DIVERGENT TRAJECTORIES AND $\mathbb{Q}$-RANK

BARAK WEISS

ABSTRACT. The author proves a conjecture of the author: If G is a semisimple real algebraic defined over $\mathbb{Q}$, Γ is an arithmetic subgroup (with respect to the given $\mathbb{Q}$-structure) and A is a diagonalizable subgroup admitting a divergent trajectory in G/Γ, then $\dim A \leq \mathrm{rank}_{\mathbb{Q}} G$.

The purpose of this paper is to prove the following result.

Theorem 1. *Let G be a semisimple real algebraic group defined over $\mathbb{Q}$, let Γ be an arithmetic lattice (with respect to the given $\mathbb{Q}$-structure), and let $\pi : G \to G/\Gamma$ be the natural map. For any $g \in G$ and any $\mathbb{R}$-diagonalizable subgroup A with $\dim A > \mathrm{rank}_{\mathbb{Q}} G$, there is a compact $K \subset G/\Gamma$ such that for any $T > 0$ there is $Y \in \mathfrak{a} = \mathrm{Lie}(A)$ satisfying $\|Y\| = T$ and $\exp(Y)\pi(g) \in K$.*

This immediately implies:

Corollary 2 ([W], Conjecture 4.11A)**.** *If G, Γ are as above, A is an $\mathbb{R}$-diagonalizable subgroup of G and $\dim A > \mathrm{rank}_{\mathbb{Q}} G$ then there are no divergent trajectories for the action of A on G/Γ.*

The case $\mathrm{rank}_{\mathbb{R}} G = \mathrm{rank}_{\mathbb{Q}} G + 1$ of Corollary 2 follows from [ToWe, Thm 1.4]. The case $\mathrm{rank}_{\mathbb{Q}} G = 1$ is easy, see [W, Prop. 4.12]. In [ChMo], Chatterjee and Morris obtained a partial result toward Corollary 2 in arbitrary $\mathrm{rank}_{\mathbb{Q}} G$, and settled the case $\mathrm{rank}_{\mathbb{Q}} G = 2$.

The result has the following geometric interpretation (see [Mo, ChMo] for more details): if X is a finite-volume locally symmetric space, then the rational rank of the corresponding lattice is equal to the maximal dimension of a closed simply connected flat in a finite cover of X.

We will be using standard terminology and notation from the theory of algebraic groups and arithmetic subgroups, see e.g. [Bo1, Bo2]. By a *real algebraic group* we mean a finite index subgroup of the real points of a semisimple real algebraic group defined over $\mathbb{Q}$. Throughout this paper, G, Γ, and $\pi : G \to G/\Gamma$ are as in the statement of Threorem 1. There is an action on G/Γ by G (and any of its subgroups) defined by $g_0\pi(g) = \pi(g_0 g)$.

The Lie algebra $\mathfrak{g}$ of G can be equipped with a $\mathbb{Q}$-structure which is compatible with the $\mathbb{Q}$-structure on G. This means that we may choose a (linear) basis of $\mathfrak{g}$ such that by definition $\mathfrak{g}(\mathbb{Z})$ is the $\mathbb{Z}$-span of this basis, and, possibly after replacing Γ with a commensurable subgroup, we have $\mathrm{Ad}(\Gamma)\mathfrak{g}(\mathbb{Z}) = \mathfrak{g}(\mathbb{Z})$. For $x = \pi(g) \in G/\Gamma$, let

$$\mathfrak{g}_x = \mathrm{Ad}(g)\mathfrak{g}(\mathbb{Z})$$

(which is independent of the choice of $g \in \pi^{-1}(x)$).

We now record some useful facts, proved in [ToWe].

Proposition 3 ([ToWe], Proposition 3.3)**.** *There is a bounded open neighborhood W of 0 in $\mathfrak{g}$ such that for any $x \in G/\Gamma$, the subalgebra generated by $W \cap \mathfrak{g}_x$ is unipotent.*

By definition, $\mathrm{rank}_{\mathbb{Q}}G$ is the dimension of a maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus in G. Let B be a minimal $\mathbb{Q}$-parabolic subgroup, and denote by $P_1, \dots, P_r$ the distinct maximal $\mathbb{Q}$-parabolic subgroups containing B. It is known that $r = \mathrm{rank}_{\mathbb{Q}}G$.

