r) \cap \mathfrak{d}_{\varepsilon,v} \neq \varnothing\}$$

is finite.

Let

$$R > \max\{\|\mathbf{a}\| : v \in S,\ \mathbf{a} \in \mathfrak{d}_{\varepsilon,v}\}, \tag{27}$$

and consider the cover of $\mathfrak{a}^+(r,R)$ by the sets $\mathfrak{d}_{\varepsilon,v}$, $v \in \mathcal{L}_1$. Again by compactness of $\mathfrak{a}^+(r,R)$ and discreteness of $\mathcal{L}_1$, this is a finite cover, i.e.,

$$\#\{v \in \mathcal{L}_1 : \mathfrak{a}^+(r,R) \cap \mathfrak{d}_{\varepsilon,v} \neq \varnothing\} < \infty. \tag{28}$$

For $i = 2, 3$, let

$$E_i = \{\mathbf{d} = (d_1, d_2, d_3) \in \mathfrak{a}^+(r,R) : d_i = 0\}.$$

Note that $\mathfrak{a}^+(r,R)$ is a quadrilateral, with $\mathfrak{a}^+(r), \mathfrak{a}^+(R)$ (respectively E_2, E_3) forming pairs of opposing edges.

Let

$$\mathcal{P} = \{\mathbf{d} \in \mathfrak{a}^+(r,R) : \exists v_1, v_2 \in \mathcal{L}_1,\ \mathbf{d} \in \mathfrak{d}_{\varepsilon,v_1} \cap \mathfrak{d}_{\varepsilon,v_2},\ v_1 \wedge v_2 \neq 0\}.$$

Suppose first that there is a connected component $\mathcal{P}_0$ of $\mathcal{P}$ such that

$$\mathcal{P}_0 \cap E_i \neq \varnothing, \quad i = 2, 3.$$

For every $\mathbf{d} \in \mathcal{P}_0$ let $v_1(\mathbf{d}), v_2(\mathbf{d}) \in \mathcal{L}_1$ such that $v_1(\mathbf{d}) \wedge v_2(\mathbf{d}) \neq 0$ and $\mathbf{d} \in \mathfrak{d}_{\varepsilon,v_1(\mathbf{d})} \cap \mathfrak{d}_{\varepsilon,v_2(\mathbf{d})}$. By (28) only finitely many pairs $v_1(\mathbf{d}), v_2(\mathbf{d})$ appear in this way. By fact c) above, and by connectedness of $\mathcal{P}_0$, $w(\mathbf{d}) = v_1(\mathbf{d}) \wedge v_2(\mathbf{d}) \in \mathcal{L}_2$ is independent of $\mathbf{d} \in \mathcal{P}_0$. In particular there is $w \in \mathcal{L}_2$ such that $\mathfrak{d}_{\varepsilon,w}$ contains points in both E_2 and E_3. That is, there are

$$(d_1, 0, d_3),\ (d_1', d_2', 0) \in \mathfrak{d}_{\varepsilon,w}.$$

Using (25) we obtain:

$$\begin{aligned}
\log \frac{\varepsilon}{|b_{12}|} &< d_3' = 0, \\
\log \frac{\varepsilon}{|b_{13}|} &< d_2 = 0, \\
\log \frac{\varepsilon}{|b_{23}|} &< d_1 = -d_3 < 0.
\end{aligned}$$

Again from (25) if follows that $0 \in \mathfrak{d}_{\varepsilon,w}$, a contradiction to a) above.

Now suppose that there is no connected component of $\mathcal{P}$ which extends from E_1 to E_2. By the Jordan–Brouwer separation theorem (cf. [Ga]), there is a connected component of $\mathfrak{a}^+ \smallsetminus \mathcal{P}$ which intersects both $\mathfrak{a}^+(r)$ and $\mathfrak{a}^+(R)$. The quadrilateral $\mathfrak{a}^+(r,R)$ is covered by the sets $\mathfrak{d}_{\varepsilon,v}$, and the boundary of each $\mathfrak{d}_{\varepsilon,v}$ is contained in $\mathcal{P}$. This implies that there is $v \in \mathcal{L}_1$ such that $\mathfrak{d}_{\varepsilon,v} \cap \mathfrak{a}^+(r) \neq \varnothing$ and $\mathfrak{d}_{\varepsilon,v} \cap \mathfrak{a}^+(R) \neq \varnothing$, contradicting (27). These two contradictions together prove (26).

So let $v \in \mathcal{L}_1$ with $Z(v) \neq \varnothing$ and $\mathfrak{a}^+ \cap \mathfrak{d}_{\varepsilon,v} \neq \varnothing$. Using a) above and (24) one easily sees that $1 \notin Z(v)$. If $Z(v) = \{2,3\}$ then v is a multiple of $\mathbf{e}_1$, and $\varrho_1(\exp(\mathbf{d})g)v = e^{d_1}\varrho_1(g)v$. By (23), this implies $\varrho_1(d_n g)v \to_{n\to\infty} 0$ for every sequence $\{d_n\} \subset A^+$ which is divergent in G, and we are done.

So suppose $Z(v) = \{2\}$ or $Z(v) = \{3\}$. Using (24) it is easy to see that if $0 < \varepsilon' < \varepsilon$ is small enough then $\mathfrak{a}^+ \cap \mathfrak{d}_{\varepsilon',v} = \varnothing$. Therefore, we can repeat the argument with ε' in place of ε, and obtain $v' \in \mathcal{L}_1$ such that $Z(v') = \{2\}$ or

$Z(v') = \{3\}$. If necessary, repeat the argument again and exchange 3 with 2, to obtain

$$Z(v) = Z(v') = \{2\}, \quad \text{where } v, v' \text{ are linearly independent.}$$

Therefore $v \wedge v'$ is a nonzero multiple of $\mathbf{e}_{12}$. Let $w = \varrho(g)^{-1}v, w' = \varrho(g)^{-1}v' \in \widehat{\mathbb{Z}^3}$. Using the fact that $w, w', \mathbf{e}_1, \mathbf{e}_2$ are primitive, there is $\gamma \in \Gamma$ such that $\varrho_1(\gamma)\mathbf{e}_1 = w$, $\varrho_1(\gamma)\mathbf{e}_2 = w'$, so for some $a \in D$ we have $\varrho_2(ag\gamma)\mathbf{e}_{12} = \pm\mathbf{e}_{12}$.

Let $h = ag\gamma$,

$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & \pm 1 \end{pmatrix} = \{x \in G : \varrho_2(x)\mathbf{e}_{12} = \pm\mathbf{e}_{12}\},$$

$$G_1 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_1^+ = A^+ \cap P = A^+ \cap G_1 = \{a_1(t) : t \geq 0\}, \quad \text{where } a_1(t) = \mathrm{diag}(e^t, e^{-t}, 1).$$

Since $\gamma \in \Gamma$, $\pi(g\gamma) = \pi(g)$ and $a \in D$ commutes with A^+, the trajectory $A^+\pi(h)$ is also divergent. It follows e.g. from [We1, Prop. 2] that $\pi(P)$ is closed in G/Γ. Hence $A_1^+\pi(h) \subset P/P \cap \Gamma$ is divergent in $P/(P \cap \Gamma)$. Again using [We1, Prop. 2], $\pi(G_1)$ is closed in $P/P \cap \Gamma$. Replacing Γ if necessary with a finite-index subgroup, we get that the homomorphism $P \to G_1$, $g_1 u \mapsto g_1$ descends to a well-defined map $P/P \cap \Gamma \to G_1/G_1 \cap \Gamma$, with compact fiber identified with $U/U \cap \Gamma$.

Write $h = g_1 u$, then it follows that $A_1^+\pi(g_1)$ is divergent in $G_1/G_1 \cap \Gamma$. Any divergent trajectory for a one-parameter diagonalizable semigroup in G_1 is obvious, as can be seen by elementary arguments (cf. also [Da, Thm. 6.1]). More precisely, it can be proved that there is a vector $w \in \mathbb{Z}^2$ such that $a_1(t)g_1 \cdot w \to_{t\to+\infty} 0$, where $g \cdot v$ denotes the standard (given) representation of G_1 on $\mathbb{R}^2$. Embed this representation as a $\varrho_1(G_1)$-invariant subspace of $\mathbb{R}^3$, by identifying $\mathbb{R}^2$ with $\mathrm{span}(\mathbf{e}_1, \mathbf{e}_2)$. In particular $w \in \mathbb{Z}^3 \cap \mathrm{span}(\mathbf{e}_1, \mathbf{e}_2)$ and $\varrho(g_1)w$ is a multiple of $\mathbf{e}_1$. Since $\varrho_1(U)$ fixes every vector in $\mathrm{span}(\mathbf{e}_1, \mathbf{e}_2)$, $\varrho(g)w$ is also a multiple of $\mathbf{e}_1$. Again, using (23), we obtain that $A^+\pi(g)$ is an obvious divergent trajectory. $\square$

4.5. **Subgroups of intermediate dimension.** Let A be a subgroup of D. We have shown in Theorem 3.9 that non-obvious divergence occurs in case $\dim A = 1$ and $\mathrm{rank}_{\mathbb{Q}} G > 1$. On the other hand [ToWe], in case of actions of the full diagonal subgroup D, there are no divergent trajectories at all if $\mathrm{rank}_{\mathbb{Q}} G < \mathrm{rank}_{\mathbb{R}} G$ and only obvious divergent trajectories in case $\mathrm{rank}_{\mathbb{Q}} G = \mathrm{rank}_{\mathbb{R}} G$. This leaves open questions about divergent trajectories for a subgroup A of D with $1 < \dim A < \dim D$. At present the following seems plausible, cf. [ToWe, §8]:

Conjecture 4.11. A. *If* $\dim A > \operatorname{rank}_{\mathbb{Q}} G$ *then there are no divergent trajectories for* A.

