CLOSED ORBITS FOR ACTIONS OF MAXIMAL TORI ON HOMOGENEOUS SPACES

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ABSTRACT. Let G be a real algebraic group defined over $\mathbb{Q}$, let Γ be an arithmetic subgroup, and let T be any torus containing a maximal $\mathbb{R}$-split torus. We prove that the closed orbits for the action of T on G/Γ admit a simple algebraic description. In particular we show that if G is reductive, an orbit $Tx\Gamma$ is closed if and only if $x^{-1}Tx$ is a product of compact torus and a torus defined over $\mathbb{Q}$, and is divergent if and only if the maximal $\mathbb{R}$-split subtorus of $x^{-1}Tx$ is defined over $\mathbb{Q}$ and $\mathbb{Q}$-split. Our analysis also yields the following:

- there is a compact $K \subset G/\Gamma$ which intersects every T-orbit.
- if $\mathrm{rank}_{\mathbb{Q}} G < \mathrm{rank}_{\mathbb{R}} G$, there are no divergent orbits for T.

1. INTRODUCTION

Let G be the group of real points of a connected $\mathbb{Q}$-algebraic group $\mathbf{G}$, and let Γ be an arithmetic subgroup of G (i.e. $\Gamma \cap \mathbf{G}(\mathbb{Z})$ has finite index in both Γ and $\mathbf{G}(\mathbb{Z})$). Any subgroup H of G acts on the homogeneous space G/Γ by left translations:

$$h\pi(g) = \pi(hg),$$

where $\pi : G \to G/\Gamma$ is the natural quotient map.

A celebrated conjecture of M. S. Raghunathan, proved in full generality by M. Ratner in the early 1990's, implies that when H is connected and generated by unipotents elements, every orbit closure is homogeneous, i.e. it coincides with the orbit of a bigger group. The resulting reduction of certain dynamical questions to algebraic questions has had many deep applications in number theory and geometry. We refer the reader to [KlShSt] for an up to date survey of these developments.

The most general conjecture regarding dynamics of such actions on homogeneous spaces was formulated by G. A. Margulis. According to Margulis' conjecture, unless the action of H admits natural factors on which H acts nontrivially as a one-parameter non-unipotent group, the orbit closures are homogeneous for groups H generated by $\mathbb{R}$-split elements [M, Conjecture 1]. (Recall that an element $g \in G$ is $\mathbb{R}$-split if all of its eigenvalues are real.) A very interesting and highly nontrivial special case of this conjecture is when H is a maximal real $\mathbb{R}$-split algebraic torus in G, that is, when $\dim H \geq 2$ and H is a maximal abelian subgroup of G generated by $\mathbb{R}$-split elements. In this paper we give an explicit algebraic description of all (topologically) closed orbits for this action. More specifically, we prove that all closed orbits are 'standard' – they correspond to $\mathbb{Q}$–subtori in $\mathbf{G}$ (Theorem 1.1).

Consider the simplest case in which $G = \mathrm{SL}(2,\mathbb{R})$, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and T is the subgroup of positive diagonal matrices. The action of T is then the geodesic flow on the unit tangent bundle to the modular surface $\mathbb{H}/\Gamma$. This is a noncompact manifold, with one cusp whose lifts correspond to all rational numbers. Any closed orbit for this flow is either periodic or divergent (an orbit $T\pi(x)$ is *divergent* if the orbit map $t \mapsto t\pi(x)$ is proper, or equivalently, if $\{t_n\pi(x)\}$ leaves compact subsets of G/Γ whenever $\{t_n\}$ leaves compact subsets of T). Since a divergent orbit must go 'into the cusp', a geodesic goes to infinity in both directions if and only if one (hence any) of its lifts to $\mathbb{H}$ has both endpoints on $\mathbb{Q} \cup \{\infty\}$. Equivalently, $T\pi(g)$ is divergent if and only if $g^{-1}Tg$ is diagonalizable over $\mathbb{Q}$. An orbit $T\pi(x)$ is periodic if and only if $\mathrm{Stab}(\pi(x)) = x^{-1}Tx \cap \mathrm{SL}(2,\mathbb{Z})$ is a cocompact subgroup of $x^{-1}Tx$. Therefore $T\pi(x)$ is periodic if and only if $x^{-1}Tx$ is defined over $\mathbb{Q}$ and it does not admit nontrivial $\mathbb{Q}$-rational homomorphisms to $\mathbb{R}^*$, i.e. $x^{-1}Tx$ is a $\mathbb{Q}$-anisotropic torus.

Our work shows that a similar description of closed and divergent orbits is valid in the general case. For example, as a special case of Theorems 1.4 and 1.5 below, we obtain:

Theorem 1.1. *Let $\mathbf{G}$ be a reductive $\mathbb{Q}$-algebraic group, $\mathbf{T}$ an $\mathbb{R}$-torus containig a maximal $\mathbb{R}$-split torus, $T = \mathbf{T}(\mathbb{R})$ and let $x \in G$. Then:*

- *$T\pi(x)$ is a closed orbit if and only if $x^{-1}\mathbf{T}x$ is a product of a $\mathbb{Q}$-subtorus and an $\mathbb{R}$-anisotropic $\mathbb{R}$-subtorus;*
- *$T\pi(x)$ is a divergent orbit if and only if the maximal $\mathbb{R}$-split subtorus of $x^{-1}\mathbf{T}x$ is defined over $\mathbb{Q}$ and $\mathbb{Q}$-split.*

According to a result of Prasad and Raghunathan [PrRa, Theorem 2.13] the set of closed T-orbits is not empty if the torus $\mathbf{T}$ is maximal. This result does not hold in general for a smaller torus.

Theorem 1.1 generalizes the following (unpublished) result of Margulis, proved in 1997 in response to a question of the second–named author.

Theorem 1.2 (Margulis)**.** *Let $G = \mathrm{SL}(n,\mathbb{R})$, $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, and let T be the group of all diagonal matrices. Then $T\pi(g)$ is divergent if and only if $g^{-1}Tg$ is a real $\mathbb{Q}$-split torus.*

We include the proof of Theorem 1.2 in the appendix.

Earlier work of S. G. Dani [Da] showed that no algebraic description of divergent trajectories is possible for the action of one–parameter diagonalizable subgroups on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$, or more generally, for actions on G/Γ when $\mathrm{rank}_{\mathbb{Q}}(\mathbf{G}) \geq 2$. Dani discusses trajectories rather than orbits, that is, the action of $T = \{d(t) : t \in \mathbb{R}\}$ is replaced by the action of the semigroup $\{d(t) : t \geq 0\}$. In [Da] actions of multidimensional semigroups or groups were not considered.

We now state our main result:

Theorem 1.3. *Let* $\mathbf{G}$ *be a* $\mathbb{Q}$*-algebraic group and let* $\mathbf{T}$ *be a torus defined over* $\mathbb{R}$ *which contains a maximal* $\mathbb{R}$*-split torus and a maximal* $\mathbb{Q}$*-split* $\mathbb{Q}$*-torus* $\mathbf{S}$*. Let* $G = \mathbf{G}(\mathbb{R})$, $T = \mathbf{T}(\mathbb{R})$, $S = \mathbf{S}(\mathbb{R})$, *and let* Γ *be an arithmetic subgroup of* G*. Then there exists a compact subset* $K \subset G/\Gamma$ *such that:*

(1) $T\pi(x) \cap K \neq \emptyset$ *for any* $x \in G$.

(2) *One of the following holds:*

(i) $x \in Z_G(S)\mathbf{G}(\mathbb{Q})\mathcal{R}_u(G)$, *where* $\mathcal{R}_u(G)$ *is the unipotent radical of* G *and* $Z_G(S)$ *is the centralizer of* S *in* G*;*

(ii) the 'set of recurrence'

$$\{d \in T : d\pi(x) \in K\}$$

is unbounded in T*.*

We apply Theorem 1.3 in order to describe the divergent orbits for T on G/Γ.

Theorem 1.4. *Let the notation be as in Theorem 1.3, and let* $x \in G$*. The following are equivalent:*

(1) *The orbit* $T\pi(x)$ *is divergent.*

(2) $x \in Z_G(S)\mathbf{G}(\mathbb{Q})\mathcal{R}_u(G)$ *and* $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} = \mathrm{rank}_{\mathbb{R}}\mathbf{G}$.

(3) *There is* $u \in \mathcal{R}_u(G)$ *such that the maximal* $\mathbb{R}$*-split* $\mathbb{R}$*-torus of* $(xu)^{-1}\mathbf{T}xu$ *is a maximal* $\mathbb{Q}$*-split* $\mathbb{Q}$*-torus of* $\mathbf{G}$.

In particular, if $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} < \mathrm{rank}_{\mathbb{R}}\mathbf{G}$ *then* G/Γ *does not contain divergent orbits for* T*.*

Our results about divergent orbits also yield information about all *closed* orbits.

Theorem 1.5. *Let the notation be as in Theorem 1.3. Given* $x \in G$ *we denote by* $\mathbf{T}_x$ *the connected component of the identity in the Zariski closure of* $x^{-1}Tx \cap \Gamma$ *in* $\mathbf{G}$*. Then the orbit* $T\pi(x)$ *is closed if and only if there exists an element* $u \in \mathcal{R}_u(Z_{\mathbf{G}}(\mathbf{T}_x))$ *such that* $(xu)^{-1}\mathbf{T}xu$ *is a product of a* $\mathbb{Q}$*-subtorus and an* $\mathbb{R}$*-anisotropic* $\mathbb{R}$*-subtorus.*

We remark that it is not difficult to explicitly describe all closed orbits of T on G/Γ in the case when Γ is *any* lattice in G and $\mathrm{rank}_{\mathbb{R}}\mathbf{G} = 1$. (See, for example, [Da, §5].) In view of Margulis' arithmeticity theorem, our assumption that Γ is *arithmetic* entails no loss of generality when $\mathrm{rank}_{\mathbb{R}}\mathbf{G} \geq 2$.

As a further application, we deduce from the first statement in Theorem 1.3:

Corollary 1.6. *Any closed T-invariant subset of G/Γ contains a minimal (w.r.t. inclusion) closed invariant subset.*

Let us briefly describe our proof of the main result, Theorem 1.3, assuming for simplicity that G is reductive. Following Margulis, we establish (1) by a 'push out' argument: we show that there is a finite subset $F \subset T$ and a neighborhood W of 0 in the Lie algebra $\mathcal{G}$ of G such that for any $g \in G$, there is $t \in F$ such that all elements of $W \cap \mathrm{Ad}(g)\mathcal{G}_{\mathbb{Z}}$ are enlarged by applying $\mathrm{Ad}(t)$ (see Proposition 4.1 for a precise formulation). Then, applying successively elements of F, after finitely many steps we obtain $t_0 \in T$ such that $\mathrm{Ad}(t_0 g)\mathcal{G}_{\mathbb{Z}} \cap W = \{0\}$, which implies that $t_0\pi(g)$ is in a compact subset of G/Γ depending only on W.