For $i = 1, \dots, r$ denote by $\mathfrak{u}_i$ the Lie algebra of $\mathrm{Rad}_u(P_i)$. Let $\mathcal{R}_i$ be the collection of all Lie algebras of unipotent radicals of maximal $\mathbb{Q}$-parabolics which are conjugate to $\mathfrak{u}_i$, and let $\mathcal{R} = \bigcup_{i=1}^r \mathcal{R}_i$.

A finite subset of $\mathfrak{g}$ is called *horospherical* if it linearly spans a subalgebra of $\mathfrak{g}$ which is conjugate to one of the $\mathfrak{u}_i$'s, and contains no proper subsets with this property.

Proposition 4 ([ToWe], Proposition 3.5)**.** *A subset $X \subset G/\Gamma$ is precompact if and only if there exists a neighborhood W of 0 in $\mathfrak{g}$ such that for all $x \in X$, $\mathfrak{g}_x \cap W$ does not contain a horospherical subset.*

We denote the unipotent radical of an algebraic group H by $\mathrm{Rad}_u(H)$.

Proposition 5 ([ToWe], Proposition 5.3)**.** *Let $B \subset G$ be a minimal $\mathbb{Q}$-parabolic in G and for $j = 1, 2$ let $U_j \subset \mathrm{Rad}_u(B)$ be a unipotent radical of a $\mathbb{Q}$-parabolic. If U_1, U_2 are conjugate then $U_1 = U_2$.*

For $i = 1, \dots, r$ let $d_i = \dim \mathfrak{u}_i$, let $V_i = \bigwedge_1^{d_i} \mathfrak{g}$ and let $\rho_i : G \to \mathrm{GL}(V_i)$ be the d_i-th exterior power of the adjoint representation. Let D be a maximal connected $\mathbb{R}$-diagonalizable subgroup of G and let $\mathfrak{d}$ denote its Lie algebra. Any connected $\mathbb{R}$-diagonalizable subgroup of G is conjugate to a subgroup of D. Let $\Psi_i \subset \mathfrak{d}^*$ be the weights of ρ_i, that is, $V_i = \bigoplus_{\chi \in \Psi_i} V_\chi$, where

$$V_\chi = \{v \in V_i : \forall Y \in \mathfrak{d},\ \rho_i(Y)v = e^{\chi(Y)}v\}$$

and

$$\Psi_i = \{\chi \in \mathfrak{d}^* : V_\chi \neq \{0\}\}.$$

For a finite $M \subset \mathfrak{g}$ let $\langle M \rangle$ denote the vector space spanned by M. For $\mathfrak{h} \in \mathcal{R}_i$ let $v_1, \dots, v_{d_i}$ be a set of generators (over $\mathbb{Z}$) of $\mathfrak{h}(\mathbb{Z})$ and let $\mathbf{p}_{\mathfrak{h}} = v_1 \wedge \dots \wedge v_{d_i}$ be the corresponding vector in V_i.

Proposition 6. *For any $x = \pi(g) \in G/\Gamma$ there is a neighborhood W of 0 in $\mathfrak{g}$ with the following property. For any $\mathfrak{h} \in \mathcal{R}_i$, $i \in \{1, \dots, r\}$, there is $\chi \in \Psi_i$ such that if M is a horospherical subset with $\langle M \rangle = \mathfrak{h}$ and $Y \in \mathfrak{d}$ is such that $\mathrm{Ad}(\exp(Y)g)M \subset W \cap \mathfrak{g}_{\exp(Y)x}$, then $\chi(Y) < 0$.*

Proof. For $i = 1, \dots, r$, let $\| \cdot \|_i$ be a norm on V_i which is the max norm with respect to a fixed basis of eigenvectors for the $\rho_i(D)$-action on V_i. By discreteness of $V_i(\mathbb{Z})$ there is a neighborhood W_i of 0 in V_i such that $W_i \cap \rho_i(g)V_i(\mathbb{Z}) = \{0\}$ and hence $\rho_i(g)\mathbf{p}_\mathfrak{h} \notin W_i$ for $\mathfrak{h} \in \mathcal{R}_i$. There is a small enough $\varepsilon > 0$ such that each W_i contains the ε-ball centered at 0 in V_i.

Now choose a neighborhood W of 0 in $\mathfrak{g}$ small enough so that if $v_1, \dots, v_{d_i} \in W$ then

$$\|v_1 \wedge \cdots \wedge v_{d_i}\|_i < \varepsilon.$$

Given $\mathfrak{h} \in \mathcal{R}_i$ let $\mathbf{p} = \rho_i(g)\mathbf{p}_\mathfrak{h}$ and choose $\chi \in \Psi_i$ so that $\|\mathbf{p}\|_i$ is equal to the absolute value of the coefficient corresponding to an eigenvector with weight χ.