B. *If* $\dim A = \operatorname{rank}_{\mathbb{Q}} G$ *then the only divergent trajectories are obvious ones.*

C. *If* $\dim A < \operatorname{rank}_{\mathbb{Q}} G$ *then there are non-obvious divergent trajectories.*

In this subsection we present some partial results supporting this conjecture. We prove part A in case $\operatorname{rank}_{\mathbb{Q}} G = 1$, and study part C in case $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, proving that there are non-obvious divergent trajectories for 'most' subgroups A with $\dim A < n - 1$.

The following was announced in [ToWe, §8]:

Proposition 4.12. *Let* G *be a semisimple* $\mathbb{Q}$*-almost simple algebraic group with* $\operatorname{rank}_{\mathbb{Q}} G = 1$ *and let* $\Gamma = G(\mathbb{Z})$*. Suppose* A *is a subgroup of* D *and* $\dim A > 1$*. Then there are no divergent trajectories for the action of* A *on* G/Γ.

Proof. It follows from [Bo1, Prop. 17.9] that there is a sequence $s_1, s_2, \ldots \in G(\mathbb{Q})$, a sequence $W_1, W_2, \ldots$ of disjoint open subsets of G, a large enough compact subset $K \subset G/\Gamma$, a $\mathbb{Q}$-representation $\varrho : G \to \mathrm{GL}(V)$ and $0 \neq v \in V(\mathbb{Q})$ such that

- $G \smallsetminus \pi^{-1}(K) = \bigcup_{i=1}^{\infty} W_i$ (disjoint union).
- For any sequence $\{g_n\} \subset W_i$ such that $\{\pi(g_n)\} \subset G/\Gamma$ is divergent, we have $\varrho(g_n s_i)v \to_{n \to \infty} 0$.

Now suppose $A\pi(g)$ is divergent and $\dim A \geq 2$. Then there is a ball $C \subset A$ such that for all $a \in A \smallsetminus C$, $a\pi(g) \notin K$ and hence $ag \in \bigcup W_i$. Since $\dim A \geq 2$, $A \smallsetminus C$ is connected and since the W_i are disjoint open sets we have $(A \smallsetminus C)g \subset W_{i_0}$ for some fixed i_0. Now let $\{a(t) : t \in \mathbb{R}\}$ be a one-parameter subgroup of A. For all large enough t, we have both $a(t) \notin C$ and $a(-t) \notin C$. Therefore

$$\varrho(a(t)gs_{i_0})v \to_{t \to \pm\infty} 0,$$

hence

$$0 \neq \varrho(gs_{i_0})v \in V^-(a(1)) \cap V^-(a(-1)),$$

which is clearly impossible. □

We now present some partial results lending credence to part C of Conjecture 4.11.

Theorem 4.13. *Let* G *be a semisimple* $\mathbb{Q}$*-algebraic group and let* $\Gamma = G(\mathbb{Z})$*. Suppose* A *is a subgroup of* D*, and for* $\ell = 1, 2$ *there are subgroups* P_ℓ *and finitely many representations* $\varrho_i^\ell : G \to \mathrm{GL}(V_i)$ *and* $v_i^\ell \in V(\mathbb{Q})$*, such that the following hold for* $\ell = 1, 2$*:*

1. *For any unbounded sequence* $\{a_n\} \subset A$ *there is a subsequence* $\{a'_n\}$ *and* i *such that* $\varrho_i^\ell(a'_n)v_i^\ell \to_{n \to \infty} 0$.
2. *For each* i*,* $\varrho_i^\ell(P_\ell)$ *leaves the line* $\mathbb{R} \cdot v_i$ *invariant.*
3. $P_\ell = \overline{P_\ell \cap G(\mathbb{Q})}$.

4. *$D \subset P_\ell$.*
5. *For any $\mathbb{R}$-root α, if $\mathcal{G}_\alpha \cap \mathrm{Lie}(P_\ell) \neq \{0\}$ then $\mathcal{G}_\alpha \subset \mathrm{Lie}(P_\ell)$.*
6. *P_1 and P_2 generate G.*

Then there is $x \in G$ such that $A\pi(x)$ is divergent, but for any one-parameter semigroup $\{a(t) = \exp(t\mathbf{a}) : t \geq 0\} \subset A$, any $\mathbb{Q}$-representation $\varrho : G \to \mathrm{GL}(V)$ and any $v \in V(\mathbb{Q})$ we have

$$\varrho(a(t)x)v \not\to_{t\to+\infty} 0.$$

In particular, $A\pi(x)$ is a non-obvious divergent trajectory.

Proof. The proof is very similar to that of Theorems 3.9 and 4.5. We sketch the required modifications.

We let $X_1, X_2, \dots$ be an enumeration of the distinct elements of $\{P_\ell g : g \in G(\mathbb{Q}),\ \ell = 1, 2\}$ and let $X_1', X_2', \dots$ be an enumeration of the distinct sets of the form (21), for some dense countable set $\{a_k(t) : k = 1, 2, \dots\}$ of one-parameter semigroups in A. Arguing as in the proof of Theorem 4.5, it suffices to verify the conditions of Theorems 2.1 and 2.8 for these choices.

Density and transversality follow by repeating verbatim the arguments given in the proof of Theorem 3.9. Local uniformity is verified using hypotheses (1,2), cf. the proofs of Proposition 4.2 and Theorem 4.5. Transversality relative to $\{X_j'\}$ is immediate, and density of level-increasing points is proved as in the proof of Theorem 3.9. Note that hypotheses (4,5) are used when carrying out the arguments of Claim 3.1. □

We now apply Theorem 4.13 and describe some intermediate subgroups admitting non-obvious divergent trajectories, in a special case.

Corollary 4.14. *Let $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, D (as before) the subgroup of positive diagonal matrices in G. Let $\chi \in X(D)$ be a rational character defined by*

$$\chi : D \to \mathbb{R}, \quad \chi(\mathrm{diag}(e^{\mathbf{d}_1}, \dots, e^{\mathbf{d}_n})) = \sum a_i \mathbf{d}_i, \quad a_i \in \mathbb{Z},$$

and let $A = \ker \chi$.

Suppose there is an index $i_0 \in \{1, \dots, n\}$ such that

$$\text{either} \quad a_{i_0} > \max_{j \neq i_0} a_j, \quad \text{or } a_{i_0} < \min_{j \neq i_0} a_j.$$

Then there are non-obvious divergent trajectories for the action of A on G/Γ.

Proof. Suppose with no loss of generality that $a_1 > \max_{2 \leq j \leq n} a_j$. Since $A \subset D$ and $\mathrm{tr}(\mathrm{diag}(\mathbf{d}_1, \dots, \mathbf{d}_n)) = \sum d_i$ vanishes on D, we can replace χ with $\chi - (a_1 - 1)\mathrm{tr}$ to assume that

$$a_1 = 1 \text{ and } a_j \leq 0, \ 2 \leq j \leq n$$

or equivalently

$$\mathrm{Lie}(A) = \{\mathrm{diag}(\mathbf{d}_1, \dots, \mathbf{d}_n) : \mathbf{d}_1 = -\sum_{j \geq 2} a_j \mathbf{d}_j = \sum_{j \geq 2} b_j \mathbf{d}_j\}, \text{ where } b_j = -a_j \geq 0.$$

We claim that for any sequence $\{a_k\} \subset A$ which is divergent in G, $a_k = \exp(\mathrm{diag}(\mathbf{d}_1^k, \dots, \mathbf{d}_n^k))$, there is at least one $i \geq 2$ and one $j \geq 2$ such that along a subsequence, $\mathbf{d}_i^k \to_{k\to\infty} -\infty$ and $\mathbf{d}_j^k \to_{k\to\infty} +\infty$. Indeed, if the first statement did not hold we would have that all the $\mathbf{d}_i^k$ are bounded below, hence by $\mathbf{d}_1 = \sum_{s\geq 2} b_s \mathbf{d}_s$ that $\mathbf{d}_1^k$ is bounded below, hence by $\mathbf{d}_1 = -\sum_{s\geq 2} \mathbf{d}_s$ that $\sum_{s\geq 2} \mathbf{d}_i^k$ is bounded above, hence all the $\mathbf{d}_i^k$ are bounded, a contradiction. The second statement is proved by a similar argument.

For $\ell = 1$ (resp. $\ell = 2$) let ϱ^ℓ be the standard action of G on $\mathbb{R}^n$ (resp. on $\bigwedge^{n-1} \mathbb{R}^n$). For $\ell = 1$ take vectors $\mathbf{e}_2, \dots, \mathbf{e}_n$ and for $\ell = 2$ take vectors

$$\mathbf{f}_i = \mathbf{e}_1 \wedge \cdots \wedge \widehat{\mathbf{e}}_i \wedge \cdots \wedge \mathbf{e}_n, \quad i \geq 2$$

(where $\widehat{\mathbf{e}}_i$ means that $\mathbf{e}_i$ is omitted in this expression). Let

$$P_1 = \begin{pmatrix} * & 0 & 0 & \cdot & 0 \\ * & * & 0 & \cdot & 0 \\ * & 0 & * & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ * & 0 & 0 & \cdot & * \end{pmatrix}, \quad P_2 = \begin{pmatrix} * & * & * & \cdot & * \\ 0 & * & 0 & \cdot & 0 \\ 0 & 0 & * & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & * \end{pmatrix}.$$

Since each $\mathbf{e}_i$ (resp., $\mathbf{f}_i$) is an eigenvector for all elements of D, with corresponding character $\mathrm{diag}(\mathbf{d}_1, \dots, \mathbf{d}_n) \mapsto \mathbf{d}_i$ (resp., $\mathrm{diag}(\mathbf{d}_1, \dots, \mathbf{d}_n) \mapsto -\mathbf{d}_i$) the previous paragraph shows that (1) holds for both values of ℓ. Statements (2–6) are easy to verify. □

Corollary 4.15. *Retain the previous notation and let $d = 4$. Then all proper algebraic subgroups of D admit non-obvious divergent trajectories, except possibly the subgroup $\{\exp(\mathrm{diag}(s, -s, t, -t)) : s, t \in \mathbb{R}\}$ and its conjugates.*

5. Rates of escape

In this section we examine the possible rates of escape for divergent trajectories. We use the Khintchine-Cassels-Dani scheme to construct both rapidly and slowly escaping non-obvious divergent trajectories. In order to make the ideas more transparent, and since this is the most interesting case from the point of view of applications to number theory, we will first consider the space $Y = G/\Gamma$ where $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. That is, Y is the so-called **space of lattices** – a parametrizing space for all unit–volume cocompact discrete subgroups of $\mathbb{R}^n$. Further below we will generalize the results to a more general setup. Throughout this section $\{a(t) : t \in \mathbb{R}\}$ is a one-parameter subgroup of G and $A = \{a(t) : t \geq 0\}$.

