On the error bounds for visible points in some cut-and-project sets

Ilya Gringlaz* Rishi Kumar† Barak Weiss‡

Abstract

We study points in cut-and-project sets which are visible from the origin, continuing a direction of inquiry initiated in [6,14], where the asymptotic density of visible points was investigated. We establish an error bound for the density of visible points in certain cases. We also prove that the set of visible points in irreducible cut-and-project sets with star-shaped windows is never relatively dense.

1 Introduction

Let $\mathcal{P} \subset \mathbf{R}^d$ be a locally finite point set, let $D \subset \mathbf{R}^d$ be a bounded measurable set with $\mathrm{vol}(D) > 0$, and for $T > 0$ let TD denote the dilated set $\{tx : x \in D, t \in [0, T]\}$. The *asymptotic density of* $\mathcal{P}$ *with respect to* D is defined to be

$$\theta(\mathcal{P}) \stackrel{\mathrm{def}}{=} \lim_{T\to\infty} \frac{\#(\mathcal{P} \cap TD)}{\mathrm{vol}(TD)}, \tag{1}$$

provided the limit exists. In general, the existence of the limit and its value may depend on the choice of the averaging set D, but this will not play a role in this paper and we suppress the dependence of θ on D from the notation. The most commonly studied case is the case in which D is the unit ball with respect to some norm on $\mathbf{R}^d$. We denote

$$\mathcal{P}_\star \stackrel{\mathrm{def}}{=} \mathcal{P} \smallsetminus \{0\}, \quad \mathcal{P}_{\mathrm{vis}} \stackrel{\mathrm{def}}{=} \{y \in \mathcal{P}_\star : ty \notin \mathcal{P}, \forall\, t \in (0,1)\},$$

the set of nonzero points and the set of points of $\mathcal{P}$ which are visible from the origin. For certain sets $\mathcal{P} \subset \mathbf{R}^d, d \geqslant 2$ for which the asymptotic density of $\mathcal{P}_{\mathrm{vis}}$ has recently been established, we will be interested in the rate of convergence of the limit in (1). We will also be interested in the question of relative density of the set $\mathcal{P}_{\mathrm{vis}}$.

*Department of Mathematics, Tel Aviv University, Israel. E-mail: redlid@gmail.com
†Department of Mathematics, Tel Aviv University, Israel. E-mail: rkumar@tauex.tau.ac.il
‡Department of Mathematics, Tel Aviv University, Israel. E-mail: barakw@tauex.tau.ac.il

2020 **Mathematics Subject Classification: Primary 52C23, Secondary 11P21.**
Key words and phrases. Cut-and-project sets, Penrose set, Amman-Beenker set, visible points, effective counting.

To set the stage we review some of what is known in case $\mathcal{P} = \mathbf{Z}^d$; i.e., the lattice of integer points. In this case, the set of visible points from the origin is given by the primitive lattice points:

$$\mathbf{Z}^d_{\text{vis}} = \left\{ (n_1, \ldots, n_d) \in \mathbf{Z}^d_\star \, : \, \gcd(n_1, \ldots, n_d) = 1 \right\}.$$

When D is the unit ball of a norm, an elementary and classical argument using Möbius inversion shows that $\theta(\mathbf{Z}^d_{\text{vis}}) = 1/\zeta(d)$ (see [2], also see the appendix of [2] for a discussion of more general averaging sets). Here $\zeta(s) = \sum_{n \in \mathbf{N}} n^{-s}$ denotes the Riemann zeta function. The question of error estimates in this convergence has been extensively studied and we give a sample of results. For the unit ball with respect to the ℓ_∞ norm we have (see [11,17])

$$\frac{\#\left(\mathbf{Z}^d_{\text{vis}} \cap [-T, T]^d\right)}{\mathrm{vol}([-T, T]^d)} = \frac{1}{\zeta(d)} + \begin{cases} O\left(\frac{(\log T)^{2/3} (\log \log T)^{1/3}}{T}\right), & d = 2, \\ O\left(\frac{1}{T}\right), & d \geqslant 3. \end{cases}$$

See [21] for the case of Euclidean balls, [3,9,10] for a discussion of more averaging sets. In this example $\mathcal{P} = \mathbf{Z}^d$, the density θ exists and is independent of D, for a large variety of sets D, but the error term depends quite delicately on D.

A subset $\mathcal{P} \subset \mathbf{R}^d$ is called *relatively dense* if there exist a constant $R > 0$ such that for every $x \in \mathbf{R}^d$, we have $d(x, \mathcal{P}) \leqslant R$. If $\mathcal{P}$ is not relatively dense, we say that it *has arbitrarily large holes*; that is, for any $R > 0$ there is a ball B of radius R for which $B \cap \mathcal{P} = \varnothing$. It was shown in [4] that the set $\mathbf{Z}^d_{\text{vis}}$ has arbitrarily large holes (see also [2, Prop. 4] and the proof of Lemma 2.4 below).

In this paper we will consider *cut-and-project sets* (also referred to as model sets), which are defined as follows (see [1,13,15]). A *grid* in $\mathbf{R}^n$ is the image of a lattice under a translation, that is a set of the form $\mathbf{y} + g\mathbf{Z}^n$ where $\mathbf{y} \in \mathbf{R}^n$ and $g \in \mathrm{GL}(n, \mathbf{R})$. Let $n = d + m$ for positive integers n, d, m, and let π_{phys} and π_{int} denote the projections of $\mathbf{R}^n$ onto the two summands in the direct sum decomposition

$$\mathbf{R}^n = \mathbf{R}^d \oplus \mathbf{R}^m. \tag{2}$$

The first and second summands in this decomposition are referred to as *physical* and *internal* space respectively, and we will continue to denote them by $\mathbf{R}^d, \mathbf{R}^m$ in the remainder of the paper. Let $\mathcal{L} \subset \mathbf{R}^n$ be a grid and $\mathcal{W} \subset \mathbf{R}^m$ be a subset referred to as the *window*. In this paper we will always assume that $\mathcal{W}$ has non-empty interior, and is Jordan measurable (that is, bounded and with boundary of zero Lebesgue measure). We will impose certain additional conditions on $\mathcal{W}$ further below. The cut-and-project set associated with $(\mathcal{W}, \mathcal{L})$ is defined as

$$\Lambda(\mathcal{W}, \mathcal{L}) = \{\pi_{\text{phys}}(y) \, : \, y \in \mathcal{L}, \, \pi_{\text{int}}(y) \in \mathcal{W}\} \subset \mathbf{R}^d.$$

We say that $\Lambda = \Lambda(\mathcal{W}, \mathcal{L})$ is *irreducible* if $\pi_{\text{int}}(\mathcal{L})$ is dense in $\mathbf{R}^m$ and $\pi_{\text{phys}}|_{\mathcal{L}}$ is injective. In this case Λ is relatively dense, and its density $\theta(\Lambda)$ exists (whenever D is Jordan-measurable) and is given by

$$\theta\left(\Lambda(\mathcal{W}, \mathcal{L})\right) = \frac{\mathrm{vol}(\mathcal{W})}{\mathrm{covol}(\mathcal{L})}$$

(see [8,13,19]). Here $\mathrm{covol}(\mathcal{L})$ denotes the *covolume* of $\mathcal{L}$, defined as $\mathrm{covol}(\mathbf{y} + g\mathbf{Z}^n) = |\det g|$. See [7,18,23,26] for some results about the rate of convergence in (1) for cut-and-project sets.

Recently, Marklof and Strömbergsson [14, Theorem 1] proved that for any irreducible cut-and-project set Λ, the density of visible points exists, is the same for all Jordan measurable D, and satisfies

$$0 < \theta(\Lambda_{\mathrm{vis}}) \leqslant \theta(\Lambda). \tag{3}$$

They also observed ([14, p.2]) that for 'generic choices' (see Proposition 3.2) of the grid $\mathcal{L}$ one has equality in the right-hand side of (3). On the other hand Hammarhjelm [6] gave certain examples of cut-and-project sets in $\mathbf{R}^2$ for which the latter inequality is strict, and computed $\theta(\Lambda_{\mathrm{vis}})$ explicitly (some of these examples had also been considered by Sing [24]). We will refer to the cases considered by Hammarhjelm as *Hammarhjelm examples* and give a definition in §4.1.

In this paper we prove the following results. In all of these results, $\Lambda = \Lambda(\mathcal{W}, \mathcal{L})$ is an irreducible cut-and-project set in $\mathbf{R}^d$, $d \geqslant 2$, and the averaging sets D are Jordan measurable. Recall that $\mathcal{W}$ is *star-shaped with respect to the origin* if for any $w \in \mathcal{W}$, the segment $\{tw : t \in [0,1]\}$ contained in $\mathcal{W}$.

Theorem 1.1. *Assume $\mathcal{W}$ is star-shaped with respect to the origin, and $\mathcal{L}$ is a lattice. Then Λ_{vis} has arbitrarily large holes.*

The special case of the set of visible points of the Amman-Beenker seems to have known before, see [24] and [1, p. 427].

Let m denote the Haar measure on the group $\mathrm{GL}(n, \mathbf{R})$.

