REL LEAVES OF THE ARNOUX-YOCOZ SURFACES
– ERRATUM

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Our arguments in §6 of the paper [HW] contain an error. In this note we explain the error and how to fix it. We are grateful to Florent Ygouf for both pointing out the mistake, and for indicating the correct argument included below. All of the results stated in the introduction of the paper remain valid, as a consequence of the amended argument which will be given below. In the recent preprint [Y], Ygouf proves related results about existence of dense rel leaves in other loci.

0.1. The error. The second assertion of Proposition 6.6 is wrong. Namely, the tangent space to $O(x)$ is not a $\mathbb{Q}$-space. This affects the validity of Lemma 6.7: while it is true that $T_r$ and $O_r$ are $g$-dimensional, it is not true that $T_r$ is a $\mathbb{Q}$-subspace. The rationality of $T_r$ is used once in the paper, in the proof of Theorem 6.9, at the end of the second paragraph of the proof.

0.2. The area form. Let $\Theta$ be the $\mathbb{R}$-valued anti-symmetric bilinear form on $H^1(S, \Sigma; \mathbb{R})$ defined by $\Theta(\beta_1, \beta_2) = \int_S \beta_1 \wedge \beta_2$. This bilinear form restricts to the intersection form on $H^1(S; \mathbb{R})$ and can also be used to compute the flat area of a surface, namely, in the notation of [HW], the area of the translation surface $x_0$ is $\Theta(\text{hol}_x(x_0), \text{hol}_y(x_0))$. See [FM, §3.3] for more information. Since $\Theta$ is defined purely in topological terms, for any homeomorphism $\varphi : (S, \Sigma) \to (S, \Sigma)$ we have $\Theta(\varphi^* \beta_1, \varphi^* \beta_2) = \Theta(\beta_1, \beta_2)$.

0.3. Correcting the proof of Theorem 6.9. Retain the notation of [HW] §6. We need to justify the claim made in the second paragraph of the proof of Theorem 6.9, that $T_r \cong P_2$, where $P_1 = \text{span}\{\text{hol}_y(x_0), 0\}$ and $P_2$ is the subspace of $P$ which is $\varphi^*$-invariant and complementary to $P_1$. Since $\Theta(\text{hol}_x(x_0), \text{hol}_y(x_0))$ is equal to the area of the surface $x_0$, it is not 0, and thus

$$P_2 = \ker F \quad \text{for} \quad F : P \to \mathbb{R}, \quad F(\beta, 0) = \Theta(\text{hol}_x(x_0), \beta). \quad (0.1)$$

We have $\dim P = g + 1$, $\dim P_2 = g$, and by Lemma 6.7, $\dim T_r = g$. 

Our proof proceeds by contradiction. Suppose $T_r \subset P_2$. Then by dimension considerations, $T_r = P_2$, and in particular, since $T_r$ is the tangent space to the torus $O_r \subset H$, $\Psi^{-1}(O_r)$ is a subtorus of $\mathbb{R}^{g+1}/\mathbb{Z}^{g+1}$, where $\Psi$ is the map defined in (6.9). Let $\tilde{\Psi}$ be the map on p. 923. The tangent space to $\Psi^{-1}(O_r)$ is then $D[\tilde{\Psi}^{-1}](T_r)$. Since this tangent space is parallel to the subtorus $\Psi^{-1}(O_r)$ of $\mathbb{R}^{g+1}/\mathbb{Z}^{g+1}$, any normal vector to $D[\tilde{\Psi}^{-1}](T_r)$ must be proportional to a rational vector.

We claim that a normal vector to $D[\tilde{\Psi}^{-1}](T_r)$ is given by $(c_0^2, c_1^2, \ldots, c_g^2)$. To see this, observe that the function $F \circ D\tilde{\Psi}$ vanishes on $D[\tilde{\Psi}^{-1}](T_r)$. By Proposition 6.1 and (6.9), we have

$$D\tilde{\Psi}(t_0, \ldots, t_g) = (t_0 c_0 C_0^* + \ldots + t_g c_g C_g^*, 0).$$

Thus,

$$F \circ D\tilde{\Psi}(t_0, \ldots, t_g) = \sum_{i=0}^{g} t_i c_i \Theta(\text{hol}_x(x_0), C_i^*).$$

For $r > 0$, the surface $x_r$ is made of $g + 1$ horizontal cylinders as in the discussion in the first paragraph of §6.1. The definition of $C_i^*$ then implies that $C_i^*(\sigma_j) = \delta_{ij}$ (Kronecker delta) and $C_i^*$ assigns 0 to all other saddle connections on boundaries of cylinders. The area of $x_r$ can be computed by adding the areas of the cylinders, and thus, by the interpretation of $\Theta$ as an area, we see that $\Theta(\text{hol}_x(x_r), C_i^*) = c_i$, the horizontal holonomy of the core curve of $C_i$ on $x_r$. Since the surfaces $x_0$ and $x_r$ assign the same horizontal holonomy to absolute periods, we also have $\Theta(\text{hol}_x(x_0), C_i^*) = c_i$, and therefore

$$F \circ D\tilde{\Psi}(t_0, \ldots, t_g) = \sum_{i=0}^{g} t_i c_i^2.$$

Since we have seen that $T_r = \ker F$, this proves the claim.

We have shown that $(c_0^2, c_1^2, \ldots, c_g^2)$ is proportional to a rational vector. On the other hand, by Theorem 3.7 we have $c_0^2 = \alpha$, and so $\alpha^2 = \frac{c_0^2}{c_0} \in \mathbb{Q}$. But, this contradicts that the fact that $\dim_{\mathbb{Q}} \mathbb{Q}(\alpha) = g \geq 3$.

REFERENCES


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