5.1. **Measuring rates of escape on the space of lattices.** A natural measure of rate of escape of a trajectory is the growth rate of the geodesic distance of a point on the trajectory to some fixed basepoint. To make this more precise, fix some inner product on $T_eG \cong \mathrm{Lie}(G)$. Right transport of this inner product gives a right-invariant Riemannian metric on the tangent space to G, which descends to a well–defined Riemannian metric on G/Γ. Let $\mathrm{dist}_{G/\Gamma}(\cdot,\cdot)$ denote the associated metric on G/Γ. For a trajectory Ay and a point $y_0 \in G/\Gamma$ define

$$D_1(t) = \mathrm{dist}_{G/\Gamma}(a(t)y, y_0). \tag{29}$$

Clearly Ay is divergent if and only if $D_1(t) \to_{t\to+\infty} +\infty$.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to **grow linearly** if there are positive constants C_1 and C_2 and $t_0 > 0$ such that

$$C_1 \le \frac{f(t)}{t} \le C_2$$

for all $t \ge t_0$. We will say that a trajectory $A\pi(x)$ **diverges with linear speed** if $D_1(t)$ grows linearly. We will be interested in the question of whether there are non-obvious divergent trajectories which diverge with linear speed.

There is an alternative way to describe rates of divergence which enables us to make our results more precise. For the remainder of this subsection let $G = \mathrm{SL}(n,\mathbb{R})$, $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, and suppose that $A \subset D$.

We define, for $x \in G$:

$$\delta(x) = \inf_{0\ne v\in\mathbb{Z}^n} \|x\cdot v\|,$$

where $x \cdot v$ denotes the standard (given) action of G on $\mathbb{R}^n$, $\|\cdot\|$ is some norm on $\mathbb{R}^n$, and

$$D_2(t) = -\log(\delta(a(t)x)).$$

By Mahler's compactness criterion, $A\pi(x)$ is divergent if and only if $D_2(t) \to_{t\to+\infty} +\infty$. The following proposition shows that for studying trajectories which diverge with linear speed, it makes no difference whether we consider $D_1(t)$ or $D_2(t)$.

Proposition 5.1. *$D_1(t)$ grows linearly if and only if $D_2(t)$ grows linearly.*

This appears to be well-known, see e.g. [KlMa, p. 342]. It is a special case of the more general Proposition 5.7 below.

From now on we will measure rates of divergence with respect to D_2. The reason is that the growth of D_2 is simple to estimate for the obvious divergent trajectories.

Example. Suppose $a(t) = \mathrm{diag}(e^{\alpha_1 t}, \dots, e^{\alpha_n t})$, with $\alpha_1 \le \cdots \le \alpha_n$ (so $\alpha_1 < 0$), and suppose for $x \in G$ that $\mathbf{e}_1$ (the first vector of the standard basis of $\mathbb{R}^n$) is an eigenvector of x. Then $\|a(t)x\cdot\mathbf{e}_1\| = e^{\alpha_1 t}\|x\cdot\mathbf{e}_1\|$, so there is a constant $C > 0$ such that

$$D_2(t) \ge -\alpha_1 t + C.$$

Let us prove the opposite inequality. Suppose that along a subsequence $t_k \to +\infty$ we had $D_2(t_k)+\alpha_1 t_k \to +\infty$. Then there are vectors $v_k \in \mathbb{Z}^n \smallsetminus \{0\}$ such that

$$e^{-\alpha_1 t_k}\|a(t)x\cdot v_k\| \to_{k\to\infty} 0.$$

With no loss of generality we can replace the norm $\|\cdot\|$ with the sup-norm on $\mathbb{R}^n$ (with respect to the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_n$); then the norm of $a(-t)$ is $e^{-\alpha_1 t}$ for $t > 0$. Thus we obtain that $x \cdot v_k \to 0$, contradicting the discreteness of $x \cdot \mathbb{Z}^n$. This proves the opposite inequality, and we obtain that the difference between $D_2(t)$ and $-\alpha_1 t$ is bounded.

Let

$$a(t) = \mathrm{diag}(e^{\alpha_1 t}, \dots, e^{\alpha_n t}), \text{ where } \sum \alpha_i = 0.$$

By conjugation with a permutation matrix, let us assume with no loss of generality that

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n.$$

Let

$$c = -\frac{\alpha_1 + \alpha_2}{2}.$$

Note that $c > 0$ whenever $n \geq 3$.

The following is the main result of this section.

Theorem 5.2. *Suppose $n \geq 3$.*

(a) *For any monotonically increasing unbounded function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ there is a non–obvious divergent trajectory $A\pi(x)$ and t_0 such that for all $t \geq t_0$,*

$$D_2(t) \geq ct - \phi(t).$$

(b) *If $x \in G/\Gamma$ is such that $ct - D_2(t)$ is bounded from above then Ax is an obvious divergent trajectory.*

Taking $\phi(t) = c't$ for any $0 < c' < c$ and applying (a) we obtain:

Corollary 5.3. *There is $x \in G$ such that the trajectory $A\pi(x)$ is a non-obvious divergent trajectory which diverges with linear speed.*

Proof of Theorem 5.2: We will deduce part (a) from Theorems 2.4 and 2.8, using Remark 2.9. We first introduce some notation.

For $k = 1, 2$ let $W_k = \bigwedge^k \mathbb{R}^n$ and let $\varrho_k : G \to \mathrm{GL}(V)$ be the k-th exterior power of the standard (given) representation of G on $\mathbb{R}^n$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$ and let $\mathbf{e}_{ij} = \mathbf{e}_i \wedge \mathbf{e}_j$, $1 \leq i < j \leq n$ be the resulting basis of W_2.

Note that for $t > 0$, e^{-2ct} is the smallest of the eigenvalues for the action of $a(t)$ on W_2.

By replacing $\phi(t)$ if necessary with a function increasing at a slower rate, we may assume with no loss of generality that $t \mapsto ct - \phi(t)$ is monotonically increasing and unbounded. Let

$$K(t) = \pi\left(\{g \in G : \delta(g) \geq e^{-(ct - \phi(t))}\}\right).$$

It is immediate that $\{K(t) : t \geq 0\}$ is a rate of growth. It is also clear that $A\pi(x)$ is divergent with rate given by $\{K(t)\}$ if and only if there is t_0 such that $D_2(t) \geq ct - \phi(t)$ for all $t \geq t_0$.

Let

$$P_1 = \begin{pmatrix} * & * & * & \cdot & * \\ 0 & * & * & \cdot & * \\ 0 & * & * & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & * & * & \cdot & * \end{pmatrix} = \{g \in G : \varrho_1(g)\mathbf{e}_1 \in \mathbb{R}\mathbf{e}_1\},$$

$$P_2 = \begin{pmatrix} * & * & * & \cdot & * \\ * & * & * & \cdot & * \\ 0 & 0 & * & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & * & \cdot & * \end{pmatrix} = \{g \in G : \varrho_2(g)\mathbf{e}_{12} \in \mathbb{R}\mathbf{e}_{12}\}.$$

Let $X_1, X_2, \ldots$ be an enumeration of the distinct elements of $\{P_i g : g \in G(\mathbb{Q}),\ i = 1, 2\}$. Then it is easy to see, as in the proof of Theorem 3.9 that the hypotheses of density and transversality hold. Let us verify the hypothesis of local uniformity with respect to $\{K(t)\}$. Let $x \in X_i = P_j g$, where $j \in \{1, 2\}$ and $g \in G(\mathbb{Q})$. If $j = 1$ let $\mathbf{e} = \mathbf{e}_1$ and if $j = 2$ let $\mathbf{e} = \mathbf{e}_{12}$. Let $v = \varrho_j(g^{-1})\mathbf{e}$. Then $\varrho(g)v$ is a multiple of $\mathbf{e}$, and the line through $\mathbf{e}$ is left invariant by $\varrho_j(P_j)$. Since $\mathbf{e}$ is an eigenvector for $\varrho_j(a(t))$, with corresponding eigenvalue either $e^{\alpha_1 t} \leq e^{-ct}$ (in case $j = 1$) or e^{-ct} (in case $j = 2$), we have

$$\varrho_j(a(t)x)v \leq e^{-ct}\varrho(x)v.$$

Since $\phi(t) \to +\infty$, there is t_0 such that for all $t \geq t_0$,

$$\|\varrho_j(a(t)x)v\| < e^{-ct+\phi(t)}. \tag{30}$$

Repeating the argument which verified local uniformity in the proof of Theorem 3.9, we find that there is a neighborhood $\mathcal{U}$ of x and t_0 such that for all $t \geq t_0$ and all $z \in \mathcal{U} \cap X_i$ we have $a(t)\pi(z) \notin K(t)$, as required.

Now define $X_{\varrho,v}$ as in (12), with $h(t) = a(t)$, and let X'_j be an enumeration of all the distinct sets $X_{\varrho,v}$. Defining the level function $L(i,j)$ as in the proof of Theorem 3.9 and repeating the arguments given there we see that the hypotheses of transversality relative to $\{X'_j\}$ and density of level-increasing points hold. Thus all conditions of Theorems 2.4 and 2.5 hold, completing the proof of part (a).