Theorem 1.2. *Assume $\mathcal{W}$ is star-shaped with respect to the origin. Then for m-a.e. $g \in \mathrm{GL}(n, \mathbf{R})$, the lattice $\mathcal{L} = g\mathbf{Z}^n$ satisfies $\theta(\Lambda_{\mathrm{vis}}) = \frac{1}{\zeta(n)}\theta(\Lambda)$ for any averaging set. Moreover for any averaging set $D \subset \mathbf{R}^d$, for any $\varepsilon > 0$, m-a.e. $g \in \mathrm{GL}(n, \mathbf{R})$, we have an error bound*

$$\left| \frac{\#\left(\Lambda_{\mathrm{vis}} \cap TD\right)}{\mathrm{vol}(TD)} - \theta(\Lambda_{\mathrm{vis}}) \right| = O\left(\mathrm{vol}(TD)^{-\frac{1}{3}+\varepsilon} \right).$$

The main ingredient in the proof of Theorem 1.2 is work of Fairchild and Han [5] which will be recalled below. We stress that Theorem 1.2 does not contradict the observation of [14] as we deal with generic *lattices*, not generic *grids*; cf. Propositions 3.2 and 3.3. Put differently, our definition of visibility concerns *visibility from the origin* but if one replaces it with *visibility from a fixed basepoint in* Λ one would see that typically, strict inequality holds in (3).

Theorem 1.3. *For the Hammarhjelm examples, if the averaging set D is convex, for any $\varepsilon > 0$ we have an error bound*

$$\left| \frac{\#\left(\Lambda_{\mathrm{vis}} \cap TD\right)}{\mathrm{vol}(TD)} - \theta(\Lambda_{\mathrm{vis}}) \right| = O\left(\mathrm{vol}(TD)^{-\frac{1}{4}+\varepsilon} \right).$$

1.1 Acknowledgements

The authors are grateful to Anish Ghosh for directing their attention to [5], and to Lior Bary-Soroker and Yaar Solomon for helpful discussions. The authors' work is supported by the grants ISF 2021/24, ISF-NCSF 3739/21 and ISF 2860/24.

2 Visible points in a lattice $\mathcal{L}$ and in $\Lambda(\mathcal{W}, \mathcal{L})$.

Denote

$$\Lambda(\mathcal{W}, \mathcal{L}_{\text{vis}}) \stackrel{\text{def}}{=} \{\pi_{\text{phys}}(y) \,:\, y \in \mathcal{L}_{\text{vis}}\,, \pi_{\text{int}}(y) \in \mathcal{W}\}.$$

The following lemma provides a relation between the visible points of a lattice and the visible points of cut-and-project sets.

Lemma 2.1. *Let $\mathcal{L}$ be a lattice in $\mathbf{R}^n$, and $\mathcal{W}$ be a window which is star-shaped with respect to the origin. Then*

$$\Lambda(\mathcal{W}, \mathcal{L})_{\text{vis}} \subset \Lambda(\mathcal{W}, \mathcal{L}_{\text{vis}}). \tag{4}$$

Proof. Let $x \in \Lambda(\mathcal{W}, \mathcal{L})_\star \smallsetminus \Lambda(\mathcal{W}, \mathcal{L}_{\text{vis}})$. By definition, there exists $y \in \mathcal{L}_\star \smallsetminus \mathcal{L}_{\text{vis}}$ such that $x = \pi_{\text{phys}}(y)$. This implies that there exists a real number $t \in (0, 1)$, such that $y' \stackrel{\text{def}}{=} ty \in \mathcal{L}$. Since $\mathcal{W}$ is star-shaped with respect to the origin and $x \in \Lambda(\mathcal{W}, \mathcal{L})$, we obtain

$$\pi_{\text{int}}(y') = t\pi_{\text{int}}(y) \in \mathcal{W}.$$

Consequently,

$$tx = t\pi_{\text{phys}}(y) = \pi_{\text{phys}}(ty) = \pi_{\text{phys}}(y') \in \Lambda(\mathcal{W}, \mathcal{L}).$$

This shows that $x \notin \Lambda(\mathcal{W}, \mathcal{L})_{\text{vis}}$. □

In [6], Hammarhjelm gave examples for which the inclusion in (4) is strict. The following result gives a condition guaranteeing equality in (4).

Proposition 2.2. *Let $\Lambda = \Lambda(\mathcal{W}, \mathcal{L})$, where $\mathcal{L}$ is a lattice and $\mathcal{W}$ is star-shaped with respect to the origin. If we have strict inclusion in (4) then there are linearly independent $y_1, y_2 \in \mathcal{L}$ such that* $\dim(\text{span}_{\mathbf{R}}(y_1, y_2) \cap \mathbf{R}^m) \geqslant 1$.

Proof. Suppose that $\Lambda(\mathcal{W}, \mathcal{L}_{\text{vis}}) \smallsetminus \Lambda(\mathcal{W}, \mathcal{L})_{\text{vis}} \neq \varnothing$. Then there exists $y_1 \in \mathcal{L}_{\text{vis}}$ such that $x \stackrel{\text{def}}{=} \pi_{\text{phys}}(y_1) \in \Lambda(\mathcal{W}, \mathcal{L}_{\text{vis}})$ and $x \notin \Lambda(\mathcal{W}, \mathcal{L})_{\text{vis}}$. If $x = 0$ then $y_1 \in \mathbf{R}^m$ and the desired conclusion holds, with y_2 any element of $\mathcal{L}$ which is not a scalar multiple of y_1. Thus we can assume that $x \neq 0$ and thus there exist $y_2 \in \mathcal{L}$ and $t \in (0, 1)$ such that

$$x' \stackrel{\text{def}}{=} \pi_{\text{phys}}(y_2) = t\pi_{\text{phys}}(y_1) = tx.$$

Clearly y_1 and y_2 are nonzero. If there was some $c \in \mathbf{R}$ such that $y_2 = cy_1$, then by applying π_{phys} we find that we must have $c = t \in (0, 1)$, in contradiction to the fact that $y_1 \in \mathcal{L}_{\text{vis}}$. Therefore y_1, y_2 are linearly independent; i.e.,

$$\dim U = 2, \text{ where } U \stackrel{\text{def}}{=} \text{span}_{\mathbf{R}}(y_1, y_2).$$

We have $\dim(\pi_{\text{phys}}(U)) = 1$, and since the kernel of π_{phys} is the space $\mathbf{R}^m$, this implies $\dim(U \cap \mathbf{R}^m) = 1$. □

We say that $\mathcal{H}$ *is a hole in* $\mathcal{P}$ if $\mathcal{H} \cap \mathcal{P} = \varnothing$. The following statement follows easily from Lemma 2.1.

Lemma 2.3. *Let $\mathcal{L}$ be a lattice in $\mathbf{R}^n$, let $\mathcal{W}$ be a window which is star-shaped around the origin, and let $\Lambda = \Lambda(\mathcal{W}, \mathcal{L})$. Suppose that $\mathcal{H}$ is a hole in $\mathcal{L}_{\mathrm{vis}}$ and $\mathcal{H}_1 \subset \mathbf{R}^d$ satisfies $\mathcal{H}_1 \times \mathcal{W} \subset \mathcal{H}$. Then $\mathcal{H}_1$ is a hole of $\Lambda(\mathcal{W}, \mathcal{L})_{\mathrm{vis}}$.*

Let $\mathcal{P} \subset \mathbf{R}^n$ and let $V \subset \mathbf{R}^n$ be a subset. We say that $\mathcal{P}$ *contains arbitrarily large holes along* V if for any $R > 0$ there is a ball B of radius R and with center in V such that $B \cap \mathcal{P} = \varnothing$.

Lemma 2.4. *Let $V \subset \mathbf{R}^n$ be a linear subspace such that $V + \mathbf{Z}^n$ is dense in $\mathbf{R}^n$. Then $\mathbf{Z}^n_{\mathrm{vis}}$ has arbitrarily large holes along V.*

Proof. Since $V + \mathbf{Z}^n$ is dense in $\mathbf{Z}^n$ we obtain by rescaling that for any $N > 0$,

$$V + N\mathbf{Z}^n = N \cdot (V + \mathbf{Z}^n) \subset \mathbf{R}^n \tag{5}$$

is a dense inclusion.

Let $\mathcal{U}$ be an open bounded subset of $\mathbf{R}^n$, and for each $A \in \mathbf{N}$ and each $\mathbf{x} = (k_1, \dots, k_n) \in \mathbf{Z}^n$, let

$$C(A, \mathbf{x}) \stackrel{\mathrm{def}}{=} \mathbf{Z}^n \cap (\mathbf{x} + [-A, A]^n) = \{(k_1 + i_1, \dots, k_n + i_n) : |i_j| \leqslant A,\ j = 1, \dots, n\}.$$

We claim that there are $N > 0$ and $\mathbf{x}_0 \in \mathbf{Z}^n$ such that for each $\mathbf{x} \in \mathbf{x}_0 + N\mathbf{Z}^n$ we have $C(A, \mathbf{x}) \cap \mathbf{Z}^n_{\mathrm{vis}} = \varnothing$. To see this, let $\mathcal{I} \stackrel{\mathrm{def}}{=} \mathbf{Z}^n \cap [-A, A]^n$, and for each tuple $(i_1, \dots, i_n) \in \mathcal{I}$, choose a prime $P_{i_1,\dots,i_n}$ so that these primes are all distinct. Then, by the Chinese remainder theorem, for each fixed $j \in \{1, \dots, n\}$ there exists a solution $k_j \in \mathbf{Z}$ to the system of congruences

$$k_j \equiv -i_j \pmod{P_{i_1,\dots,i_n}}, \qquad (i_1, \dots, i_n) \in \mathcal{I}. \tag{6}$$

Define

$$\mathbf{x}_0 = (k_1, \dots, k_n) \in \mathbf{Z}^n \quad \text{and} \quad N \stackrel{\mathrm{def}}{=} \prod_{(i_1,\dots,i_n) \in \mathcal{I}} P_{i_i,\dots,i_n}.$$

Then (6) remains true if $(k_1, \dots, k_n)$ are the coordinates of any $\mathbf{x} \in \mathbf{x}_0 + N\mathbf{Z}^n$, and the choice (6) ensures that for such vectors, $\gcd(k_1 + i_1, \dots, k_n + i_n) \geqslant P_{i_1,\dots,i_n}$. This proves the claim.

