THE SET OF BADLY APPROXIMABLE VECTORS IS STRONGLY C^1 INCOMPRESSIBLE

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Abstract. We prove that the countable intersection of C^1-diffeomorphic images of certain Diophantine sets has full Hausdorff dimension. For example, we show this for the set of badly approximable vectors in $\mathbb{R}^d$, improving earlier results of Schmidt and Dani. To prove this, inspired by ideas of McMullen, we define a new variant of Schmidt's (α, β)-game and show that our sets are *hyperplane absolute winning* (HAW), which in particular implies winning in the original game. The HAW property passes automatically to games played on certain fractals, thus our sets intersect a large class of fractals in a set of positive dimension. This extends earlier results of Fishman to a more general set-up, with simpler proofs.

1. Introduction

We begin with a definition introduced by S.G. Dani [6]: a subset S of $\mathbb{R}^d$ is called *incompressible* if for any nonempty open $U \subset \mathbb{R}^d$ and any uniformly bi-Lipschitz sequence $\{f_i\}$ of maps of U onto (possibly different) open subsets of $\mathbb{R}^d$, the set

$$\bigcap_{i=1}^{\infty} f_i^{-1}(S) \tag{1.1}$$

has Hausdorff dimension d. Here a sequence of maps $\{f_i : U \to \mathbb{R}^d\}$ is called *uniformly bi-Lipschitz* if

$$\sup_i \operatorname{Lip}(f_i) < \infty\,, \tag{1.2}$$

where $\operatorname{Lip}(f)$ is the bi-Lipschitz constant of f, defined to be the infimum of $C \geq 1$ such that $C^{-1}\|\mathbf{x} - \mathbf{y}\| \leq \|f(\mathbf{x}) - f(\mathbf{y})\| \leq C\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in U$. Here and below $\|\cdot\|$ stands for the Euclidean norm on $\mathbb{R}^d$. One of the main interesting examples of incompressible sets is the set $\mathbf{BA}_d$ of *badly approximable vectors*, namely those $\mathbf{x} \in \mathbb{R}^d$ satisfying

$$\left\|\mathbf{x} - \frac{\mathbf{p}}{q}\right\| \geq \frac{c}{q^{1+1/d}} \tag{1.3}$$

for some $c = c(\mathbf{x}) > 0$ and for any $\mathbf{p} \in \mathbb{Z}^d$, $q \in \mathbb{N}$. Its incompressibility is a consequence of a stronger property: namely, of the fact that $\mathbf{BA}_d$ is a winning set of a game introduced in the 1960s by W.M. Schmidt [26, 28] (see §2 for definitions and a discussion).

Our goals in this paper are to remove the hypothesis of uniformity in (1.2) and to consider intersections of bi-Lipschitz images of S with certain closed subsets $K \subset \mathbb{R}^d$. Let us say that $S \subset \mathbb{R}^d$ is *strongly* (resp., *strongly C^1*, *strongly affinely*) *incompressible on K* if one has

$$\dim\left(\bigcap_{i=1}^{\infty} f_i^{-1}(S) \cap K\right) = \dim(U \cap K) \tag{1.4}$$

for any open $U \subset \mathbb{R}^d$ with $U \cap K \neq \varnothing$ and any sequence $\{f_i\}$ of bi-Lipschitz maps (resp., C^1 diffeomorphisms, affine nonsingular maps) of U onto (possibly different) open subsets of $\mathbb{R}^d$ (here and hereafter dim stands for the Hausdorff dimension).

It is easy to see that strong incompressibility (in $\mathbb{R}^d$) is shared by subsets of full Lebesgue measure in $\mathbb{R}^d$: indeed, if $\mathbb{R}^d \smallsetminus S$ has Lebesgue measure zero, then the set (1.1) has full

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measure in U, and hence full Hausdorff dimension. See also §6.2 for an example of a strongly incompressible residual subset of $\mathbb{R}^d$ of Lebesgue measure zero. On the other hand, using Schmidt games one can exhibit sets with strong intersection properties which are Lebesgue-null and meager, i.e. are small from the point of view of both measure and category. Indeed, it follows from Schmidt's work that any winning subset of $\mathbb{R}^d$ is incompressible, and strongly C^1 incompressible if $d = 1$ (the reason being that diffeomorphisms in dimension 1 are, in Schmidt's terminology, *local isometries*).