We now prove (b). Let κ be such that $D_2(t) \geq ct - \kappa$. Since $D_2(t) \to +\infty$ the trajectory $A\pi(x)$ is divergent. We have

$$\delta(a(t)x) = e^{-D_2(t)} \leq e^{\kappa}e^{-ct}.$$

By the definition of δ, for each large enough t there is a nonzero vector $v = v(t) \in \mathbb{Z}^n$ such that

$$\|\varrho_1(a(t)x)v\| \leq \kappa_1 e^{-ct}.$$

Suppose the divergence is non–obvious. Then there is no fixed v_0 such that

$$\{t : v(t) = v_0\}$$

is unbounded. Hence there is an infinite sequence of distinct nonzero vectors $v_k \in \mathbb{Z}^n$ and an unbounded sequence $T_1 < T_2 < \cdots$ such that

$$t \in [T_k, T_{k+1}] \Longrightarrow \|\varrho_1(a(t)x)v_k\| \leq \kappa_1 e^{-ct}.$$

Replacing if necessary each v_k by a shorter primitive vector we get that all the v_k are primitive and hence for each k, v_k and v_{k+1} are linearly independent, and satisfy, for $t = T_{k+1}$:

$$\|\varrho_1(a(t)x)v_k\| \leq \kappa_1 e^{-ct}$$

and

$$\|\varrho_1(a(t)x)v_{k+1}\| \leq \kappa_1 e^{-ct}.$$

Therefore, for any norm on W_2 there is κ_2 such that

$$\|\varrho_2(a(t)x)v_k \wedge v_{k+1}\| \leq \kappa_2 e^{-2ct}. \tag{31}$$

We define a norm on W_2 by

$$\|\sum_{1 \leq i < j \leq n} a_{ij}\mathbf{e}_{ij}\| = \max |a_{ij}|.$$

Since $\mathbf{e}_{ij}$ are eigenvectors for the action of A on W_2, and since the minimal eigenvalue for the action of $a(t)$ on W_2 is e^{-2ct}, we obtain for every $w \in W_2$,

$$\|\varrho_2(a(t))w\| \geq e^{-2ct}\|w\|. \tag{32}$$

From (31) and (32) we obtain that for every k,

$$\|\varrho_2(x)v_k \wedge v_{k+1}\| \leq \kappa_2. \tag{33}$$

From the discreteness of $\{u \wedge v : u, v \in \mathbb{Z}^n\}$ we obtain that the set of elements $v = v_1 \wedge v_2 \in W_2$ satisfying (33) is finite, hence for some k there is an unbounded sequence $\{T_j\}$ for which

$$\|\varrho_2(a(T_j)x)v_k \wedge v_{k+1}\| \leq \kappa_2 e^{-2cT_j}.$$

Hence

$$\|\varrho_2(a(t)x)v_k \wedge v_{k+1}\| \to_{t \to +\infty} 0,$$

and $A\pi(x)$ is an obvious divergent trajectory. □

We now apply Theorem 2.5 to prove the existence of divergent trajectories which do not diverge too quickly.

Theorem 5.4. *For any rate of growth $\{K(t)\}$ there is $x \in G$ such that $A\pi(x)$ diverges but does not diverge with rate given by $\{K(t)\}$. In particular $A\pi(x)$ can be chosen to be a non-obvious divergent trajectory.*

Proof. We first explain why the second assertion follows from the first. We can change $\{K(t)\}$ to a slower rate of growth by defining $K_1(t) = K(\phi(t))$ for any monotonically increasing unbounded function $\phi(t)$, $\phi(t) < t$. Thus there is no loss of generality in assuming that a trajectory which does not diverge with rate given by $\{K(t)\}$ does not diverge with linear speed.

It is easily seen using Propositions 5.1, 5.7 that any obvious divergent trajectory diverges with linear speed. Hence the second assertion follows from the first.

Define $\{X_i\}$ as in the proof of Theorem 5.2. Let us show that the conditions of Theorem 2.5 hold. Since for any $x \in \bigcup X_i$, the trajectory $A\pi(x)$ diverges with linear speed, by making $\{K(t)\}$ slower in the previous paragraph we may assume that for any $x \in \bigcup X_i$, $A\pi(x)$ diverges with rate given by $\{K(t)\}$. We have verified above that the $\{X_i\}$ satisfy the conditions of Theorem 2.4. It is well-known that the action of $\{a(t)\}$ on G/Γ is ergodic w.r.t. the natural measure (see e.g. [St, §9]) hence topologically transitive.

It remains to verify the hypothesis of density of connected components. Let P_1, P_2 be as in the proof of Theorem 5.2. We claim that G is **boundedly generated by** P_1, P_2, that is, there is r such that every $g \in G$ can be written as $g = p_1 q_1 \cdots p_r q_r$ with $p_i \in P_1$, $q_j \in P_2$. Indeed, let B be a minimal $\mathbb{R}$-parabolic subgroup contained in $P_1 \cap P_2$ and write

$$G = \bigcup_{w \in \overline{W}} BwB \tag{34}$$

for the Bruhat decomposition (over $\mathbb{R}$) of G, see [BoTi1, §5], where w ranges over a set $\overline{W}$ of representatives of elements of the $\mathbb{R}$-Weyl group $W = N_G(D)/C_G(D)$. The finite set $\overline{W}$ is generated by elements of P_1, P_2 since P_1, P_2 are both maximal parabolic subgroups and hence generate G. Since W is finite, there is a bound r on the length of words required to express $\overline{W}$, and this bound is sufficient for all elements of G by (34).

Thus the r-fold multiplication map

$$\mu^r : (P_1 \times P_2)^r \to G, \quad \mu^r(p_1, q_1, \dots , p_r, q_r) = p_1 \cdots q_r$$

is onto. It is also an algebraic morphism, and hence real analytic. Thus for any neighborhoods $\mathcal{U}_1, \mathcal{U}_2$ of e in P_1, P_2 respectively, the image of $\mu^r|_{(\mathcal{U}_1 \times \mathcal{U}_2)^r}$ contains a neighborhood of e in G. Given a neighborhood $\mathcal{U}$ of $x \in G$ we take $\mathcal{U}_1, \mathcal{U}_2$ small enough connected neighborhoods of the identity so that

$$\mu^r((\mathcal{U}_1 \times \mathcal{U}_2)^r) \subset \mathcal{U}x^{-1}.$$

Write $P_i(\mathbb{Q}) = P_i \cap G(\mathbb{Q})$, $\mathcal{U}_i(\mathbb{Q}) = \mathcal{U}_i \cap G(\mathbb{Q})$. By Proposition 3.11, $\overline{P_i(\mathbb{Q})}$ is dense in P_i with respect to the Lie group topology. Therefore $\mathcal{U}_i = \overline{\mathcal{U}_i(\mathbb{Q})}$ for $i = 1, 2$, and hence the closure of

$$M = \mu^r((\mathcal{U}_1(\mathbb{Q}) \times \mathcal{U}_2(\mathbb{Q}))^r)$$

contains a neighborhood of the identity. On the other hand it is easily checked that any two points in Mx are connected by $\{X_i\}$ in $\mathcal{U}$. Therefore the density of connected components hypothesis is valid and the proof is complete. □

5.2. **Generalizations.** The results of the previous subsection may be generalized in several directions – e.g. general semisimple algebraic groups G, groups satisfying the condition $\mathrm{rank}_{\mathbb{Q}}G = \mathrm{rank}_{\mathbb{R}}G$, different ways of measuring rates of escape, and acting semigroups $\{h(t)\}$ which are non-quasi-unipotent. Note that Theorem 5.4 generalizes in all the directions above, all that was used in its proof was that $\mathrm{rank}_{\mathbb{Q}}G \geq 2$ and that G is boundedly generated by any two maximal parabolic subgroups. In order to generalize Theorem 5.2 and Corollary 5.3 we introduce the required terminology, and then sketch the necessary modifications to our argument.

5.2.1. *Siegel sets and rates of escape.* We now let G be a semisimple $\mathbb{Q}$-algebraic group and $\Gamma = G(\mathbb{Z})$. In order to generalize the results of the previous subsection, one needs a suitable way to measure the rate of escape, that is, find a replacement for the function δ used above. We use reduction theory, referring the reader to [Bo1] for additional details.

Let C be a finite subset of $G(\mathbb{Q})$ as in [Bo1, Thm. 13.1]. Let B be a minimal $\mathbb{Q}$-parabolic subgroup containing D, let $\varrho : G \to \mathrm{GL}(V)$ be a $\mathbb{Q}$-irreducible representation defined over $\mathbb{Q}$ and $0 \neq v \in V(\mathbb{Q})$ such that $\varrho(B)$ leaves invariant the line $\mathbb{R} \cdot v$ (in the terminology of [BoTi1], ϱ is 'strongly rational'). This means that there is $\bar{\chi} \in \Lambda(\varrho)$ such that for all $d = \exp(X) \in D$, $\varrho(d)v = e^{\bar{\chi}(X)}v$. We denote the restriction of $\bar{\chi}$ to S by χ, and call (ϱ, v) satisfying these hypotheses a **coördinate pair**, with $\bar{\chi}$ and χ the **weights associated to** v. Fix some norm on V and define

$$\widetilde{\delta}(g) = \widetilde{\delta}_{\varrho,v}(g) = \min_{\gamma \in \Gamma \cdot C} \|\varrho(g\gamma)v\|.$$

Note that δ is obtained as a special case by taking for ϱ the standard (given) representation of $G = \mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}^n$ and $v = \mathbf{e}_1$, and taking $C = \{e\}$. Note also that functions such as $g \mapsto \|\varrho(g)v\|$ are described in [Bo1, §14].

We have the following generalization of Mahler's compactness criterion:

Proposition 5.5. *Let (ϱ, v) be a coördinate pair and let $X \subset G$. Then $\pi(X) \subset G/\Gamma$ is precompact if and only if $\inf_{x \in X} \widetilde{\delta}_{\varrho,v}(x) > 0$.*

Let $\lambda_1, \ldots, \lambda_r$ be a set of simple $\mathbb{Q}$-roots, let $\bar{\lambda}, \ldots, \bar{\lambda}_t$ be a set of simple $\mathbb{R}$-roots for a compatible order, and for $\tau \in \mathbb{R}$ let

$$S_\tau = \{s \in S : \forall i,\ \lambda_i(s) \leq \tau\}.$$

We will need the following:

Lemma 5.6. *Suppose (ϱ, v) is a coördinate pair, with $\bar{\chi}, \chi$ the weights associated to v. Then $\chi = \sum a_i \lambda_i$ where $a_i > 0$ for all i, and $\bar{\chi} = \sum b_j \bar{\lambda}_j$, where $b_j > 0$ for all j. In particular, for each $\tau, \eta \in \mathbb{R}$, the set*

$$\{s \in S_\tau : \chi(s) \geq \eta\}$$

is compact.

