# MODIFIED SCHMIDT GAMES AND DIOPHANTINE APPROXIMATION WITH WEIGHTS

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Abstract. We show that the sets of weighted badly approximable vectors in $\mathbb{R}^n$ are winning sets of certain games, which are modifications of $(\alpha, \beta)$-games introduced by W. Schmidt in 1966. The latter winning property is stable with respect to countable intersections, and is shown to imply full Hausdorff dimension.

## 1. Introduction

A classical result of Dirichlet states that for any $\mathbf{x} \in \mathbb{R}^n$ there are infinitely many $q \in \mathbb{N}$ such that $\|q\mathbf{x}-\mathbf{p}\| < q^{-1/n}$ for some $\mathbf{p} \in \mathbb{Z}^n$. One says that $\mathbf{x} \in \mathbb{R}^n$ is *badly approximable* if the right hand side of the above inequality cannot be improved by an arbitrary positive constant. In other words, if there is $c > 0$ such that for any $\mathbf{p} \in \mathbb{Z}^n$, $q \in \mathbb{N}$ one has

$$\|q\mathbf{x} - \mathbf{p}\| \geq \frac{c}{q^{1/n}} \,. \tag{1.1}$$

Here $\|\cdot\|$ can be any norm on $\mathbb{R}^n$, which unless otherwise specified will be chosen to be the supremum norm. We denote the set of all badly approximable vectors in $\mathbb{R}^n$ by $\mathbf{Bad}_n$, or $\mathbf{Bad}$ if the dimension is clear from the context. It is well known that Lebesgue measure of $\mathbf{Bad}$ is zero; but nevertheless this set is quite large. Namely it is *thick*, that is, its intersection with every open set in $\mathbb{R}^n$ has full Hausdorff dimension (Jarnik [J] for $n = 1$, Schmidt [S1, S3] for $n > 1$). In fact Schmidt established a stronger property of the set $\mathbf{Bad}$: that it is a so-called winning set for a certain game which he invented for that occasion, see §2 for more detail. In particular, the latter property implies that for any countable sequence of similitudes (compositions of translations and homotheties) $f_i : \mathbb{R}^n \to \mathbb{R}^n$, the intersection $\cap_i f_i(\mathbf{Bad})$ is thick as well.

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Our purpose in this paper is to introduce a modification of Scmidt's game, and apply it to similarly study a weighted generalization of the notion of badly approximable vectors. Take a vector $\mathbf{r} = (r_i \mid 1 \le i \le n)$ such that

$$r_i > 0 \quad \text{and} \quad \sum_{i=1}^{m} r_i = 1\,, \tag{1.2}$$

thinking of each $r_i$ as of a weight assigned to $x_i$. It is easy to show that the following multiparameter version of the aforementioned Dirichlet's result holds: for $\mathbf{r}$ as above and any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ there are infinitely many $q \in \mathbb{N}$ such that

$$\max_{1\le i\le n} |qx_i - p_i|^{1/r_i} < q^{-1} \text{ for some } \mathbf{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n\,. \tag{1.3}$$

This motivates the following definition: say that $\mathbf{x}$ is $\mathbf{r}$-*badly approximable* if the right hand side of (1.3) cannot be improved by an arbitrary positive constant; in other words, if there is $c > 0$ such that for any $\mathbf{p} \in \mathbb{Z}^n$, $q \in \mathbb{N}$ one has

$$\max_{1\le i\le n} |qx_i - p_i|^{1/r_i} \ge \frac{c}{q}\,. \tag{1.4}$$

Following [PV] and [KTV], denote by $\mathbf{Bad}(\mathbf{r})$ the set of $\mathbf{r}$-badly approximable vectors. It is not hard to make sense of the above definition when one or more of the components of $\mathbf{r}$ are equal to zero: one simply needs to ignore these components following a convention $a^\infty = 0$ when $0 \le a < 1$. For example, $\mathbf{Bad}(1,0) = \mathbf{Bad}_1 \times \mathbb{R}$ and $\mathbf{Bad}(0,1) = \mathbb{R} \times \mathbf{Bad}_1$. Also it is clear that $\mathbf{Bad}_n = \mathbf{Bad}(\mathbf{n})$ where

$$\mathbf{n} = (1/n, \dots, 1/n)\,. \tag{1.5}$$

One of the main results of [PV] states that the set $\mathbf{Bad}(\mathbf{r})$ is thick for any $\mathbf{r}$ as above (this was conjectured earlier in [K3]). A complete proof is given in [PV] for the case $n = 2$, but the method, based on some ideas of Davenport, straightforwardly extends to higher dimensions as noted by the authors of [PV]. A slightly different proof can be found in [KTV]. In this paper we present a modification (in our opinion, a simplification) of the argument from the aforementioned papers which yields a stronger result. Namely, in §§2–3 we describe a variation of Schmidt's game, which we call *modified Schmidt game* (to be abbreviated by MSG) induced by a family of contracting automorphisms of $\mathbb{R}^n$, and study properties of winning sets of those modified games. We show that winning sets of MSGs are thick (Corollary 3.4), and a countable intersection of sets winning for the same game is winning as well (Theorem 2.4). In §4 we prove

**Theorem 1.1.** *Let* $\mathbf{r}$ *be as in* (1.2), *and let* $\mathcal{F}^{(\mathbf{r})} = \{\Phi_t^{(\mathbf{r})} : t > 0\}$ *be the one-parameter semigroup of linear contractions of* $\mathbb{R}^n$ *defined by*

$$\Phi_t^{(\mathbf{r})} = \operatorname{diag}(e^{-(1+r_1)t}, \ldots, e^{-(1+r_n)t})\,. \tag{1.6}$$

*Then the set* $\mathbf{Bad}(\mathbf{r})$ *is a winning set for the modified Schmidt game (to be abbreviated by MSG) induced by* $\mathcal{F}^{(\mathbf{r})}$*; in particular, it is thick.*

Note that the original Schmidt's game can be viewed as a MSG induced by the family of homotheties of $\mathbb{R}^n$; thus Schmidt's theorem on **Bad** being a winning set is a special case of Theorem 1.1. The countable intersection property of winning sets of MSGs makes it possible to intersect $\mathbf{Bad}(\mathbf{r})$ with its countably many dilates and translates (see a remark after Theorem 4.2), as well as establish, in a simpler way, another result of [PV], namely that the set

$$\mathbf{Bad}(r_1, r_2) \cap \mathbf{Bad}(1, 0) \cap \mathbf{Bad}(0, 1) \tag{1.7}$$

is thick for any $0 < r_1, r_2 < 1$ with $r_1 + r_2 = 1$. This and other concluding remarks are made in §5.

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## 2. Modified Schmidt Games

2.1. **Schmidt's game.** Let $(E, d)$ be a complete metric space, and let $\Omega \stackrel{\text{def}}{=} E \times \mathbb{R}_+$ (the set of formal balls in $E$). Following [S1], define a partial ordering (Schmidt's containment) on $\Omega$ as follows:

$$(x', r') \le_s (x, r) \quad \Longleftrightarrow \quad d(x', x) + r' \le r\,. \tag{2.1}$$

To each pair $(x, r) \in \Omega$ we associate a closed ball in $E$ via the 'ball' function $B$: $B(x, r) \stackrel{\text{def}}{=} \{y \in E : d(x, y) \le r\}$. Note that $(x', r') \le_s (x, r)$ implies $B(x', r') \subset B(x, r)$; while in Euclidean space these conditions are in fact equivalent, in a general metric space the converse need not hold.

Now pick $0 < \alpha, \beta < 1$ and consider the following game, commonly referred to as *Schmidt's game*, played by two players, whom we will call[1] Alice and Bob. The game starts with Bob choosing $x_1 \in E$ and $r > 0$, hence specifying a pair $\omega_1 \stackrel{\text{def}}{=} (x_1, r)$. Alice may now choose any point $x_1' \in E$ provided that $\omega_1' \stackrel{\text{def}}{=} (x_1', \alpha r) \leq_s \omega_1$. Next, Bob chooses a point $x_2 \in E$ such that $\omega_2 \stackrel{\text{def}}{=} (x_2, \alpha\beta r) \leq_s \omega_1'$, and so on. Continuing in the same manner, one obtains a nested sequence of balls in $E$:

$$B(\omega_1) \supset B(\omega_1') \supset B(\omega_2) \supset B(\omega_2') \supset \ldots \supset B(\omega_k) \supset B(\omega_k') \supset \ldots$$

A subset $S$ of $E$ is called $(\alpha, \beta)$-*winning* if Alice can play in such a way that the unique point of intersection

$$\bigcap_{k=1}^{\infty} B(\omega_k) = \bigcap_{k=1}^{\infty} B(\omega_k') \tag{2.2}$$

lies in $S$, no matter how Bob plays. $S$ is called $\alpha$-*winning* if it is $(\alpha, \beta)$-winning for all $\beta > 0$, and *winning* if it is $\alpha$-winning for some $\alpha > 0$. We will denote balls chosen by Bob (resp., Alice) by $B_k \stackrel{\text{def}}{=} B(\omega_k)$ and $A_k \stackrel{\text{def}}{=} B(\omega_k')$.

The following three theorems are due to Schmidt [S1].

**Theorem 2.1.** *Let $S_i \subset E$, $i \in \mathbb{N}$, be a sequence of $\alpha$-winning sets for some $0 < \alpha < 1$; then $\cap_{i=1}^{\infty} S_i$ is also $\alpha$-winning.*

**Theorem 2.2.** *Suppose the game is played on $E = \mathbb{R}^n$ with the Euclidean metric; then any winning set is thick.*

**Theorem 2.3.** *For any $n \in \mathbb{N}$, $\mathbf{Bad}_n$ is $(\alpha, \beta)$-winning whenever $2\alpha < 1 + \alpha\beta$; in particular, it is $\alpha$-winning for any $0 < \alpha \leq 1/2$.*

It can also be shown that for various classes of continuous maps of metric spaces, the images of winning sets are also winning for suitably modified values of constants. See [S1, Theorem 1] and [D3, Proposition 5.3] for details.