Suppose $M' = \{v_1, \dots, v_{d_i}\} \subset \mathfrak{g}_x$ and $Y \in \mathfrak{d}$ are such that

$$\mathrm{Ad}(\exp(Y))M' \subset W$$

and $\langle M \rangle = \mathfrak{h}$, where $M = \mathrm{Ad}(g^{-1})M' \subset \mathfrak{g}(\mathbb{Z})$.

Then by the definition of W_i and W we find that

$$\|\rho_i(\exp(Y))\mathbf{p}\|_i < \varepsilon \quad \text{and} \quad \|\mathbf{p}\|_i \geq \varepsilon.$$

Thus by definition of $\| \cdot \|_i$, $\chi(Y) < 0$. □

The following topological result is crucial for the proof of Theorem 1. Despite its simplicity we were unable to find it in the literature.

Proposition 7. *Let S be an n-dimensional sphere centered at 0 in $\mathbb{R}^{n+1}$. Suppose $\mathcal{V}$ is a cover of S by open sets such that for any $V \in \mathcal{V}$ there is a linear functional $\chi : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\chi(s) < 0$ for any $s \in V$. Then there is $s \in S$ such that*

$$\#\{V \in \mathcal{V} : s \in V\} \geq n + 1.$$

Proof. Since S is compact we can assume that $\mathcal{V} = \{V_1, \dots, V_k\}$ is finite. We recall the standard construction of the *nerve* of $\mathcal{V}$. Let $\mathbf{e}_1, \dots, \mathbf{e}_k$ be the vectors of the standard basis of $\mathbb{R}^k$ and let $\mathcal{C}$ be the simplicial complex whose vertices are $\mathbf{e}_1, \dots, \mathbf{e}_k$ and whose $(\ell - 1)$-dimensional subsimplices are the convex hulls of any set of vertices $\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_\ell}$ for which $V_{i_1} \cap \cdots \cap V_{i_\ell} \neq \varnothing$. Suppose by contradiction that the proposition is false; then $\dim \mathcal{C} \leq n - 1$.

Let $f_1, \ldots, f_k$ be a partition of unity subordinate to $\mathcal{V}$, i.e. each $f_i : S \to [0,1]$ is continuous, $\sum f_i \equiv 1$ and

$$\text{supp}\, f_i = \{s \in S : f_i(s) \neq 0\} \subset V_i.$$

For each $s \in S$ define $F : S \to \mathcal{C}$ by

$$F(s) = \sum f_i(s)\mathbf{e}_i.$$

We claim that we can modify $\mathcal{C}$ and F to obtain a simplicial complex $\widetilde{\mathcal{C}}$ and $\widetilde{F} : S \to \widetilde{\mathcal{C}}$ with the following properties:

(i) $\dim \widetilde{\mathcal{C}} \leq \dim \mathcal{C}$.
(ii) $\widetilde{F}$ is onto.
(iii) For any $b \in \widetilde{\mathcal{C}}$ there is i such that for any simplex Δ of $\widetilde{\mathcal{C}}$,

$$b \in \Delta \quad \Longrightarrow \quad \widetilde{F}^{-1}(\Delta) \subset V_i.$$

Let us assume the validity of this claim and complete the proof of the proposition. Let B be the closed ball centered at 0 such that $S = \partial B$. We define $g : \widetilde{\mathcal{C}} \to B$ as follows. For each vertex b of $\widetilde{\mathcal{C}}$ let $g(b) \in \widetilde{F}^{-1}(b)$ (note that $\widetilde{F}^{-1}(b) \neq \varnothing$ by (ii)). All other points of $\widetilde{\mathcal{C}}$ are convex combinations of vertices and we extend g to all of $\widetilde{\mathcal{C}}$ in an affine manner. Note that $0 \notin g(\widetilde{\mathcal{C}})$; in fact, for every $b \in \widetilde{\mathcal{C}}$, by (iii) there is i such that $g(b)$ is a convex combination of elements of V_i, and in particular $\chi(g(b)) < 0$ for the linear functional χ which assumes negative values on V_i. Moreover, for every $s \in S$ there is i such that $s \in V_i$ and $g(\widetilde{F}(s))$ is contained in the convex hull of V_i. Thus, applying radial projection to the image of g we obtain a continuous function $G : \widetilde{\mathcal{C}} \to S$ such that for each $s \in S$, s and $G \circ \widetilde{F}(s)$ are in the same (open) hemisphere. In particular s and $G \circ \widetilde{F}(s)$ determine a unique great circle, and moving along it defines a homotopy between $G \circ \widetilde{F}$ and the identity map $S \to S$. We have shown that a map homotopic to the identity map from the n-sphere to itself factors through a simplicial complex of dimension at most $n-1$. Thus the identity map induces the zero map on the nontrivial n-th dimensional homology group of S, a contradiction.