Proof. It is well-known (see e.g. [Hu, §10]) that any dominant weight is a linear combination of the simple roots with all coefficients positive. This proves the first two assertions. The third assertion follows easily from the first. □

Proof of Proposition 5.5. The implication ⇒ is immediate from discreteness of $\varrho(\Gamma \cdot C)v$, see the proof of Proposition 3.1.

For the converse, suppose $x_n \in X$ and $\{\pi(x_n)\} \subset G/\Gamma$ has no convergent subsequence. Passing to a subsequence and applying [Bo1, Thm. 13.1, Lem. 12.2], we may write $x_n = k_n s_n c \gamma_n$ where k_n belongs to a compact subset of G, $\{s_n\} \subset S_\tau$ has no convergent subsequence, $c \in C$ and $\gamma_n \in \Gamma$. It follows from Lemma 5.6 that $\chi(s_n) \to -\infty$. Therefore, letting $u_n = \varrho(\gamma_n^{-1} c)v$ we have

$$\widetilde{\delta}(x_n) \leq \|\varrho(x_n)u_n\| = \|\varrho(k_n s_n)v\| \to 0,$$

so $\inf_{x \in X} \widetilde{\delta}(x) = 0$. □

We define

$$\widetilde{D}(t) = \widetilde{D}_{\varrho,v}(t) = -\log(\widetilde{\delta}_{\varrho,v}(a(t)x)).$$

In view of Proposition 5.5, $A\pi(x)$ is divergent if and only if $\widetilde{D}(t) \to +\infty$.

Let $D_1(t)$ be as in (29). We have the following generalization of Proposition 5.1:

Proposition 5.7. *Let (ϱ, v) be a coördinate pair. Then $D_1(t)$ grows linearly if and only if $\widetilde{D}_{\varrho,v}(t)$ grows linearly.*

Proof. Let M be any noncompact set. For functions $f, g : M \to \mathbb{R}$ we will write $f \simeq g$ if there is a compact subset $M_0 \subset M$ and a positive constant C such that

$$m \in M \smallsetminus M_0 \implies \frac{1}{C} \leq \frac{f(m)}{g(m)} \leq C.$$

For each t, using [Bo1, Thm. 13.1], let $a(t)x = k(t)s(t)\gamma(t)$, where $k(t)$ belongs to a compact subset of G, $s(t) \in S$, $\gamma(t) \in C \cdot \Gamma$. Let $\mathrm{dist}_G(x, y)$ denote the distance in G of two points with respect to some right-invariant Riemannian metric on G. Let χ be the weight on S corresponding to v. We have seen in Lemma 5.6 that $\chi = \sum a_i \lambda$, with $a_i > 0$ for all i.

The proof consists of three steps:

a. $D_1(t) \simeq \mathrm{dist}_G(s(t))$.
b. $\mathrm{dist}_G(s(t)) \simeq -\log \chi(s(t))$.
c. $-\log \chi(s(t)) \simeq \widetilde{D}_{\varrho,v}(t)$.

Part a. follows from work of Siegel (see [Ab] for a discussion of this and more delicate questions). Part b. can be deduced from [Ab, §6]. Part c. can be obtained by direct computation using the fact that $k(t)$ is bounded in G. □

5.2.2. *A generalization of Theorem 5.2.* Retain the terminology of the previous subsection. Let $\{h(t) : t \in \mathbb{R}\}$ be a one-parameter non-quasi-unipotent subgroup and let $a(t)$ be the diagonalizable component of the Jordan decomposition of $h(t)$ (see Proposition 3.7). Applying a conjugation, assume $\{a(t)\} \subset D$. By a further conjugation, assume $\{a(t)\}$ is in the closed Weyl chamber determined by the choice of some minimal $\mathbb{R}$-parabolic subgroup whose opposing parabolic is contained in B. In particular, using the second assertion in Lemma 5.6, this implies that $\varrho(a(t))v \to 0$.

Let (ϱ, v) be a coördinate pair, and for a subspace $V' \subset V$ let $V'(\mathbb{Q})$ be the rational vectors in V'. Let $a = a(1)$,

$$\Lambda = \Lambda_\varrho, \ \ \Lambda(a) = \{\chi(a) : \chi \in \Lambda\}$$
$$\Delta = \{\lambda \in \Lambda_\varrho : V_\lambda(\mathbb{Q}) \neq \{0\}\}, \ \ \Delta(a) = \{\lambda(a) : \lambda \in \Delta\}.$$

We list the elements of $\Lambda(a)$ (resp. $\Delta(a)$) by size, with multiplicity. That is, we write $\Lambda(a) = \{\alpha_1, \dots, \alpha_s\}, \ \Delta(a) = \{\beta_1, \dots, \beta_t\}$ with

$$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_s \ \text{ and } \ \#\{i : \alpha_i = b\} = \sum_{\chi(a)=b} \dim V_\chi$$
$$\beta_1 \le \beta_2 \le \cdots \le \beta_t \ \text{ and } \ \#\{i : \beta_i = b\} = \sum_{\lambda(a)=b} \dim V_\lambda(\mathbb{Q}).$$

Now let

$$c_1 = -\frac{\alpha_1 + \alpha_2}{2}, \quad c_2 = -\frac{\beta_1 + \beta_2}{2}. \tag{35}$$

We have:

Proposition 5.8. *$c_1 \ge c_2$ and $c_1 = c_2$ if* $\mathrm{rank}_{\mathbb{Q}} G = \mathrm{rank}_{\mathbb{R}} G$.
If G is almost $\mathbb{Q}$-simple and $\mathrm{rank}_{\mathbb{Q}} G \ge 2$ *then $c_2 > 0$.*

Proof. The first assertion follows immediately from the fact that $\Delta \subset \Lambda$ with $\Delta = \Lambda$ when $\mathrm{rank}_{\mathbb{Q}} G = \mathrm{rank}_{\mathbb{R}} G$.

Since $\{a(t)\}$ is in the closed Weyl chamber with respect to an order for which χ is dominant, $\chi(a) = \alpha_1 = \beta_1$. Let $W = N_G(S)/C_G(S)$, the $\mathbb{Q}$-Weyl group of G. By [Bo2, 21.4], $N_G(D) \cap G(\mathbb{Q})$ contains representatives for W, and it is clear that $N_G(D) \cap G(\mathbb{Q})$ preserves Δ. W acts on S and hence also on $\mathrm{Lie}(S)$ and $\mathrm{Lie}(S)^*$. Since G is almost $\mathbb{Q}$-simple, $\Phi_{\mathbb{Q}}$ is an irreducible root system and hence W acts irreducibly on $\mathrm{Lie}(S)^*$. By considering the kernel of $\sum_{w \in W} w\beta$ for any $\beta \in X(S)$ we obtain $\sum_{w \in W} w\beta = 0$. Therefore $\sum_{i=1}^t \beta_i = 0$, so to prove the second assertion it suffices to show that $t \ge 3$, and this will follow from the inequality $\#W\bar{\chi} \ge 3$.

By the above $\sum_{w \in W} w\bar{\chi} = 0$ and by irreducibility, $\mathrm{span} W\bar{\chi} = \mathrm{Lie}(S)^*$. This means that $0 \in \mathrm{int\,conv} W\bar{\chi}$ and hence $\#W\bar{\chi} \ge \dim S + 1 \ge 3$, proving the claim. □

Theorem 5.9. *Preserve the above notation, and assume G is almost $\mathbb{Q}$-simple. Let $u(t)$ be the unipotent part of $h(t)$. We have:*

(a) *Suppose B is a maximal $\mathbb{Q}$-parabolic and $u(t)$ is trivial. Then for any monotonically increasing unbounded function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ there is a non–obvious divergent trajectory $A\pi(x)$ and t_0 such that for all $t \geq t_0$,*

$$\widetilde{D}_{\varrho,v}(t) \geq c_2 t - \phi(t).$$

(b) *Suppose that $u(t)$ is trivial. If $x \in G/\Gamma$ is such that $c_1 t - \widetilde{D}_{\varrho,v}(t)$ is bounded from above then Ax is an obvious divergent trajectory.*

(c) *Suppose B is a maximal $\mathbb{Q}$-parabolic and $c < c_2$. Then there is a non-obvious divergent trajectory $A\pi(x)$ and t_0 such that for all $t \geq t_0$,*

$$\widetilde{D}_{\varrho,v}(t) \geq ct.$$

Sketch of proof. We repeat the proof of Theorem 5.2, with minor modifications.

For part (a), let $\varrho_1 = \varrho$, $P_1 = B$, $\varrho_2 = \bigwedge^2 \varrho_1$. Let $v' \in V(\mathbb{Q})$ be an eigenvector for the action of $\varrho_1(D)$ such that $\varrho_1(a)v' = e^{\beta_2}v'$. Let

$$P_2 = \{g \in G : \varrho_2(g)v \wedge v' = \mathbb{R}v \wedge v'\}.$$

Then P_2 is defined over $\mathbb{Q}$, contains D, and is not contained in P_1. Since P_1 is a maximal $\mathbb{Q}$-parabolic subgroup, P_1 and P_2 generate G. This is all that is required for the arguments used in proving Theorem 5.2, and we obtain (a).

For part (b), repeat the argument of Theorem 5.2. Here it is important that c_1 be the smallest eigenvalue for $\varrho_2(a)$, which is guaranteed by our assumption that $\{u(t)\}$ is trivial.