2.2. **A modification.** We now introduce a variant of this game, which is in fact a special case of the general framework of $(\mathfrak{F}, \mathfrak{S})$-games described by Schmidt in [S1]. As before, let $E$ be a complete metric space, and let $\mathcal{C}(E)$ stand for the set of nonempty compact subsets of $E$. Fix

[1] Schmidt originally named his players 'white' and 'black'; in the subsequent literature letters $A$ and $B$ were often used instead. We are grateful to Andrei Zelevinsky for suggesting the Alice/Bob nomenclature following a convention common in computer science.

$t_* \in \mathbb{R} \cup \{-\infty\}$ and define $\Omega = E \times (t_*, \infty)^2$. Suppose in addition that we are given

(a) a partial ordering $\leq$ on $\Omega$, and
(b) a monotonic function $\psi : (\Omega, \leq) \to \big(\mathcal{C}(E), \subset\big)$.

Here monotonicity means that $\omega' \leq \omega$ implies $\psi(\omega') \subset \psi(\omega)$. Now fix $a_* \geq 0$ and suppose that the following property holds:

(MSG0) For any $(x, t) \in \Omega$ and any $s > a_*$ there exists $x' \in E$ such that $(x', t + s) \leq (x, t)$.

Pick two numbers $a$ and $b$, both bigger than $a_*$. Now Bob begins the $\psi$-$(a, b)$-game by choosing $x_1 \in E$ and $t_1 > t_*$, hence specifying a pair $\omega_1 \stackrel{\text{def}}{=} (x_1, t_1)$. Alice may now choose any point $x_1' \in E$ provided that $\omega_1' \stackrel{\text{def}}{=} (x_1', t_1 + a) \leq \omega_1$. Next, Bob chooses a point $x_2 \in E$ such that $\omega_2 \stackrel{\text{def}}{=} (x_2, t_1 + a + b) \leq \omega_1'$, and so on. Continuing in the same manner, one obtains a nested sequence of compact subsets of $E$:

$$B_1 = \psi(\omega_1) \supset A_1 = \psi(\omega_1') \supset \ldots \supset B_k = \psi(\omega_k) \supset A_k = \psi(\omega_k') \supset \ldots$$

where $\omega_k = (x_k, t_k)$ and $\omega_k' = (x_k', t_k')$ with

$$t_k = t_1 + (k-1)(a+b) \text{ and } t_k' = t_1 + (k-1)(a+b) + a\,. \tag{2.3}$$

Note that Bob and Alice can always make their choices by virtue of (MSG0), and that the intersection

$$\bigcap_{k=1}^{\infty} \psi(\omega_k) = \bigcap_{k=1}^{\infty} \psi(\omega_k') \tag{2.4}$$

is nonempty and compact. Let us say that $S \subset E$ is $(a, b)$-*winning* for the *modified Schmidt game corresponding to* $\psi$, to be abbreviated as $\psi$-MSG, if Alice can proceed in such a way that the set (2.4) is contained in $S$ no matter how Bob plays. Similarly, say that $S$ is an $a$-*winning* set of the game if $S$ is $(a, b)$-winning for any choice of $b > a_*$, and that $S$ is *winning* if it is $a$-winning for some $a > a_*$. Note that we are suppressing $a_*$ and $t_*$ from our notation, hopefully this will cause no confusion.

Clearly the game described above coincides with the original $(\alpha, \beta)$-game if we let

$$\begin{gathered} \psi(x,t) = B(x, e^{-t}), \quad (x', t') \leq (x, t) \Leftrightarrow (x', e^{-t'}) \leq_s (x, e^{-t}), \\ a = -\log\alpha,\ b = -\log\beta,\ a_* = 0,\ t_* = -\infty\,. \end{gathered} \tag{2.5}$$

[2]Note that everywhere one could replace $\mathbb{R}$ with some fully ordered semigroup. This more general setup presents no additional difficulties but we omit it to simplify notation.

Here is some more notation which will be convenient later. For $t > t_*$ we let

$$\Omega_t \stackrel{\text{def}}{=} \{(x,t) : x \in E\}\,,$$

so that $\Omega$ is a disjoint union of the 'slices' $\Omega_t$, $t > t_*$. Then for $s > 0$ and $\omega \in \Omega_t$ define

$$I_s(\omega) \stackrel{\text{def}}{=} \{\omega' \in \Omega_{t+s} : \omega' \le \omega\}\,.$$

In other words, $I_a(\omega)$ and $I_b(\omega)$ are the sets of allowed moves of Alice and Bob respectively starting from position $\omega$. Using this notation condition (MSG0) can be reworded as

(MSG0) $I_s(\omega) \ne \varnothing$ for any $\omega \in \Omega$, $s > a_*$.

2.3. **General properties.** Remarkably, even in the quite general setup described in §2.2, an analogue of Theorem 2.1 holds and can be proved by a verbatim repetition of the argument from [S1]:

**Theorem 2.4.** *Let a metric space $E$, partially ordered $\Omega = X \times (t_*, \infty)$ and $\psi$ be as above, let $a > a_*$, and let $S_i \subset E$, $i \in \mathbb{N}$, be a sequence of $a$-winning sets of the $\psi$-MSG. Then $\cap_{i=1}^{\infty} S_i$ is also $a$-winning.*

*Proof.* Take an arbitrary $b > a_*$, and make Alice play according to the following rule. At the first, third, fifth ... move Alice will make a choice according to an $(a, 2a+b, S_1)$-strategy (that is, will act as if playing an $(a, 2a+b)$-game trying to reach $S_1$). At the second, sixth, tenth ... move she will use an $(a, 4a+3b, S_2)$-strategy. In general, at the $k$th move, where $k \equiv 2^{i-1} (\operatorname{mod} 2^i)$, she will play the $\big(a, a+(2^i-1)(a+b)\big)$-game trying to reach a point in $S_i$. It is easy to see that, playing this way, Alice can enforce that the intersection of the chosen sets belongs to $S_i$ for each $i$. □

Here are two more general observations about MSGs and their winning sets.

**Lemma 2.5.** *Let $E$, $\Omega$ and $\psi$ be as above, and suppose that $S \subset E$, $a, b > a_*$ and $t_0 > t_*$ are such that whenever Bob initially chooses $\omega_1 \in \Omega_t$ with $t \ge t_0$, Alice can win the game. Then $S$ is an $(a,b)$-winning set of the $\psi$-MSG.*

*Proof.* Regardless of the initial move of Bob, Alice can make arbitrary (dummy) moves waiting until $t_k$ becomes at least $t_0$, and then apply the strategy he/she is assumed to have. □

This lemma shows that the collection of $(a,b)$-winning sets of a given $\psi$-MSG depends only on the 'tail' of the family $\{\Omega_t\}$ and not on the value of $t_*$.

**Lemma 2.6.** *Let $E_1, E_2$ be complete metric spaces, and consider two games corresponding to $\psi_i : \Omega_i \to \mathcal{C}(E_i)$, where $\Omega_i = E_i \times (t_*, \infty)$. Suppose that $S_i \subset E_i$ is an $(a,b)$-winning set of the $\psi_i$-MSG, $i = 1, 2$. Then $S_1 \times S_2$ is an $(a,b)$-winning set of the $\psi$-MSG played on $E = E_1 \times E_2$ with the product metric, where $\psi$ is defined by*

$$\psi(x_1, x_2, t) = \psi_1(x_1, t) \times \psi_2(x_2, t)\,.$$

*Proof.* Play a game in the product space by playing two separate games in each of the factors. □

It is also possible to write down conditions on $f : E \to E$, quite restrictive in general, sending winning sets of the $\psi$-MSG to winning sets. We will exploit this theme in §3.3.

2.4. **Dimension estimates.** Our next goal is to generalize Schmidt's lower estimate for the Hausdorff dimension of winning sets in $\mathbb{R}^n$. Note that in general it is not true, even for original Schmidt's game (2.5) played on an arbitrary complete metric space, that winning sets have positive Hausdorff dimension: see Proposition 5.2 for a counterexample. We are going to make some assumptions that will be sufficient to ensure that a winning set for the $\psi$-MSG is big enough. Namely we will assume:

(MSG1) For any open $\varnothing \neq U \subset E$ there is $\omega \in \Omega$ such that $\psi(\omega) \subset U$.

(MSG2) There exist $C, \sigma > 0$ such that $\operatorname{diam}\big(\psi(\omega)\big) \le Ce^{-\sigma t}$ for all $t \ge t_*$, $\omega \in \Omega_t$.

We remark that it follows from (MSG1) that any $(a,b)$-winning set of the game is dense, and from (MSG2) that the intersection (2.4) consists of a single point.

To formulate two additional assumptions, we suppose that we are given a locally finite Borel measure $\mu$ on $E$ satisfying the following conditions:

($\mu$1) $\mu\big(\psi(\omega)\big) > 0$ for any $\omega \in \Omega$.

($\mu$2) For any $a > a_*$ there exist $c, \rho > 0$ with the following property: $\forall\, \omega \in \Omega$ with $\operatorname{diam}\big(\psi(\omega)\big) \le \rho$ and $\forall\, b > a_*$ $\exists\, \theta_1, \dots, \theta_N \in I_b(\omega)$ such that $\psi(\theta_i)$, $i = 1, \dots, N$, are essentially disjoint, and that for every $\theta_i' \in I_a(\theta_i)$, $i = 1, \dots, N$, one has

$$\mu\Big(\bigcup_i \psi(\theta_i')\Big) \ge c\mu\big(\psi(\omega)\big)\,.$$

The utility of the latter admittedly cumbersome condition will become clear in the sequel, see Proposition 5.1. Here and hereafter we say that $A, B \subset E$ are *essentially disjoint* if $\mu(A \cap B) = 0$. In particular, it follows from ($\mu$1) and (MSG1) that such a measure $\mu$ must have full support (this will be our standing assumption from now on). Also, note that (MSG0) is a consequence of ($\mu 2$).