Now given $R > 0$, let A be large enough so that for any $\mathbf{x} \in \mathcal{U} + V$, the set $C(A, \mathbf{x})$ contains the ball $B(\mathbf{x}_1, R)$ for some $\mathbf{x}_1 \in V$. Such an A exists because $\mathcal{U}$ is bounded. Since the inclusion in (5) is dense, we have $\mathcal{U} + V + N\mathbf{Z}^n = \mathbf{R}^n$, and thus

$$\mathbf{x}_0 = u + v - \mathbf{x}, \quad \text{where } u \in \mathcal{U},\ v \in V,\ -\mathbf{x} \in N\mathbf{Z}^n.$$

In particular

$$\mathbf{x}_0 + \mathbf{x} \in (\mathbf{x}_0 + N\mathbf{Z}^n) \cap (\mathcal{U} + V).$$

Now using the claim, we see that $\mathbf{Z}^n_{\mathrm{vis}}$ contains arbitrarily large holes along V. □

Proof of Theorem 1.1. Write $\Lambda = \Lambda(\mathcal{W}, \mathcal{L})$, where $\mathcal{W}$ is star-shaped with respect to the origin, $\mathcal{L}$ is a lattice, and $\pi_{\mathrm{int}}(\mathcal{L})$ is a dense subset of $\mathbf{R}^m$. By Lemma 2.3, it suffices to show that $\mathcal{L}_{\mathrm{vis}}$ has arbitrarily large holes along $\mathbf{R}^d$.

To this end, write $\mathcal{L} = g\mathbf{Z}^n$ for $g \in \mathrm{GL}(n, \mathbf{R})$ and let $V \stackrel{\text{def}}{=} g^{-1}\mathbf{R}^d$. Since $\pi_{\text{int}}(\mathcal{L})$ is a dense subset of $\mathbf{R}^m$, and the standard topolgy on $\mathbf{R}^m$ is the quotient topology for the projection $\pi_{\text{int}} : \mathbf{R}^n \to \mathbf{R}^m$, we see that $\mathbf{R}^d + \mathcal{L}$ is dense in $\mathbf{R}^n$. Since linear transformations are homeomorphisms, this implies that $V + \mathbf{Z}^n$ is dense in $\mathbf{R}^n$. Applying Lemma 2.4, we see that $\mathbf{Z}^n_{\text{vis}}$ has arbitrarily large holes along V. Since linear transformations are bi-Lipschitz maps, and send visible points to visible points, this implies that $\mathcal{L}_{\text{vis}}$ has arbitrarily large holes along $\mathbf{R}^d$, as required. □

3 Random cut-and-project sets and point counting

In this section we discuss probability measures on discrete subsets of $\mathbf{R}^d$, explain what we mean by a 'random' cut-and-project set, and prove Theorem 1.2. In fact we will prove the stronger Theorem 3.4 below. For a complete metric space X, we denote by $\mathrm{Cl}(X)$ the space of closed subsets of X, equipped with the Chabauty-Fell topology (see [18, §2.2] and references therein).

Proposition 3.1. *Let ν be any measure on $\mathrm{Cl}(\mathbf{R}^n)$ which is supported on discrete countable sets and invariant under translations. Then for ν-a.e. $\mathcal{P}$, $\mathcal{P}_\star = \mathcal{P}_{\text{vis}}$.*

Proof. Suppose to the contrary that the set

$$K \stackrel{\text{def}}{=} \{\mathcal{P} : \mathcal{P} \text{ is discrete and countable, and } \mathcal{P}_{\text{vis}} \neq \mathcal{P}_\star\}$$

satisfies $\nu(K) > 0$. The group $\mathbf{R}^n$ acts on $\mathrm{Cl}(\mathbf{R}^n)$ by translations, and preserves ν. By the Birkhoff ergodic theorem applied to the indicator function $\mathbf{1}_K$, there is $\mathcal{P}_0 \in K$ such that

$$\mathrm{vol}(\mathbf{T}) > 0, \quad \text{where } \mathbf{T} \stackrel{\text{def}}{=} \{t \in \mathbf{R}^n : t + \mathcal{P}_0 \in K\}. \tag{7}$$

Write $(\mathcal{P}_0)_\star = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$. For any $i \neq j$, the set

$$\mathbf{T}_{ij} \stackrel{\text{def}}{=} \{t \in \mathbf{R}^n : 0 \text{ is on the line through } t + \mathbf{x}_i \text{ and } t + \mathbf{x}_j\}$$

is a line in $\mathbf{R}^n$, so satisfies $\mathrm{vol}(\mathbf{T}_{ij}) = 0$. But whenever $\mathcal{P}_{\text{vis}} \neq \mathcal{P}_\star$, there are two distinct nonzero points in $\mathcal{P}$ such that the line containing them passes through the origin. Thus we have $\mathbf{T} \subset \bigcup_{i \neq j} \mathbf{T}_{ij}$, contradicting (7). □

Let $\mathrm{SL}_n(\mathbf{R})$ and $\mathrm{ASL}_n(\mathbf{R})$ denote the groups of orientation- and volume-preserving linear and affine transformation on $\mathbf{R}^n$ respectively. Consider the associated homogeneous spaces

$$\mathscr{X}_n \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbf{R})/\mathrm{SL}_n(\mathbf{Z}), \qquad \mathscr{Y}_n \stackrel{\text{def}}{=} \mathrm{ASL}_n(\mathbf{R})/\mathrm{ASL}_n(\mathbf{Z}),$$

of covolume-one lattices and grids in $\mathbf{R}^n$. Both spaces are equipped with the quotient topology, or equivalently, the Chabauty-Fell topology. Let $m_{\mathscr{X}_n}$ and $m_{\mathscr{Y}_n}$ denote the Haar-Siegel measures on $\mathscr{X}_n$ and $\mathscr{Y}_n$, respectively; i.e., the unique Borel probability measures invariant under the transitive action of $\mathrm{SL}_n(\mathbf{R})$ and $\mathrm{ASL}_n(\mathbf{R})$. Then we have embeddings $\mathscr{X}_n \subset \mathscr{Y}_n \subset \mathrm{Cl}(\mathbf{R}^n)$.

Fixing the direct sum decomposition (2), the corresponding projections π_{phys}, π_{int}, and a window $\mathcal{W}$, following [13] we define a map

$$\Psi : \mathscr{Y}_n \to \mathrm{Cl}(\mathbf{R}^d), \quad \Psi(\mathcal{L}) \stackrel{\mathrm{def}}{=} \Lambda(\mathcal{W}, \mathcal{L}).$$

Finally let $\bar{\mu}$ and μ denote respectively the pushforwards of $m_{\mathscr{Y}_n}$ and $m_{\mathscr{X}_n}$ under Ψ. That is, these measures construct a random cut-and-project set by fixing the direct sum decomposition and window, and randomly choosing a grid or lattice $\mathcal{L}$. It was noted in [13] that $\bar{\mu}$ (respectively, μ) is invariant and ergodic under the action of $\mathrm{ASL}_d(\mathbf{R})$ (respectively, $\mathrm{SL}_d(\mathbf{R})$) on $\mathrm{Cl}(\mathbf{R}^d)$, and gives full mass to the collection of irreducible cut-and-project sets. Measures satisfying these properties are called *RMS measures*; they have been completely classified, see [13,18]. When referring to 'generic' cut-and-project sets, we have in mind cut-and-project sets which form a set of full measure with respect to μ.

A crucial difference between μ and $\bar{\mu}$ is that $\bar{\mu}$ is invariant under translations, but μ is not. Applying Proposition 3.1, we immediately obtain:

Proposition 3.2. *For $\bar{\mu}$-a.e. cut-and-project set, $\Lambda_\star = \Lambda_{\mathrm{vis}}$, and thus $\theta(\Lambda) = \theta(\Lambda_{\mathrm{vis}})$.*

The situation for μ is very different, as the following shows:

Proposition 3.3. *Suppose the window $\mathcal{W}$ is star-shaped with respect to the origin. Let m denote the Haar measure on $\mathrm{GL}_n(\mathbf{R})$. Then for m-a.e. $g \in \mathrm{GL}_n(\mathbf{R})$ we have*

$$\Lambda(\mathcal{W}, g\mathbf{Z}^n_{\mathrm{vis}}) = \Lambda(\mathcal{W}, g\mathbf{Z}^n)_{\mathrm{vis}}. \tag{8}$$

In particular, for μ-a.e. cut-and-project set Λ, if $\Lambda = \Lambda(\mathcal{W}, \mathcal{L})$ then $\Lambda_{\mathrm{vis}} = \Lambda(\mathcal{W}, \mathcal{L}_{\mathrm{vis}})$.