Note that a statement similar to Proposition 4.1 is established in [KaMa], where it is shown that F as above exists in G. We show that F can be found inside T.

We then show that in case a compact $C \subset T$ is given and $x \notin Z_G(S)\mathbf{G}(\mathbb{Q})$ then an element t_0 as above can be found, which furthermore does not belong to C. In the case $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$, using Mahler's compactness criterion, Margulis showed this by proving Proposition A.2: For $g \notin T\mathrm{SL}(n,\mathbb{Q})$, any finite set L of nonzero vectors in $g\mathbb{Z}^n$, and any neighborhood W of 0 in $\mathbb{R}^n$, there is $t_0 \in T \setminus C$ such that $t_0 L \cap W = \emptyset$.

In the general case, an analogue of Proposition A.2 in which the action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}^n$ is replaced with the adjoint representation of G on $\mathcal{G}$, turns out to be false. Remedying this is one of the difficult parts of the proof, and requires a more subtle compactness criterion involving what we call 'horospherical subsets' – finite sets of vectors spanning the unipotent radical of a maximal $\mathbb{Q}$-parabolic subalgebra – rather than individual vectors (see Definition 3.4 and Proposition 3.5). We prove Proposition 5.1, which is an analogue of Proposition A.2 for horospherical subsets. Thus it appears that horospherical subsets share some of the advantageous properties of the individual vectors in the proof of Theorem 1.2 for $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$.

In the concluding section of this paper we give some examples illustrating the necessity of our hypotheses, and formulate some questions for further research.

The results of this paper were announced in [To].

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2. Preliminaries

2.1. **Notation and terminology.** As usual $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ denote the complex, real, rational and integer numbers, respectively.

We shall use freely the standard notions of the theory of linear algebraic groups [Bo1].

We use boldface letters in order to denote the k-algebraic groups (where k is a field) and if $k \subset \mathbb{R}$ we will use the corresponding uppercase letters to denote the group of $\mathbb{R}$-rational points of these k-algebraic groups. So, we have $H = \mathbf{H}(\mathbb{R})$ where $\mathbf{H}$ is a k-algebraic group. The group H is called a *real algebraic group*.

In this paper $\mathbf{G}$ denotes a $\mathbb{Q}$-algebraic group, $\mathbf{S}$ a maximal $\mathbb{Q}$-split subtorus in $\mathbf{G}$ and $\mathbf{T}$ denotes an $\mathbb{R}$-torus containing both a maximal $\mathbb{R}$-split subtorus in $\mathbf{G}$ and $\mathbf{S}$. Note that $\mathbf{T}$ is an almost direct product (over $\mathbb{R}$) of a $\mathbb{R}$-split subtorus and a $\mathbb{R}$-anisotropic subtorus and the group of $\mathbb{R}$-points of the latter is compact. Also recall that if $\mathbf{H}$ is any $\mathbb{R}$-torus in $\mathbf{G}$ then $\mathbf{H}$ contains a unique maximal (in $\mathbf{H}$) $\mathbb{Q}$-subtorus and $\mathbf{H}$ contains a unique maximal (in $\mathbf{H}$) $\mathbb{R}$-split subtorus.

We consider $\mathbf{G}$ as a $\mathbb{Q}$-subgroup of $\mathrm{GL}(n, \mathbb{C})$ and we denote by $\mathbf{G}(\mathbb{Z})$ the group of integer matrices of $\mathbf{G}$. If k is a subfield of $\mathbb{C}$ then the k-rank of $\mathbf{G}$ (notation: $\mathrm{rank}_k\mathbf{G}$) is by definition the dimension of the maximal k-split subtori of $\mathbf{G}$. We denote by $\mathcal{R}_u(\mathbf{G})$ (resp., $\mathcal{R}_u(G)$) the (real points of) the unipotent radical of $\mathbf{G}$.

If H is a Lie group we denote the identity component of H by H^0. We preserve the same notation for the identity component of an algebraic group $\mathbf{H}$ (with respect to the Zariski topology). The above notation does not lead to confusion because if $\mathbf{H} \subset \mathrm{GL}(n, \mathbb{C})$ then the identity component of the *Lie group* $\mathbf{H}$ coincides with the identity component of the *algebraic group* $\mathbf{H}$.

If H and L are subgroups of G then $N_L(H)$ denotes the normalizer of H in L and $Z_L(H)$ denotes the centralizer of H in L.

The Lie algebra Lie($\mathbf{G}$) of $\mathbf{G}$ is equipped with a $\mathbb{Q}$-structure which is compatible with the $\mathbb{Q}$-structure of $\mathbf{G}$ [Bo1, Theorem 3.4]. We denote $\mathcal{G} = \mathrm{Lie}(\mathbf{G})(\mathbb{R})$ and $\mathcal{G}_{\mathbb{Z}} = \mathrm{Lie}(\mathbf{G})(\mathbb{Z})$. We have $\mathcal{G} = \mathrm{Lie}(G)$. We fix a *norm* $\|\cdot\|$ on $\mathcal{G}$ and use the same notation for the restriction of $\|\cdot\|$ to any subalgebra of $\mathcal{G}$. We say that a Lie subalgebra $\mathcal{U}$ of $\mathcal{G}$ is *unipotent* if it corresponds to a Zariski closed unipotent subgroup of G.

Let Γ be an arithmetic subgroup of G. Then $\mathrm{Ad}(\Gamma)$ is an arithmetic subgroup of $\mathrm{Ad}(\mathbf{G})$ (cf. [Bo3]). So there is an arithmetic subgroup $\Gamma_0 \subset G$ such that $\mathrm{Ad}(\Gamma_0)\mathcal{G}_{\mathbb{Z}} = \mathcal{G}_{\mathbb{Z}}$. Since Γ and Γ_0 are *commensurable* (that is, $\Gamma \cap \Gamma_0$ is of finite index in both Γ and Γ_0) and the validity of all assertions we will prove is unaffected by a passage from Γ to a commensurable subgroup (see Lemma 6.1 below), from now on we will (as we can) assume that

$$\mathrm{Ad}(\Gamma)\mathcal{G}_{\mathbb{Z}} = \mathcal{G}_{\mathbb{Z}}.$$

Let $\pi : G \to G/\Gamma$ be the natural quotient map. For a closed subgroup H of G, we let

$$\Gamma_H = \Gamma \cap H, \quad \text{and } \pi_H : H \to H/\Gamma_H$$

be the natural quotient map.

For any $x = \pi(g) \in G/\Gamma$, we will let

$$\mathcal{G}_x = \mathrm{Ad}(g)\mathcal{G}_{\mathbb{Z}}$$

(which makes sense in view of the above hypothesis).

2.2. **k-roots.** The facts of this subsection will be essential for the proof of Proposition 5.2. Let k be any field, $\mathbf{H}$ be a reductive k-algebraic group and $\mathbf{S}$ a maximal k-split algebraic torus. Let Φ be the set of k-roots with respect to $\mathbf{S}$, Φ^+ the set of positive k-roots (corresponding to a chosen minimal k-parabolic subgroup $\mathbf{B}$ containing $\mathbf{S}$) and Δ the subset of simple k-roots in Φ^+.(When k is not clear from the context we will write Φ_k, Φ_k^+ and Δ_k instead of Φ, Φ^+ and Δ, respectively.) We refer to [Bo1, §21] for the standard definitions related to the k-roots.

Let

$$\Phi_0 = \{\alpha \in \Phi : \frac{1}{2}\alpha \notin \Phi\}.$$

It is well known that Φ_0 is a *reduced* root system and the root systems Φ and Φ_0 have the same Weyl chambers, bases of simple roots and Weyl group (cf. [ViO, ch. 4, §2]).

For every $\alpha \in \Delta$ we define a projection $\pi_\alpha : \Phi \to \mathbb{Z}$ by $\pi_\alpha(\lambda) = n_\alpha$ where $\lambda = \sum_{\beta \in \Delta} n_\beta \beta$.

Let $\mathbf{P}$ be a standard parabolic subgroup, i.e. $\mathbf{B} \subset \mathbf{P}$. There exists $\Psi \subset \Delta$ such that $\mathbf{P}$ is a semi-direct product of $\mathcal{R}_u(\mathbf{P})$ and of its Levi subgroup $Z_{\mathbf{G}}(\mathbf{S}_\Psi)$, where $\mathbf{S}_\Psi = \bigcap_{\alpha \in \Delta \setminus \Psi} \ker(\alpha)$. Denote by $\mathcal{G}_\chi$ the root–space corresponding to $\chi \in \Phi$. Recall that

$$\text{Lie}(\mathcal{R}_u(\mathbf{P})) = \bigoplus_{\exists \alpha \in \Psi,\ \pi_\alpha(\chi) > 0} \mathcal{G}_\chi. \tag{1}$$

and

$$\text{Lie}(Z_{\mathbf{G}}(\mathbf{S}_\Psi)) = \text{Lie}(Z_{\mathbf{G}}(\mathbf{S})) \oplus \bigoplus_{\forall \alpha \in \Psi,\ \pi_\alpha(\chi) = 0} \mathcal{G}_\chi, \tag{2}$$

cf. [Bo1, §21.12].

3. Horospherical subsets and compactness criterion

In this section $\mathbf{G}$ is a connected *reductive* $\mathbb{Q}$-algebraic group.

3.1. **Siegel sets.** Let us fix a minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ and a maximal $\mathbb{Q}$-split $\mathbb{Q}$-subtorus $\mathbf{S}$ of $\mathbf{P}$. Denote by M the connected component of the identity in the (unique) maximal $\mathbb{Q}$-anisotropic subgroup of $Z_G(S)$, by K a maximal compact subgroup of G and by Δ the set of simple $\mathbb{Q}$-roots of $\mathbf{G}$ corresponding to the choice of $\mathbf{P}$. For every $r > 0$ we denote $B_r = \{x \in \mathcal{G} : \|x\| \leq r\}$. Also for every $\eta > 0$ we put

$$S_\eta = \{s \in S : \forall \alpha \in \Delta,\ \alpha(s) \leq \eta\}. \tag{3}$$

Following [Bo2, 12.3], by a *Siegel set* with respect to K, P and S we mean the set

$$\Sigma = \Sigma_{\eta,\omega} = K S_\eta \omega \tag{4}$$

where ω is a compact neighborhood of e in $M\mathcal{R}_u(P)$.