Now recall that the *lower pointwise dimension* of $\mu$ at $x \in E$ is defined by[3]

$$\underline{d}_\mu(x) \stackrel{\mathrm{def}}{=} \liminf_{r \to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r},$$

and for $U \subset E$ let us put

$$\underline{d}_\mu(U) \stackrel{\mathrm{def}}{=} \inf_{x \in U} \underline{d}_\mu(x).$$

It is known, see e.g. [Fa, Proposition 4.9(a)] or [Pe, Theorem 7.1(a)], that $\underline{d}_\mu(U)$ is a lower bound for the Hausdorff dimension of $U$ for any nonempty open $U \subset E$, and very often it is possible to choose $\mu$ such that $\underline{d}_\mu(x)$ is equal to $\dim(E)$ for every $x$. For instance this is the case when *$\mu$ satisfies a power law*, that is, if there exists $\gamma, c_1, c_2, r_0 > 0$ such that

$$c_1 r^\gamma \le \mu\big(B(x,r)\big) \le c_2 r^\gamma \text{ whenever } r \le r_0 \text{ and } x \in E \tag{2.6}$$

(then necessarily $\dim(U) = \gamma$ for any nonempty open $U \subset E$).

**Theorem 2.7.** *Suppose that $E$, $\Omega$, $\psi$ and a measure $\mu$ on $E$ are such that* (MSG0–2) *and* ($\mu$1–2) *hold. Take $a, b > a_*$ and let $S$ be an $(a,b)$-winning set of the $\psi$-MSG. Then for any open $\varnothing \ne U \subset E$, one has*

$$\dim(S \cap U) \ge \underline{d}_\mu(U) + \frac{1}{\sigma}\left(\frac{\log c}{a+b}\right), \tag{2.7}$$

*where $\sigma$ is as in* (MSG2) *and $c$ as in* ($\mu$2)*. In particular,* $\dim(S \cap U)$ *is not less than $\underline{d}_\mu(U)$ whenever $S$ is winning.*

Before proving this theorem let us observe that it generalizes Theorem 2.2, with Lebesgue measure playing the role of $\mu$. Indeed, conditions (MSG0–2) are trivially satisfied in the case (2.5). It is also clear that ($\mu$1) holds and that $\underline{d}_\mu(x) = n$ for all $x \in \mathbb{R}^n$. As for ($\mu$2), note that there exists a constant $\bar{c}$, depending only on $n$, such that for any $0 < \beta < 1$, the unit ball in $\mathbb{R}^n$ contains a disjoint collection of closed balls $D_i'$ of radius $\beta$ of relative measure at least $\bar{c}$; and no matter how balls $D_i \subset D_i'$ of radius $\alpha\beta$ are chosen, their total relative measure will not be less than $\bar{c}\alpha^n$. Rescaling, one obtains ($\mu$2). See Lemma 3.2 and Proposition 5.1 for further generalizations.

For the proof of Theorem 2.7 we will use a construction suggested in [Mc, U] and formalized in [KM]. Let $E$ be a complete metric space

[3]This and other properties, such as the Federer property introduced in §5.1, are usually stated for open balls, but versions with closed balls are clearly equivalent, modulo a slight change of constants if necessary.

equipped with a locally finite Borel measure $\mu$. Say that a countable family $\mathcal{A}$ of compact subsets of $E$ of positive measure is *tree-like* (or *tree-like with respect to* $\mu$) if $\mathcal{A}$ is the union of finite subcollections $\mathcal{A}_k$, $k \in \mathbb{Z}_+$, such that $\mathcal{A}_0 = \{A_0\}$ and the following four conditions are satisfied:

(TL0) $\mu(A) > 0$ for any $A \in \mathcal{A}$;

(TL1) $\forall\, k \in \mathbb{N} \quad \forall\, A, B \in \mathcal{A}_k \quad$ either $A = B \quad$ or $\quad \mu(A \cap B) = 0$;

(TL2) $\forall\, k \in \mathbb{N} \quad \forall\, B \in \mathcal{A}_k \quad \exists\, A \in \mathcal{A}_{k-1} \quad$ such that $\quad B \subset A$;

(TL3) $\forall\, k \in \mathbb{N} \quad \forall\, A \in \mathcal{A}_{k-1} \quad \exists\, B \in \mathcal{A}_k \quad$ such that $\quad B \subset A$.

Then one has $A_0 \supset \cup\mathcal{A}_1 \supset \cup\mathcal{A}_2 \ldots$, a decreasing intersection of nonempty compact sets (here and elsewhere we denote $\cup\mathcal{A}_k = \bigcup_{A \in \mathcal{A}_k} A$), which defines the (nonempty) *limit set* of $\mathcal{A}$,

$$\mathbf{A}_\infty = \bigcap_{k \in \mathbb{N}} \cup\mathcal{A}_k \,.$$

Let us also define the *kth stage diameter* $d_k(\mathcal{A})$ of $\mathcal{A}$:

$$d_k(\mathcal{A}) \stackrel{\text{def}}{=} \max_{A \in \mathcal{A}_k} \operatorname{diam}(A)\,,$$

and say that $\mathcal{A}$ is *strongly tree-like* if it is tree-like and in addition

(STL) $\qquad \lim_{k\to\infty} d_k(\mathcal{A}) = 0$.

Finally, for $k \in \mathbb{Z}_+$ let us define the $k$th stage 'density of children' of $\mathcal{A}$ by

$$\Delta_k(\mathcal{A}) \stackrel{\text{def}}{=} \min_{B \in \mathcal{A}_k} \frac{\mu\big(\cup\, \mathcal{A}_{k+1} \cap B\big)}{\mu(B)}\,,$$

the latter being always positive due to (TL3). The following lemma, proved in [KW1] and generalizing results of C. McMullen [Mc, Proposition 2.2] and M. Urbanski [U, Lemma 2.1], provides a needed lower estimate for the Hausdorff dimension of $\mathbf{A}_\infty$:

**Lemma 2.8.** *Let $\mathcal{A}$ be a strongly tree-like (relative to $\mu$) collection of subsets of $A_0$. Then for any open $U$ intersecting $\mathbf{A}_\infty$ one has*

$$\dim(\mathbf{A}_\infty \cap U) \ge \underline{d}_\mu(U) - \limsup_{k\to\infty} \frac{\sum_{i=0}^k \log \Delta_i(\mathcal{A})}{\log\, d_k(\mathcal{A})}\,.$$

Note that even though [KW1, Lemma 2.5] is stated for $E = \mathbb{R}^n$, its proof, including the Mass Distribution Principle on which the lower estimate for the Hausdorff dimension is based, is valid in the generality of an arbitrary complete metric space.

*Proof of Theorem 2.7.* Our goal is to find a strongly tree-like collection $\mathcal{A}$ of sets whose limit set is a subset of $S \cap U$. It will be constructed by considering possible moves for Bob at each stage of the game, and the corresponding counter-moves specified by Alice's winning strategy. Fix $a, b > a_*$ for which $S$ is $(a,b)$-winning. By assumption (MSG1), Bob may begin the game by choosing $t_1 > t_*$ and $\omega_1 \in \Omega_{t_1}$ such that $\psi(\omega_1) \subset U$ and $\operatorname{diam}\big(\psi(\omega_1)\big) < \rho$, where $\rho$ is as in ($\mu$2). Since $S$ is winning, Alice can choose $\omega_1' \in I_a(\omega)$ such that $A_0 \stackrel{\text{def}}{=} \psi(\omega_1')$ has nonempty intersection with $S$; it will be the ground set of our tree-like family.

Now let $\theta_1, \dots, \theta_N \in I_b(\omega_1')$ be as in ($\mu$2) for $\omega = \omega_1'$. Each of these could be chosen by Bob at the next step of the game. Since $S$ is $(a,b)$-winning, for each of the above choices $\theta_i$ Alice can pick $\theta_i' \in I_a(\theta_i)$ such that every sequence of possible further moves of Bob can be counteracted by Alice resulting in her victory in the game. The collection of images $\psi(\theta_i')$ of these choices of Alice, essentially disjoint in view of ($\mu$2), will comprise the first level $\mathcal{A}_1$ of the tree. Repeating the same for each of the choices we obtain $\mathcal{A}_2$, $\mathcal{A}_3$ etc. Property (TL0) follows from ($\mu$1), and (TL1–3) are immediate from the construction. Also, in view of (MSG2) and (2.3), the $k$th stage diameter $d_k$ is not bigger than $Ce^{-\sigma(t_1+k(a+b)+a)}$, hence (STL). Since Alice makes choices using her winning strategy, the limit set $\mathbf{A}_\infty$ of the collection must lie in $S$. Assumption ($\mu$2) implies that $\Delta_k(\mathcal{A})$ is bounded below by a positive constant $c$ independent of $k$ and $b$. Applying Lemma 2.8 we find

$$\begin{aligned}\dim(\mathbf{A}_\infty \cap U) &\geq \underline{d}_\mu(U) - \limsup_{k\to\infty} \frac{(k+1)\big(\log c\big)}{\log C - \sigma(t_1 + k(a+b) + a)} \\ &= \underline{d}_\mu(U) + \frac{1}{\sigma}\left(\frac{\log c}{a+b}\right) \to_{b\to\infty} \underline{d}_\mu(U)\,.\end{aligned}$$

□

## 3. Games induced by contracting automorphisms

3.1. **Definitions.** In this section we take $E = H$ to be a connected Lie group with a right-invariant Riemannian metric $d$, and assume that it admits a one-parameter group of automorphisms $\{\Phi_t : t \in \mathbb{R}\}$ such that $\Phi_t$ is contracting for $t > 0$ (recall that $\Phi : H \to H$ is *contracting* if for every $g \in H$, $\Phi^k(g) \to e$ as $k \to \infty$). It is not hard to see that $H$ must be simply connected and nilpotent, and the differential of each $\Phi_t$, $t > 0$, must be a linear isomorphism of the Lie algebra $\mathfrak{h}$ of $H$ with the modulus of all eigenvalues strictly less than 1. In other words, $\Phi_t = \exp(tX)$ where $X \in \operatorname{End}(\mathfrak{h})$ and the real parts of all eigenvalues of $X$ are negative. Note that $X$ is not assumed to be diagonalizable, although this will be the case in our main example.