In a more recent development, it was shown by the second-named author [12] that the set $\mathbf{BA}_d$ is strongly affinely incompressible. Moreover, the main result of [12] establishes strong affine incompressibility of $\mathbf{BA}_d$ on various fractal subsets of $\mathbb{R}^d$, characterized by their ability to support measures satisfying certain decay conditions. In particular it is proved there that $\mathbf{BA}_d$ is strongly affinely incompressible on $K = \operatorname{supp}\mu$ whenever μ is a measure on $\mathbb{R}^d$ which is *absolutely decaying* and *Ahlfors regular*. Examples of such sets include limit sets of irreducible families of self-similar contractions of $\mathbb{R}^d$ satisfying the open set condition; see §5 for more examples and precise definitions.

In this paper we generalize the aforementioned result as follows:

Theorem 1.1. *Let μ be a measure on $\mathbb{R}^d$ which is absolutely decaying and Ahlfors regular. Then $\mathbf{BA}_d$ is strongly C^1 incompressible on $K = \operatorname{supp}\mu$.*

Our results are in fact much more general. We use ideas from a recent paper [23] of McMullen, where, among other things, it was proved that $\mathbf{BA}_1$ is strongly incompressible, and introduce a modification of games considered therein. More precisely, we show that the set $\mathbf{BA}_d$ has a property which we refer to as *hyperplane absolute winning*, abbreviated by HAW; see §2 for a definition. This property is stable under countable intersection and implies winning in Schmidt's sense. One of the main results of the present paper (Theorem 2.4) is that the HAW property is preserved by C^1 diffeomorphisms; this immediately implies the case $K = \mathbb{R}^d$ of Theorem 1.1. Furthermore, a rather elementary observation, see Proposition 4.9, shows that the HAW property implies a similar property defined with respect to a game played on K, where K is a closed subset of $\mathbb{R}^d$ satisfying much less restrictive assumptions that in Theorem 1.1. More precisely, in §4 we introduce a class of *hyperplane diffuse* sets which includes those satisfying assumptions of Theorem 1.1, and prove the following

Theorem 1.2. *Let $K \subset \mathbb{R}^d$ be hyperplane diffuse, U an open subset of $\mathbb{R}^d$ with $U \cap K \neq \varnothing$, $S \subset \mathbb{R}^d$ hyperplane absolute winning, and let $\{f_i\}$ be a sequence of C^1 diffeomorphisms of U onto (possibly different) open subsets of $\mathbb{R}^d$. Then the set*

$$\bigcap_{i=1}^{\infty} f_i^{-1}(S) \cap K \tag{1.5}$$

has positive Hausdorff dimension.

This theorem in particular applies to $S = \mathbf{BA}_d$; other examples of HAW sets are discussed later in the paper.

The structure of the paper is as follows: in §2 we first describe Schmidt's game in its original form, then its modification introduced by McMullen, and then introduce a new variant. In §3 we prove Theorem 2.4, which establishes the invariance of the new winning property under nonsingular smooth maps, and also show that $\mathbf{BA}_d$ is hyperplane absolute winning, thereby proving the statement made in the title of the paper. A remarkable feature of hyperplane absolute winning sets, is that they have a large intersection with any set from a large class of fractals. To this end, in §4 we explain how the games defined in §2 are played on proper subsets of $\mathbb{R}^d$, introduce all the necessary definitions to distinguish a class of diffuse fractals on which those games can be played successfully, and prove Proposition 4.9 which implies that diffuse sets intersect hyperplane absolute winning sets nontrivially. From this we derive Theorem 1.2. A discussion of measures whose supports can be shown to have the diffuseness property takes place in §5 and leads to the proof of Theorem 1.1. The last section is devoted to some concluding remarks and several open questions.

2. Games

Consider the game, originally introduced by Schmidt [26] where two players, whom we will call Bob and Alice, successively choose nested closed balls in $\mathbb{R}^d$

$$B_1 \supset A_1 \supset B_2 \supset \dots \tag{2.1}$$

satisfying

$$\rho(A_i) = \alpha\rho(B_i), \quad \rho(B_{i+1}) = \beta\rho(A_i) \tag{2.2}$$

for all i, where $\rho(B)$ denotes the radius of B, and $0 < \alpha, \beta < 1$ are fixed. A set $S \subset \mathbb{R}^d$ is said to be (α, β)-*winning* if Alice has a strategy guaranteeing that the unique point of intersection $\bigcap_{i=1}^\infty B_i = \bigcap_{i=1}^\infty A_i$ of all the balls belongs to S, regardless of the way Bob chooses to play. It is said to be α-*winning* if it is (α, β)-winning for all $\beta > 0$, and *winning* if it is α-winning for some α.