The Siegel sets are related to the fundamental sets. A subset $\Sigma \subset G$ is *fundamental for an arithmetic subgroup* Γ of G if the following conditions are fulfilled : $K\Sigma = \Sigma$, $\Sigma\Gamma = G$ and, for each $b \in \mathbf{G}(\mathbb{Q})$, the set

$$\{\gamma \in \Gamma : \Sigma b \cap \Sigma\gamma \neq \emptyset\}$$

is finite.

Recall the following classical result [Bo2, Theorem 15.5]:

Theorem 3.1 (Borel and Harish-Chandra)**.** *With the above notations there exists a Siegel set* Σ *and a finite subset* $C \subset \mathbf{G}(\mathbb{Q})$ *such that* $\Omega = \Sigma C$ *is a fundamental subset for* Γ *in* G*. Furthermore,* Ω *is compact if and only if* $rank_{\mathbb{Q}}\mathbf{G} = 0$ *and it has finite Haar measure if and only if* $\mathbf{G}^0$ *does not admit non-trivial* $\mathbb{Q}$*-characters.*

Note that if A is a precompact subset of $\mathcal{R}_u(P)$ and $\eta > 0$ then $\{sus^{-1} : s \in S_\eta,\ u \in A\}$ is also precompact [Bo2, Lemma 12.2]. Since S centralizes M it follows from (4), the definition of S_η and Theorem 3.1 that the following assertion holds.

Proposition 3.2. *There exist* $\eta_0 > 0$*, a compact subset* $L_0 \subset G$ *and a finite subset* $C_0 \subset \mathbf{G}(\mathbb{Q})$ *such that*

$$G = \Sigma_0 \Gamma, \quad \text{where } \Sigma_0 = L_0 S_{\eta_0} C_0.$$

The next proposition represents an infinitesimal analog (in the arithmetic case) of the classical result of Zassenhaus and Margulis-Kazhdan [Ra, Theorem 8.16].

Proposition 3.3. *There exists a compact neighborhood* W *of* 0 *in* $\mathcal{G}$ *such that for every* $x \in G/\Gamma$ *the subalgebra generated by* $W \cap \mathcal{G}_x$ *is unipotent.*

Proof. Let P^- be the parabolic subgroup of G opposite to P. Then

$$\mathcal{G} = \mathcal{U} \oplus \mathcal{Z} \oplus \mathcal{U}^- \tag{5}$$

where $\mathcal{U} = \mathrm{Lie}(\mathcal{R}_u(P))$, $\mathcal{U}^- = \mathrm{Lie}(\mathcal{R}_u(P^-))$ and $\mathcal{Z} = \mathrm{Lie}(Z_G(S))$. Note that the decomposition in (5) is defined over $\mathbb{Q}$ and therefore, after replacing $\mathcal{G}_{\mathbb{Z}}$ with a finite index lattice,

$$\mathcal{G}_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}} \oplus \mathcal{Z}_{\mathbb{Z}} \oplus \mathcal{U}^-_{\mathbb{Z}}, \tag{6}$$

where $\mathcal{U}_{\mathbb{Z}}$, $\mathcal{Z}_{\mathbb{Z}}$, $\mathcal{U}^-_{\mathbb{Z}}$ are the lattices of integer vectors in $\mathcal{U}$, $\mathcal{Z}$, $\mathcal{U}^-$ respectively.

We choose the norm $\|\cdot\|$ in such a way that

$$\|v\| = \max\{\|u\|, \|z\|, \|u^-\|\} \tag{7}$$

for all $v = u + z + u^- \in \mathcal{G}$, $u \in \mathcal{U}$, $z \in \mathcal{Z}$ and $u^- \in \mathcal{U}^-$.

Since $\mathrm{Ad}(q)\mathcal{G}_{\mathbb{Z}}$ is commensurable with $\mathcal{G}_{\mathbb{Z}}$ if $q \in \mathbf{G}(\mathbb{Q})$, there exists a positive integer n such that

$$\frac{1}{n}\mathcal{G}_{\mathbb{Z}} \supset \mathrm{Ad}(g)\mathcal{G}_{\mathbb{Z}} \tag{8}$$

for all $g \in C_0$.

Using Proposition 3.2 and (3) one can easily prove the existence of a constant $0 < c < 1$ such that

$$\|\mathrm{Ad}(h)w\| \geq c\|w\| \tag{9}$$

if $w \in \mathcal{G}$ and $h \in L_0$, or if $w \in \mathcal{U}^- \oplus \mathcal{Z}$ and $h \in S_{\eta_0}$.

Let $r > 0$ be such that $B_r \cap \mathcal{G}_{\mathbb{Z}} = \{0\}$ and $W = B_\epsilon$ where $\epsilon < \frac{rc^2}{n}$. Let $x = \pi(g)$, where $g = ksq$, $k \in L_0$, $s \in S_{\eta_0}$ and $q \in C_0$, and let $\mathrm{Ad}(g)v \in W$, where $v \in \mathcal{G}_{\mathbb{Z}}$, $v \neq 0$. Write $\mathrm{Ad}(q)v = v_1 + v_2$, where $v_1 \in \mathcal{U}^- \oplus \mathcal{Z}$ and $v_2 \in \mathcal{U}$. By (6) and (8) we have $nv_i \in \mathcal{G}_{\mathbb{Z}}$, $i = 1, 2$. Assume that $v_1 \neq 0$. By the choice of r we then have $\|v_1\| \geq r/n$ and in view of (7) and (9),

$$\begin{aligned}
\|\mathrm{Ad}(g)v\| &= \|\mathrm{Ad}(ks)(v_1 + v_2)\| \\
&\geq c\|\mathrm{Ad}(s)(v_1 + v_2)\| \\
&\geq c\|\mathrm{Ad}(s)v_1\| \\
&\geq c^2\|v_1\| \\
&\geq \frac{rc^2}{n} > \epsilon,
\end{aligned}$$

a contradiction. Therefore $v_1 = 0$, i.e. $\mathrm{Ad}(q)v = v_2 \in \mathcal{U}$ for all $v \in W \cap \mathcal{G}_x$, which completes the proof. □

3.2. **Horospherical Subsets.** Let us introduce the following

Definition 3.4. By a *horospherical subset* we mean a minimal (with respect to inclusion) finite subset of $\mathcal{G}$ which spans a subalgebra conjugate to the unipotent radical of a maximal parabolic $\mathbb{Q}$-subalgebra of $\mathcal{G}$.

Proposition 3.5 (Compactness Criterion). *A subset $A \subset G/\Gamma$ is precompact if and only if there exists a neighborhood W of 0 in $\mathcal{G}$ such that for all $x \in A$, $\mathcal{G}_x \cap W$ does not contain a horospherical subset.*

Proof. Suppose first that A is precompact. Thus there is a compact subset $K \subset G$ such that $A \subset \pi(K)$. From the continuity of Ad, there is a neighborhood W of 0 in $\mathcal{G}$ such that $\mathrm{Ad}(K)(\mathcal{G}_{\mathbb{Z}}) \cap W = \{0\}$. In particular for any $x \in A$, $\mathcal{G}_x \cap W = \{0\}$ does not contain a horospherical subset.

Now suppose A is not precompact. In view of Theorem 3.1 and Proposition 3.2 there exists a sequence $\{g_n\}$ such that $\pi(g_n) \in A$, $g_n =$

$k_n s_n f_n$ where $k_n \in L_0$, $s_n \in S_{\eta_0}$ and $f_n \in C_0$, and $g_n \to \infty$. Passing to a subsequence, we find a simple $\mathbb{Q}$-root α such that $\alpha(s_n) \to -\infty$, and assume that $f_n = f$ for some fixed $f \in C_0$. Let $\mathbf{P}_\alpha$ be the maximal $\mathbb{Q}$-parabolic corresponding to α, and let $\mathcal{V}_\alpha$ be the Lie algebra of $\mathcal{R}_u(P_\alpha)$. It follows from (1) that $\mathrm{Ad}(s_n)u \to 0$ for every $u \in \mathcal{V}_\alpha$. Let $u_1, \ldots, u_r$ be a basis for $\mathcal{V}_\alpha$ which is contained in $\mathcal{G}_{\mathbb{Z}}$. Multiplying all of the u_i by the common denominator of the coordinates of $\mathrm{Ad}(f^{-1})$, we may assume that $\mathrm{Ad}(f^{-1})u_i \in \mathcal{G}_{\mathbb{Z}}$ for all $i = 1, \ldots, r$. Since L_0 is compact we get

$$\lim_{n \to \infty} \mathrm{Ad}(g_n)(\mathrm{Ad}(f^{-1})u_i) = 0$$

for $i = 1, \ldots, r$. In particular any neighborhood W of 0 in $\mathcal{G}$ contains the horospherical subset $\{\mathrm{Ad}(g_n)(\mathrm{Ad}(f^{-1})u_1), ..., \mathrm{Ad}(g_n)(\mathrm{Ad}(f^{-1})u_r)\}$ for all sufficiently large n. $\square$

Remark 3.6. Let us compare Proposition 3.5 with other compactness criteria, retaining the same notation. Mahler's compactness criterion (cf. [Ra, Corollary 10.9]) is the statement that for $G = \mathrm{SL}(n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, a subset $\pi(A) \subset G/\Gamma$ is precompact if and only if there is a neighborhood W of 0 in $\mathbb{R}^n$ such that for all $x \in A$, $x\mathbb{Z}^n \cap W = \{0\}$.

Let $\mathbf{G}$ be an arbitrary semisimple $\mathbb{Q}$-algebraic group. Then the following holds: A subset $\pi(A) \subset G/\Gamma$ is precompact if and only if there is a neighborhood W of 0 in $\mathcal{G}$ such that for all $x \in A$, $\mathcal{G}_x \cap W = \{0\}$. This may be seen by identifying (up to commensurability) G/Γ with $\mathrm{Ad}(G)/\mathrm{Ad}(G) \cap \mathcal{G}_{\mathbb{Z}} \subset \mathrm{SL}(\mathcal{G})/\mathrm{SL}(\mathcal{G}_{\mathbb{Z}})$ and applying Mahler's compactness criterion to $\mathrm{SL}(\mathcal{G})/\mathrm{SL}(\mathcal{G}_{\mathbb{Z}})$.

For the problems considered in this paper, we need to use the observation that $\mathcal{G}_x$ contains a "small" vector (i.e. $\pi(x)$ is "far" from $\pi(e)$) if and only if $\mathcal{G}_x$ contains a finite set of "small" vectors spanding the unipotent radical of a parabolic $\mathbb{Q}$-subalgebra. Thus we arrive to Proposition 3.5 which reflects better than the above criteria the algebraic structure of G.

4. Pushing out

In the present section we assume that $\mathbf{T}$ is a maximal $\mathbb{R}$-split torus.