Say that a subset $D_0$ of $H$ is *admissible* if it is compact and has non-empty interior. For such $D_0$ and any $t \in \mathbb{R}$ and $x \in H$, define

$$\psi(x,t) = \Phi_t(D_0)x\,, \tag{3.1}$$

and then introduce a partial ordering on $\Omega \stackrel{\text{def}}{=} H \times \mathbb{R}$ by

$$(x',t') \leq (x,t) \quad \Longleftrightarrow \quad \psi(x',t') \subset \psi(x,t)\,. \tag{3.2}$$

Monotonicity of $\psi$ is immediate from the definition, and we claim that, with $t_* = -\infty$ and some $a_*$, it satisfies conditions (MSG0–2). Indeed, let $\sigma > 0$ be any number such that the real parts of all the eigenvalues of $X$ are smaller than $-\sigma$. Then, since $D_0$ is bounded, it follows that for some $c_0 > 0$ one has

$$d\big(\Phi_t(g), \Phi_t(h)\big) \leq c_0 e^{-\sigma t} \tag{3.3}$$

for all $g, h \in D_0$, thus (MSG2) is satisfied (recall that the metric is chosen to be right-invariant, so all the elements of $\psi(\Omega_t)$ are isometric to $\Phi_t(D_0)$). For the same reasons, for any open $U \subset H$ there exists $s = s(U) > 0$ such that $U$ contains a translate of $\Phi_t(D_0)$ for any $t \geq s$, which implies (MSG1). Since $D_0$ is assumed to have nonempty interior, (MSG0) follows as well, with $a_* = s(\operatorname{Int} D_0)$.

We denote $\mathcal{F} \stackrel{\text{def}}{=} \{\Phi_t : t > 0\}$ and refer to the game determined by (3.1) and (3.2) as the modified Schmidt game *induced* by $\mathcal{F}$. Note that in this situation the map $\psi$ is injective, i.e. the pair $(x,t)$ is uniquely determined by $D_0$ and the translate $\Phi_t(D_0)x$. Consequently, without loss of generality we can describe the game in the language of choosing translates of $\Phi_a(D)$ or $\Phi_b(D)$ inside $D$, where $D$ is a domain chosen at some stage of the game. Clearly when $H = \mathbb{R}^n$, $D_0$ is a closed unit ball and $\Phi_t = e^{-t}\mathrm{Id}$, we recover Schmidt's original game.

Note also that we have suppressed $D_0$ from the notation. This is justified in light of the following proposition:

**Proposition 3.1.** *Let $D_0, D_0'$ be admissible, and define $\psi$ and $\psi'$ as in (3.1) using $D_0$ and $D_0'$ respectively. Let $s > 0$ be such that for some $x, x' \in H$,*

$$\Phi_s(D_0)x \subset D_0' \text{ and } \Phi_s(D_0')x' \subset D_0 \tag{3.4}$$

*(such an $s$ exists in light of (3.3) and the admissibility of $D_0, D_0'$). Suppose that $b > 2s$ and $S \subset H$ is $(a,b)$-winning for the $\psi$-MSG; then it is $(a+2s, b-2s)$-winning for the $\psi'$-MSG. In particular, if $S$ is $a$-winning for the $\psi$-MSG, then it is $(a+2s)$-winning for the $\psi'$-MSG.*

*Proof.* We will show how, using an existing $(a,b)$-strategy of the $\psi$-MSG, one can define an $(a+2s, b-2s)$-strategy for the $\psi'$-MSG. Given a translate $B'_k$ of $\Phi_t(D'_0)$ chosen by Bob at the $k$th step, pick a translate $B_k$ of $\Phi_{t+s}(D_0)$ contained in it, and then, according to the given $\psi$-winning strategy, a translate $A_k$ of $\Phi_{t+s+a}(D_0)$. In the latter one can find a translate of $\Phi_{t+2s+a}(D'_0)$; this will be the next choice $A'_k$ of Alice. Indeed, any move that could be made by Bob in response, that is, a translate $B'_{k+1}$ of $\Phi_{t+2s+a+b-2s}(D'_0) = \Phi_{t+a+b}(D'_0)$ inside $A'_k$, will contain a translate of $\Phi_{t+a+b+s}(D_0)$, and the latter can be viewed as a move responding to $A_k$ according to the $\psi$-strategy. Thus the process can be continued, eventually yielding a point from $S$ in the intersection of all the chosen sets. □

3.2. **A dimension estimate.** Choose a Haar measure $\mu$ on $H$ (note that $\mu$ is both left- and right-invariant since $H$ is unimodular). Our next claim is that conditions ($\mu$1–2) are also satisfied. Indeed, since $D_0$ is admissible, $\mu(D_0)$ is positive, and one has

$$\mu\big(\Phi_t(D_0)x\big) = e^{-\delta t}\mu(D_0) \quad \text{for any } x \in H,\ t \in \mathbb{R}\,, \tag{3.5}$$

hence ($\mu$1), where $\delta = -\operatorname{Tr}(X)$. Also, in view of (3.5) and the definition of $\psi$, to verify ($\mu$2) it suffices to show

**Lemma 3.2.** *Let $D_0$ be admissible and let $a_*$ be as in* (MSG0)*. Then there exists $\bar{c} > 0$ such that for any $b > a_*$, $D_0$ contains essentially disjoint right translates $D_1, \dots, D_N$ of $\Phi_b(D_0)$ such that*

$$\mu\Big(\bigcup_i D_i\Big) \ge \bar{c}\mu(D_0)\,. \tag{3.6}$$

Indeed, if this holds, then, in view of (3.5) the conclusion of the lemma holds with $D_0$ replaced by $\Phi_t(D_0)x$ for every $x$ and $t$, and then, by (MSG0) and (3.5), ($\mu$2) holds with $c = \bar{c}e^{-\delta a}$.

For the proof of Lemma 3.2 we use the following result from [KM]:

**Proposition 3.3.** *Let $H$ be a connected simply connected nilpotent Lie group. Then for any $r > 0$ there exists a neighborhood $V$ of identity in $H$ with piecewise-smooth boundary and with* $\operatorname{diam}(V) < r$*, and a countable subset $\Delta \subset H$ such that $H = \bigcup_{\gamma\in\Delta} \overline{V}\gamma$ and*

$$V\gamma_1 \cap V\gamma_2 = \varnothing \text{ for different } \gamma_1, \gamma_2 \in \Delta\,. \tag{3.7}$$

For example, if $H = \mathbb{R}^n$ one can take $V$ to be the unit cube,

$$V = \big\{(x_1, \dots, x_n) : |x_j| < 1/2\big\}\,,$$

and $\Delta = \mathbb{Z}^n$, or rescale both $V$ and $\Delta$ to obtain domains of arbitrary small diameter. See [KM, Proposition 3.3] for a proof of the above proposition.

*Proof of Lemma 3.2.* First note that, since $b$ is assumed to be greater than $a_*$, $D_0$ contains at least one translate of $\Phi_b(D_0)$, thus the left hand side of (3.6) is not less than $e^{-\delta b}\mu(D_0)$. Now, in view of (3.5), while proving the lemma one can replace $D_0$ by $\Phi_t(D_0)$ for any $t \geq 0$. Thus without loss of generality one can assume that $D_0$ is contained in $V$ as in Proposition 3.3 with $r \leq 1$, and that (3.3) holds $\forall\, g, h \in V$. Now choose a nonempty open ball $B \subset D_0$. We are going to estimate from below the measure of the union of sets of the form $\Phi_b(D_0\gamma)$, where $\gamma \in \Delta$, contained in $B$; they are disjoint in view of (3.7).

Since

$$\bigcup_{\gamma\in\Delta,\, \Phi_b(\overline{V}\gamma)\subset B} \Phi_b(\overline{V}\gamma) = \bigcup_{\gamma\in\Delta,\, \Phi_b(\overline{V}\gamma)\cap B\neq\varnothing} \Phi_b(\overline{V}\gamma) \smallsetminus \bigcup_{\gamma\in\Delta,\, \Phi_b(\overline{V}\gamma)\cap\partial B\neq\varnothing} \Phi_b(\overline{V}\gamma)\,,$$

we can conclude that the measure of the set in the left hand side is not less than

$$\mu(B) - \mu\left(\big\{\mathrm{diam}\big(\Phi_b(V)\big)\text{-neighborhood of } \partial B\big\}\right).$$

Clearly for any $0 < \varepsilon < 1$ the measure of the $\varepsilon$-neighborhood of $\partial B$ is bounded from above by $c'\varepsilon$ where $c'$ depends only on $B$. In view of (3.3) and (3.5), it follows that

$$\begin{aligned}\mu\left(\bigcup_{\gamma\in\Delta,\, \Phi_b(D_0\gamma)\subset B} \Phi_b(D_0\gamma)\right) &\geq \frac{\mu(D_0)}{\mu(V)}\big(\mu(B) - c_0c'e^{-\sigma b}\mathrm{diam}(V)\big)\\ &= \mu(D_0)\left(\frac{\mu(B)}{\mu(V)} - \frac{c_0c'\mathrm{diam}(V)}{\mu(V)}e^{-\sigma b}\right),\end{aligned}$$

which is not less than $\frac{\mu(B)}{2\mu(V)}\mu(D_0)$ if $e^{-\sigma b} \leq \mu(B)/2c_0c'\mathrm{diam}(V)$. Combining this with the remark made in the beginning of the proof, we conclude that (3.6) holds with

$$\bar{c} = \min\left(\frac{\mu(B)}{2\mu(V)}, \left(\frac{\mu(B)}{2c_0c'\mathrm{diam}(V)}\right)^{\delta/\sigma}\right).$$