Proof. Denote

$$\mathscr{B} \stackrel{\mathrm{def}}{=} \{g \in G \,:\, \exists u_1, u_2 \in \mathbf{Z}^n \text{ s.t. } \dim(\mathrm{span}_{\mathbf{R}}(gu_1, gu_2) \cap \mathbf{R}^m) \geqslant 1\}.$$

In light of Proposition 2.2, it suffices to prove that $m(\mathscr{B}) = 0$. We have

$$\mathscr{B} = \bigcup_{u_1, u_2} \mathscr{B}(u_1, u_2),$$

where the union ranges over pairs of vectors in $\mathbf{Z}^n$, and

$$\mathscr{B}(u_1, u_2) = \{g \in G \,:\, \dim(\mathrm{span}_{\mathbf{R}}(gu_1, gu_2) \cap \mathbf{R}^m) \geqslant 1\}.$$

Thus it is enough to prove $m(\mathscr{B}(u_1, u_2)) = 0$, for fixed u_1, u_2. The set $\mathscr{B}(u_1, u_2)$ is a submanifold (in fact, an algebraic subvariety) of $\mathrm{GL}_n(\mathbf{R})$. By our assumption $d \geqslant 2$, there is $g \in \mathrm{GL}_n(\mathbf{R})$ for which gu_1, gu_2 both belong to $\mathbf{R}^d$ and in particular $\mathscr{B}(u_1, u_2)$ is a proper submanifold of $\mathrm{GL}_n(\mathbf{R})$. Recalling that m assigns zero measure to proper submanifolds of $\mathrm{GL}_n(\mathbf{R})$, we obtain that $m(\mathscr{B}(u_1, u_2)) = 0$.

Now let m' denote the Haar measure on $\mathrm{SL}_n(\mathbf{R})$. Since the validity of (8) is not affected if one replaces $\mathbf{Z}^n$ by its dilate $c\mathbf{Z}^n$, (8) also holds for m'-a.e. $g \in \mathrm{SL}_n(\mathbf{R})$. Since the measure $m_{\mathscr{X}_n}$ is the restriction of m' to a fundamental domain for the action of $\mathrm{SL}_n(\mathbf{R})$, the second assertion follows. ∎

Following [20], we say that a collection of Borel subsets $\{\Omega_T : T > 0\}$ of $\mathbf{R}^d$ is an *unbounded ordered family* if

- $0 \leqslant T_1 \leqslant T_2 \Rightarrow \Omega_{T_1} \subset \Omega_{T_2}$;
- For all $T > 0$, $\mathrm{vol}(\Omega_T) < \infty$;
- $\mathrm{vol}(\Omega_T) \xrightarrow[T\to\infty]{} \infty$; and
- For all large enough $V > 0$ there is T such that $\mathrm{vol}(\Omega_T) = V$.

Theorem 3.4. *Let $d \geqslant 2$, and let the window $\mathcal{W} \subset \mathbf{R}^m$ be star-shaped with respect to the origin. Fix an unbounded ordered family $\{\Omega_T : T > 0\}$ in $\mathbf{R}^d$. Then, for any $\varepsilon > 0$, for μ-a.e. Λ,*

$$\#(\Omega_T \cap \Lambda(\mathcal{W}, \mathcal{L})_{\mathrm{vis}}) = \frac{\mathrm{vol}(\mathcal{W})}{\zeta(n)} \cdot \mathrm{vol}(\Omega_T) + O\left(\mathrm{vol}(\Omega_T)^{\frac{2}{3}+\varepsilon}\right). \tag{9}$$

Proof. The proof will proceed by passing to the larger space $\mathbf{R}^n$ and applying a result of Fairchild and Han [5]. Recall we have assumed $d \geqslant 2$ and thus $n = d + m \geqslant 3$. Let $G \stackrel{\mathrm{def}}{=} \mathrm{SL}_n(\mathbf{R})$, and as before, let m' denote the Haar measure on G. Let $\{\Omega'_T : T > 0\}$ denote an unbounded ordered family in $\mathbf{R}^n$, and let $\varepsilon > 0$. It follows from the case $N = 1$ of [5, Theorem 1.3], that for m'-a.e. $g \in G$,

$$\#(g\mathbf{Z}^n_{\mathrm{vis}} \cap \Omega'_T) = \frac{\mathrm{vol}(\Omega'_T)}{\zeta(n)} + O\left(\mathrm{vol}(\Omega'_T)^{\frac{2}{3}+\varepsilon}\right).$$

In particular, if we specialize to $\Omega'_T \stackrel{\mathrm{def}}{=} \Omega_T \times \mathcal{W}$, then we obtain for μ-a.e. $\mathcal{L} \in \mathscr{X}_n$,

$$\#(\mathcal{L}_{\mathrm{vis}} \cap \Omega'_T) = \frac{\mathrm{vol}(\Omega_T) \cdot \mathrm{vol}(\mathcal{W})}{\zeta(n)} + O\left(\mathrm{vol}(\Omega_T)^{\frac{2}{3}+\varepsilon}\right)$$

(where the implicit constant depends on $\mathcal{L}$, $\mathcal{W}$, ε and the family $\{\Omega_T\}$). Note that when $\pi_{\mathrm{phys}}|_{\mathcal{L}}$ is injective we have

$$\#(\mathcal{L}_{\mathrm{vis}} \cap \Omega'_T) = \#(\Lambda(\mathcal{W}, \mathcal{L}_{\mathrm{vis}}) \cap \Omega_T).$$

Thus, restricting further to the set of $\mathcal{L}$ for which the conclusion of Proposition 3.3 holds, and for which $\pi_{\mathrm{phys}}|_{\mathcal{L}}$ is injective, we obtain that for μ-a.e. $\mathcal{L}$, (9) holds. □

Remark 3.5. *It would be interesting to extend Theorem 3.4 to other RMS measures which are not invariant under translations (the case of translation invariant RMS measures follows from Proposition 3.1 and [18]).*

4 Quadratic number fields and Hammarhjelm examples

In this section, we will recall relevant information about real quadratic number fields and review the results of Hammarhjelm [6]. Let $K = \mathbf{Q}(\sqrt{d})$ with $d > 1$, be a real quadratic number field, and let $\mathcal{O}_K$ be its ring of integers. The *norm* of an integer $x \in \mathcal{O}_K$ is defined as

$$N(x) = x\sigma(x),$$

where σ is the non-trivial Galois automorphism of K. The *unit group* $\mathcal{O}_K^\times$ consists of the elements $x \in \mathcal{O}_K$ such that $N(x) = \pm 1$. The unique element $\lambda = \lambda_K \in \mathcal{O}_K^\times$ such that

$$\mathcal{O}_K^\times = \{\pm 1\} \times \{\lambda^i \,:\, i \in \mathbf{Z}\}, \qquad \lambda > 1,$$

is called the *fundamental unit* of K. For an ideal $I \in \mathcal{O}_K$, the *norm* of I is defined as

$$N(I) = |\mathcal{O}_K/I|\,,$$

which is always finite. If $\mathcal{O}_K$ is a principal ideal domain, we have $N(I) = |N(x)|$, where I is the principal ideal generated by x. Throughout the paper, we will consider fields K for which $\mathcal{O}_K$ is a principal ideal domain. For any $x_1, \ldots, x_d \in \mathcal{O}_K$, let $\gcd(x_1, \ldots, x_d)$ denote a fixed generator of the ideal generated by $x_1, \ldots, x_d$. We write $\gcd(x_1, \ldots, x_d) = 1$ when $\gcd(x_1, \ldots, x_d)$ is a unit. We say that $x \in \mathcal{O}_K$ is *prime* if $x = ab,\ a, b \in \mathcal{O}_K \implies a \in \mathcal{O}_K^\star$ or $b \in \mathcal{O}_K^\star$.

The Dedekind zeta function associated with $\mathcal{O}_K$ is given by

$$\zeta_{\mathcal{O}_K}(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{N(I)^s},$$

where the summation is over the nonzero ideals $I \subset \mathcal{O}_K$ and the series converges whenever $\Re(s) > 1$ (see [12, p.131] for details). The Möbius function μ is defined for ideals $I \subset \mathcal{O}_K$ as follows:

$$\mu(I) = \begin{cases} 1, & N(I) = 1, \\ (-1)^r, & I = p_1 \cdots p_r, \text{ for distinct prime ideals } p_1, \ldots, p_r, \\ 0, & I \subset p^2 \text{ for some prime ideal p.} \end{cases}$$

Using the Möbius function, the reciprocal of the Dedekind zeta function can be expressed as

$$\frac{1}{\zeta_{\mathcal{O}_K}(s)} = \sum_{I \subset \mathcal{O}_K} \frac{\mu(I)}{N(I)^s}, \tag{10}$$

and this series converges absolutely when $\Re(s) > 1$. Since any ideal in $\mathcal{O}_K$ can be written as $I = g\mathcal{O}_K$, where $g \in \mathcal{O}_K$, and we will sometimes write $\mu(I)$ as $\mu(g)$. The *Minkowski embedding* of $\mathcal{O}_K$ into $\mathbf{R}^2$ is defined as

$$\mathcal{L}_{\mathcal{O}_K} \stackrel{\text{def}}{=} \{(x, \sigma(x)) \,:\, x \in \mathcal{O}_K\}.$$

We say that $\mathcal{W} \subset \mathbf{R}^m$ is *centrally symmetric* if $\mathcal{W} = -\mathcal{W}$. With these preliminaries, we can state the result of Hammarhjelm [6].