Proposition 4.1. *Suppose* $\mathbf{G}$ *is a reductive* $\mathbb{Q}$*-group. Then there exist a compact neighborhood* W *of* 0 *in* $\mathcal{G}$*, a constant* $c > 1$ *and a finite subset* F *of* T^0 *such that for every* $x \in G/\Gamma$ *there is* $f \in F$ *such that for all* $v \in \mathrm{span}(W \cap \mathcal{G}_x)$*,*

$$\|\mathrm{Ad}(f)v\| \geq c\|v\|.$$

Let $U_0 \subset G$ be a maximal unipotent subgroup normalized by T. Let $\mathcal{U}_0$ be the Lie algebra of U_0 and $r = \dim U_0$. Let $\wedge^r \mathcal{G}$ be the r-th exterior power of $\mathcal{G}$, $\mathrm{P}(\wedge^r \mathcal{G})$ the linear projective space corresponding to $\wedge^r \mathcal{G}$ and $\mathrm{Gr}_r(\mathcal{G})$ the Grassmannian subvariety of $\mathrm{P}(\wedge^r \mathcal{G})$. $\mathrm{Gr}_r(\mathcal{G})$ is compact and its points correspond bijectively to r-dimensional linear subspaces of $\mathcal{G}$. We fix a nonzero vector $a \in \wedge^r \mathcal{U}_0$ and denote by $[a]$ the point in $\mathrm{Gr}_r(\mathcal{G})$ corresponding to $\mathcal{U}_0$. The group G acts on $\mathrm{Gr}_r(\mathcal{G})$ via the adjoint representation. Put $X_0 = \mathrm{Ad}(G)[a]$. From the fact that all minimal parabolic subgroups are conjugate, it follows that X_0 is the *space of all maximal unipotent subalgebras of* $\mathcal{G}$. Since $P_0 = \{g \in G : \mathrm{Ad}(g)[a] = [a]\}$ is a (minimal) parabolic subgroup, the quotient G/P_0 is compact and therefore so is X_0. A simple argument using the continuity of the action of G on $\mathrm{Gr}_r(\mathcal{G})$ proves the following

Lemma 4.2. *Let $g \in G$ and $\mathcal{U} \in X_0$. Assume that there exists $c > 1$ such that*

$$\|\mathrm{Ad}(g)v\| \geq c\|v\|$$

for all vectors $v \in \mathcal{U}$, and let $1 < c' < c$.

(i) Then there exists a neighborhood W of $\mathcal{U}$ in X_0 such that for all $\mathcal{U}' \in W$ and all $v' \in \mathcal{U}'$

$$\|\mathrm{Ad}(g)v'\| \geq c'\|v'\|;$$

(ii) Assume in addition that $\mathcal{U} = \lim_{n \to +\infty} \mathrm{Ad}(g^n)\mathcal{U}_1$ where $\mathcal{U}_1 \in X_0$. Then for every $c_1 > 1$ there exists $n_1 > 0$ such that

$$\|\mathrm{Ad}(g^n)v_1\| \geq c_1\|v_1\|$$

for all $n > n_1$ and all $v_1 \in \mathcal{U}_1$.

We also need

Lemma 4.3. *Let $\mathcal{U} \in X_0$ and $c > 1$. Then there exists $t \in T^0$ such that*

$$\|\mathrm{Ad}(t)v\| \geq c\|v\|$$

for all $v \in \mathcal{U}$.

Proof. Let $\mathcal{U}_0$ and a be as above. We choose an order on the roots $\Phi_{\mathbb{R}}$ (equivalently, a basis of simple roots $\Delta_{\mathbb{R}}$), so that $\mathcal{U}_0$ is spanned by the root subspaces corresponding to all positive roots. Let

$$\wedge^r \mathcal{G} = \oplus V_\lambda \tag{10}$$

be the decomposition of $\wedge^r \mathcal{G}$ into a direct sum of weight subspaces, where λ_0 is the highest weight in (10). Then a spans V_{λ_0}. There exists $g \in G$ such that $\mathrm{Ad}(g)\mathcal{U}_0 = \mathcal{U}$. Let $g = unp$, where $u \in U_0$,

$n \in N_G(T)$ and $p \in P_0$ be the Bruhat decomposition of g. Denote by w the projection of n into the Weyl group $N_G(T)/Z_G(T)$. Clearly, the element $v_0 = \mathrm{Ad}(np)a$ belongs to $V_{w\lambda_0}$.

Let us say that a vector $v \in \wedge^r \mathcal{G}$ *dominates* v_0 if $v = v_0 + v_1$, where

$$v_1 \in \check{V} = \bigoplus_{\lambda > w\lambda_0} V_\lambda.$$

We now claim that if v dominates v_0 then so does $\mathrm{Ad}(u)(v)$.

To see this, we write $u = u_1 u_2 \cdots u_s$, where each u_i belongs to a root subgroup, that is, $u_i \in \exp(\mathcal{G}_{\chi_i}),\ \chi_i > 0$. By induction on s it suffices to prove our claim in case $s = 1$, that is in case $u = \exp(X)$ with $X \in \mathcal{G}_\chi,\ \chi > 0$. From the representation theory of sl(2) we know that for $v' \in V_\lambda,\ \ \mathrm{ad}_X^k(v') \in V_{\lambda + k\chi}$.

Now we compute:

$$\begin{aligned} \mathrm{Ad}(u)(v_0 + v_1) &= \mathrm{Ad}(\exp(X))(v_0) + \mathrm{Ad}(\exp(X))(v_1) \\ &= v_0 + \sum_{k \geq 1} \frac{1}{k!} \mathrm{ad}_X^k(v_0) + \sum_{k \geq 0} \frac{1}{k!} \mathrm{ad}_X^k(v_1) \\ &= v_0 + v_2, \end{aligned}$$

where $v_2 \in \check{V}$. This proves our claim. In particular we obtain that $\mathrm{Ad}(g)(a) = \mathrm{Ad}(u)(v_0)$ dominates v_0.

Let t be an element from the interior of the Weyl chamber corresponding to $w\Delta_{\mathbb{R}}$. The highest weight for the action of $\mathrm{Ad}(t)$ on $\wedge^r \mathcal{G}$ is $w\lambda_0$. Therefore $[v_0]$ is an attracting fixed point for the induced action of t on $P(\wedge^r \mathcal{G})$. The basin of attraction consists of all $[v]$ for which $v \in \wedge^r \mathcal{G}$ has a nonzero $V_{w\lambda_0}$ component. Since we have proved that $\mathrm{Ad}(g)(a)$ dominates v_0, it has a nonzero $V_{w\lambda_0}$ component, and hence

$$\lim_{m \to +\infty} \mathrm{Ad}(t^m)(\mathrm{Ad}(g)[a]) = \mathrm{Ad}(n)[a]$$

in X_0. Now the lemma follows from Lemma 4.2 (ii). □

Proof of Proposition 4.1. Using Lemma 4.3, Lemma 4.2 (i), and the compactness of X_0, we obtain for any $c > 1$ a finite subset $F \subset T^0$ such that for each unipotent subalgebra $\mathcal{U} \subset \mathcal{G}$ there exists an $f \in F$ with

$$\|\mathrm{Ad}(f)v\| \geq c\|v\|$$

for all $v \in \mathcal{U}$.

The proposition now follows by taking W as in Proposition 3.3. □

5. Characterization of $Z_G(S)\mathbf{G}(\mathbb{Q})$

The goal of this section is the proof of

Proposition 5.1. *Let* $\mathbf{G}$ *be a reductive* $\mathbb{Q}$*-algebraic group, and let* $g \in G$ *with* $\mathrm{Stab}_S(\pi(g))$ *finite. Suppose there are a compact subset* $C \subset \mathcal{G}$ *and* $r > 0$ *such that the following holds: for every* $d \in S$ *with* $\|d\| \geq r$ *there exists a horospherical subset* $\mathcal{H} \subset \mathrm{Ad}(g)\mathcal{G}_{\mathbb{Z}} \cap C$ *such that* $\mathrm{Ad}(d)\mathcal{H} \subset C$*. Then* $g \in Z_G(S)\mathbf{G}(\mathbb{Q})$*.*

The proof of the proposition relies on two propositions about parabolic subgroups, true in the context of any reductive k-algebraic groups, and on a certain rationality criterion.

5.1. **Intersections of Parabolic Subgroups.** In the following two propositions we suppose that $\mathbf{G}$ is a connected reductive algebraic group defined over an *arbitrary* field k. We use the notation from 2.2.

Proposition 5.2. *For every minimal parabolic* k*-subgroup* $\mathbf{B}$ *containing* $\mathbf{S}$ *we fix a proper parabolic* k*-subgroup* $\mathbf{P_B}$ *containing* $\mathbf{B}$*. Then*

$$\bigcap_{\mathbf{B}} \mathbf{P_B} = Z_{\mathbf{G}}(\mathbf{S}). \tag{11}$$

Proof. Let $\mathbf{J}$ be the subgroup of $\mathbf{G}$ defined by the left-hand side of (11). First we will show that $\mathrm{Lie}(\mathbf{J}) = \mathrm{Lie}(Z_{\mathbf{G}}(\mathbf{S}))$ which means that $Z_{\mathbf{G}}(\mathbf{S})$ has finite index in $\mathbf{J}$.

Note that $\mathrm{Lie}(\mathbf{J})$ is a sum of root spaces with respect to $\mathbf{S}$ (because $\mathbf{S} \subset \mathbf{J}$). Suppose by contradiction that λ is a nontrivial root with $\mathcal{G}_\lambda \subset \mathrm{Lie}(\mathbf{J})$. Let $\mathbf{B}$ be a minimal parabolic k-subgroup, and let $\mathbf{B}^-$ be the opposite minimal parabolic k-subgroup. Then $\mathcal{G}_\lambda$ is contained in the Levi factor of either $\mathrm{Lie}(\mathbf{P_B})$ or $\mathrm{Lie}(\mathbf{P_B^-})$, otherwise λ would be simultaneously positive and negative with respect to the order determined by $\mathbf{B}$.

It follows from the above and (2) that for any base of simple roots Δ there is $\beta \in \Delta$ such that $\pi_\beta(\lambda) = 0$. But all roots of a given length are conjugate [Hu, 10.4, Lemma C and 10.3, Theorem]. Therefore there is a basis of simple roots Δ_0 for which λ is a maximal long or a maximal short root in the reduced root system Φ_0. In order to obtain a contradiction it is enough to show that in this case $\pi_\beta(\lambda) \neq 0$ for all $\beta \in \Delta_0$. If λ is a maximal long root the fact is proved in [Hu, 10.4, Lemma A]. Let λ be a maximal short root. Then its dual $\lambda^\vee$ is a maximal long root in the dual root system $\Phi_0^\vee$ [Hu, Ex.11, p.55]. Applying again [Hu, 10.4, Lemma A], we get that $\pi_{\beta^\vee}(\lambda^\vee) \neq 0$ for all

$\beta^\vee \in \Delta^\vee$. Since $\gamma^\vee = \frac{2\gamma}{(\gamma,\gamma)}$ for any character γ of $\mathbf{S}$, we obtain that $\pi_\beta(\lambda) \neq 0$ for all $\beta \in \Delta_0$, as required.