□

In view of the discussion preceding Proposition 3.2, an application of Theorem 2.7 yields

**Corollary 3.4.** *Any winning set for the MSG induced by $\mathcal{F}$ as above is thick.*

3.3. **Images of winning sets.** One of the nice features of the original Schmidt's game is the stability of the class of its winning sets under certain maps, see e.g. [S1, Theorem 1] or [D3, Proposition 5.3]. We close this section by describing some self-maps of $H$ which send winning sets of the game induced by $\mathcal{F}$ to winning sets:

**Proposition 3.5.** *Let $\varphi$ be an automorphism of $H$ commuting with $\Phi_t$ for all $t$. Then there exists $s > 0$ (depending on $\varphi$ and the choice of an admissible $D_0$) such that the following holds. Take $t_0 \in \mathbb{R}$ and $x_0 \in H$, and consider*

$$f : H \to H,\ x \mapsto \Phi_{t_0}\big(\varphi(x)\big)x_0\,. \tag{3.8}$$

*Then for any $a > a_*$, $b > a_* + 2s$ and any $S \subset H$ which is $(a, b)$-winning for the MSG induced by $\mathcal{F}$, the set $f(S)$ is $(a + 2s, b - 2s)$-winning for the same game.*

*Proof.* Since $D_0$ is admissible and $\varphi$ is a homeomorphism, there exists $s > 0$ such that some translates of both $\varphi(D_0)$ and $\varphi^{-1}(D_0)$ contain $\Phi_s(D_0)$. Then, for $f$ as in (3.8), since $\varphi$ is a group homomorphism and $\Phi_t \circ \varphi = \varphi \circ \Phi_t$ for all $t$, it follows that

$$\begin{gathered}\text{for any } t \in \mathbb{R} \text{ and } x \in H \quad \exists\, x', x'' \in H \text{ such that}\\ f\big(\Phi_t(D_0)x\big) \supset \Phi_{t+t_0+s}(D_0)x',\ f^{-1}\big(\Phi_t(D_0)x\big) \supset \Phi_{t-t_0+s}(D_0)x''\,.\end{gathered} \tag{3.9}$$

Suppose that Alice and Bob are playing the game with parameters $(a + 2s, b - 2s)$ and target set $f(S)$. Meanwhile their clones $\widetilde{\text{Alice}}$ and $\widetilde{\text{Bob}}$ are playing with parameters $(a, b)$, and we are given a strategy for $\widetilde{\text{Alice}}$ to win on $S$. Let $B_k = \Phi_t(D_0)x$ be a move made by Bob at the $k$th stage of the game. Thus by (3.9), $f^{-1}(B_k)$ contains a set $\widetilde{B}_k = \Phi_{t-t_0+s}(D_0)y$ for some $y \in H$. Then, in response to $\widetilde{B}_k$ as if it were $\widetilde{\text{Bob}}$'s choice, $\widetilde{\text{Alice}}$'s strategy specifies $\widetilde{A}_k = \Phi_{t-t_0+s+a}(D_0)y' \subset \widetilde{B}_k$, a move which ensures convergence to a point of $S$. Again by (3.9), the set $f(\widetilde{A}_k)$ contains $A_k = \Phi_{t+a+2s}(D_0)x'$ for some $x' \in H$, which can be chosen by Alice as her next move. Now for any choice made by Bob of

$$B_{k+1} = \Phi_{t+a+2s+b-2s}(D_0)z = \Phi_{t+a+b}(D_0)z \subset A_k$$

Alice can proceed as above, since $f^{-1}(B_{k+1})$ will contain a valid move for $\widetilde{\text{Bob}}$ in response to $\widetilde{A}_k$. Continuing this way, Alice can enforce

$$\bigcap_{k=1}^{\infty} A_k = \bigcap_{k=1}^{\infty} f(\widetilde{A}_k) = f\left(\bigcap_{k=1}^{\infty} \widetilde{A}_k\right) \in f(S)\,,$$

winning the game. $\square$

## 4. $\mathbf{Bad}(\mathbf{r})$ IS WINNING

In this section we take $H = \mathbb{R}^n$ and prove Theorem 1.1, that is, exhibit a strategy for the MSG induced by $\mathcal{F}^{(\mathbf{r})}$ as in (1.6) which ensures that Alice can always zoom to a point in $\mathbf{Bad}(\mathbf{r})$. Our argument is similar to that from [PV], which in turn is based on ideas of Davenport. In view of remarks made at the end of the previous section, we can make an arbitrary choice for the initial admissible domain $D_0$, and will choose it to be the unit cube in $\mathbb{R}^n$, so that translates of $\Phi_t^{(\mathbf{r})}(D_t)$ are boxes with sidelengths $e^{-(1+r_1)t}, \dots, e^{-(1+r_n)t}$. The main tool will be the so-called 'simplex lemma', the idea of which is attributed to Davenport in [PV]. Here is a version suitable for our purposes.

**Lemma 4.1.** *Let $D \subset \mathbb{R}^n$ be a box with sidelengths $\rho_1, \dots, \rho_n$, and for $N > 0$ denote by $\mathcal{Q}(N)$ the set of rational vectors $\mathbf{p}/q$ written in lowest terms with $0 < q < N$. Also let $f$ be a nonsingular affine transformation of $\mathbb{R}^n$ and $J$ the Jacobian of $f$ (that is, the absolute value of the determinant of its linear part). Suppose that*

$$\rho_1 \cdots \rho_n < \frac{J}{n! N^{n+1}} \,. \tag{4.1}$$

*Then there exists an affine hyperplane $\mathcal{L}$ such that $f\big(\mathcal{Q}(N)\big) \cap D \subset \mathcal{L}$.*

*Proof.* Apply [KTV, Lemma 4] to the set $f^{-1}(D)$. □

Now let us state a strengthening of Theorem 1.1:

**Theorem 4.2.** *Let $\mathbf{r}$ be as in (1.2), $\Phi_t = \Phi_t^{(\mathbf{r})}$ as in (1.6), and $\psi$ as in (3.1) where $D_0 = [-1/2, 1/2]^n$. Then for any nonsingular affine transformation $f$ of $\mathbb{R}^n$ whose linear part commutes with $\Phi_1$ and any $a > \max_i \frac{\log 2}{1+r_i}$, $f\big(\mathbf{Bad}(\mathbf{r})\big)$ is an $a$-winning set for the $\psi$-MSG.*

We remark that in this case $a$ can be chosen independently of the linear part of $f$; note that this is not guaranteed by a general result as in Proposition 3.5, but relies on special properties of the set $\mathbf{Bad}(\mathbf{r})$. Consequently, in view of Theorem 2.4, for any sequence $\{L_i\}$ of nonsingular diagonal matrices and a sequence $\{\mathbf{y}_i\}$ of vectors in $\mathbb{R}^n$, the intersection $\bigcap_{i=1}^{\infty} \big(L_i\big(\mathbf{Bad}(\mathbf{r})\big) + \mathbf{y}_i\big)$ is also $a$-winning, hence thick.

*Proof of Theorem 4.2.* We first claim that $\mathbf{y} \in f\big(\mathbf{Bad}(\mathbf{r})\big)$ if and only if there is $c' > 0$ such that

$$\max_{1 \le i \le n} \left| y_i - f\left(\frac{\mathbf{p}}{q}\right)_i \right| \ge \frac{c'}{q^{1+r_i}} \tag{4.2}$$

for all $\mathbf{p} \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ (here $f(\mathbf{x})_i$ denotes the $i$th coordinate of $f(\mathbf{x})$).

To see this, let $\mathbf{r} = (r_i)$ be as in (1.2), and let

$$\mathbb{R}^n = \bigoplus V_j \tag{4.3}$$

be the eigenspace decomposition for $\Phi_1$. Letting $\{\mathbf{e}_i\}$ denote the standard basis of $\mathbb{R}^n$, we have that $V_j = \operatorname{span}(\mathbf{e}_i : i \in I_j)$ where $I_j$ is a maximal subset of $\{1, \ldots, n\}$ with $r_i$ the same for all $i \in I_j$. Since the linear part of $f$ commutes with $\Phi_1$, it preserves each $V_j$, so that we may write

$$f(\mathbf{x}) = \mathbf{x}_0 + \sum_j A_j P_j(\mathbf{x}),$$

where $\mathbf{x}_0 \in \mathbb{R}^n$, $P_j$ is the projection onto $V_j$ determined by the direct sum decomposition (4.3), and $A_j : V_j \to V_j$ is an invertible linear map. Let $K$ be a positive constant such that for each $\mathbf{x} \in V_j$,

$$\frac{1}{K}\|\mathbf{x}\| \le \|A_j\mathbf{x}\| \le K\|\mathbf{x}\|.$$

With these choices it is easy to show that (1.4) implies (4.2) for $\mathbf{y} = f(\mathbf{x})$ with $c' = c^{\max r_i}/K$, and similarly that (4.2) for $\mathbf{y} = f(\mathbf{x})$ implies (1.4) with $c = (c'/K)^{1+\max r_i^{-1}}$.

