REL LEAVES OF THE ARNOUX-YOCCOZ SURFACES – ERRATUM

W. PATRICK HOOPER AND BARAK WEISS

Our arguments in §6 of the paper [HW] contain an error. In this note we explain the error and how to fix it. We are grateful to Florent Ygouf for both pointing out the mistake, and for indicating the correct argument included below. All of the results stated in the introduction of the paper remain valid, as a consequence of the amended argument which will be given below. In the recent preprint [Y], Ygouf proves related results about existence of dense rel leaves in other loci.

0.1. **The error.** The second assertion of Proposition 6.6 is wrong. Namely, the tangent space to $\mathcal{O}(x)$ is not a $\mathbb{Q}$-space. This affects the validity of Lemma 6.7: while it is true that T_r and $\mathcal{O}_r$ are $\mathbf{g}$-dimensional, it is not true that T_r is a $\mathbb{Q}$-subspace. The rationality of T_r is used once in the paper, in the proof of Theorem 6.9, at the end of the second paragraph of the proof.

0.2. **The area form.** Let Θ be the $\mathbb{R}$-valued anti-symmetric bilinear form on $H^1(S, \Sigma; \mathbb{R})$ defined by $\Theta(\beta_1, \beta_2) = \int_S \beta_1 \wedge \beta_2$. This bilinear form restricts to the intersection form on $H^1(S; \mathbb{R})$ and can also be used to compute the flat area of a surface, namely, in the notation of [HW], the area of the translation surface x_0 is $\Theta(\mathrm{hol}_x(x_0), \mathrm{hol}_y(x_0))$. See [FM, §3.3] for more information. Since Θ is defined purely in topological terms, for any homeomorphism $\varphi : (S, \Sigma) \to (S, \Sigma)$ we have $\Theta(\varphi^*\beta_1, \varphi^*\beta_2) = \Theta(\beta_1, \beta_2)$.

0.3. **Correcting the proof of Theorem 6.9.** Retain the notation of [HW, §6]. We need to justify the claim made in the second paragraph of the proof of Theorem 6.9, that $T_r \not\subset P_2$, where $P_1 = \mathrm{span}\{(\mathrm{hol}_y(x_0), 0)\}$ and P_2 is the subspace of P which is φ^*-invariant and complementary to P_1. Since $\Theta(\mathrm{hol}_x(x_0), \mathrm{hol}_y(x_0))$ is equal to the area of the surface x_0, it is not 0, and thus

$$P_2 = \ker F \quad \text{for} \quad F : P \to \mathbb{R}, \qquad F(\beta, 0) = \Theta(\mathrm{hol}_x(x_0), \beta). \qquad (0.1)$$

We have $\dim P = \mathbf{g} + 1$, $\dim P_2 = \mathbf{g}$, and by Lemma 6.7, $\dim T_r = \mathbf{g}$.

Our proof proceeds by contradiction. Suppose $T_r \subset P_2$. Then by dimension considerations, $T_r = P_2$, and in particular, since T_r is the tangent space to the torus $\mathcal{O}_r \subset \mathcal{H}$, $\Psi^{-1}(\mathcal{O}_r)$ is a subtorus of $\mathbb{R}^{\mathbf{g}+1}/\mathbb{Z}^{\mathbf{g}+1}$, where Ψ is the map defined in (6.9). Let $\widetilde{\Psi}$ be the map on p. 923. The tangent space to $\Psi^{-1}(\mathcal{O}_r)$ is then $D[\widetilde{\Psi}^{-1}](T_r)$. Since this tangent space is parallel to the subtorus $\Psi^{-1}(\mathcal{O}_r)$ of $\mathbb{R}^{\mathbf{g}+1}/\mathbb{Z}^{\mathbf{g}+1}$, any normal vector to $D[\widetilde{\Psi}^{-1}](T_r)$ must be proportional to a rational vector.

We claim that a normal vector to $D[\widetilde{\Psi}^{-1}](T_r)$ is given by $\left(c_0^2, c_1^2, \ldots, c_{\mathbf{g}}^2\right)$. To see this, observe that the function $F \circ D\widetilde{\Psi}$ vanishes on $D[\widetilde{\Psi}^{-1}](T_r)$. By Proposition 6.1 and (6.9), we have

$$D\widetilde{\Psi}(t_0, \ldots, t_{\mathbf{g}}) = \left(t_0 c_0 C_0^* + \ldots + t_{\mathbf{g}} c_{\mathbf{g}} C_{\mathbf{g}}^*, 0\right).$$

Thus,

$$F \circ D\widetilde{\Psi}(t_0, \ldots, t_{\mathbf{g}}) = \sum_{i=0}^{\mathbf{g}} t_i c_i \, \Theta\left(\mathrm{hol}_x(x_0), C_i^*\right).$$

For $r > 0$, the surface x_r is made of $\mathbf{g} + 1$ horizontal cylinders as in the discussion in the first paragraph of §6.1. The definition of C_i^* then implies that $C_i^*(\sigma_j) = \delta_{ij}$ (Kronecker delta) and C_i^* assigns 0 to all other sadddle connections on boundaries of cylinders. The area of x_r can be computed by adding the areas of the cylinders, and thus, by the interpretation of Θ as an area, we see that $\Theta\left(\mathrm{hol}_x(x_r), C_i^*\right) = c_i$, the horizonal holonomy of the core curve of C_i on x_r. Since the surfaces x_0 and x_r assign the same horizontal holonomy to absolute periods, we also have $\Theta\left(\mathrm{hol}_x(x_0), C_i^*\right) = c_i$, and therefore

$$F \circ D\widetilde{\Psi}(t_0, \ldots, t_{\mathbf{g}}) = \sum_{i=0}^{\mathbf{g}} t_i c_i^2.$$

Since we have seen that $T_r = \ker F$, this proves the claim.

We have shown that $\left(c_0^2, c_1^2, \ldots, c_{\mathbf{g}}^2\right)$ is proportional to a rational vector. On the other hand, by Theorem 3.7 we have $\frac{c_1}{c_0} = \alpha$, and so $\alpha^2 = \frac{c_1^2}{c_0^2} \in \mathbb{Q}$. But, this contradicts that the fact that $\dim_{\mathbb{Q}} \mathbb{Q}(\alpha) = \mathbf{g} \geqslant 3$.

References

[FM] G. Forni and C. Matheus, *Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards,* in J. Mod. Dyn. **8** No. 3-4, p. 271–436 (2014).

[HW] W. P. Hooper and B. Weiss, *Rel leaves of the Arnoux-Yoccoz surfaces*, with an appendix by Lior Bary-Soroker, Mark Shusterman and Umberto Zannier, Selecta Math., (2018) 24, no. 2, 875–934.

[Y] F. Ygouf, *A criterion for density of the isoperiodic leaves in rank 1 affine-invariant orbifolds* , preprint (2020) `https://arxiv.org/abs/2002.01186`

City College of New York, whooper@ccny.cuny.edu

Tel Aviv University, barakw@post.tau.ac.il