In [23] McMullen introduced the following modification: Bob and Alice choose balls B_i, A_i so that

$$B_1 \supset B_1 \smallsetminus A_1 \supset B_2 \supset B_2 \smallsetminus A_2 \supset B_3 \supset \cdots,$$

and

$$\rho(A_i) \le \beta\rho(B_i), \quad \rho(B_{i+1}) \ge \beta\rho(A_i),$$

where $\beta < 1/3$ is fixed. One says that $S \subset \mathbb{R}^d$ is β-*absolute winning* if Alice has a strategy which leads to

$$\bigcap_{i=1}^{\infty} B_i \cap S \neq \varnothing \tag{2.3}$$

regardless of how Bob chooses to play; S is said to be *absolute winning* if it is β-absolute winning for all $0 < \beta < 1/3$. Note a significant difference between this modification and the original rule: now Alice has rather limited control over the situation, since she can block fewer of Bob's possible moves at the next step. Also, in the new version the radii of balls do not have to tend to zero, therefore $\cap_i B_i$ does not have to be a single point (however the outcome with radii not tending to 0 is clearly winning for Alice as long as S is dense).

The following proposition summarizes some important properties of winning and absolute winning subsets of $\mathbb{R}^d$:

Proposition 2.1. (a) *Winning sets are dense and have Hausdorff dimension d.*
(b) *Absolute winning implies α-winning for all $\alpha < 1/2$.*
(c) *The countable intersection of α-winning (resp., absolute winning) sets is again α-winning (resp., absolute winning).*
(d) *The image of an α-winning set under a bi-Lipschitz map f is α'-winning, where α' depends only on α and* $\mathrm{Lip}(f)$.
(e) *The image of an absolute winning set under a quasisymmetric map is absolute winning.*

For the proofs, see [26] and [23]. We recall that f is called M-*quasisymmetric* if

$$\begin{gathered}\forall\, r > 0\ \exists\, s > 0 \text{ such that for all balls } B(\mathbf{x}, r) \subset \mathbb{R}^d \\ \text{one has } B\big(f(\mathbf{x}), s\big) \subset f\big(B(\mathbf{x}, r)\big) \subset B\big(f(\mathbf{x}), Ms\big)\,,\end{gathered} \tag{2.4}$$

and *quasisymmetric* if it is M-quasisymmetric for some M. Note that being quasisymmetric is equivalent to being quasiconformal when $d \ge 2$, see [15, Theorem 7.7]; also, bi-Lipschitz clearly implies quasisymmetric, but not vice versa. The above proposition can be used to establish the following incompressibility properties of winning sets:

Proposition 2.2. (a) *Winning sets are incompressible.*
(b) *Absolute winning sets are strongly incompressible.*

Part (a) was proved by Dani in [6], and (b) follows from one of the main results of [23]. One can see now that one of the advantages of proving a set to be absolute winning is a possibility to intersect it with its countably many bi-Lipschitz images without worrying about a uniform bound on bi-Lipschitz constants.

Note that winning sets arise naturally in many settings in dynamics and Diophantine approximation [4, 5, 6, 8, 10, 11, 19, 20, 24, 26, 27, 30, 31]. Several examples of absolute winning sets were exhibited by McMullen in [23], most notably the set of badly approximable numbers in $\mathbb{R}$. However absolute winning does not occur as frequently; see [23] for examples of sets in $\mathbb{R}$ which are winning but not absolute winning. Another example is the set of badly approximable vectors in $\mathbb{R}^d$ for $d > 1$. Indeed, the hyperplane $\{0\} \times \mathbb{R}^{d-1}$ is disjoint from $\mathbf{BA}_d$, thus Alice must avoid it in order to win the game. However it is clear that Bob can play the absolute game in such a way that his balls are always centered on this hyperplane.