It remains to prove the claim. Consider the sets $\mathcal{U}_i = \mathcal{C} \smallsetminus F(S \smallsetminus V_i)$, $i = 1, \ldots, k$. These are open sets since $S \smallsetminus V_i$ is compact, they cover $\mathcal{C}$ and $F^{-1}(\mathcal{U}_i) \subset V_i$. Take a decomposition of $\mathcal{C}$ into simplices sufficiently small so that for any $b \in \mathcal{C}$ there is i such that for any simplices Δ, Δ',

$$b \in \Delta, \ \Delta \cap \Delta' \neq \varnothing \quad \Longrightarrow \quad \Delta' \subset \mathcal{U}_i. \tag{1}$$

We write the simplices in this decomposition of $\mathcal{C}$ as $\Delta_1, \ldots, \Delta_t$, where $\dim \Delta_j \geq \dim \Delta_\ell$ whenever $j < \ell$, and inductively define simplicial complexes $\mathcal{C}_0, \ldots, \mathcal{C}_t$ with each $\mathcal{C}_\ell$ a union of simplices of $\mathcal{C}$, and $F_\ell : S \to \mathcal{C}_\ell$, as follows. We first set $F = F_0$, $\mathcal{C} = \mathcal{C}_0$, and supposing F_ℓ and $\mathcal{C}_\ell$ have been defined, consider three cases. If $\Delta_{\ell+1} \subset F_\ell(S)$ then $\mathcal{C}_{\ell+1} = \mathcal{C}_\ell$ and $F_{\ell+1} = F_\ell$. If $F_\ell(S) \cap \Delta_{\ell+1} = \varnothing$ then $\mathcal{C}_{\ell+1} = \mathcal{C}_\ell \smallsetminus \operatorname{int} \Delta_{\ell+1}$ and $F_{\ell+1} = F_\ell|_{\mathcal{C}_{\ell+1}}$. Finally, if $\Delta_{\ell+1} \not\subset F_\ell(S)$ then by compactness of S there is $\delta \in \operatorname{int} \Delta_{\ell+1} \smallsetminus F_\ell(S)$ and we set $\mathcal{C}_{\ell+1} = \mathcal{C}_\ell \smallsetminus \operatorname{int}\Delta_{\ell+1}$ and define $F_{\ell+1}$ on $F_\ell^{-1}(\operatorname{int} \Delta_{\ell+1})$ by radially retracting from δ to $\partial \Delta_{\ell+1}$, and without changing the map F_ℓ outside of $F_\ell^{-1}(\operatorname{int} \Delta_{\ell+1})$. We set $\widetilde{F} = F_t$ and $\widetilde{\mathcal{C}} = \mathcal{C}_t$. It is clear that (i) holds for $\widetilde{F}$. To verify (ii), suppose $y \in \widetilde{\mathcal{C}} \smallsetminus \widetilde{F}(S)$, and let j be the largest index for which $y \in \operatorname{int} \Delta_j$. Then by construction $y \notin F_j(S)$. This implies by construction that $\operatorname{int} \Delta_j \cap \mathcal{C}_j = \varnothing$, so $y \notin \widetilde{\mathcal{C}}$, a contradiction. Finally, to prove (iii), let $b \in \widetilde{\mathcal{C}}$, let i satisfy (1) and suppose $b \in \Delta$. By induction one sees that if $F(s) \in \Delta'$ then also $\widetilde{F}(s) \in \Delta'$, so if $\widetilde{F}(s) \in \Delta$ and $F(s) \in \Delta'$ then $\Delta \cap \Delta' \neq \varnothing$, i.e.

$$\widetilde{F}^{-1}(\Delta) \subset \bigcup_{\Delta \cap \Delta' \neq \varnothing} F^{-1}(\Delta'),$$

and (iii) follows from (1). □

Proof of Theorem 1. Let A be as in the statement of the theorem. Applying a conjugation we may assume that $A \subset D$. Let $x = \pi(g)$ for $g \in G$, and let W be a small enough neighborhood of 0 in $\mathfrak{g}$ so that the conclusions of Propositions 3 and 6 are satisfied. Let

$$K = \{z \in G/\Gamma : \mathfrak{g}_z \cap W \text{ does not contain a horospherical subset}\},$$

a compact subset of G/Γ by Proposition 4. Thus if $Y \in \mathfrak{a}$ with $\exp(Y)x \notin K$ then for some $M \subset \mathfrak{g}(\mathbb{Z})$ we have $\langle M \rangle \in \mathcal{R}$ and $\operatorname{Ad}(\exp(Y)g)M \subset W \cap \mathfrak{g}_{\exp(Y)x}$.