To prove that $\mathbf{J} = Z_{\mathbf{G}}(\mathbf{S})$ we make the following observation: if $\mathbf{P}$ and $\mathbf{Q}$ are parabolic k-subgroups of $\mathbf{G}$ and $\mathbf{S} \subset \mathbf{P} \cap \mathbf{Q}$ then $(\mathbf{P} \cap \mathbf{Q})\mathcal{R}_u(\mathbf{P})$ is a parabolic k-subgroup [Bo1, Prop. 14.22 (i)] and $\mathbf{P} \cap \mathbf{Q}$ contains a Levi k-subgroup of $(\mathbf{P} \cap \mathbf{Q})\mathcal{R}_u(\mathbf{P})$ containing $\mathbf{S}$. Applying the observation successively to the subgroups $\mathbf{P_B}$ we obtain that there exists a parabolic k-subgroup $\mathbf{B}_0$ such that $\mathbf{B}_0 \supset \mathbf{J}$ and $\mathbf{J}$ contains a Levi subgroup of $\mathbf{B}_0$. Since $\mathrm{Lie}(\mathbf{J}) = \mathrm{Lie}(Z_{\mathbf{G}}(\mathbf{S}))$ and $Z_{\mathbf{G}}(\mathbf{S})$ is a connected reductive group [Bo1, Cor. 11.12], $Z_{\mathbf{G}}(\mathbf{S})$ is a Levi subgroup of $\mathbf{B}_0$, that is $\mathbf{B}_0$ is the semi-direct product of $Z_{\mathbf{G}}(\mathbf{S})$ and $\mathcal{R}_u(\mathbf{B}_0)$. In particular [Bo1, Corollary 14.19] its action on $\mathcal{R}_u(\mathbf{B}_0)$ by conjugation has no fixed points. On the other hand, it normalizes the finite subgroup $\mathbf{J} \cap \mathcal{R}_u(\mathbf{B}_0)$, and by connectedness centralizes it. So $\mathbf{J} \cap \mathcal{R}_u(\mathbf{B}_0)$ is trivial and therefore, $\mathbf{J} = Z_{\mathbf{G}}(\mathbf{S})$. □

Proposition 5.3. *Let* $\mathbf{B}$ *be a minimal parabolic* k*-subgroup of* $\mathbf{G}$ *and* $\mathbf{P}$ *be a parabolic subgroup of* $\mathbf{G}$ *such that* $\mathbf{P}$ *is conjugate to a* k*-subgroup of* $\mathbf{G}$ *and* $\mathcal{R}_u(\mathbf{P}) \subset \mathbf{B}$*. Then:*
(i) $\mathbf{B} \subset \mathbf{P}$ *and* $\mathbf{P}$ *is a* k*-group.*
(ii) If $g \in \mathbf{G}$ *and* $g\mathcal{R}_u(\mathbf{P})g^{-1} \subset \mathbf{B}$ *then* $g \in \mathbf{P}$*.*

Proof. Let $\mathbf{Q}$ be a parabolic k-subgroup containing $\mathbf{B}$ and conjugate to $\mathbf{P}$. Since $\mathbf{Q} \supset \mathcal{R}_u(\mathbf{P})$ it follows from [Bo1, Prop. 14.22 (iii)] that $\mathbf{P} = \mathbf{Q}$. This proves assertion (i).

Now let g be as in (ii). From (i) we have $\mathbf{B} \subset \mathbf{P}$ and applying (i) to $g\mathbf{P}g^{-1}$ in place of $\mathbf{P}$ we obtain $\mathbf{B} \subset g\mathbf{P}g^{-1}$. It follows from [Bo1, Cor. 11.17 (i)] that $\mathbf{P} = g\mathbf{P}g^{-1}$, hence $g \in N_{\mathbf{G}}(\mathbf{P}) = \mathbf{P}$. □

5.2. **Rationality Criterion.** Because of lack of reference we provide a short proof of the following apparently well known rationality criterion.

Proposition 5.4. *Let* $\mathbf{V}$ *be an affine* $\mathbb{Q}$*-algebraic variety and* $\mathbf{W}$ *be an* $\mathrm{Aut}(\mathbb{C})$*-invariant closed algebraic subvariety of* $\mathbf{V}$*. Then* $\mathbf{W}$ *is defined over* $\mathbb{Q}$*.*

Proof. It follows from the classical Hilbert basis theorem that $\mathbf{W}$ is defined over a finite extension of the field of rational functions of r variables $\mathbb{Q}(t)$ where $t = (t_1, ..., t_r)$. Since the restriction homomorphism $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(t)) \to \mathrm{Gal}(\overline{\mathbb{Q}(t)}/\mathbb{Q}(t))$ (where $\overline{\mathbb{Q}(t)}$ is the algebraic closure of $\mathbb{Q}(t)$ in $\mathbb{C}$) is surjective, in view of the proposition hypothesis $\mathbf{W}(\overline{\mathbb{Q}(t)})$ is $\mathrm{Gal}(\overline{\mathbb{Q}(t)}/\mathbb{Q}(t))$-stable. It follows from [Bo1, AG.14.4] that $\mathbf{W}$ is defined over $\mathbb{Q}(t)$. Let $\mathbf{V} \subset \mathbb{C}^n$ and let $f \in \mathbb{Q}(t)[x]$, where $x = (x_1, ..., x_n)$, be a polynomial which vanishes on $\mathbf{W}$. Multiplying f by a polynomial from $\mathbb{Q}[t]$ we can (and will) assume that $f \in \mathbb{Q}[t, x]$. In order to prove the proposition it is enough to show that

$$\{x_0 \in \mathbb{C}^n : f(t, x_0) = 0\} = \{x_0 \in \mathbb{C}^n : f(t_0, x_0) = 0, \ \forall t_0 \in \mathbb{Q}^r\}.$$

The inclusion "$\subset$" is obvious. On the other hand, if $f(t, x_0) \neq 0$ for some $x_0 \in \mathbb{C}^n$ then, since $\mathbb{Q}^r$ is Zariski dense in $\mathbb{C}^r$, there exists $x_0 \in \mathbb{Q}^r$ such that $f(t_0, x_0) \neq 0$. This proves the opposite inclusion. □

We also record the following well known fact.

Proposition 5.5. [Bo2, Proposition 7.7] *Suppose* $\mathbf{H}$ *is a reductive* $\mathbb{Q}$-*subgroup of* $\mathbf{G}$. *Then there is a* $\mathbb{Q}$-*representation* $\rho : \mathbf{G} \to \mathbf{GL}(\mathbf{V})$ *and* $v \in \mathbf{V}(\mathbb{Q})$ *such that* $\mathbf{H} = \{g \in \mathbf{G} : \rho(g)v = v\}$. *In particular,* $\rho(\mathbf{G}(\mathbb{Z}))v$ *is closed in* V *and* $H\pi(e)$ *is a closed orbit.*

Proof of Proposition 5.1. Let $\mathbf{B}$ be a minimal parabolic $\mathbb{Q}$-subgroup containing $\mathbf{S}$ and $\mathcal{B}$ be its Lie algebra. We claim that $\mathcal{B} \cap \mathrm{Ad}(g)\mathcal{G}_{\mathbb{Z}}$ contains a horospherical subset $\mathcal{H}$. Indeed, let $d \in S$ be an element in the interior of the positive Weyl chamber correpsonding to $\mathbf{B}$. This means that for any $v \notin \mathcal{B}$ we have

$$\lim_{n \to +\infty} \mathrm{Ad}(d^n)v = \infty. \tag{12}$$

Let r and C be as in the statement of the proposition. For every n with $\|d^n\| \geq r$ there is a horospherical subset $\mathcal{H}_n \subset \mathrm{Ad}(g)\mathcal{G}_{\mathbb{Z}} \cap C$ with $\mathrm{Ad}(d^n)\mathcal{H}_n \subset C$. Since $\mathrm{Ad}(g)\mathcal{G}_{\mathbb{Z}} \cap C$ is finite, the family of subsets $\{\mathcal{H}_n : n \in \mathbb{N}\}$ is finite. Therefore there is a horospherical subset $\mathcal{H} \subset \mathrm{Ad}(g)\mathcal{G}_{\mathbb{Z}} \cap C$ such that $\mathrm{Ad}(d^n)\mathcal{H} \subset C$ for infinitely many $n \in \mathbb{N}$. Now, using (12), we get that $\mathcal{H} \subset \mathcal{B}$, as claimed.

Let $\mathcal{V}$ denote the subalgebra generated by $\mathcal{H}$, and let $\sigma \in \mathrm{Aut}(\mathbb{C})$. Since $\mathcal{B}$ is defined over $\mathbb{Q}$, ${}^{\sigma}\mathcal{H} \subset \mathcal{B}$. On the other hand, $\mathrm{Ad}(g^{-1})\mathcal{H} \subset \mathcal{G}_{\mathbb{Z}}$ and σ acts trivially on $\mathcal{G}_{\mathbb{Z}}$. Therefore, ${}^{\sigma}\mathcal{H} = \mathrm{Ad}({}^{\sigma}gg^{-1})(\mathcal{H})$. So, $\mathrm{Ad}({}^{\sigma}gg^{-1})\mathcal{V} \subset \mathcal{B}$. Denote by $\mathbf{P_B}$ the normalizer of $\mathcal{V}$ in $\mathbf{G}$. Then, in view of Proposition 5.3, $\mathbf{B} \subset \mathbf{P_B}$, $\mathbf{P_B}$ is defined over $\mathbb{Q}$, and ${}^{\sigma}gg^{-1} \in \mathbf{P_B}$ for every minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{B}$. Using Proposition 5.2 we get that ${}^{\sigma}gg^{-1} \in Z_{\mathbf{G}}(\mathbf{S})$. Hence, ${}^{\sigma}(g^{-1}\mathbf{S}g) = g^{-1}\mathbf{S}g$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. Therefore $g^{-1}\mathbf{S}g$ is defined over $\mathbb{Q}$ (Proposition 5.4). It follows from Proposition 5.5 that $Sg\Gamma$ is closed. Since $\mathrm{Stab}_S(\pi(g))$ is finite (by assumption) we get that $g^{-1}\mathbf{S}g$ is a maximal $\mathbb{Q}$-split torus. By [Bo1, Theorem 20.9] there exists an $h \in \mathbf{G}(\mathbb{Q})$ such that $g^{-1}\mathbf{S}g = h^{-1}\mathbf{S}h$, i.e. $g \in N_G(S)\mathbf{G}(\mathbb{Q})$. Recall that $(N_G(S) \cap \mathbf{G}(\mathbb{Q}))Z_G(S) = N_G(S)$ (cf. [BoT, Theorem 5.3]). Therefore, $g \in Z_G(S)\mathbf{G}(\mathbb{Q})$. □