Theorem 4.1. *Let $\mathcal{W} \subset \mathbf{R}^2$ be star-shaped with respect to the origin and centrally symmetric, let K be one of $\mathbf{Q}(\sqrt{2}), \mathbf{Q}(\sqrt{5})$, and let*

$$\mathcal{L} \stackrel{\text{def}}{=} \{(x_1, x_2, \sigma(x_1), \sigma(x_2)) : x_1, x_2 \in \mathcal{O}_K\} \cong \mathcal{L}_{\mathcal{O}_K} \oplus \mathcal{L}_{\mathcal{O}_K}.$$

Then the density of $\Lambda(\mathcal{W},\mathcal{L})_{\mathrm{vis}}$ *exists with respect to any Jordan measurable averaging set* D*, and we have*

$$\theta(\Lambda(\mathcal{W},\mathcal{L})_{\mathrm{vis}}) = \left(1-\frac{1}{\lambda_K^2}\right)\frac{1}{\zeta_{\mathcal{O}_K(2)}}\,\theta(\Lambda(\mathcal{W},\mathcal{L})).$$

Remark 4.2. *1. For specific choices of* $\mathcal{W}$*, one obtains the Amman-Beenker point set and sets associated with the Penrose tiling vertex set. The density of visible points in the Amman-Beenker point set had been established earlier by Sing [24].*

2. Hammarhjelm's results are stated with a slightly less restrictive condition than central symmetry, namely, for an explicit constant $c > 1$ *depending on the field* K*, Hammarhjelm requires* $-\mathcal{W} \subset c\mathcal{W}$*.*

3. The proof of Theorem 4.1 works for other fields which satisfy a condition we will discuss below.

4.1 The Hammarhjelm condition

Let $K = \mathbf{Q}(\sqrt{d})$ be a real quadratic number field with $d \geqslant 2$ square free. In the range $2 \leqslant d \leqslant 100$, for

$$d = 2,3,5,6,7,11,13,14,17,19,21,22,23,29,31,33,37,38,41,43,$$
$$46,47,53,57,59,61,62,67,69,71,73,77,83,86,89,93,94,97$$

the ring of integers $\mathcal{O}_K$ are principal ideal domains. It is conjectured that there are infinitely many real quadratic number fields K with $\mathcal{O}_K$ principal ideal domain (see [16, p.37]). Denote

$$\mathcal{P} = \{\pi \in \mathcal{O}_K \,:\, \pi \text{ is prime and } 1 < \pi < \lambda\},$$

where λ is the fundamental unit. We say that a number field K for which $\mathcal{O}_K$ is principal ideal domain *satisfies the Hammarhjelm condition* if

$$|\sigma(\pi)| > 1, \qquad \pi \in \mathcal{P}.$$

Hammarhjelm [6] verified that the condition holds for $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{5})$.

The condition may be interpreted graphically as follows. Let $\mathcal{L}_{\mathcal{O}_K}$ be the Minkowski embedding of $\mathcal{O}_K$. Then the Hammarhjelm condition holds precisely when

$$\mathcal{L}_{\mathcal{O}_K} \cap ((1,\lambda)\times[-1,1]) = \varnothing. \tag{11}$$

Note that the left-hand side of (11) is always finite, being the intersection of a lattice and a bounded set. Furthermore, generators for $\mathcal{L}_{\mathcal{O}_K}$ can be easily computed. Thus one can check (11) by hand (see Figure 1). In the range $1 \leqslant d \leqslant 100$, the Hammarhjelm condition is satisfied only for $d = 2,5,13,29,53$.

We will say that a cut-and-project set $\Lambda(\mathcal{W},\mathcal{L})$ is a *Hammarhjelm example* if $\mathcal{W}$ is convex and centrally symmetric, and $\mathcal{L} = \mathcal{L}_{\mathcal{O}_K} \oplus \mathcal{L}_{\mathcal{O}_K}$, where K is a real quadratic field K satisfying the Hammarhjelm condition.

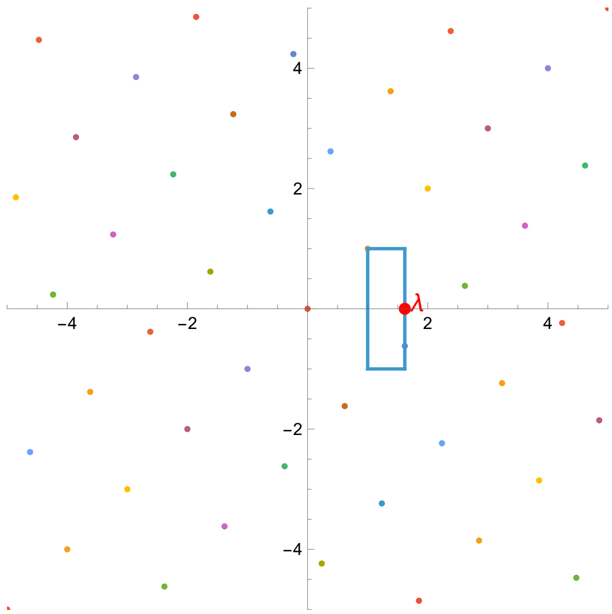

Figure 1: The Minkowski embedding of $\mathcal{O}_K$, where $K = \mathbf{Q}(\sqrt{5})$ and $\lambda = \frac{1}{2}(1+\sqrt{5})$.

5 Effective inclusion-exclusion

In this section we prove Theorem 1.3. Once more, we prove a stronger result, namely:

Theorem 5.1. *Let $d \geqslant 2$, let $m = d$, and let $\mathcal{W} \subset \mathbf{R}^d$ be a convex centrally symmetric window. Let K be a real quadratic field satisfying the Hammarhjelm condition, and let*

$$\mathcal{L} \overset{\text{def}}{=} \{(x_1, \ldots, x_d, \sigma(x_1), \ldots, \sigma(x_d)) \, : \, x_i \in \mathcal{O}_K\} \cong \bigoplus^{d} \mathcal{L}_{\mathcal{O}_K} \subset \mathbf{R}^{2d}. \tag{12}$$

Let $\Lambda = \Lambda(\mathcal{W}, \mathcal{L})$, and let $D \subset \mathbf{R}^d$ be a convex averaging set. Then we have the following asymptotic estimate as $T \to \infty$

$$\frac{\#(\Lambda_{\text{vis}} \cap TD)}{\text{vol}(TD)} = \left(1 - \frac{1}{\lambda^d}\right) \cdot \frac{\theta(\Lambda)}{\zeta_{\mathcal{O}_K}(d)} + \begin{cases} O\left(\frac{\log T}{\sqrt{T}}\right), & d = 2, \\ O\left(\frac{1}{\sqrt{T}}\right), & d \geqslant 3. \end{cases}$$

It is possible to relax the convexity assumption on $\mathcal{W}$ and D in Theorem 5.1 as follows. If there is some $s \in \mathbf{N}$ so that D, $\mathcal{W}$ and $D \times t\mathcal{W}$ (for every t), are all of *narrow class* s then in the proof below, we may apply [22, Lemma 1] (see also [27, Thm. 2.3]) as a substitute for Proposition 5.2 below. We leave the details to the dedicated reader.

In the proof we will use the following counting result, see [22]:

Proposition 5.2. *For any $n \in \mathbf{N}$ there is $C > 0$ so that the following holds. Let $\mathcal{S} \subset \mathbf{R}^n$, let $\mathcal{L} \subset \mathbf{R}^n$ be a lattice, and let $c > 0$ and $T_0 \geqslant 1$ such that:*

(1) $\mathcal{S}$ is convex and its diameter is bounded above by T_0;

(2) The lattice $\mathcal{L}$ contains n linearly independent vectors of length at most T_0, and $n-1$ linearly independent vectors of length at most c.

Then

$$\left|\#(\mathcal{S}\cap\mathcal{L})-\frac{\mathrm{vol}(\mathcal{S})}{\mathrm{covol}(\mathcal{L})}\right|\leqslant CcT_0^{n-1}. \tag{13}$$

Remark 5.3. *Note that in [22] it is further assumed that $\mathcal{S}$ is compact; however for convex sets, the general case can be obtained by approximating $\mathcal{S}$ from inside and outside by compact convex sets.*

For $g\in\mathcal{O}_K$, let a_g be the diagonal matrix

$$a_g\stackrel{\mathrm{def}}{=}\mathrm{diag}(\underbrace{g,\dots,g}_{d\text{ times}},\underbrace{\sigma(g),\dots,\sigma(g)}_{d\text{ times}}),$$

and let

$$\mathcal{L}_g\stackrel{\mathrm{def}}{=}a_g\mathcal{L}=\{(gx_1,\dots,gx_d,\sigma(gx_1),\dots,\sigma(gx_d))\,:\,x_i\in\mathcal{O}_K\}. \tag{14}$$

Since $\mathcal{O}_K$ is a ring, $\mathcal{L}_g\subset\mathcal{L}$, and we have

$$\mathrm{covol}(\mathcal{L}_g)=|N(g)|^d\,\mathrm{covol}(\mathcal{L}). \tag{15}$$

Notation. In the remainder of this section, K and $\mathcal{W}$ satisfy the hypotheses of Theorem 5.1, $\mathcal{L}$ is given by (12), and $\beta\in\mathcal{O}_K^\times$.