The above example suggests that in order to build a version of an absolute game adapted for treatment of the set of badly approximable vectors, one needs to allow Alice to 'black out' neighborhoods of hyperplanes. This is precisely how we proceed to define our new version of the game. Namely, fix $k \in \{0, 1, \ldots, d-1\}$ and $0 < \beta < 1/3$, and define the *k-dimensional β-absolute game* in the following way. Bob initially chooses $\mathbf{x}_1 \in \mathbb{R}^d$ and $\rho_1 > 0$, thus defining a closed ball $B_1 = B(\mathbf{x}_1, \rho_1)$. Then in each stage of the game, after Bob chooses $\mathbf{x}_i \in \mathbb{R}^d$ and $\rho_i > 0$, Alice chooses an affine subspace $\mathcal{L}$ of dimension k and removes its ε-neighborhood $A_i \stackrel{\text{def}}{=} \mathcal{L}^{(\varepsilon)}$ from $B_i \stackrel{\text{def}}{=} B(\mathbf{x}_i, \rho_i)$ for some $0 < \varepsilon \le \beta\rho_i$ (note that ε is allowed to depend on i). Then Bob chooses $\mathbf{x}_{i+1}$ and $\rho_{i+1} \ge \beta\rho_i$ such that

$$B_{i+1} \stackrel{\text{def}}{=} B(\mathbf{x}_{i+1}, \rho_{i+1}) \subset B_i \smallsetminus A_i\,.$$

The set S is said to be *k-dimensionally β-absolute winning* if Alice has a strategy guaranteeing that $\bigcap_i B_i$ intersects S. Note that our convention on *intersecting* S is in agreement with McMullen's approach and simplifies our proofs, but is slightly different from Schmidt's definition of general games in [26], which requires the intersection of balls to be a subset of S. Also note that

$$\begin{gathered}k\text{-dimensional } \beta\text{-absolute winning implies } k\text{-dimensional } \beta'\text{-absolute winning}\\ \text{whenever } \beta' \ge \beta, \text{ as long as } \beta' < 1/3\,.\end{gathered} \tag{2.5}$$

We will say that S is *k-dimensionally absolute winning* if it is k-dimensionally β-absolute winning for every $0 < \beta < 1/3$ (equivalently, for arbitrary small positive β). Observe that the strongest case, $k = 0$, is precisely McMullen's absolute winning property. We will be mostly interested in the weakest case, $k = d-1$, in other words, the case when Alice removes neighborhoods of affine hyperplanes. For the sake of brevity, $(d-1)$-dimensionally absolute winning sets will be called *hyperplane absolute winning*, or *HAW* sets.

The following proposition summarizes some properties of sets winning in this new version of the game:

Proposition 2.3. (a) *HAW (and thus k-dimensional absolute winning $\forall\, k \le d-1$) implies α-winning for all $\alpha < 1/2$; hence* $\dim(S \cap U) = d$ *for any $k \le d-1$, any k-dimensionally absolute winning $S \subset \mathbb{R}^d$ and any nonempty open $U \subset \mathbb{R}^d$.*

(b) *The countable intersection of k-dimensionally absolute winning sets is k-dimensionally absolute winning.*

(c) *The image of a k-dimensionally absolute winning set under a C^1 diffeomorphism of $\mathbb{R}^d$ is k-dimensionally absolute winning.*

Parts (a) and (b) are straightforward, and part (c) will be proved in the next section in the following stronger form:

Theorem 2.4. *Let $S \subset \mathbb{R}^d$ be k-dimensionally absolute winning, $U \subset \mathbb{R}^d$ open, and $f : U \to \mathbb{R}^d$ a C^1 nonsingular map. Then $f^{-1}(S) \cup U^c$ is k-dimensionally absolute winning. Consequently, any k-dimensionally absolute winning set is strongly C^1 incompressible.*

To deduce the statement made in the title of the paper, it remains to establish the HAW property for the set of badly approximable vectors:

Theorem 2.5. $\mathbf{BA}_d \subset \mathbb{R}^d$ *is hyperplane absolute winning.*

We point out that $\mathbf{BA}_d$ is not k-dimensionally absolute winning for $k < d-1$, since, as in the case $k = 0$, Bob can play in such a way that his balls are always centered on $\{0\} \times \mathbb{R}^{d-1}$.