For part (c), note that the unipotent part $u(t)$ only changes norms by an amount which is polynomial in t. Hence, using the rate of growth

$$K(t) = \{g \in G : \widetilde{\delta}_{\varrho,v}(g) \geq e^{-ct}\}.$$

the proof of Theorem 5.2, part (a) still works. □

Corollary 5.10. *Suppose G is semisimple and almost $\mathbb{Q}$-simple with* $\mathrm{rank}_{\mathbb{R}} G = \mathrm{rank}_{\mathbb{Q}} G \geq 2$, $\Gamma = G(\mathbb{Z})$, $\{a(t) : t \in \mathbb{R}\}$ *a one-parameter $\mathbb{R}$-diagonalizable subgroup, $A = \{a(t) : t \geq 0\}$, (ϱ, v) a coördinate pair with B a maximal $\mathbb{Q}$-parabolic. Let $c = c_1 = c_2$ be as in (35). Then*

(a) *For any monotonically increasing unbounded function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ there is a non–obvious divergent trajectory $A\pi(x)$ and t_0 such that for all $t \geq t_0$,*

$$\widetilde{D}_{\varrho,v}(t) \geq ct - \phi(t).$$

(b) *If $x \in G/\Gamma$ is such that $ct - \widetilde{D}_{\varrho,v}(t)$ is bounded from above then Ax is an obvious divergent trajectory.*

6. Quadratic differential spaces

6.1. **Obvious divergence.** In this section we discuss divergent trajectories for the Teichmüller geodesic flow, which is a flow on the moduli space of unit area quadratic differentials on a surface of finite type S. We first briefly introduce terminology and notation. For more details and references to the literature, the reader is referred to [MasTa] and [MiWe, §4].

Let S be an orientable surface of genus $g \geq 2$, so that S admits a hyperbolic structure. Let $\widetilde{\mathcal{Q}}$ be the space of quadratic differentials on S and let $\widetilde{\mathcal{Q}}_1$ be the subspace of unit-area quadratic differentials. Both are bundles over $\mathrm{Teich}(S)$ (the space of complex structure on S) and the latter is naturally identified with the unit co-tangent bundle of $\mathrm{Teich}(S)$ and in particular is a fiber bundle over $\mathrm{Teich}(S)$ with compact fiber. Let $\mathrm{Mod}(S)$ be the mapping class group, let $\mathcal{Q}_1 = \widetilde{\mathcal{Q}}_1/\mathrm{Mod}(S)$ the moduli space of quadratic differentials, and let $\pi : \widetilde{\mathcal{Q}}_1 \to \mathcal{Q}_1$ denote the quotient map. There is a structure of a manifold (resp. orbifold) on $\widetilde{\mathcal{Q}}_1$ (resp., on $\mathcal{Q}_1$), of dimension $12g - 13$. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\widetilde{\mathcal{Q}}_1$, and the action descends to a well-defined action on $\mathcal{Q}_1$, admitting a finite smooth invariant measure. The action of matrices

$$g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

is called the **Teichmüller geodesic flow.**

The space $\mathcal{Q}_1$ is noncompact. Let Γ^S be the set of nontrivial free homotopy classes of unoriented non-peripheral simple closed curves on S, and for every $\gamma \in \Gamma^S$, and every $q \in \widetilde{\mathcal{Q}}_1$, let $l_{q,\gamma}$ denote the length of a minimal representative of γ with respect to the flat metric defined by q. For any $q \in \widetilde{\mathcal{Q}}_1$, the set

$$\{l_{q,\gamma} : \gamma \in \Gamma^S\}$$

is discrete and hence attains a minimum, which we denote by $l_{\min}(q)$.

The following is an analogue of Mahler's compactness criterion:

Proposition 6.1 (Compactness Criterion)**.** *Let $X \subset \widetilde{\mathcal{Q}}_1$. Then*

$$\overline{\pi(X)} \subset \mathcal{Q}_1 \text{ is compact} \quad \Longleftrightarrow \quad \inf\{l_{\min}(q) : q \in X\} > 0.$$

In particular, for $q \in \widetilde{\mathcal{Q}}_1$, the trajectory $\{g_t\pi(q) : t \geq 0\}$ is divergent if and only if $l_{\min}(g_t q) \to_{t\to+\infty} 0$.

Every $q \in \widetilde{\mathcal{Q}}_1$ determines a finite set of singularities $\Sigma = \Sigma(q)$ and a pair of transverse measured foliations on $S \smallsetminus \Sigma$, called the **horizontal and vertical foliations of** q. For each q there is a natural identification of $\Sigma(q)$ with $\Sigma(gq)$ for any $g \in \mathrm{SL}(2,\mathbb{R})$. For any t, the horizontal and vertical foliations for $g_t q$ are topologically the same as those for q, but the measure transverse to the horizontal (resp. vertical) leaves is multiplied by $e^{-t/2}$ (resp., by $e^{t/2}$). For any x_1, $x_2 \in \Sigma(q)$ (we allow $x_1 = x_2$) and any segment δ in $S \smallsetminus \Sigma$ connecting x_1 and x_2, which is a straight segment with respect to the Euclidean structure determined by q, the integrals of the measure transverse to the vertical (resp. horizontal) foliation along δ give a vector

$$u(\delta, q) = \begin{pmatrix} x(\delta, q) \\ y(\delta, q) \end{pmatrix},$$

well-defined up to sign.

It follows from the above that for all t,

$$u(\delta, g_t q) = \begin{pmatrix} e^{t/2} x(\delta, q) \\ e^{-t/2} y(\delta, q) \end{pmatrix}. \tag{36}$$

Each $\gamma \in \Gamma^S$ has a shortest representative, with respect to the flat metric corresponding to q, consisting of finitely many line segments $\delta_1, \ldots, \delta_r$ joined end to end. A shortest representative is not unique but can only change by homotopy through a metric cylinder. A representative of γ which is linear on $S \smallsetminus \Sigma(q)$, and for which the difference between incoming and outgoing angle at each visit to $\Sigma(q)$ is at least π, must be shortest. This description implies that for any $g \in \mathrm{SL}(2, \mathbb{R})$, the concatenation of the δ_i is also a shortest representative for γ with respect to gq. We then have $l_{q,\gamma} = \sum_1^r \|u(\delta_i, q)\|$. In particular, if each of the δ_i's is contained entirely in leaves of the vertical foliation of q, then

$$\begin{aligned} l_{\min}(g_t q) &\leq l_{g_t q, \gamma} \\ &= \sum_1^r \|u(\delta_i, g_t q)\| \\ &= e^{-t/2} \sum_1^r l_{q, \delta_i} \longrightarrow_{t \to +\infty} 0. \end{aligned}$$

Using Proposition 6.1 we obtain:

Proposition 6.2. *Suppose $q \in \widetilde{\mathcal{Q}}_1$ and suppose there is an element of Γ^S with a representative consisting of line segments contained entirely in leaves of the vertical foliation of q. Then $\{g_t \pi(q) : t \geq 0\}$ is divergent.*

Similarly, if there is an element of Γ^S with a representative consisting of line segments contained entirely in leaves of the horizontal foliation of q, then $\{g_t \pi(q) : t \leq 0\}$ is divergent.

Definition 6.3. *The trajectory $\{g_t \pi(q)\}$ is called an* **obvious divergent trajectory** *if the hypothesis of Proposition 6.2 holds.*

6.2. **Remarks.** The **Teichmüller horocycle flow** is obtained by applying the one-parameter subgroup of upper-triangular unipotent matrices in $\mathrm{SL}(2, \mathbb{R})$, and the **circle flow** is obtained by applying the one-parameter subgroup $\{r_\theta : \theta \in \mathbb{R}\}$ of rotation matrices in $\mathrm{SL}(2, \mathbb{R})$. It was proved by Veech [Ve] (see also [MiWe]) that there are no divergent trajectories for the Teichmüller horocycle flow, and the same is true for the disc flow since all orbits for this flow are periodic. Note that every one-parameter subgroup of $\mathrm{SL}(2, \mathbb{R})$ is conjugate to one of the three subgroups above and that any two-dimensional subgroup contains a one-parameter unipotent subgroup; therefore none of the flows arising from the $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{Q}_1$, except the Teichmüller geodesic flow, admit divergent trajectories.

We will see below that it is easy to construct obvious divergent trajectories for the Teichmüller geodesic flow. Non-obvious divergent trajectories (which have been called **spiraling** trajectories) also exist. This is a consequence of work of Masur [Mas1], who shows that there are quadratic differentials q for which the set of θ for which $r_\theta q$ is divergent has positive Hausdorff dimension. Since the set of θ for which $\{g_t r_\theta \pi(q)\}$ is an obvious divergent trajectory is countable, the result follows.

Much of the interest in divergent trajectories for the Teichmüller geodesic flow is due to their connection with minimal non-uniquely ergodic interval exchange transformations and rational billiards. See [Mas2] and [MasTa] for surveys.

The terminology of §6.1, and the results of §6.3 – 6.4, generalize to the case that S has n punctures and $3g-3+n \geq 2$. We have chosen to omit this case as it requires some additional arguments in the proofs. If $3g-3+n=1$, that is for the punctured torus or the sphere with four punctures, the Teichmüller geodesic flow on $\mathcal{Q}_1$ coincides with the action of the diagonal subgroup on $\mathrm{SL}(2,\mathbb{R})/\Gamma$, where Γ is a non-uniform lattice, and in these cases, as remarked in §4, only obvious divergent trajectories exist.

6.3. **Rates of escape.** There are a number of different ways to define rates of escape of divergent trajectories on quadratic differential spaces. In order to obtain precise results, we make a definition compatible with Proposition 6.1. That is, we define

$$D(t) = -\log(l_{\min}(g_t q)).$$

It follows from Proposition 6.1 that $\{g_t\pi(q) : t \geq 0\}$ is divergent if and only if $D(t) \to_{t\to+\infty} +\infty$.