Lemma 5.4. *Let $\mathcal{L}_g$ be as in (14), and let $D\subset\mathbf{R}^d$ be a convex averaging set. Then, there is a constant C_1 such that for any $g\in\mathcal{O}_K$, and any $T\geqslant|N(g)|$,*

$$\left|\#\left(\Lambda(\beta\mathcal{W},\mathcal{L}_g)\cap TD\right)-\frac{\mathrm{vol}(TD)\cdot\mathrm{vol}(\mathcal{W})}{\mathrm{covol}(\mathcal{L})}\cdot\frac{\beta^d}{|N(g)|^d}\right|\leqslant C_1\left(\frac{T}{|N(g)|}\right)^{d-\frac{1}{2}}.$$

Proof. We have

$$\begin{aligned}\#\left(\Lambda(\beta\mathcal{W},\mathcal{L}_g)\cap TD\right)&=\#\left((TD\times\beta\mathcal{W})\cap\mathcal{L}_g\right).\\&=\#\left(a_g^{-1}\left(TD\times\beta\mathcal{W}\right)\cap\mathcal{L}\right).\end{aligned} \tag{16}$$

This reduces our problem to the problem of counting lattice points in a convex set. Our goal will be to ensure that the hypotheses of Proposition 5.2 are satisfied, with T_0 as small as possible. To this end we will apply diagonal elements which fix the lattice $\mathcal{L}$ but change the convex body in question.

Choose positive numbers c, T_1 so that assumption (2) of Proposition 5.2 holds, for c and for any $T_0\geqslant T_1$. Let A_1 denote the one-parameter diagonal group

$$A_1\stackrel{\mathrm{def}}{=}\left\{\mathrm{diag}(\underbrace{e^s,\dots,e^s}_{d\text{ times}},\underbrace{e^{-s},\dots,e^{-s}}_{d\text{ times}})\,:\,s\in\mathbf{R}\right\}.$$

Now suppose $g_0\in\mathcal{O}_K^\times$, $g_0>1$ is a unit of norm one. Then multiplication by g_0 permutes the integers of K and hence $a_{g_0}\mathcal{L}=\mathcal{L}$. Moreover $a_{g_0}\in A_1$. It follows that

$$A_0\stackrel{\mathrm{def}}{=}\{a_0\in A_1\,:\,a_0\mathcal{L}=\mathcal{L}\}$$

is a co-compact subgroup of A_1, and for any $a_0 \in A_0$,

$$\#\left(\Lambda(\beta\mathcal{W}, \mathcal{L}_g) \cap TD\right) \overset{(16)}{=} \#a_0\left(a_g^{-1}(TD \times \beta\mathcal{W}) \cap \mathcal{L}\right) = \#\left(a_0 a_g^{-1}(TD \times \beta\mathcal{W}) \cap \mathcal{L}\right).$$

In the remainder of the proof, we will say that two quantities X, Y are *comparable* if their ratio X/Y is bounded above and below by positive constants which do not depend on g and T (but may depend on the window $\mathcal{W}$, the averaging set D, the number field K, and the number β). In this case we will write $X \asymp Y$.

For $a_0 \in A_0$ we will write

$$\mathcal{S}(a_0) \overset{\text{def}}{=} a_0 a_g^{-1}(TD \times \beta\mathcal{W}) = \mathcal{S}_1(a_0) \times \mathcal{S}_2(a_0), \qquad \text{where } \mathcal{S}_i(a_0) \subset \mathbf{R}^d$$

(note that these sets also depend on T but we suppress this dependence in the notation). Since A_0 is cocompact in A_1, and the linear action of A_1 scales the first factor of the decomposition $\mathbf{R}^{2d} = \mathbf{R}^d \oplus \mathbf{R}^d$ by a constant factor while scaling the second factor by its reciprocal, we can find $a_0 \in A_0$ so that $\mathcal{S}_1(a_0)$ and $\mathcal{S}_2(a_0)$ have comparable diameters D_1, D_2. Moreover each diameter D_i is comparable to the dth root of the volume $\mathrm{vol}(\mathcal{S}_i(a_0))$. Since

$$D_1^d\, D_2^d \asymp \mathrm{vol}(\mathcal{S}(a_0)) = \mathrm{vol}(a_0 a_g^{-1}(TD \times \beta\mathcal{W})) = \mathrm{vol}(a_g^{-1}(TD \times \beta\mathcal{W})) \asymp \frac{T^d}{|N(g)|^d},$$

this implies that

$$D_1 \asymp D_2 \asymp \sqrt{\frac{T}{|N(g)|}} \asymp \mathrm{diam}(\mathcal{S}(a_0)). \tag{17}$$

Thus both assumptions of Proposition 5.2 hold, with $n = 2d$ and with

$$T_0 \overset{\text{def}}{=} \max\left(T_1, \mathrm{diam}(\mathcal{S}(a_0))\right). \tag{18}$$

Note that $T_0 \asymp \sqrt{\frac{T}{|N(g)|}}$. Indeed, when the maximum in (18) is attained by the term $\mathrm{diam}(\mathcal{S}(a_0))$, we have this from (17), and when the maximum is attained by T_1, we have this from $1 \leqslant \sqrt{\frac{T}{|N(g)|}} \leqslant T_1 = T_0$. Now the desired conclusion follows from the conclusion of Proposition 5.2. □

The following lemma was proved in the case $d = 2$ in [6, Lemma 4.6]).

Lemma 5.5. *Let $g \in \mathcal{O}_K$, let $T > 0$ and let B_T be the ball of radius T around the origin. Then for $\beta \in \mathcal{O}_K^\times$, there is a constant L such that*

$$\#(\Lambda(\beta\mathcal{W}), \mathcal{L}_g)_\star \cap B_T) \leqslant \frac{L \cdot T^d}{|N(g)|^d},$$

where the constant L depends only on $\mathcal{W}$ and β.

The proof of Lemma 5.5 again uses the invariance of $\mathcal{L}$ under the group A_0 and a rescaling argument, as in the proof of Lemma 5.4. The proof in [6] easily generalizes to arbitrary $d \geqslant 2$ and we omit it.

Lemma 5.6. *Write* $\sigma(x) \stackrel{\text{def}}{=} (\sigma(x_1), \ldots, \sigma(x_d))$. *Then we have*

$$\Lambda(\mathcal{W}, \mathcal{L})_{\text{vis}} = \left\{ (x_1, \ldots, x_d) \in \Lambda(\mathcal{W}, \mathcal{L})_\star \,:\, \gcd(x_1, \ldots, x_d) = 1,\ \sigma(x) \notin \frac{1}{\lambda}\mathcal{W} \right\}. \tag{19}$$

Similar results were proved in [6, Propositions 4.8 & 4.14] for $d = 2$, $K = \mathbf{Q}(\sqrt{2})$, and $K = \mathbf{Q}(\sqrt{5})$. We provide the proof for completeness.

Proof. We start by proving the inclusion $\subset$ in (19). Suppose that $x = (x_1, \ldots, x_d) \in \Lambda(\mathcal{W}, \mathcal{L})_\star$, and that $\gcd(x_1, \ldots, x_d) \neq 1$. Then there is a prime $\pi \in \mathbb{P}$ which divides $x_1, \ldots, x_d$. Consequently we have $\pi^{-1}x \in \mathcal{O}_K^d$. Replacing π if necessary by its multiple by a unit, we may assume that $\pi \in (1, \lambda)$, and hence, by the Hammarhjelm condition, $|\sigma(\pi)| > 1$. Since $\mathcal{W}$ is convex and centrally symmetric, it follows that

$$\sigma(\pi^{-1}x) = \sigma(\pi)^{-1}\sigma(x) \in \mathcal{W}.$$

Thus, we conclude that $\pi^{-1}x \in \Lambda(\mathcal{W}, \mathcal{L})$, and $x \notin \Lambda(\mathcal{W}, \mathcal{L})_{\text{vis}}$. If $\sigma(x) \in \frac{1}{\lambda}\mathcal{W}$ then

$$\sigma(\lambda^{-1}x) = \pm\lambda\sigma(x) \in \mathcal{W},$$

where we have used that $\lambda\sigma(\lambda) = \pm 1$ and $\mathcal{W}$ is convex and centrally symmetric. It follows that $\lambda^{-1}x \in \Lambda(\mathcal{W}, \mathcal{L})$, and again $x \notin \Lambda(\mathcal{W}, \mathcal{L})_{\text{vis}}$.