Here is one more class of winning sets which we show to be HAW. Let $\mathbb{T}^d \stackrel{\mathrm{def}}{=} \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional torus, and let $\pi : \mathbb{R}^d \to \mathbb{T}^d$ denote the natural projection. Given a nonsingular semisimple matrix $R \in \mathrm{GL}_d(\mathbb{Q}) \cap M_d(\mathbb{Z})$ and a point $y \in \mathbb{T}^d$, define the set

$$\tilde{E}(R, y) \stackrel{\mathrm{def}}{=} \left\{ \mathbf{x} \in \mathbb{R}^d : y \notin \overline{\{\pi(R^k\mathbf{x}) : k \in \mathbb{N}\}} \right\} ;$$

in other words, a lift to $\mathbb{R}^d$ of the set of points of $\mathbb{T}^d$ whose orbits under the endomorphism f_R of $\mathbb{T}^d$ induced by R,

$$f_R(x) \stackrel{\mathrm{def}}{=} \pi(R\mathbf{x}) \text{ where } \mathbf{x} \in \pi^{-1}(\mathbf{x}) ,$$

do not approach y. This is a set which has Lebesgue measure zero when f_R is ergodic and, in the special case $y \in \mathbb{Q}^d/\mathbb{Z}^d$, was proved to be winning by Dani [5] (see [3] for more general results). We have

Theorem 2.6. *For every R as above and any $y \in \mathbb{T}^d$, the set $\tilde{E}(R, y)$ is hyperplane absolute winning.*

We prove the three theorems stated above in the next section, and then in §4 study games played on proper subsets of $\mathbb{R}^d$.

3. Proof of Theorems 2.4, 2.5 and 2.6

Proof of Theorem 2.4. The idea is to 'pull back' the strategy via the map f. We first describe this informally. Alice is playing Bob with target set $f^{-1}(S) \cup U^c$ and parameter $\beta \in (0, 1/3)$; let us call this Game 1. She will define an appropriate constant $\beta' \in (0, 1/3)$, and consider a Game 2, played with target set S and parameter β', for which it is assumed she has a winning strategy. For each move B_i made by Bob in Game 1, Alice will use f to construct a set B_i' which is a legal move in Game 2. Using her strategy, she will have a move A_i' in Game 2, and she will use f^{-1} to construct from this set a legal move A_i in Game 1. The fact that she is assured to win in Game 2 will be shown to imply that this will also result in her victory in Game 1. We now proceed to the details.

No matter what β is, note that if the diameters of the balls B_i do not tend to zero, then $\cap B_i$ has nonempty interior. Since S is winning, it is dense, so $f^{-1}(S) \cup U^c$ is dense as well, and $\cap B_i$ must intersect it nontrivially. Thus, we may assume the diameters of B_i do tend to zero. Furthermore, we may assume that there exists some $i \in \mathbb{N}$ such that $B_i \subset U$, since otherwise $B_i \cap U^c$ is a sequence of nonempty, nested closed subsets of the closed set U^c, so that $\cap_i B_i \cap U^c$ is nonempty.

We will need to control the distortion of the map f. Denote by $J_{\mathbf{z}} \stackrel{\mathrm{def}}{=} d|_{\mathbf{z}} f$ the derivative of f at $\mathbf{z}$. First we note that for any closed ball $B \subset U$ and any $\mathbf{x}, \mathbf{y} \in B$, we have

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \le \sup_{\mathbf{z} \in B} \|J_{\mathbf{z}}\|_{op} \, \|\mathbf{x} - \mathbf{y}\|,$$

where $\|J_{\mathbf{z}}\|_{op}$ denotes the operator norm of $J_{\mathbf{z}}$. For any $\mathbf{x}_0 \in B$, denote by

$$L_{\mathbf{x}_0}(\mathbf{x}) \stackrel{\mathrm{def}}{=} f(\mathbf{x}_0) + J_{\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0)$$

the linear approximation to f at $\mathbf{x}_0$, and similarly let $\bar{L}_{\mathbf{y}'}$ denote the linear approximation to f^{-1} at $\mathbf{y}' \in f(B)$. We claim that

$$\sup_{\mathbf{y}, \mathbf{y}_0 \in f(B)} \frac{\|\bar{L}_{\mathbf{y}_0}(\mathbf{y}) - f^{-1}(\mathbf{y})\|}{\|\mathbf{y} - \mathbf{y}_0\|} \tag{3.1}$$

tends to zero when the diameter of B tends to zero, B being a subset of a fixed compact set. To see this, define

$$H_{\mathbf{y}_0}(\mathbf{z}) \stackrel{\mathrm{def}}{=} f^{-1}(\mathbf{z}) - J_{\mathbf{y}_0}^{-1}(\mathbf{z}) ,$$