6. Proof of Theorem 1.3

We first make some reductions toward the proof of the theorem. The following standard proposition justifies passing from Γ to any commensurable subgroup:

Lemma 6.1. *Let Γ and Γ' be discrete commensurable subgroups of G, $\pi' : G \to G/\Gamma'$ the natural quotient map, and H be a closed subgroup of G. Then the following hold:*

1. *For any subset $A \subset G$, $\pi(A) \subset G/\Gamma$ is precompact if and only if $\pi'(A) \subset G/\Gamma'$ is precompact.*
2. *For any $g \in G$ the orbit $H\pi(g) \subset G/\Gamma$ is divergent (respectively, closed) if and only if the orbit $H\pi'(g) \subset G/\Gamma'$ is divergent (respectively, closed).*

Proof. Since Γ and Γ' are commensurable it is enough to prove the lemma for $\Gamma \subset \Gamma'$. Note that the natural map $\phi : G/\Gamma \to G/\Gamma'$, $\phi(\pi(g)) = \pi'(g)$, is proper and G-equivariant. This implies all the statements of the lemma, except the implication

$$H\pi'(g) \text{ is closed} \implies H\pi(g) \text{ is closed.}$$

In order to prove this implication note that $Hg\Gamma'$ is closed and $Hg\Gamma' = \bigcup_{\gamma\in\Gamma'} Hg\gamma$ is a countable union of right cosets of H. It follows from Baire's category theorem that one, and therefore each, of these cosets is open in $Hg\Gamma'$, and hence the complement of $Hg\Gamma$ in $Hg\Gamma'$ is open. This implies that $H\pi(g)$ is closed.

□

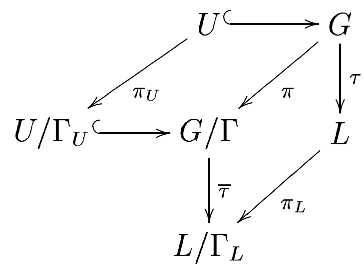
The following will be useful in reducing the proofs of Theorems 1.3 and 1.4 to the case that $\mathbf{G}$ is reductive.

Proposition 6.2 (Levi decomposition over $\mathbb{Q}$)**.** *Let $\mathbf{G}$ be a connected $\mathbb{Q}$-algebraic group, and let $\mathbf{H}$ be a reductive $\mathbb{Q}$-algebraic subgroup. Then there is a reductive $\mathbb{Q}$-subgroup $\mathbf{L}$ containing $\mathbf{H}$ such that $\mathbf{G}$ is a semidirect product defined over $\mathbb{Q}$ of $\mathbf{L}$ and $\mathcal{R}_u(\mathbf{G})$.*

Proof. See [BoS], Proposition 5.1. □

Applying Proposition 6.2 and using the facts that $\mathbf{T}$ is contained in a maximal $\mathbb{Q}$-subtorus of $\mathbf{G}$ (cf. [PrRa]) and Γ_N is cocompact in N for any unipotent $\mathbb{Q}$-group $\mathbf{N}$ [Ra, Chap. 3], we deduce:

Proposition 6.3. *Let $\tau : \mathbf{G} \to \mathbf{L}$ be the natural map, and let $\mathbf{U} = \mathcal{R}_u(\mathbf{G})$. Possibly after passing to a finite index subgroup, we have the following commutative diagram:*



Furthermore:

- $\Gamma_L\Gamma_U$ *is of finite index in* Γ.
- G/Γ *carries the structure of a fiber bundle, with* L/Γ_L *as base and* U/Γ_U *as a compact fiber.*
- *The orbit* $T\pi(x)$ *is divergent in* G/Γ *if and only if the orbit* $T\pi_L(\tau(x))$ *is divergent in* L/Γ_L.

Proof of Theorem 1.3. The torus T is a product of a compact real torus T_a and a maximal $\mathbb{R}$-diagonalizable real torus T_i. Assume that the theorem is true for T_i and let K be the corresponding compact subset. Then replacing K by T_aK one easily sees that the theorem is also true for T. Therefore we may (as we will) assume with no loss of generality that $\mathbf{T}$ is $\mathbb{R}$-split.

Assume first that $\mathbf{G}$ is a reductive group. Let W, c and F be the same as in the formulation of Proposition 4.1. Let $W_0 \subset \mathcal{G}$ be a ball centered at 0 and contained in $\bigcap_{f\in F} \mathrm{Ad}(f)W$. It is easy to see that:

(*) If $v \in \mathcal{G}$ and $v \notin W$ then $\mathrm{Ad}(f)v \notin W_0$ for any $f \in F$.

Let K be the closure of

$$\{x \in G/\Gamma : \mathcal{G}_x \cap W_0 \text{ does not contain a horospherical subset}\}.$$

In view of Proposition 3.5, K is compact. We will prove that K satisfies the conclusions of Theorem 1.3.

For every real $r > 0$ we denote by T_r the ball of radius r in T^0 centered at 1 and by C_r the smallest closed ball in $\mathcal{G}$ centered at 0 which contains

$$W \cup \bigcup_{d\in T_r^{-1}F} \mathrm{Ad}(d)(W).$$

(Recall that $\mathcal{G}$ and T^0 are endowed with norms. See §2.1.)

Let us fix an element $x \in G/\Gamma$. Note that only the following three mutually exclusive cases are possible :

(a) $\mathrm{Stab}_S(x)$ is finite and there exists $r > 0$ such that for every $d \in S^0$ with $\|d\| > r$, $\mathcal{G}_x \cap C_r$ contains a horospherical subset $\mathcal{H}$ such that $\mathrm{Ad}(d)\mathcal{H} \subset C_r$;

(b) $\mathrm{Stab}_S(x)$ is finite and for every $r > 0$ there exists $d(r) \in S^0$ with $\|d(r)\| > r$ such that $\mathrm{Ad}(d(r))\mathcal{H} \not\subset C_r$ for every horospherical subset $\mathcal{H}$ of $\mathcal{G}_x \cap C_r$;

(c) $\mathrm{Stab}_S(x)$ is infinite.

Now let us prove that $K \cap Tx \neq \emptyset$. Let $r > 0$. In each of the cases (a), (b) and (c) we will construct inductively a finite sequence $d_0, d_1, \ldots, d_n$ in T^0. (In fact, the sequence we construct will depend on r, only in case (b).) We put $d_0 = 1$ in the cases (a) and (c) and we put $d_0 = d(r)$ in the case (b). If $d_0 x \in K$ our sequence consists only of d_0, i.e. $n = 0$. Now assume that $d_0, \ldots, d_i,\ i \geq 0$, have already been chosen. Let $\tilde{d}_i = d_i d_{i-1} \cdots d_0$. Using Proposition 4.1, we fix an element $d_{i+1} \in F$ such that for every $w \in W \cap \mathrm{Ad}(\tilde{d}_i)(\mathcal{G}_x)$ we have

$$\|\mathrm{Ad}(d_{i+1})w\| \geq c\|w\|. \tag{13}$$

It follows from (*) and (13) that if w_i (resp. w_{i+1}) is a shortest nonzero vector in $W_0 \cap \mathrm{Ad}(\tilde{d}_i)(\mathcal{G}_x)$ (resp. $W_0 \cap \mathrm{Ad}(\tilde{d}_{i+1})(\mathcal{G}_x)$) then $\|w_{i+1}\| \geq c\|w_i\|$. Therefore, there exists an index n with the property: d_n is the first element in our sequence such that $W_0 \cap \mathrm{Ad}(\tilde{d}_n)(\mathcal{G}_x)$ does not contain a horospherical subset (equivalently, n is the first natural number for which $\tilde{d}_n x \in K$). This proves that $Tx \cap K \neq \emptyset$.

Now to complete the proof let us consider the cases (a), (b) and (c) separately and show that in each case at least one of the conditions (i), (ii) holds. In case (a), applying Proposition 5.1 (with $C = C_r$), we get that $g \in Z_G(S)\mathbf{G}(\mathbb{Q})$. Thus (a) implies (i). It remains to show that (b) implies (ii) and that (c) also implies (ii).

For case (b), since $\tilde{d}_n x \in K$ and $r > 0$ is arbitrary it is enough to show that $\tilde{d}_n \notin T_r$. If $n = 0$ then $\tilde{d}_n = d(r)$ and there is nothing to prove. Let $n > 0$. Assume that $\tilde{d}_n \in T_r$. By the choice of n, $W_0 \cap \mathrm{Ad}(\tilde{d}_{n-1})(\mathcal{G}_x)$ contains a horospherical subset $\mathcal{H}$. Denote $\mathcal{H}_1 = \mathrm{Ad}(\tilde{d}_{n-1}^{-1})\mathcal{H}$. Then $\mathcal{H}_1$ is a horospherical subset of $\mathcal{G}_x$. Since $\tilde{d}_{n-1}^{-1} = \tilde{d}_n^{-1} d_n \in T_r^{-1}F$ and $\mathcal{H} \subset W_0$, it follows from the definition of C_r that $\mathcal{H}_1 \subset C_r$. In view of the choice of d_0 and the definition of C_r, we have that $\mathrm{Ad}(d_0)\mathcal{H}_1 \not\subset W$. On the other hand, it follows from (*), (13) and the choice of the d_i's that if $v \in \mathcal{G}_x$ and $\mathrm{Ad}(d_0)(v) \notin W$ then $\mathrm{Ad}(\tilde{d}_{n-1})v \notin W_0$. Therefore, $\mathcal{H} = \mathrm{Ad}(\tilde{d}_{n-1})\mathcal{H}_1$ is not a subset of W_0. Contradiction.

Finally, suppose (c) holds (note that it is possible to construct examples where (c) holds using [To, Proposition 3.2]). Then there exists an element of infinite order $t \in \mathrm{Stab}_S(x)$. The sequence

$$\{\tilde{d}_n t^k : k \in \mathbb{Z}\}$$

is unbounded in T and satisfies $\tilde{d}_n t^k x \in K$ for all k. This completes the proof of the theorem in case $\mathbf{G}$ is reductive.