Next, we prove the inclusion $\supset$. Suppose that $x = (x_1, \ldots, x_d) \neq 0$ does not belong to the left-hand side of (19), that is

$$x \in \Lambda(\mathcal{W}, \mathcal{L})_\star \smallsetminus \Lambda(\mathcal{W}, \mathcal{L})_{\text{vis}}.$$

Then there is some $t \in (0, 1)$ such that $tx \in \Lambda(\mathcal{W}, \mathcal{L})$. Let i be an index such that $x_i \neq 0$. Since the coordinates x_i and tx_i are both in $\mathcal{O}_K$, we have that $t \in K$. Since $\Lambda(\mathcal{W}, \mathcal{L})$ is locally finite, by taking t as small as possible we may assume that

$$y = (y_1, \ldots, y_d) \stackrel{\text{def}}{=} tx \in \Lambda(\mathcal{W}, \mathcal{L})_{\text{vis}}.$$

Since $t \in K$, we can write $t = b/a$, with $a, b \in \mathcal{O}_K$, with a, b co-prime. If b is not a unit, then from $ay = bx$ we see that $\gcd(y_1, \ldots, y_d) \neq 1$, which is a contradiction to the direction $\subset$ we have already proved. Thus we can assume $a = t^{-1} \in \mathcal{O}_K$, and so for each i, a divides $x_i = t^{-1}y_i$. Thus, either $\gcd(x_1, \ldots, x_d) \neq 1$, in which case x does not belong to the right-hand side of (19), or a is a unit. If a is a unit then $t = \lambda^{-k}$ for some $k \in \mathbf{Z}$, and since $t \in (0, 1)$ and $\lambda > 1$ we have $k \in \mathbf{N}$. We have that $\mathcal{W}$ contains $\sigma(x)$ as well as $\sigma(y) = \sigma(\lambda^{-k}x) = \pm\lambda^k\sigma(x)$. Since $\mathcal{W}$ is centrally symmetric and convex, it follows that $\mathcal{W}$ also contains $\lambda\sigma(x)$, that is, $\sigma(x) \in \frac{1}{\lambda}\mathcal{W}$. So in this case again we have that x does not belong to the right-hand side of (19). □

For the lattice $\mathcal{L}$ as in (12) and a window $\mathcal{W}$, define the set of *primitive points* as

$$\Lambda_{\text{pr}}(\mathcal{W}, \mathcal{L}) \stackrel{\text{def}}{=} \{x = (x_1, \ldots, x_d) \in \Lambda(\mathcal{W}, \mathcal{L})_\star \,:\, \gcd(x_1, \ldots, x_d) = 1\}.$$

Note that for the integer lattice, with the standard gcd, the set of primitive points coincides with the set of visible points. But here we work with the gcd of the quadratic field K and this is no longer the case. Lemma 5.6 shows that under the Hammarshjelm condition,

$$\#\left(\Lambda(\mathcal{W}, \mathcal{L})_{\text{vis}} \cap TD\right) = \#\left(\Lambda_{\text{pr}}(\mathcal{W}, \mathcal{L}) \cap TD\right) - \#\left(\Lambda_{\text{pr}}\left(\frac{1}{\lambda}\mathcal{W}, \mathcal{L}\right) \cap TD\right). \tag{20}$$

Given $T > 0$, $\beta \in \mathcal{O}_K^\times$, and an averaging set D, let

$$C \stackrel{\text{def}}{=} \{\pi \in \mathbb{P} : \Lambda(\beta\mathcal{W}, \mathcal{L})_\star \cap \Lambda(\beta\mathcal{W}, \mathcal{L}_\pi)_\star \cap TD \neq \varnothing\}\,.$$

It follows from the local finiteness of cut-and-project sets (see [6, Lemma 4.4] for the detailed argument) that C is finite, and we write $C = \{\pi_1, \ldots, \pi_n\}$.

We have

$$\begin{aligned} \Lambda_{\mathrm{pr}}(\beta\mathcal{W}, \mathcal{L}) \cap TD &= (\Lambda(\beta\mathcal{W}, \mathcal{L})_\star \cap TD) \smallsetminus \bigcup_{\pi\in\mathbb{P}} (\Lambda(\beta\mathcal{W}, \mathcal{L})_\star \cap \Lambda(\beta\mathcal{W}, \mathcal{L}_\pi)_\star) \\ &= (\Lambda(\beta\mathcal{W}, \mathcal{L})_\star \cap TD) \smallsetminus \bigcup_{i=1}^{n} (\Lambda(\beta\mathcal{W}, \mathcal{L})_\star \cap \Lambda(\beta\mathcal{W}, \mathcal{L}_{\pi_i})_\star \cap TD)\,, \end{aligned} \tag{21}$$

For a finite subset $F \subset \mathbb{P}$, let $\prod_F$ denote the product of elements of F. We have

$$\Lambda(\beta\mathcal{W}, \mathcal{L}) \cap \left(\bigcap_{\pi\in F} \Lambda(\beta\mathcal{W}, \mathcal{L}_\pi) \right) = \Lambda\left(\beta\mathcal{W}, \mathcal{L}_{\prod_F}\right). \tag{22}$$

Since $\mathcal{L}_g = \mathcal{L}_{ug}$ for any unit u and $g \in \mathcal{O}_K$, we have

$$\Lambda(\mathcal{W}, \mathcal{L}_{ug}) = \Lambda(\mathcal{W}, \mathcal{L}_g). \tag{23}$$

By (21), (22), (23), and the inclusion and exclusion principle, we have

$$\begin{aligned} \#(\Lambda_{\mathrm{pr}}(\beta\mathcal{W}, \mathcal{L}) \cap TD) &= \#\left((\Lambda(\beta\mathcal{W}, \mathcal{L})_\star \cap TD)\right) \\ &\quad + \sum_{i=1}^{n} (-1)^i \sum_{F_i \subset C : |F_i| = i} \# \left(\left(\left(\Lambda(\beta\mathcal{W}, \mathcal{L})_\star \cap \bigcap_{\pi\in F_i} \Lambda(\beta\mathcal{W}, \mathcal{L}_\pi)_\star \right) \cap TD \right) \right) \\ &= \#\left((\Lambda(\beta\mathcal{W}, \mathcal{L})_\star \cap TD)\right) \\ &\quad + \sum_{i=1}^{n} (-1)^i \sum_{F_i \subset C : |F_i| = i} \# \left(\left(\Lambda\left(\beta\mathcal{W}, \mathcal{L}_{\prod_{F_i}}\right)_\star \cap TD \right) \right) \\ &= \sum_{g \in [1,\lambda) \bigcap \mathcal{O}_K} \mu(g) \cdot \#\left((\Lambda(\beta\mathcal{W}, \mathcal{L}_g)_\star \cap TD)\right). \end{aligned} \tag{24}$$

Lemma 5.7. *We have*

$$\left| \#(\Lambda_{\mathrm{pr}}(\beta\mathcal{W}, \mathcal{L}) \cap TD) - \frac{\mathrm{vol}(\mathcal{W}) \cdot \mathrm{vol}(TD)}{\mathrm{covol}(\mathcal{L})} \cdot \frac{\beta^d}{\zeta_{\mathcal{O}_K}(d)} \right| \leqslant \begin{cases} C_2 T^{3/2} \log T, & d = 2, \\ C_d T^{d-1/2}, & d \geqslant 3, \end{cases}$$

where C_2 and C_d are constants depending only on $\mathcal{W}$ and β.

Proof. To simplify notation, denote

$$M_T \stackrel{\text{def}}{=} \frac{\mathrm{vol}(\mathcal{W}) \cdot \mathrm{vol}(TD)}{\mathrm{covol}(\mathcal{L})}. \tag{25}$$

Let H_n be the number of ideals in $\mathcal{O}_K$ of norm n. It follows from [12, Theorem 39] that

$$H_n \leqslant H \cdot n^{1/2}, \tag{26}$$

where $H > 0$ is a constant independent of n. Since (10) converges absolutely and (24) is a finite sum, we have

$$\left|\#(\Lambda_{\mathrm{pr}}(\beta\mathcal{W},\mathcal{L}) \cap TD) - \frac{M_T\beta^d}{\zeta_{\mathcal{O}_K}(d)}\right| = \left|\sum_{g\in[1,\lambda)\bigcap\mathcal{O}_K} \mu(g)\left(\#(\Lambda(\beta\mathcal{W},\mathcal{L}_g)_\star \cap TD) - \frac{M_T\beta^d}{|N(g)|^d}\right)\right|. \tag{27}$$