The main result of this section follows:

Theorem 6.4. (a) *For any monotonically increasing unbounded function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ there is a non–obvious divergent trajectory $\{g_t\pi(q) : t \geq 0\}$ and t_0 such that for all $t \geq t_0$,*

$$D(t) \geq t/2 - \phi(t).$$

(b) *If $q \in \widetilde{\mathcal{Q}}_1$ is such that $t/2 - D(t)$ is bounded from above then $\{g_t\pi(q)\}$ is an obvious divergent trajectory.*

We first introduce some terminology which will be used in the proof. We again refer the reader to [MiWe, §4] for definitions and references. Let $\mathcal{ML}(S)$ be the space of measured laminations on S, and let $\mathcal{PML}(S)$ be its projectivization. Recall that $\mathcal{ML}(S)$ is the space of measured geodesic laminations on S with respect to some (any) complete hyperbolic structure on S. *We identify* $\mathcal{PML}(S)$ *with a subset of* $\mathcal{ML}(S)$ as follows. Fix σ_0, a hyperbolic structure on S, and identify $\mathcal{PML}(S)$ with the set of $\lambda \in \mathcal{ML}(S)$ for which $\ell(\lambda, \sigma_0) = 1$, where ℓ denotes the length. This amounts to choosing a section to the map $\mathcal{ML}(S) \to \mathcal{PML}(S)$. When there is no risk of confusion, λ will also denote the underlying topological lamination, and for $\gamma \in \Gamma^S$, γ will also denote the corresponding measured lamination.

Let $\mathcal{MF}(S)$ denote the space of measured foliations on S and let $\mathcal{PMF}(S)$ denote its projectivization. Hubbard and Masur showed that $\widetilde{\mathcal{Q}}$ can be identified with $\mathrm{Teich}(S) \times \mathcal{MF}(S)$. The projection $\widetilde{\mathcal{Q}}_1 \to \mathcal{PMF}(S)$ maps q to the equivalence class of its vertical foliation. Note that in the original Hubbard-Masur construction, q is mapped to its horizontal foliation but this is merely a convention. There is "leaf-straightening" map $\tau : \mathcal{MF}(S) \to \mathcal{ML}(S)$ which was defined and shown to be a homeomorphism by Thurston. The composition gives a homeomorphism $I : \widetilde{\mathcal{Q}} \to \mathrm{Teich}(S) \times \mathcal{ML}(S)$, which intertwines the $\mathrm{Mod}(S)$-action on $\widetilde{\mathcal{Q}}$ with the product of the natural $\mathrm{Mod}(S)$-actions on each factor. If $t > 0$ and $q \in \widetilde{\mathcal{Q}}$ then $I(q)$ and $I(tq)$ differ by multiplication by t, and hence $I(\widetilde{\mathcal{Q}}_1)$ is identified with $\mathrm{Teich}(S) \times \mathcal{PML}(S)$.

Let $i : \mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R}$ denote the geometric intersection number. Via the section chosen above we can and will write $i : \mathcal{PML}(S) \times \mathcal{PML}(S) \to \mathbb{R}$. Although the number $i(\lambda_1, \lambda_2)$, $\lambda_i \in \mathcal{PML}(S)$ depends on the section, the condition $i(\lambda_1, \lambda_2) = 0$ is well-defined. For each $\gamma \in \Gamma^S$, let $X_\gamma \subset \widetilde{\mathcal{Q}}_1$ denote the set of all q for which γ has a representative contained in the vertical foliation corresponding to q. We use the maps of Hubbard-Masur and Thurston defined above to describe X_γ.

Lemma 6.5. *For each* $\gamma \in \Gamma^S$,

$$I(X_\gamma) = \mathrm{Teich}(S) \times \{\lambda \in \mathcal{PML}(S) : i(\lambda, \gamma) = 0\}.$$

In particular, each X_γ *is a submanifold of* $\widetilde{\mathcal{Q}}_1$*, with boundary, of codimension* 1.

Proof. Since the condition $q \in X_\gamma$ depends only on the vertical foliation $\mathcal{F}$ determined by q, we need only prove that γ has a representative contained in the leaves of $\mathcal{F} \in \mathcal{MF}(S)$ if and only if $i(\tau(\mathcal{F}), \gamma) = 0$. The latter condition is equivalent to the assertion that γ is either disjoint from, or contained in, $\mathrm{supp}\, \tau(\mathcal{F})$.

A leaf of $\mathcal{F}$ is called *regular* if it does not pass through the singularity set Σ and *singular* otherwise. From our description of length-minimizing curves it follows that if γ has a representative in $\mathcal{F}$ then it has a representative which is contained in a singular leaf. In [Lev], the leaves of $\tau(\mathcal{F})$ are explicitly described, and it is shown that two leaves in $\mathcal{F}$ intersect essentially if and only if the corresponding leaves in $\mathcal{F}$ do. Moreover the leaves of $\tau(\mathcal{F})$ correspond to either regular leaves of $\mathcal{F}$, or singular leaves which may be homotoped off of the singular leaves and in particular have no essential intersection with any singular leaves. Thus γ has a representative contained in the singular leaves of $\mathcal{F}$ if and only if the corresponding geodesic on S is either contained in $\tau(\mathcal{F})$ or does not essentially intersect any of the geodesics in $\tau(\mathcal{F})$. The first assertion of the Lemma follows.

For the second assertion, let $S' = S \smallsetminus \gamma$. Then S' is a surface with boundary (possibly disconnected), and X_γ is homeomorphic to $\mathcal{ML}(S') \times [0, \infty)$. The homeomorphism is defined by sending λ to (λ', c), where λ' is the restriction of λ to S', and c is the weight on γ. The dimension of the Teichmüller space of a surface of genus g with n boundary components is known to be $6g - 6 + 2n$, so a dimension count completes the proof of the assertion. $\square$

Proof of Theorem 6.4. We will deduce part (a) from Theorem 2.4. By replacing $\phi(t)$ if necessary with a function increasing at a slower rate, we may assume with no loss of generality that $t \mapsto t/2 - \phi(t)$ is monotonically increasing and unbounded. Let

$$K(t) = \pi \left(\{ q \in \widetilde{\mathcal{Q}}_1 : l_{\min}(q) \geq e^{-(t/2 - \phi(t))} \} \right).$$

It is immediate that $\{K(t) : t \geq 0\}$ is a rate of growth. It is also clear that $\{g_t \pi(q) : t \geq 0\}$ is divergent with rate given by $\{K(t)\}$ if and only if there is t_0 such that $D(t) \geq t/2 - \phi(t)$ for all $t \geq t_0$.

Let $X_1, X_2, \ldots$ be an enumeration of the sets $\{X_\gamma : \gamma \in \Gamma^S\}$. We verify the hypotheses of Theorem 2.4:

- **Density.** Let $X_i = X_\gamma$ for $\gamma \in \Gamma^S$ and let

  $$\widetilde{X}_\gamma = \{\lambda \in \mathcal{PML}(S) : i(\lambda, \gamma) = 0\}.$$

  By Lemma 6.5 it is enough to prove that

  $$\bigcup_{\eta \neq \gamma} \widetilde{X}_\gamma \cap \widetilde{X}_\eta$$

  is dense in $\widetilde{X}_\gamma$. Let $S' = S \smallsetminus \gamma$. Let $\lambda_0 \in \widetilde{X}_\gamma$, and recall that by the section we have chosen, we have $\lambda_0 \in \mathcal{ML}(S)$. Let $\lambda = \lambda_0|_{S'}$ and let c be the weight of γ. Note that λ can be thought of as a measured geodesic lamination on S'. Since the maximal number of disjoint simple closed curves on S is $3g - 3 + n$, which by assumption is at least 3, there are simple closed curves on S', so $\mathcal{ML}(S')$ is nontrivial. Weighted simple closed curves are dense in $\mathcal{ML}(S')$, and any weighted simple closed curve η' on S' can be transformed into $\eta \in \widetilde{X}_\gamma \cap \widetilde{X}_{\eta'}$, by assigning the transverse measure given by η' to paths in $S \smallsetminus \gamma$ and assigning weight c to γ. Thus λ_0 can be approximated arbitrarily well by laminations in $\widetilde{X}_\gamma \cap \widetilde{X}_{\eta'}$, as required.

- **Transversality.** If γ, γ' are disjoint then $\widetilde{X}_\gamma \cap \widetilde{X}_{\gamma'} = \{\lambda \in \mathcal{PML}(S) : i(\lambda, \gamma) = i(\lambda, \gamma') = 0\}$ — a codimension one submanifold of $\widetilde{X}_\gamma$. Hence $X_\gamma \cap X_{\gamma'}$ is a codimension one submanifold of X_γ.

  Suppose γ and γ' intersect, and suppose that $\lambda \in X_\gamma$. If γ is contained in the support of λ then $\lambda \notin X_{\gamma'}$, and if γ is not contained in the support of λ then λ may be perturbed slightly to a lamination supported on $\mathrm{supp}\,\lambda \cup \gamma$, by adding a small weight to γ. This gives laminations in $X_\gamma \smallsetminus X_{\gamma'}$ arbitrarily close to λ, proving the assertion.

- **Local uniformity with respect to $\{K(t)\}$.** Let $q \in X_\gamma$. By continuity of the length function, for all q' in a sufficiently small neighborhood $\mathcal{U}$ of q, we have
$$l_{q',\gamma} < 2l_{q,\gamma}.$$
The calculation given in the proof of Proposition 6.2 shows that for all $q' \in X_\gamma$,
$$l_{g_tq',\gamma} = e^{-t/2}l_{q',\gamma}.$$
Since $\phi(t) \to +\infty$ there is therefore t_0 be large enough so that for all $t \geq t_0$,
$$l_{g_tq,\gamma} = e^{-t/2}l_{q,\gamma} \leq e^{-(t/2-\phi(t))}/2,$$
hence for all $q' \in \mathcal{U} \cap X_\gamma$ we have
$$l_{\min}(g_tq') \leq l_{g_tq',\gamma} \leq e^{-(t/2-\phi(t))}.$$

We now prove (b). Let $q \in \widetilde{\mathcal{Q}}_1$ and let κ be such that $D(t) > t/2 - \kappa$ for all t. Then $D(t) \to +\infty$ so $\{g(t)\pi(q)\}$ is a divergent trajectory.