Now let $\mathbf{G}$ be an arbitrary $\mathbb{Q}$-algebraic group and $\mathbf{T}$ and $\mathbf{S}$ be as in the formulation of the theorem. Let $\mathbf{G}' = \mathbf{G}/\mathcal{R}_u(\mathbf{G})$, $\phi : \mathbf{G} \to \mathbf{G}'$ be the natural $\mathbb{Q}$-rational homomorphism, $\Gamma' = \phi(\Gamma)$, $\mathbf{T}' = \phi(\mathbf{T})$ and $\mathbf{S}' = \phi(\mathbf{S})$. We will also use the notation ϕ for the restricted map $\phi : G \to G'$. The homomorphism ϕ induces a natural surjective G-equivariant map $\psi : G/\Gamma \to G'/\Gamma'$. Let K' be a compact subset of G'/Γ' which satisfies the conclusions of the theorem for the reductive group $\mathbf{G}'$ and the tori $\mathbf{S}'$ and $\mathbf{T}'$. Since $\Gamma \cap \mathcal{R}_u(G)$ is a cocompact lattice of $\mathcal{R}_u(G)$, the map ψ is proper. Furthermore, $\phi(\mathbf{G}(\mathbb{Q})) = \mathbf{G}'(\mathbb{Q})$ (see 6.2). This implies readily that the compact $K = \psi^{-1}(K')$ has the required properties.

Proof of Corollary 1.6. Let Z be a closed T-invariant subset and let $Z_1 \supset Z_2 \supset \cdots$ be a descending sequence of closed invariant subsets of Z. By Zorn's lemma it suffices to show that $Z_\infty = \bigcap_i Z_i \neq \emptyset$. To see this, let K be the compact subset as in the first statement of Theorem 1.3. Then $K \cap Z_i$ is nonempoty for every i. So $K \cap Z_1 \supset K \cap Z_2 \supset \cdots$ is a descending sequence of compact sets. Therefore $\emptyset \neq \bigcap_i (K \cap Z_i) \subset Z_\infty$, as required.

□

7. Proofs of theorems 1.4 and 1.5

Proof of Theorem 1.4. Since the maximal $\mathbb{R}$-split torus of $\mathbf{T}$ is cocompact, we will assume without loss of generality that $\mathbf{T}$ is $\mathbb{R}$-split.

Using the facts that maximal $\mathbb{Q}$-split $\mathbb{Q}$-tori in $\mathbf{G}$ are conjugate under $\mathbf{G}(\mathbb{Q})$, and that $N_G(\mathbf{S}) \subset Z_G(\mathbf{S})\mathbf{G}(\mathbb{Q})$ we obtain that (2) and (3) are equivalent. Let us prove the equivalence of (1) and (2).

Using Proposition 6.3 we may assume in proving the theorem that $\mathbf{G}$ is reductive. Suppose first that $x \in Z_G(S)\mathbf{G}(\mathbb{Q})$ and $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} = \mathrm{rank}_{\mathbb{R}}\mathbf{G}$. Then $S = T$, and $\mathbf{T}_0 = x^{-1}\mathbf{T}x$ is defined over $\mathbb{Q}$ and $\mathbb{Q}$-split. By Proposition 5.5 $T_0\Gamma$ is closed, and hence the orbit map $T_0/T_0 \cap \Gamma \to G/\Gamma$ is proper. Since $\mathbf{T}_0$ is $\mathbb{Q}$-split, and using the fact that $\chi(T_0 \cap \Gamma) \subset \{\pm 1\}$ for any $\mathbb{Q}$-rational character χ on T_0, we see that $T_0 \cap \Gamma$ is finite. Hence $T_0\pi(e)$ is divergent. Therefore so is $T\pi(x) = xT_0\pi(e)$.

Now suppose that $T\pi(x)$ is divergent. Let $\mathbf{H} = Z_{\mathbf{G}}(\mathbf{S})$. Since alternative (ii) in the second statement of Theorem 1.3 does not hold, we have $x \in H\mathbf{G}(\mathbb{Q})$. Let us write $x = hq$ where $h \in H$ and $q \in \mathbf{G}(\mathbb{Q})$. The groups Γ and $q\Gamma q^{-1}$ being commensurable, we get that $T\pi(h)$ is also divergent in G/Γ. Since $\mathbf{H}$ is a reductive $\mathbb{Q}$-group, $H\pi(e)$ is closed in G/Γ and therefore $T\pi_H(h)$ is divergent in H/Γ_H. Note that $\mathbf{H}$ is an almost direct product over $\mathbb{Q}$ of $\mathbf{S}$ and a $\mathbb{Q}$-anisotropic subgroup $\mathbf{H}'$. Put $\mathbf{T} = \mathbf{S}\mathbf{T}'$, where $\mathbf{T}'$ is a maximal $\mathbb{R}$-split torus of $\mathbf{H}'$. Write $h = ys$ where $y \in H'$ and $s \in S$. Then $T'\pi_{H'}(y)$ is divergent in $H'/\Gamma_{H'}$. By Theorem 3.1, $H'/\Gamma_{H'}$ is compact and hence $T'\pi_{H'}(y)$ is a compact divergent orbit. This can only occur if $\mathbf{T}'$ is finite, thus $\mathbf{S} = \mathbf{T}$. This completes the proof. □

Proof of Theorem 1.5. Let $\mathbf{H} = Z_{\mathbf{G}}(\mathbf{T}_x)$ where $\mathbf{T}_x$ is as in the statement of the theorem. Let $\mathbf{H}_1 = \mathbf{H}/\mathbf{T}_x$, $\phi : \mathbf{H} \to \mathbf{H}_1$ be the natural $\mathbb{Q}$-homomorphism of $\mathbb{Q}$-algebraic groups and $\Gamma_1 = \phi(\Gamma_H)$. Note that Γ_1 is an arithmetic group (cf. [Bo2]). Also let $\mathbf{T}_1 = \phi(x^{-1}\mathbf{T}x)$, $\pi_1 : H_1 \to H_1/\Gamma_1$ the natural projection and $\bar{\phi} : H/\Gamma_H \to H_1/\Gamma_1$ the natural map induced by ϕ. Since $\Gamma \cap T_x$ is a cocompact lattice in T_x, the map $\bar{\phi}$ is proper.

Since $\bar{\phi}$ is proper and $x^{-1}Tx\pi_H(e) = \bar{\phi}^{-1}(T_1\pi_1(e))$, the orbit $x^{-1}Tx\pi_H(e)$ is closed in H/Γ_H if and only if the orbit $T_1\pi_1(e)$ is closed in H_1/Γ_1. Also, by the definition of $\mathbf{T}_x$, the orbit $T_1\pi_1(e)$ is divergent if it is closed. Applying Theorem 1.4 we get that $T_1\pi_1(e)$ is closed if and only if there exists $u_1 \in \mathcal{R}_u(H_1)$ such that $u_1^{-1}\mathbf{T}_1u_1$ is a product of a $\mathbb{Q}$-torus and an $\mathbb{R}$-anisotropic $\mathbb{R}$-torus. Since $\mathbf{T}_x$ is defined over $\mathbb{Q}$ and $\phi(\mathcal{R}_u(H)) = \mathcal{R}_u(H_1)$ (because $\mathbf{T}_x$ is a torus), we obtain that $x^{-1}Tx\pi_H(e)$ is closed if and only if there exists $u \in \mathcal{R}_u(H)$ such that $(xu)^{-1}\mathbf{T}xu$ is a product of a $\mathbb{Q}$-torus and an $\mathbb{R}$-anisotropic $\mathbb{R}$-torus. It is easy to see (by using, for example, Proposition 5.5) that the natural map $H/\Gamma_H \to G/\Gamma$ is proper and injective. Therefore, $T\pi(x)$ is closed if and only if $x^{-1}Tx\pi_H(e)$ is closed which, in view of of the above equivalences, implies that $T\pi(x)$ is closed if and only if there exists $u \in \mathcal{R}_u(H)$ such that $(xu)^{-1}\mathbf{T}xu$ is a product of a $\mathbb{Q}$ torus and an $\mathbb{R}$-anisotropic $\mathbb{R}$-torus. □

8. Examples and Open Questions

In this section $\mathbf{G}$ is always a *semisimple* $\mathbb{Q}$-algebraic group and $\mathbf{T}$ is a maximal $\mathbb{R}$-split torus.

Example 1. First we define a quaternion division algebra Δ over $\mathbb{Q}$ as follows. Put $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $j = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}$, $k = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$, and let 1 denote the identity matrix in $M(2, \mathbb{R})$.

Then it is easy to see by direct computation that $\Delta = \mathbb{Q}.1 + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ is a division algebra over $\mathbb{Q}$ and $\mathcal{O} = \mathbb{Z}.1 + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ is a ring. Since $\Delta \otimes_{\mathbb{Q}} \mathbb{R} \cong M(2, \mathbb{R})$ the matrix algebra $M(2, \Delta)$ is naturally imbedded in $M(4, \mathbb{R})$ and $M(2, \Delta) \otimes_{\mathbb{Q}} \mathbb{R} \cong M(4, \mathbb{R})$. Denote by $\mathbf{G}$ the $\mathbb{Q}$-algebraic group defined by $\mathbf{G}(\mathbb{Q}) = \{x \in M(2, \Delta) : \det(x) = 1\}$. Then $\mathbf{G}(\mathbb{R}) = \mathrm{SL}(4, \mathbb{R})$. Note that $\mathcal{G}_{\mathbb{Z}} = \{a \in M(2, \mathcal{O}) : \mathrm{trace}(a) = 0\}$ and $\Gamma = \mathrm{SL}(4, \mathbb{R}) \cap M(2, \mathcal{O})$ is an arithmetic subgroup of $\mathbf{G}(\mathbb{R})$.

In view of the above identifications the set of all matrices

$$d(x) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x^{-1} & 0 \\ 0 & 0 & 0 & x^{-1} \end{pmatrix}, x \in \mathbb{R}^*,$$

represents the group $\mathbf{S}(\mathbb{R})$ where $\mathbf{S}$ is a maximal $\mathbb{Q}$-split $\mathbb{Q}$-subtorus of $\mathbf{G}$. Let

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G, \ \alpha = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{G}.$$

Denote $g(x) = \sigma d(x)$. Then

$$\mathrm{Ad}(g(x))\alpha = \begin{pmatrix} 0 & -x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{G}.$$

Let $O(x)$ be the orbit $S\pi(g(x))$. Since $\mathrm{Ad}(S)$ fixes $\mathrm{Ad}(g(x))\alpha$, it follows from the Mahler compactness criterion (see Remark 3.6) that if K is a compact in G/Γ then there exists a positive ϵ depending on K such that $O(x) \cap K = \emptyset$ for all $0 < |x| < \epsilon$. Therefore if we act on G/Γ with S instead of T then in contrast to Theorem 1.3 there is no compact $K \subset G/\Gamma$ which intersects all S-orbits. The example also shows that the conclusion of Proposition 4.1 is generally false for the action of S instead of T.