Let us fix $a > \max_i \frac{\log 2}{1+r_i}$ and $t_0 > 0$ such that

$$e^{-t_0(n+1)} < \frac{J}{2^n n!}\,, \tag{4.4}$$

where $J$ is the Jacobian of $f$. We will specify a strategy for Alice. Bob makes a choice of (arbitrarily large) $b$ and an initial rectangle, that is, a translate $B_1$ of $\Phi_{t_1}(D_0)$ for some $t_1$ which we demand to be at least $t_0$ (the latter is justified by Lemma 2.5). We then choose a positive constant $c'$ such that

$$e^{(a+b)(1+r_i)}c' < \left(\tfrac{1}{2} - e^{-a(1+r_i)}\right) e^{-t_0(1+r_i)} \tag{4.5}$$

for each $i$ (we remark that $\frac{1}{2} - e^{-a(1+r_i)}$ is positive because of the choice of $a$). Our goal will be to prove the following

**Proposition 4.3.** *For any choices of $B_1, \ldots, B_k$ made by Bob it is possible for Alice to choose $A_k \subset B_k$ such that whenever $\mathbf{y} \in A_k$, inequality (4.2) holds for all $\mathbf{p}, q$ with $0 < q < e^{(k-1)(a+b)}$.*

If the above claim is true, then the intersection point $\mathbf{y}$ of all balls will satisfy (4.2) for all $\mathbf{p} \in \mathbb{Z}^n$ and $q \in \mathbb{N}$, that is, will belong to $f\big(\mathbf{Bad}(\mathbf{r})\big)$. $\square$

*Proof of Proposition 4.3.* We proceed by induction on $k$. In case $k = 1$, the statement is trivially true since the set of $q \in \mathbb{N}$ with $0 < q < 1$ is empty. Now suppose that $A_1, \ldots, A_{k-1}$ are chosen according to the claim, and Bob picks $B_k \subset A_{k-1}$. Note that $B_k$ is a box with sidelengths $\rho_i \stackrel{\text{def}}{=} e^{-(t_1+(k-1)(a+b))(1+r_i)}$, $i = 1, \dots, n$.

By induction, for each $\mathbf{y} \in B_k \subset A_{k-1}$ (4.2) holds for all $\mathbf{p}, q$ with $0 < q < e^{(k-2)(a+b)}$. Thus we need to choose $A_k \subset B_k$ such that the same is true for

$$e^{(k-2)(a+b)} \le q < e^{(k-1)(a+b)} . \tag{4.6}$$

Let $\mathbf{p}/q$, written in lowest terms, be such that (4.6) holds, and that (4.2) does not hold for some $\mathbf{y} \in B_k$; in other words, for each $i$ one has

$$|y_i - f(\mathbf{p}/q)_i| < \frac{c'}{q^{1+r_i}} \le \frac{e^{(a+b)(1+r_i)}c'}{e^{(k-1)(a+b)(1+r_i)}}$$

for some $\mathbf{y} \in B_k$. Denote by $\tilde{\mathbf{y}}$ the center of $B_k$, so that

$$|y_i - \tilde{y}_i| \le \frac{\rho_i}{2} ;$$

then for each $i$,

$$|\tilde{y}_i - f(\mathbf{p}/q)_i| < \left(e^{(a+b)(1+r_i)}c' + e^{-t_0(1+r_i)}/2\right) e^{-(k-1)(a+b)(1+r_i)} \underset{(4.5)}{<} \rho_i .$$

Thus, if we denote by $D$ the box centered at $\tilde{\mathbf{y}}$ with sidelengths $2\rho_i$, we can conclude that $f(\mathbf{p}/q) \in D$; but also $\mathbf{p}/q \in \mathcal{Q}(e^{(k-1)(a+b)})$, and

$$2^n\rho_1 \cdots \rho_n = 2^n e^{-(t_0+(k-1)(a+b))(n+1)} \underset{(4.4)}{<} \frac{J}{n!\left(e^{(k-1)(a+b)}\right)^{n+1}} .$$

Therefore, by Lemma 4.1, there exists an affine hyperplane $\mathcal{L}$ containing all $f(\mathbf{p}/q)$ as above.

Clearly it will be advantageous for Alice is to stay as far from all those vectors as possible, i.e., choose $A_k \subset B_k$ to be a translate of $\Phi_{t_1+(k-1)(a+b)+a}(D_0)$ which maximizes the distance from $\mathcal{L}$. A success is guaranteed by the assumption $a > \max_i \frac{\log 2}{1+r_i}$, which amounts to saying that for each $i$, the ratio of the length of the $i$th side of the new box to the length of the $i$th side of $B_k$ is $e^{-a(1+r_i)} < 1/2$. This implies that for each $\mathbf{x} \in A_k$ chosen this way and any $\mathbf{x}' \in \mathcal{L}$, there exists $i$ such that $|x_i - x'_i|$ is not less than the length of the $i$th side of $B_k$ times $(\frac{1}{2} - e^{-a(1+r_i)})$. Therefore, whenever (4.6) holds and $\mathbf{x} \in A_k$, for some $i$ one has

$$\begin{aligned}
|x_i - f(\mathbf{p}/q)_i| &\ge e^{-(t_0+(k-1)(a+b))(1+r_i)}\left(\tfrac{1}{2} - e^{-a(1+r_i)}\right) \\
&= e^{-t_0(1+r_i)}\left(\tfrac{1}{2} - e^{-a(1+r_i)}\right)e^{-(a+b)(1+r_i)}e^{-((k-2)(a+b))(1+r_i)} \\
&\ge e^{-t_0(1+r_i)}\left(\tfrac{1}{2} - e^{-a(1+r_i)}\right)e^{-(a+b)(1+r_i)}q^{-(1+r_i)} \underset{(4.5)}{\ge} c'q^{-(1+r_i)} ,
\end{aligned}$$

establishing (4.2). □

## 5. Concluding remarks

5.1. **Dimension of winning sets for Schmidt's game.** The formalism developed in §§2–3 appears to be quite general, and we expect it to be useful in a wide variety of situations. In particular, new information can be extracted even for the original Schmidt's game. Namely, here we state a condition on a metric space sufficient to conclude that any winning set of Schmidt's game (2.5) has big enough dimension. This will be another application of Theorem 2.7. Recall that a locally finite Borel measure $\mu$ on a metric space $X$ is called *Federer*, or doubling, if there is $K > 0$ and $\rho > 0$ such that for all $x \in \operatorname{supp}\mu$ and $0 < r < \rho$,

$$\mu\big(B(x, 3r)\big) \le K\mu\big(B(x, r)\big)\,. \tag{5.1}$$

**Proposition 5.1.** *Let $E$ be a complete metric space which is the support of a Federer measure $\mu$. Then there exist $c_1, c_2 > 0$, depending only on $K$ as in* (5.1)*, such that whenever $0 < \alpha < 1$, $0 < \beta < 1/2$, and $S$ is an $(\alpha, \beta)$-winning set for Schmidt's game as in* (2.5) *played on $E$ and $\varnothing \ne U \subset E$ is open, one has*

$$\dim(S \cap U) \ge \underline{d}_\mu(U) - \frac{c_1|\log\alpha| + c_2}{|\log\alpha| + |\log\beta|}\,. \tag{5.2}$$

*In particular,* $\dim(S \cap U) \ge \underline{d}_\mu(U)$ *if $S$ is winning.*

Clearly Theorem 2.2 is a special case of the above result. In addition, Proposition 5.1 generalizes a recent result of L. Fishman [Fi, Thm. 3.1 and Cor. 5.3] that for a measure $\mu$ satisfying a power law (see (2.6); this condition obviously implies Federer) a winning set for Schmidt's original game (2.5) played on $E = \operatorname{supp}\mu$ has full Hausdorff dimension. See [KW1, Example 7.5] for an example of asubset of $\mathbb{R}$ (a similar construction is possible in $\mathbb{R}^n$ for any $n$) supporting a measure of full Hausdorff dimension which is Federer but does not satisfy a power law.

*Proof of Proposition 5.1.* We need to check the assumptions of Theorem 2.7. Conditions (MSG0–2) are immediate, and ($\mu$1) holds since $\operatorname{supp}\mu = E$. Thus it only suffices to verify ($\mu$2). It will be convenient to switch back to Schmidt's multiplicative notation of §2.1. Fix $0 < \alpha < 1$; we claim that there exists $c' > 0$ such that for any $x, x' \in E$ and $0 < r < \rho$ one has

$$B(x', \alpha r) \subset B(x, r) \implies \mu\big(B(x', \alpha r)\big) \ge c'\mu\big(B(x, r)\big)\,. \tag{5.3}$$

Indeed, choose $m \in \mathbb{N}$ and $c' > 0$ such that

$$\alpha/6 < 3^{-m} \le \alpha/2 \text{ and } c' = K^{-m}\,. \tag{5.4}$$

Iterating (5.1) $m$ times, we find $\mu\big(B(x',\alpha r)\big) \ge c'\mu\big(B(x',2r)\big)$ for any $x' \in E$ and $r > 0$. Since for any $x' \in B(x,r)$ the latter ball is contained in $B(x',2r)$, (5.3) follows.