Note the conclusion of Lemma 5.5 remains valid for TD instead of B_T (perhaps at the cost of changing the constant L). Thus, using Lemmas 5.4 and 5.5, the right-hand side of (27) is bounded by

$$\begin{aligned}
&C_1 \sum_{\substack{g\in[1,\lambda)\bigcap\mathcal{O}_K\\ |N(g)|\leqslant T}} \frac{T^{d-1/2}}{|N(g)|^{d-1/2}} + L \sum_{\substack{g\in[1,\lambda)\bigcap\mathcal{O}_K\\ |N(g)|>T}} \frac{T^d}{|N(g)|^d} + M_T\beta^d \sum_{\substack{g\in[1,\lambda)\bigcap\mathcal{O}_K\\ |N(g)|>T}} \frac{1}{|N(g)|^d}\\
\leqslant &C_1 \sum_{\substack{g\in[1,\lambda)\bigcap\mathcal{O}_K\\ |N(g)|\leqslant T}} \frac{T^{d-1/2}}{|N(g)|^{d-1/2}} + L \sum_{\substack{g\in[1,\lambda)\bigcap\mathcal{O}_K\\ |N(g)|>T}} \frac{T^d}{|N(g)|^d} + C_{M,\beta}\cdot T^d \sum_{\substack{g\in[1,\lambda)\bigcap\mathcal{O}_K\\ |N(g)|>T}} \frac{1}{|N(g)|^d} \qquad (28)\\
\leqslant &C_1 \sum_{\substack{g\in[1,\lambda)\bigcap\mathcal{O}_K\\ |N(g)|\leqslant T}} \frac{T^{d-1/2}}{|N(g)|^{d-1/2}} + C'_{M,\beta}\cdot T^d \sum_{\substack{g\in[1,\lambda)\bigcap\mathcal{O}_K\\ |N(g)|>T}} \frac{1}{|N(g)|^d},
\end{aligned}$$

where $C_{M,\beta}$ is a constant depending only on $\mathcal{W}$ and β such that $M_T\beta^d \leqslant C_{M,\beta}\cdot T^d$, and $C'_{M,\beta} = C_{M,\beta} + L$. By (26), (27), and (28), we have

$$\begin{aligned}
\left|\#(\Lambda_{\mathrm{pr}}(\beta\mathcal{W},\mathcal{L}) \cap TD) - \frac{M_T\beta^d}{\zeta_{\mathcal{O}_K}(d)}\right| &\leqslant C_1 \cdot \sum_{k=1}^{\lfloor T\rfloor} \frac{H_k\cdot T^{d-1/2}}{k^{d-1/2}} + C'_{M,\beta}T^d \sum_{k=\lfloor T\rceil}^{\infty} \frac{H_k}{k^d}\\
&\leqslant C_1HT^{d-1/2}\int_{x=1}^{T} \frac{x^{1/2}}{x^{d-1/2}}dx + C'_{M,\beta}\cdot HT^d\int_{x=T}^{\infty}\frac{x^{1/2}}{x^d}dx.
\end{aligned} \tag{29}$$

In case $d = 2$, the right-hand side of (29) is

$$\begin{aligned}
&C_1HT^{3/2}\int_{x=1}^{T}\frac{1}{x}dx + C'_{M,\beta}T^2H\int_{x=T}^{\infty}\frac{1}{x^{3/2}}dx\\
\leqslant &C_1HT^{3/2}\log T + C'_{M,\beta}HT^{3/2}\\
\leqslant &C_2(\beta)T^{3/2}\log T,
\end{aligned} \tag{30}$$

where $C_2(\beta)$ is a constant depending only on $\mathcal{W}$ and β. In case $d \geqslant 3$, the right-hand side of (29) is

$$\begin{aligned}
&C_1HT^{d-1/2}\int_{x=1}^{T}\frac{x^{1/2}}{x^{d-1/2}}dx+C'_{M,\beta}T^dH\int_{x=T}^{\infty}\frac{x^{1/2}}{x^d}dx\\
=&C_1HT^{d-1/2}\int_{x=1}^{T}\frac{1}{x^{d-1}}dx+C'_{M,\beta}T^dH\int_{x=T}^{\infty}\frac{1}{x^{d-1/2}}dx\\
\leqslant&C_1HT^{d-1/2}+C'_{M,\beta}HT^{3/2}\leqslant C_d(\beta)T^{d-1/2},
\end{aligned}\tag{31}$$

where $C_d(\beta)$ is a constant depending only on $\mathcal{W}$ and β. Now the required statement follows by combining (29), (30), and (31). □

Finally we are ready for the

Proof of Theorem 5.1. By equation (20), Lemma 5.7, and for M_T as in (25), we have

$$\begin{aligned}
\left|\#(\Lambda(\mathcal{W},\mathcal{L})_{\text{vis}}\cap TD)-\left(1-\frac{1}{\lambda^d}\right)\frac{M_T}{\zeta_{\mathcal{O}_K}(d)}\right|&\leqslant\left|\#(\Lambda_{\text{pr}}(\mathcal{W},\mathcal{L})\cap TD)-\frac{M_T}{\zeta_{\mathcal{O}_K}(d)}\right|\\
&\quad+\left|\#\left(\Lambda_{\text{pr}}\left(\frac{1}{\lambda}\mathcal{W},\mathcal{L}\right)\cap TD\right)-\frac{1}{\lambda^d}\frac{M_T}{\zeta_{\mathcal{O}_K}(d)}\right|\\
&\leqslant\begin{cases}(C_2(1)+C_2(1/\lambda))T^{3/2}\log T, & d=2,\\ (C_d(1)+C_d(1/\lambda))T^{d-1/2}, & d\geqslant 3.\end{cases}
\end{aligned}\tag{32}$$

By (32), we have

$$\left|\#(\Lambda(\mathcal{W},\mathcal{L})_{\text{vis}}\cap TD)-\left(1-\frac{1}{\lambda^d}\right)\frac{M_T}{\zeta_{\mathcal{O}_K}(d)}\right|=\begin{cases}O(T^{3/2}\log T), & d=2,\\ O(T^{d-1/2}), & d\geqslant 3.\end{cases}\tag{33}$$

□

References

[1] M. Baake and U. Grimm, **Aperiodic Order Vol. 1: A Mathematical Invitation (Encyclopedia of Mathematics and its Applications, 149)**, Cambridge University Press, Cambridge, 2013.

[2] M. Baake, R. V. Moody, and P. A. B. Pleasants, *Diffraction from visible lattice points and k-th power free integers*, Discrete Math. **221(1-3)** (2000), 3–42.

[3] C. Burrin, A. Nevo, R. Rühr and B. Weiss, *Effective counting for discrete lattice orbits in the plane via Eisenstein series,* Enseign. Math. **66** (2020), no. 3-4, 259–304.

[4] P. Erdős, *On an elementary problem in number theory,* Canadian Mathematical Bulletin **1** (1958), 5–8.

[5] S. Fairchild and J. Han, *Mean value theorems for the S-arithmetic primitive Siegel transforms,* (2024), preprint `arXiv:2310.03459`

[6] G. Hammarhjelm, *The density and minimal gap of visible points in some planar quasicrystals,* Discrete Math. **345(12)** (2022).

[7] A. Haynes, M. Kelly, and B. Weiss, *Equivalence relations on separated nets arising from linear toral flows,* Proc. Amer. Math. Soc. (3) **109(5)** (2014), 1203–1228.

[8] A. Hof, *Uniform distribution and the projection method,* in **Quasicrystals and discrete geometry (Toronto, ON, 1995),** Fields Inst. Monogr. 10, (1998), 201-–206.

[9] T. Horesh and Y. Karasik, *Equidistribution of primitive vectors, and the shortest solution of their GCD equations*, preprint (2021) `https://arxiv.org/pdf/1903.01560`

[10] M. N. Huxley and W. G. Nowak, *Primitive lattice points in convex planar domains,* Acta Arith. **LXXVI** (1996) 271–283.

[11] H. Q. Liu, *On Euler's function,* Proc. Roy. Soc. Edinburgh Sect. A Math. **146(4)** (2016), 769–775.

[12] D. A. Marcus, **Number Fields**, second ed., Springer-Verlag, 1995.

[13] J. Marklof and A. Strömbergsson, *Free path lengths in quasicrystals,* Commun. Math. Phys. **330** (2014), 723–755.

[14] J. Marklof and A. Strömbergsson, *Visibility and directions in quasicrystals,* Int. Math. Res. Not. **2015(15)** (2014), 6588-6617.

[15] Y. Meyer, **Algebraic numbers and harmonic analysis**, North-Holland Mathematical Library. Vol. 2. 274 p. (1972).

[16] J. Neukirch, **Algebraic Number Theory**, Grundlehren der mathematischen Wissenschaften **322**, Springer-Verlag, Berlin, 1999.

[17] J. Nymann, *On the probability that k positive integers are relatively prime,* J. Number Theory. **4(5)** (1972), 469–473.

[18] R. Rühr, Y. Smilansky, and B. Weiss, *Classification and statistics of cut-and-project sets,* J. Eur. Math. Soc. **26(9)** (2023), 3575–3638.

[19] M. Schlottmann, *Cut-and-project sets in locally compact abelian groups,* in **Quasicrystals and discrete geometry (Toronto, ON, 1995),** Fields Inst. Monogr. 10, (1998), 247-–264.

[20] W. M. Schmidt, *A metrical theorem in geometry of numbers,* Trans. Am. Math. Soc., **95** (1960) 516–529.

[21] W. M. Schmidt, *Asymptotic formulae for point lattices of bounded determinant and subspaces of bounded height*, Duke Math. J. **35**, 327–339 (1968).

[22] W. M. Schmidt, *Northcott's theorem on heights. II. The quadratic case,* Acta Arith. **70** (1995), 343-–375.

[23] S. Schmieding and R. Treviño, *Self affine Delone sets and deviation phenomena,* Commun. Math. Phys. 357, No. 3, 1071–1112.

[24] B. Sing, Visible points lecture notes, `https://www.bb-math.com/bernd/home.html`

[25] M. M. Skriganov, *Ergodic theory on* $\mathrm{SL}(n)$*, Diophantine approximations and anomalies in the lattice point problem*, Invent. Math. **132**, No. 1, 1–72 (1998).

[26] Y. Soloman, *A simple condition for bounded displacement,* J. Math. Anal. Appl. **414(1)** 2014, 134–148.

[27] M. Widmer, *Lipschitz class, narrow class, and counting lattice points,* Proc. Amer. Math. Soc. **140(2)** (2012), 677-689.