Lemma 8.1. *With the above notation assume that* $\mathbf{T}$ *is a* $\mathbb{Q}$*-torus. Then*

$$N_G(S)\mathbf{G}(\mathbb{Q}) \cap N_G(T) \subset N_G(S).$$

Proof. Let $w = nq$, where $w \in N_G(T)$, $n \in N_G(S)$ and $q \in \mathbf{G}(\mathbb{Q})$. Then

$$w^{-1}\mathbf{S}w = q^{-1}\mathbf{S}q \subset \mathbf{T}.$$

Hence $w^{-1}\mathbf{S}w$ is a maximal $\mathbb{Q}$-split subtorus in $\mathbf{T}$. Since $\mathbf{T}$ has only one maximal $\mathbb{Q}$-split subtorus [Bo2, 8.15] we get that $\mathbf{S} = w^{-1}\mathbf{S}w$. $\square$

It is clear that if $x \in Z_G(S)\mathbf{G}(\mathbb{Q})$ then $S\pi(x)$ is divergent. Using Lemma 8.1, one can easily show that if $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} \neq \mathrm{rank}_{\mathbb{R}}\mathbf{G}$ then, in contrast with Corollary 1.1, there might exist divergent orbits $S\pi(x)$ such that $x \notin Z_G(S)\mathbf{G}(\mathbb{Q})$ (equivalently, such that $x^{-1}\mathbf{S}x$ is not a $\mathbb{Q}$-split $\mathbb{Q}$-torus).

Example 2. Let $\mathbf{G}$ be a $\mathbb{Q}$-algebraic group of type A_2 or G_2 such that $\mathrm{rank}_{\mathbb{R}}\mathbf{G} = 2$ and $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} = 1$ (cf. [Ti]). We fix a maximal $\mathbb{Q}$-subtorus $\mathbf{T}$ in $\mathbf{G}$ such that $\mathbf{T} = \mathbf{S} \times \mathbf{S}'$, where $\mathbf{S}$ is a maximal $\mathbb{Q}$-split subtorus of $\mathbf{G}$ and $\mathbf{S}'$ is a maximal $\mathbb{Q}$-anisotropic subtorus of $\mathbf{G}$. Let $W_{\mathbb{R}}$ be the Weyl group with respect to the $\mathbb{R}$-split torus $\mathbf{T}$. It is well known (and easy to see) that $W_{\mathbb{R}}$ contains an element which acts on the vector space of roots $\mathbb{R}^2$ as a rotation with an angle of $\frac{\pi}{3}$. Therefore there exists $w \in N_G(T)$ such that $wSw^{-1} \not\subseteq S \cup S'$. In view of the above lemma $w \notin N_G(S)\mathbf{G}(\mathbb{Q})$. Since $\mathbf{T}$ is defined over $\mathbb{Q}$ the orbit $T\pi(e)$ is closed and homeomorphic to T/Γ_T. But

$$T/\Gamma_T \cong S/\Gamma_S \times S'/\Gamma_{S'},$$

$S \cap \Gamma$ is finite, and $\pi_T(wSw^{-1}) \subset T/\Gamma_T$ is a closed non-compact subgroup. Hence the orbit $S\pi(w) \subset G/\Gamma$ is divergent although $w \notin N_G(S)\mathbf{G}(\mathbb{Q})$.

8.1. **Questions.** In view of Theorem 1.4 we have a satisfactory description of all divergent S-orbits if $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} = \mathrm{rank}_{\mathbb{R}}\mathbf{G}$. Let $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} \neq \mathrm{rank}_{\mathbb{R}}\mathbf{G}$. Comparing Example 2 with [Da, Theorem 6.1] it remains possible that all divergent orbits for S admit a simple description. In order to formulate a precise question we first make a definition generalizing the one in [Da]:

Definition 8.2. Let D be a subgroup of G, and let $g \in G$. We say that the orbit $D\pi(g)$ is a *degenerate divergent orbit* if there is a finite set of representations $\rho_i : \mathbf{G} \to \mathbf{GL}(\mathbf{V_i})$, $i = 1, \ldots, r$, defined over $\mathbb{Q}$, and $v_i \in \mathbf{V_i}(\mathbb{Q})$, such that for any divergent sequence $\{d_n\} \subset D$ there is a subsequence $\{d_{n_k}\}$ and $i \in \{1, \ldots, r\}$ such that

$$\lim_{k \to \infty} \rho_i(d_{n_k}g)v_i = 0.$$

It is easy to see that a degenerate divergent orbit is divergent. Note that the definition in [Da] is more restrictive as it describes explicitly the representations which occur.

We now ask:

Question 1. Is every divergent orbit for the action of S on G/Γ a degenerate divergent orbit?

We have seen that if $\dim S < \dim T$ then there are no divergent orbits for T, where T is a maximal $\mathbb{R}$-split torus. This raises:

Question 2. Suppose D is an $\mathbb{R}$-split torus with $\dim S < \dim D$. Are there any divergent orbits for D?

Appendix A. Proof of Margulis' result

We expose Margulis' proof of Theorem 1.2. In this section $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, and T is the group of diagonal matrices in G.

We begin with two facts about the action of T on $\mathbb{R}^n$.

Proposition A.1. *There is a ball $W \subset \mathbb{R}^n$, centered at 0, a finite set $F \subset T$, and $c > 1$ such that for every $g \in G$ there is $f \in F$ such that for every $w \in g\mathbb{Z}^n \cap W$ we have:*

$$\|fw\| \geq c\|w\|.$$

Proof. Every $g \in G$ has determinant equal to 1 and therefore preserves the volume element in $\mathbb{R}^n$. It follows that there is a small enough neighborhood W of 0 such that for every g, $\mathrm{span}(W \cap g\mathbb{Z}^n)$ is a proper linear subspace of $\mathbb{R}^n$. So it suffices to show that there is a finite $F \subset T$ and $c > 1$ such that for every proper linear subspace $V \subset \mathbb{R}^n$ there is $f \in F$ such that for all $v \in V$,

$$\|fv\| \geq c\|v\|.$$

By the compactness of the Grassmannian variety it suffices to show that for every proper subspace $V \subset \mathbb{R}^n$ there is $t \in T$ such that for every nonzero $v \in V$ we have $\|tv\| > \|v\|$. This is a simple exercise. □

Proposition A.2. *If $g \in G$ and $g \notin T\mathrm{SL}(n, \mathbb{Q})$ then for any neighborhood W of 0 in $\mathbb{R}^n$, any finite $J \subset g\mathbb{Z}^n - \{0\}$ and any compact $C \subset T$, there is $t \in T - C$ such that*

$$tJ \cap W = \emptyset.$$

Proof. Let $\{\mathbf{e}_i,\ i = 1, \ldots, n\}$ be the standard basis of $\mathbb{R}^n$. It is easy to verify that if $g \notin T\,\mathrm{SL}(n, \mathbb{Q})$ then there is some i such that

$$\mathbb{R}\mathbf{e}_i \cap g\mathbb{Z}^n = \{0\}.$$

Let $\alpha_i(t)$ be the diagonal matrix with $e^{-(n-1)t}$ in the i-th diagonal entry and e^t in all other diagonal entries. Then for any nonzero $w \in g\mathbb{Z}^n$, we have

$$\alpha_i(s)w \to_{s\to\infty} \infty.$$

Thus for all large enough s, we will have

$$\alpha_i(s)J \cap W = \emptyset.$$

□

Proof of Theorem 1.2. It is well known (see, for example, Proposition 5.5) that $T\pi(g)$ is divergent if $g \in T\,\mathrm{SL}(n,\mathbb{Q})$. Suppose $g \notin T\,\mathrm{SL}(n,\mathbb{Q})$. We will find a compact $K \subset G/\Gamma$ such that for every compact $C \subset T$, there is $t \in T - C$ such that $t\pi(g) \in K$, contradicting divergence.

Let W, F, c be as in Proposition A.1. Suppose with no loss of generality that $1 \in F$, and let

$$W_0 \subset \bigcap_{f\in F^{-1}} fW$$

be a ball around 0. It satisfies

$$\forall f \in F, \quad v \in W_0 \Longrightarrow fv \in W. \tag{14}$$

Define

$$K = \pi(\{x \in G : x\mathbb{Z}^n \cap W_0 = \{0\}\}).$$

By Mahler's compactness criterion, K is a compact subset of G/Γ.

Let $J = g\mathbb{Z}^n \cap C^{-1}W$, and using Proposition A.2, let t_0 be an element of $T - C$ such that $t_0 J \cap W = \{0\}$. Define inductively a sequence $t_0, t_1, \ldots$ as follows. If $t_0, \ldots, t_k$ have already been chosen, let $\tilde{t_k} = t_k t_{k-1} \cdots t_0$ and using Proposition A.1 let $t_{k+1} \in F$ be such that

$$w \in W \cap \tilde{t}_k g\mathbb{Z}^n \quad \Longrightarrow \quad \|t_{k+1}w\| \geq c\|w\|.$$

It follows from (14) that

$$W_0 \cap \tilde{t}_{k+1} g\mathbb{Z}^n \subset t_{k+1}(W \cap \tilde{t}_k g\mathbb{Z}^n)$$

and therefore the length of the shortest nonzero vector in $W_0 \cap \tilde{t}_{k+1} g\mathbb{Z}^n$ is at least c times the length of the shortest nonzero vector in $W_0 \cap \tilde{t}_k g\mathbb{Z}^n$. Thus for large enough k, we will have $\tilde{t}_k g\mathbb{Z}^n \cap W_0 = \{0\}$. Let k be the smallest index for which this is true. Clearly $\tilde{t}_k \pi(g) \in K$, and it remain to show that $\tilde{t}_k \notin C$. If $k = 0$ this follows from the choice of t_0. Suppose $k \geq 1$ and $\tilde{t}_k \in C$. By minimality of k, there is a nonzero vector $v \in W_0 \cap \tilde{t}_{k-1} g\mathbb{Z}^n$. By (14) and using induction on k, $v = \tilde{t}_{k-1} v_0$ for some nonzero $v_0 \in W_0 \cap g\mathbb{Z}^n$ and $\tilde{t}_j v_0 \in W_0$ for $j = 0, 1, \ldots, k-1$.

In particular $t_0 v_0 \in W_0$. Also by (14), $t_k v \in W$. So $\tilde{t}_k v_0 = t_k v \in W$ and hence $v_0 \in C^{-1}W$. Thus $v_0 \in J$ and $t_0 v_0 \in W_0$, contradicting the choice of t_0. □

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