Now take $0 < \beta < 1/2$ and $\omega = (x,r) \in E \times \mathbb{R}_+$ with $r < \rho$, and let $x_i$, $i = 1,\dots,N$, be a maximal collection of points such that $\theta_i \stackrel{\text{def}}{=} (x_i,\beta r) \le_s \omega$ and balls $B(\theta_i)$ are pairwise disjoint. By maximality,

$$B\big(x,(1-\beta)r\big) \subset \bigcup_{i=1}^{N} B(x_i,3\beta r)\,.$$

(Indeed, otherwise there exists $y \in B\big(x,(1-\beta)r\big)$ with $d(y,x_i) > 3\beta r$, which implies that $(y,\beta r) \le_s \omega$ and $B(y,\beta r)$ is disjoint from $B(\theta_i)$ for each $i$.) In view of (5.3), for any choices of $\theta_i' \stackrel{\text{def}}{=} (x_i',\alpha\beta r) \le_s \theta_i$ one has $\mu\big(B(\theta_i')\big) \ge c'\mu\big(B(\theta_i)\big)$. This implies

$$\begin{aligned}
\mu\big(\bigcup B(\theta_i')\big) &= \sum \mu\big(B(\theta_i)\big) \ge c' \sum \mu\big(B(\theta_i)\big) \\
&\ge \frac{c'}{K} \sum \mu\big(B(x_i,3\beta r)\big) \ge \frac{c'}{K}\mu\big(B(x,(1-\beta)r\big) \\
&\ge \frac{c'}{K}\mu\big(B(x,r/2)\big) \ge \frac{c'}{K^2}\mu\big(B(\omega)\big)\,.
\end{aligned}$$

Hence ($\mu 2$) holds with $c = c'/K^2$, and (5.2), with explicit $c_1$ and $c_2$, follows from (2.7) and (5.4). $\square$

The next proposition shows that, as was mentioned in §2.4, without additional assumptions on a metric space the conclusion of Theorem 2.2 could fail:

**Proposition 5.2.** *There exists a complete metric space $E$ of positive Hausdorff dimension containing a countable (hence zero-dimensional) winning set $S$ for the game* (2.5)*.*

*Proof.* Let $X = \{0,1,2\}^{\mathbb{N}}$, equipped with the metric

$$d\big((x_n),(y_n)\big) = 3^{-k}, \quad \text{where } k = \min\{j : x_j \ne y_j\}.$$

Let $E \subset X$ be the subset of sequences in which the digit 0 can only be followed by 0; i.e.

$$x_\ell = 0,\ k \ge \ell \implies x_k = 0.$$

Then $E$ is a closed subset of $X$ so is a complete metric space when equipped with the restriction of $d$.

Let $S$ be the set of sequences in $E$ for which the digit 0 appears. Then $S$ is a countable dense subset in $E$ but no point in $S$ is an accumulation point of $E \smallsetminus S$. In particular $\dim(S) = 0$, and it is easily checked that $\dim(E) = \log 2/\log 3 > 0$.

Let $\alpha = 1/27$, and let $\beta$ be arbitrary. Suppose that Bob chooses $\omega = (x, r)$, where $x = (x_n)$. Letting Alice play arbitrarily we can assume that $r < 1$. Let $\ell \in \mathbb{N}$ be chosen so that $3^{-(\ell+1)} < r \le 3^{-\ell}$. Note that $B(\omega)$ contains all sequences $(y_n)$ with $y_i = x_i$ for all $i \le \ell$, and in particular the sequence $z = (x_1, \ldots, x_\ell, x_{\ell+1}, 0, 0, \ldots) \in S$. Now Alice chooses $\omega' = (z, \alpha r)$; it is easy to see that $\omega' \le_s \omega$ and that $B(\omega') = \{z\}$ (a singleton), since any other sequence in this ball must begin with $(x_1, \ldots, x_\ell, x_{\ell+1}, 0)$. Thus the outcome of the game is $z$ and Alice is the winner. □

It is not hard to see that such an example can be realized as a compact subset of $\mathbb{R}$ with the induced metric (e.g. by identifying sequences $(x_n)$ with real numbers $0.x_1x_2\ldots$ expanded in base 3).

It is also worth remarking that another special case of our general framework is an $(\alpha, \beta)$-game played on arbitrary metric space $E$ but with Schmidt's containment relation (2.1) replaced by

$$(x', r') \le (x, r) \quad \Longleftrightarrow \quad B(x', r') \subset B(x, r)\,, \tag{5.5}$$

similarly to the way it was done in (3.2). The two conditions are equivalent when $E$ is a Euclidean space. However in general, e.g. when $E$ is a proper closed subset of $\mathbb{R}$ or $\mathbb{R}^n$ such as those considered in [Fi], (5.5) is weaker, and the classes of winning sets for the two games could differ. Still, by modifying the argument of this subsection one can show that the conclusions of both propositions hold when the game is played according to the weaker containment relationship.

5.2. **Sets of the form** (1.7) **and their generalizations.** Take $E = \mathbb{R}^n$ and let $\mathcal{F}$ be a one-parameter semigroup of its linear contracting transformations. Suppose that $E = E_1 \oplus E_2$ where both $E_1$ and $E_2$ are invariant under $\mathcal{F}$, denote by $\mathcal{F}_1$ the restriction of $\mathcal{F}$ to $E_1$, and suppose that $S_1 \subset E_1$ is a winning subset of the MSG induced by $\mathcal{F}_1$. Then it immediately follows from Lemma 2.6 that $S_1 \times E_2$ is a winning subset of the MSG induced by $\mathcal{F}$. Applying it to $\mathcal{F} = \mathcal{F}^{(\mathbf{r})}$ as in (1.6) we obtain

**Proposition 5.3.** *For* $\mathbf{r}$ *as in* (1.2) *and* $1 \le k \le n$, *define* $\mathbf{s} \in \mathbb{R}^k$ *by*

$$s_i = \frac{1 + (k+1)r_i - \sum_{l=1}^k r_l}{k + \sum_{l=1}^k r_l}\,, \quad i = 1, \ldots, k\,. \tag{5.6}$$

*Then* $\mathbf{Bad}(\mathbf{s}) \times \mathbb{R}^{n-k}$ *is a winning set for the MSG induced by* $\mathcal{F}^{(\mathbf{r})}$, *and therefore so is its intersection with* $\mathbf{Bad}(\mathbf{r})$.

*Proof.* Note that $\mathbf{s}$ is defined so that $\sum_i s_i$ is equal to 1, and the vector $(1+s_1,\dots,1+s_k)$ is proportional to $(1+r_1,\dots,1+r_k)$. Therefore the semigroup $\mathcal{F}^{(\mathbf{s})}$ is simply a reparameterization of the restriction of $\mathcal{F}^{(\mathbf{r})}$ to $\mathbb{R}^k$, and the claim follows from Lemma 2.6. □

It is clear that the winning property of the set (1.7) follows from a special case of the above proposition. The same scheme of proof, which seems to be much less involved than that of [PV], is applicable to multiple intersections of sets of weighted badly approximable vectors. E.g. given $\mathbf{r}\in\mathbb{R}^3$ with $r_1+r_2+r_3=1$, equation (5.6) can be used to define $s_{ij}$ for $i,j=1,\dots,3$, $i\neq j$, such that

$$\sum_i s_{ij} = 1 \text{ for } j=1,2,3\,, \tag{5.7}$$

and that

$$\begin{aligned}\mathbf{Bad}(\mathbf{r})\cap\mathbf{Bad}(s_{13},s_{23},0)\cap\mathbf{Bad}(s_{12},0,s_{32})\cap\mathbf{Bad}(0,s_{21},s_{31})\\ \cap\,\mathbf{Bad}(1,0,0)\cap\mathbf{Bad}(0,1,0)\cap\mathbf{Bad}(0,0,1)\end{aligned} \tag{5.8}$$

is a winning set for the MSG induced by $\mathcal{F}^{(\mathbf{r})}$, and therefore is thick. Take for example $\mathbf{r}=(\frac12,\frac13,\frac16)$; our conclusion is that

$$\begin{aligned}\mathbf{Bad}(\mathbf{r})\cap\mathbf{Bad}(\tfrac{10}{17},\tfrac{7}{17},0)\cap\mathbf{Bad}(\tfrac{9}{16},0,\tfrac{7}{16})\cap\mathbf{Bad}(0,\tfrac23,\tfrac13)\\ \cap\,\mathbf{Bad}(1,0,0)\cap\mathbf{Bad}(0,1,0)\cap\mathbf{Bad}(0,0,1)\end{aligned}$$

is thick. We remark that the assertion made in [PV, p. 32], namely that given $\mathbf{r}$ as above, the set (5.8) is thick for an *arbitrary* choice of $s_{ij}$ satisfying (5.7), does not seem to follow from either our methods of proof or those of [PV].

5.3. **Games and dynamics.** The appearance of the semigroup $\mathcal{F}^{(\mathbf{r})}$ in our analysis of the set $\mathbf{Bad}(\mathbf{r})$ can be naturally explained from the point of view of homogeneous dynamics. Let $G=\mathrm{SL}_{n+1}(\mathbb{R})$, $\Gamma=\mathrm{SL}_{n+1}(\mathbb{Z})$. The homogeneous space $G/\Gamma$ can be identified with the space of unimodular lattices in $\mathbb{R}^{n+1}$. To a vector $\mathbf{x}\in\mathbb{R}^n$ one associates a unipotent element $\tau(\mathbf{x})=\begin{pmatrix} I_n & \mathbf{x}\\ 0 & 1\end{pmatrix}$ of $G$, which gives rise to a lattice

$$\tau(\mathbf{x})\mathbb{Z}^{n+1}=\left\{\begin{pmatrix} q\mathbf{x}-\mathbf{p}\\ q\end{pmatrix} : q\in\mathbb{Z},\ \mathbf{p}\in\mathbb{Z}^n\right\}\in G/\Gamma\,.$$

Then, given $\mathbf{r}$ as in (1.2), consider the one-parameter subgroup $\{g_t^{(\mathbf{r})}\}$ of $G$, where

$$g_t^{(\mathbf{r})}\stackrel{\text{def}}{=}\mathrm{diag}(e^{r_1t},\dots,e^{r_nt},e^{-t})\,. \tag{5.9}$$

It was observed by Dani [D1] for $\mathbf{r} = \mathbf{n}$ and by the first named author [K2] for arbitrary $\mathbf{r}$ that $\mathbf{x} \in \mathbf{Bad}(\mathbf{r})$ if and only if the trajectory

$$\left\{g_t^{(\mathbf{r})}\tau(\mathbf{x})\mathbb{Z}^{n+1} : t \geq 0\right\}$$

is bounded in $G/\Gamma$. Note that the $g_t^{(\mathbf{r})}$-action on $G/\Gamma$ is partially hyperbolic, and it is straightforward to verify that the $\tau(\mathbb{R}^n)$-orbit foliation is $g_t^{(\mathbf{r})}$-invariant, and that the action on the foliation induced by the $g_t^{(\mathbf{r})}$-action on $G/\Gamma$ is realized by $\mathcal{F}^{(\mathbf{r})}$. Namely, one has

$$g_t^{(\mathbf{r})}\tau(\mathbf{x})y = g_t^{(\mathbf{r})}\tau\big(\Phi_{-t}^{(\mathbf{r})}(\mathbf{x})\big)y$$

for any $y \in G/\Gamma$.