We have
$$l_{\min}(g_tq) = e^{-D(t)} < e^{\kappa}e^{-t/2},$$
so for each $t > 0$ there is $\gamma = \gamma(t) \in \Gamma^S$ such that
$$l_{g_tq,\gamma} < \kappa_1 e^{-t/2}.$$

Suppose the divergence is non–obvious. For each $\gamma \in \Gamma^S$ take a finite concatenation of line segments $\delta_i = \delta_i(\gamma),\ i = 1, \ldots, r(\gamma)$ which form a shortest representative of γ with respect to q. This concatenation forms a shortest representative of γ with respect to g_tq for all t. Since the divergence is non-obvious, at least one of the δ_i is not contained in the leaves of the vertical foliation, and hence $l_{g_tq,\gamma} \to_{t\to+\infty} +\infty$. Therefore there is an infinite sequence of distinct $\gamma_k \in \Gamma^S$ and a sequence $t_1, t_2, \ldots$ such that
$$l_{g_{t_k}q,\gamma_k} = \sum_{i=1}^{r(\gamma_k)} l_{g_{t_k}q,\delta_i(\gamma_k)} < \kappa_1 e^{-t/2}.$$

Applying the element g_{-t_k} and using (36) we obtain that for each i
$$l_{q,\delta_i} \leq \sqrt{2}\max\left(e^{t_k/2}|x(\delta_i, g_{t_k}q)|,\ e^{-t_k/2}|y(\delta_i, g_{t_k}q)|\right) \leq \sqrt{2}e^{t_k/2}l_{g_{t_k}q,\delta_i}.$$

Hence
$$l_{q,\gamma_k} = \sum_{i=1}^{r} l_{q,\delta_i} \leq \sqrt{2}e^{t_k/2}\sum_{i=1}^{r} l_{g_{t_k}q,\delta_i} < \sqrt{2}\kappa_1.$$
This contradicts the discreteness of
$$\{l_{q,\gamma} : \gamma \in \Gamma^S\}.$$

$\square$

6.4. **Pinching several disjoint geodesics.** In this section we produce divergent trajectories in which more than one geodesic is being pinched. Say that two elements of Γ^S are **disjoint** if they have disjoint representatives. Equivalently, their shortest representatives with respect to some (any) hyperbolic structure σ_0 are disjoint. For $r \leq 3g-3$, let $\mathcal{E}_r$ denote the collection of sets of r disjoint elements of Γ^S. For $M \in \mathcal{E}_r$, let

$$l^r_{q,M} = \max_{\gamma \in M} l_{q,\gamma}, \qquad l^r_{\min}(q) = \min_{M \in \mathcal{E}_r} l^r_{q,M}.$$

For a trajectory $\{g_t\pi(q) : t \geq 0\}$, let

$$D^r(t) = -\log(l^r_{\min}(g_t q)).$$

Let us say that $\{g_t\pi(q)\}$ is an r-**divergent trajectory** if $D^r(t) \to +\infty$.

Thus $\{g_t\pi(q)\}$ is r-divergent if and only if for any $\varepsilon > 0$ and any large enough t there are r disjoint curves on S of length less than ε. The case $r = 1$ was discussed in the previous section, namely $l_{q,\gamma} = l^1_{q,\{\gamma\}}$, $l^1_{\min}(q) = l_{\min}(q)$ and $D^1(t) = D(t)$. In particular an r-divergent trajectory is divergent.

Repeating the computation of Proposition 6.2 we obtain that if there are r disjoint curves all contained in the vertical foliation corresponding to q, then $\{g_t\pi(q)\}$ is an r-divergent trajectory. We will now show that the scheme presented in §2 is useful for producing r-divergent trajectories which are not obvious. Furthermore we will control the rate of escape, generalizing part (a) of Theorem 6.4.

Theorem 6.6. *Suppose $r + 1 \leq 3g - 3$. For any monotonically increasing unbounded function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ there is a trajectory $\{g_t\pi(q) : t \geq 0\}$ and t_0 such that for all $t \geq t_0$,*

$$D^r(t) \geq t/2 - \phi(t),$$

and $\{g_t\pi(q)\}$ is not an obvious divergent trajectory.

Proof. We will deduce the result from Theorems 2.4, 2.8 and Remark 2.9. By replacing $\phi(t)$ if necessary with a function increasing at a slower rate, we may assume with no loss of generality that $t \mapsto t/2 - \phi(t)$ is monotonically increasing and unbounded. Let

$$K(t) = \pi\left(\{q \in \widetilde{\mathcal{Q}}_1 : l^r_{\min}(q) \geq e^{-(t/2-\phi(t))}\}\right).$$

It is immediate that $\{K(t) : t \geq 0\}$ is a rate of growth. Note however that $K(t)$ is not compact if $r \geq 2$. It is also clear that $\{g_t\pi(q) : t \geq 0\}$ is divergent with rate given by $\{K(t)\}$ if and only if there is t_0 such that $D^r(t) \geq t/2 - \phi(t)$ for all $t \geq t_0$.

For any $M \in \mathcal{E}_r$ we let

$$X_M = \{q \in \widetilde{\mathcal{Q}}_1 : \forall \gamma \in M,\ \gamma \text{ is contained in the vertical foliation of } q\}.$$

It follows from Lemma 6.5 that

$$
\begin{aligned}
(37) \qquad I(X_M) &= I(\bigcap_{\gamma \in M} X_\gamma) = \bigcap_{\gamma \in M} I(X_\gamma) \\
&= \mathrm{Teich}(S) \times \{\lambda \in \mathcal{PML}(S) : \forall \gamma \in M,\, i(\lambda, \gamma) = 0\}.
\end{aligned}
$$

It follows from arguments as in the proof of Lemma 6.5 that each X_M is a submanifold of $\widetilde{\mathcal{Q}}_1$ of codimension r, and that if $M = M' \cup M''$, $M' \cap M'' = \varnothing$ then $X_{M'} \cap X_{M''}$ is a submanifold of $X_{M'}$ of codimension $\# M''$. Moreover $X_M \subset X_\gamma$ if and only if $\gamma \in M$.

Let $M_1, M_2, \ldots$, be an enumeration of $\mathcal{E}_r$, let $X_i = X_{M_i}$, let $\gamma_1, \gamma_2, \ldots$ be an enumeration of Γ^S, and let $X'_j = X_{\gamma_j}$.

We define a level function $L(i,j)$ as follows:

$$
L(i,j) = \begin{cases} \infty & \gamma_j \notin M_i \\ 0 & \gamma_j \in M_i \end{cases}
$$

It is clear that L is a level function for $(\{X_i\}, \{X'_j\})$. Let us first verify the hypotheses of Theorem 2.8.

- **Transversality with respect to $\{X'_j\}$.** If $X_i \not\subset X'_j$ then $\gamma = \gamma_j \notin M = M_i$. Suppose first that γ is disjoint from all elements of M. Then the argument of Lemma 6.5 applies to show that $X_\gamma \cap X_M$ is a codimension one submanifold of X_M.

  Now suppose γ intersects a curve $\gamma' \in M$. Using Lemma 6.5, let $\widetilde{X}_\gamma$, $\widetilde{X}_M \subset \mathcal{PML}(S)$ such that X_γ (resp. X_M) is identified with $\mathrm{Teich}(S) \times \widetilde{X}_\gamma$ (resp. $\mathrm{Teich}(S) \times \widetilde{X}_M$). We continue to identify $\mathcal{PML}(S)$ with a subset of $\mathcal{ML}(S)$. We need to show that $\widetilde{X}_M = \overline{\widetilde{X}_M \smallsetminus \widetilde{X}_\gamma}$. Let $\lambda \in \widetilde{X}_M$, so λ is a lamination on S whose intersection with all elements of M is trivial. If $\gamma' \subset \operatorname{supp} \lambda$ then $\lambda \notin X_\gamma$. If $\gamma \not\subset \operatorname{supp} \lambda$ then arguing as in the proof of transversality above we obtain that λ is arbitrarily close to elements of $\widetilde{X}_M \smallsetminus X_\gamma$. The assertion is proved.
- **Density of level increasing points.** Suppose $X_i \subset X'_j$. Write $M = M_i$, $\gamma = \gamma_j$, so $\gamma \in M$. Let $\widetilde{X}_M$, $\widetilde{X}_\gamma$ be as above. Suppose $\lambda_0 \in \widetilde{X}_M$, and let $S' = S \smallsetminus M$. By assumption on r, the dimension of $\mathcal{ML}(S')$ is positive, hence by Thurston's theorem there is $\lambda \in \mathcal{ML}(S')$ arbitrarily close to $\lambda_0|_{S'}$ in $\mathcal{ML}(S')$ which is supported on a simple closed curve η on S'. Let $M' = M \cup \{\eta\} \smallsetminus \{\gamma\}$, $M' = M_k$, and let $\widetilde{\lambda}$ be a measured lamination on S such that $\widetilde{\lambda}|_{S'} = \lambda$ and $\widetilde{\lambda}$ and λ_0 have the same transverse measures on curves of M. Such a $\widetilde{\lambda}$ can be chosen arbitrarily close to λ_0. By construction $\widetilde{\lambda}$ is in $\widetilde{X}_M \cap \widetilde{X}_{M'} = \widetilde{X}_i \cap \widetilde{X}_k$ and $\gamma \notin M'$, so that $L(k,j) = \infty$. The assertion follows.

We now have to verify the hypotheses of Theorem 2.4. For this, note that density (resp. transversality) follows immediately from density of level increasing points (resp. transversality w.r.t. $\{X'_j\}$), and local uniformity w.r.t. $\{K(t)\}$ is proved just as in the proof of Theorem 6.4 above. □

Question 6.7. *We have not shown the existence of geodesic trajectories which diverge arbitrarily slowly. Using Theorem 2.5, it is sufficient to verify the hypothesis of density of connected components.*

Question 6.8. *The space $\mathcal{Q}_1$ is stratified according to the cardinality and structure of $\Sigma(q)$ for $q \in \mathcal{Q}_1$. It would be interesting to see whether the results of the two previous subsections remain valid if one is only interested in divergent trajectories on a fixed stratum.*

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