Let $\mathfrak{a}$ denote the Lie algebra of A, equipped with a Euclidean norm. Assume the theorem is false; then for some $T > 0$, the $(\dim A - 1)$–dimensional sphere

$$S = \{Y \in \mathfrak{a} : \|Y\| = T\}$$

is covered by the sets $\mathcal{V} = \{V(\mathfrak{h}) : \mathfrak{h} \in \mathcal{R}\}$, where

$$V(\mathfrak{h}) = \{Y \in \mathfrak{a} : \exists M \subset \mathfrak{g} \text{ s.t. } \langle M \rangle = \mathfrak{h},\ \operatorname{Ad}(\exp(Y)g)M \subset W \cap \mathfrak{g}_{\exp(Y)x}\}.$$

Applying Propositions 6 and 7 we find that there is $Y_0 \in S$ which is contained in $V(\mathfrak{h}_1) \cap \cdots \cap V(\mathfrak{h}_{r+1})$ with $\mathfrak{h}_j$ distinct for $j = 1, \ldots, r+1$.

After re-ordering the indices, this means that for $j = 1, 2$ and some $g_0 \in G$ there are $M_j \subset \mathfrak{g}$ such that

$$\mathfrak{h}_1 \neq \mathfrak{h}_2, \ \mathrm{Ad}(g_0)\mathfrak{h}_1 = \mathfrak{h}_2, \quad \text{where } \mathfrak{h}_j = \langle M_j \rangle \tag{2}$$

and

$$\mathrm{Ad}(z)M_j \subset W \cap \mathfrak{g}_{\pi(z)} \qquad \text{where } z = \exp(Y_0)g.$$

Let $\mathfrak{f}$ be the Lie algebra spanned by $W \cap \mathfrak{g}_{\pi(z)}$. By Proposition 3 it is unipotent. Note that $\mathfrak{f}' = \mathrm{Ad}(z^{-1})\mathfrak{f}$ is generated by elements of $\mathfrak{g}(\mathbb{Z})$, therefore it is contained in the Lie algebra $\mathfrak{f}''$ of the unipotent radical of a minimal $\mathbb{Q}$-parabolic. Then by construction we have $\mathfrak{h}_1 \subset \mathfrak{f}''$ and $\mathrm{Ad}(g_0)\mathfrak{h}_1 = \mathfrak{h}_2 \subset \mathfrak{f}''$. Hence by Proposition 5 we have that $\mathfrak{h}_1 = \mathfrak{h}_2$, a contradiction to (2). □

Acknowledgements. I am very grateful to Curt McMullen and Michael Levin for explaining basic topology to me. The proof of Proposition 7 is due to Levin, and an earlier version of this paper used ideas of McMullen. I am also grateful to Dave Morris and George Tomanov for useful discussions, and to the Max Planck Institute in Bonn for its hospitality.

References

[Bo1] A. Borel, **Introduction aux groupes arithmetiques**, Hermann, Paris, 1969.

[Bo2] A. Borel, **Linear algebraic groups, second enlarged edition.** Springer, 1991.

[ChMo] P. Chatterjee and D. Morris, *Divergent torus orbits in homogeneous spaces of $\mathbb{Q}$-rank two,* (2004), preprint.

[Mo] D. (Witte) Morris, **Introduction to arithmetic groups** (preliminary version), available online at:
http://people.uleth.ca/~dave.morris/LectureNotes.shtml#ArithmeticGroups

[ToWe] G. Tomanov and B. Weiss, *Closed orbits for actions of maximal tori on homogeneous spaces*, Duke Math. J. 119 (2003), no. 2, 367–392.

[W] B. Weiss, *Divergent trajectories on noncompact parameter spaces*, Geom. Funct. Anal. **14** no. 1 (2004) 94–149.

Ben Gurion University, Be'er Sheva, Israel 84105 barakw@math.bgu.ac.il