Dani used Schmidt's result on the winning property of the set **Bad** and the aforementioned correspondence to prove that the set of points of $G/\Gamma$ with bounded $g_t^{(\mathbf{n})}$-trajectories, where $\mathbf{n}$ is as in (1.5), is thick. Later [KM] this was established for arbitrary flows $(G/\Gamma, g_t)$ 'with no non-trivial quasiunipotent factors'. In fact the following was proved: denote by $H^+$ the $g_1$-expanding horospherical subgroup of $G$, that is,

$$H^+ = \{h \in G : g_{-t}hg_t \to e \text{ as } t \to \infty\}\,;$$

then for any $y \in G/\Gamma$ the set

$$\big\{h \in H^+ : \{g_t hy : t \geq 0\} \text{ is bounded in } G/\Gamma\big\} \tag{5.10}$$

is thick. The main result of the present paper strengthens the above conclusion in the case $G = \mathrm{SL}_{n+1}(\mathbb{R})$, $\Gamma = \mathrm{SL}_{n+1}(\mathbb{Z})$ and $g_t = g_t^{(\mathbf{r})}$ as in (5.9). Namely, consider the subgroup $H = \tau(\mathbb{R}^n)$ of $H^+$ (the latter for generic $\mathbf{r}$ is isomorphic to the group of all upper-triangular unipotent matrices). Then for any $y \in G/\Gamma$, the intersection of the set (5.10) with an arbitrary coset $Hh'$ of $H$ in $H^+$ is winning for a certain MSG determined only by $\mathbf{r}$ (hence is thick). In particular, in view of Theorem 2.1 and Proposition 3.5, this implies that for an arbitrary countable sequence of points $y_k \in G/\Gamma$, the intersection of all sets $\big\{g \in G : \{g_t g y_k\} \text{ is bounded in } G/\Gamma\big\}$ is thick.

We note that the proof in [KM] is based on mixing of the $g_t$-action on $G/\Gamma$, while to establish the aforementioned stronger winning property mixing does not seem to be enough, and additional arithmetic considerations are necessary. In a recent work [KW3], for any flow $(G/\Gamma, g_t)$ with no nontrivial quasiunipotent factors we describe a class of subgroups $H$ of the $g_1$-expanding horospherical subgroup of $G$ which are normalized by $g_t$ and have the property that for any $y \in G/\Gamma$, the set

$$\big\{h \in H : \{g_t hy : t \geq 0\} \text{ is bounded in } G/\Gamma\big\} \tag{5.11}$$

is winning for the MSG induced by contractions $h \mapsto g_{-t}hg_t$. The argument is based on reduction theory for arithmetic groups, that is, on an analysis of the structure of cusps of arithmetic homogeneous spaces.

Another result obtained in [KW3] is that for $G$, $\Gamma$, $\{g_t\}$, $H$, $y$ as above and any $z \in G/\Gamma$, sets

$$\left\{h \in H : z \notin \overline{\{g_t hy : t \geq 0\}}\right\} \tag{5.12}$$

are also winning for the same MSG. Again this is a strengthening of results on the thickness of those sets existing in the literature, see [K1]. Combining the two statements above and using the intersection property of winning sets (5.11) and (5.12), one finds a way to construct orbits which are both bounded and stay away from a given countable subset of $G/\Gamma$, which settles a conjecture made by Margulis in [Ma].

5.4. **Systems of linear forms.** A special case of the general theorem mentioned in the previous subsection is a generalization of the main result of the present paper to the case of systems of linear forms. Namely, let $m, n$ be positive integers, denote by $M_{m,n}$ the space of $m \times n$ matrices with real entries (system of $m$ linear forms in $n$ variables), and say that $Y \in M_{m,n}$ is $(\mathbf{r},\mathbf{s})$-*badly approximable* if

$$\inf_{\mathbf{p}\in\mathbb{Z}^m,\, \mathbf{q}\in\mathbb{Z}^n\smallsetminus\{0\}} \max_i |Y_i\mathbf{q} - p_i|^{1/r_i} \cdot \max_j |q_j|^{1/s_j} > 0\,,$$

where $Y_i$, $i = 1, \dots, m$ are rows of $Y$ and $\mathbf{r} \in \mathbb{R}^m$ and $\mathbf{s} \in \mathbb{R}^n$ are such that

$$r_i, s_j > 0 \quad \text{and} \quad \sum_{i=1}^m r_i = 1 = \sum_{j=1}^n s_j\,. \tag{5.13}$$

(Here the components of vectors $\mathbf{r},\mathbf{s}$ can be thought of as weights assigned to linear forms $Y_i$ and integers $q_j$ respectively.) The correspondence described in the previous subsection extends to the matrix set-up, with $G = \mathrm{SL}_{m+n}(\mathbb{R})$ and $\Gamma = \mathrm{SL}_{m+n}(\mathbb{Z})$ and

$$g_t^{(\mathbf{r},\mathbf{s})} = \mathrm{diag}(e^{r_1 t}, \dots, e^{r_m t}, e^{-s_1 t}, \dots, e^{-s_n t})$$

acting on $G/\Gamma$. This way one can show that the set $\mathbf{Bad}(\mathbf{r},\mathbf{s}) \subset M_{m,n}$ of $(\mathbf{r},\mathbf{s})$-badly approximable systems is winning for the MSG induced by the semigroup of contractions $\Phi_t : (y_{ij}) \mapsto (e^{-(r_i+s_j)t}y_{ij})$ of $M_{m,n}$ (a special case where all weights are equal is a theorem of Schmidt [S2]). This generalizes Theorem 1.1 and strengthens [KW2, Corollary 4.5] where it was shown that $\mathbf{Bad}(\mathbf{r},\mathbf{s})$ is thick for any choice of $\mathbf{r},\mathbf{s}$ as in (5.13).

5.5. **Playing games on other metric spaces.** The paper [KTV], where it was first proved that the set of weighted badly approximable vectors in $\mathbb{R}^n$ has full Hausdorff dimension, contains a discussion of analogues of the sets $\mathbf{Bad}(\mathbf{r})$ over local fields other than $\mathbb{R}$. In [KTV, §§5.3–5.5] it is explained how to apply the methods of [KTV, §§2–4] to studying weighted badly approximable vectors in vector spaces over $\mathbb{C}$ as well as over non-Archimedean fields[4]. Similarly one can apply the methods of the present paper to replace Theorems 17–19 of [KTV] by stronger statements that the corresponding sets are winning sets of certain MSGs. For that one needs to generalize the set-up of §3 and consider modified Schmidt games induced by contracting automorphisms of arbitrary locally compact topological groups (not necessarily real Lie groups).

Another theme of the papers [KTV] and [KW1] is intersecting the set of badly approximable vectors with some nice fractals in $\mathbb{R}^n$. For example [KTV, Theorem 11], slightly generalized in [KW1, Theorem 8.4], states the following: let $\mu = \mu_1 \times \cdots \times \mu_d$, where each $\mu_i$ is a measure on $\mathbb{R}$ satisfying a power law (called 'condition (A)' in [KTV]); then $\dim\big(\mathbf{Bad}(\mathbf{r}) \cap \operatorname{supp}\mu\big) = \dim(\operatorname{supp}\mu)$. Following an approach developed recently by Fishman [Fi], it seems possible to strengthen this result; in particular, one can consider a modified Schmidt game played on $E = \operatorname{supp}\mu$, with $\mu$ as above, and prove that the intersection of $E$ with $\mathbf{Bad}(\mathbf{r})$ is a winning set of this game.

5.6. **Schmidt's Conjecture.** Finally we would like to mention a question posed by W. Schmidt [S4] in 1982: is it true that for $\mathbf{r} \neq \mathbf{r}'$, the intersection of $\mathbf{Bad}(\mathbf{r})$ and $\mathbf{Bad}(\mathbf{r}')$ is nonempty? Schmidt conjectured that the answer is affirmative in the special case $n = 2$, $\mathbf{r} = (\frac{1}{3}, \frac{2}{3})$ and $\mathbf{r}' = (\frac{2}{3}, \frac{1}{3})$, pointing out that disproving his conjecture would amount to proving Littlewood's Conjecture (see [EKL] for its statement, history and recent developments). Unfortunately, the results of the present paper do not give rise to any progress related to Schmidt's Conjecture. Indeed, each of the weight vectors $\mathbf{r}$ comes with its own set of rules for the corresponding modified Schmidt game, and there are no reasons to believe that winning sets of different games must have nonempty intersection. One can also observe that $\mathbf{Bad}(\frac{2}{3}, \frac{1}{3}) = f\big(\mathbf{Bad}(\frac{1}{3}, \frac{2}{3})\big)$ where $f$ is a reflection of $\mathbb{R}^2$ around the line $y = x$. This reflection however does not commute with $\mathcal{F}^{(1/3,2/3)}$, hence Theorem 4.2 cannot be used

[4]See also [Kr] where Schmidt's result on the winning property of the set of badly approximable systems of linear forms is extended to the field of formal power series.

to conclude[5] that $f\big(\mathbf{Bad}(\frac{1}{3},\frac{2}{3})\big)$ is a winning set of the MSG induced by $\mathcal{F}^{(1/3,2/3)}$.

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