so that $H_{\mathbf{y}_0}$ is differentiable with $d_{\mathbf{z}}H_{\mathbf{y}_0} = J_{\mathbf{z}}^{-1} - J_{\mathbf{y}_0}^{-1}$. Since f is C^1 and nonsingular, $\sup_{\mathbf{z},\mathbf{y}_0\in f(B)} \|d_{\mathbf{z}}H_{\mathbf{y}_0}\|_{op}$ tends to zero with the diameter of $f(B)$, where again $f(B)$ is any subball of a fixed compact set. Since f is continuous and bi-Lipschitz on compact sets, this sup will tend to zero as the diameter of B tends to zero, B being any subball of a fixed compact set. But

$$\frac{\|\bar{L}_{\mathbf{y}_0}(\mathbf{y}) - f^{-1}(\mathbf{y})\|}{\|\mathbf{y}-\mathbf{y}_0\|} = \frac{\|H_{\mathbf{y}_0}(\mathbf{y}_0) - H_{\mathbf{y}_0}(\mathbf{y})\|}{\|\mathbf{y}-\mathbf{y}_0\|} \le \sup_{\mathbf{z}\in B} \|d_{\mathbf{z}}H_{\mathbf{y}_0}\|_{op}\,,$$

which proves the claim.

Let B_1 be the initial ball chosen by Bob, and put

$$C_1 \stackrel{\text{def}}{=} \sup_{\mathbf{x}\in B_1} \|J_{\mathbf{x}}\|_{op}, \quad C_2 \stackrel{\text{def}}{=} \sup_{\mathbf{x}\in f(B_1)} \|(J_{\mathbf{x}})^{-1}\|_{op}.$$

Let $n \in \mathbb{N}$ be large enough that

$$C(\beta+1)\beta^{n-2} < 1, \quad \text{where } C \stackrel{\text{def}}{=} 2C_1\,C_2, \tag{3.2}$$

and let $\beta' = \beta^n < 1/3$. By making enough dummy moves in the beginning of the game so that Bob's ball B becomes small enough, Alice can force the quantity in (3.1) to be less than $C_2(1+1/\beta)\beta'$. Moreover, as discussed above, after sufficiently many dummy moves we may assume Bob's ball B is contained in U. We renumber balls so that such a sufficiently small ball is labelled B_1.

From this point on we will call our game (played with target set $f^{-1}(S)\cup U^c$ and parameter β) Game 1, and use induction to construct a legal sequence of moves

$$B_1' \supset B_1' \smallsetminus A_1' \supset B_2' \supset \dots$$

in a Game 2, played with target set S and parameter β'. In Game 2 Alice will play according to the strategy we have assumed she has. At the same time we will describe a strategy for Alice in Game 1 and a sequence $1 = j_1 < j_2 < \cdots$, such that

$$f(B_{j_i}) \subset B_i' \tag{3.3}$$

for all i. This will imply that

$$f\big(\bigcap_i B_i\big) = f\big(\bigcap_i B_{j_i}\big) = \bigcap_i f(B_{j_i}) \subset \bigcap_i B_i' \subset S,$$

so that $\bigcap_i B_i \in f^{-1}(S) \cup U^c$ and Alice wins.

In order to make this construction possible, in our induction we will have to guarantee that some additional properties hold. Namely we will choose $j_1 = 1 < j_2 < \cdots$, and sets A_i, B_i', A_i' so that together with Bob's chosen moves B_i, we will have for each ℓ:

(i) All sets chosen are valid moves for Games 1 and 2 respectively.

(ii) Writing

$$B_{j_i} = B(\mathbf{x}_i, \rho_i),\ B_i' = B\big(f(\mathbf{x}_i), \rho_i'\big),\ \rho_i' = C_1\rho_i,$$

we have

$$\beta^n \le \frac{\rho_i}{\rho_{i-1}} < \beta^{n-1} \qquad (n \text{ as in (3.2)}). \tag{3.4}$$

(iii) $A_i' = \mathcal{L}_i'^{(\varepsilon_i')}$ are dictated by Alice's strategy for Game 2, and

$$A_{j_i} \supset f^{-1}\left(\mathcal{L}_i'^{(\eta\rho_i')}\right), \quad \text{where } \eta = (1+1/\beta)\beta'.$$

Note that (3.3) follows from (ii) and the definition of C_1, thus specifying choices satisfying (i)–(iii) will prove the theorem.

Suppose we have carried out these choices up to stage $\ell-1$. Alice will play dummy moves until the first time the radius ρ of Bob's ball B_m satisfies $\rho/\rho_{\ell-1} < \beta^{n-1}$, and set $j_\ell = m$. Note that j_ℓ is well-defined since the radii chosen by Bob tend to zero, and at each step ρ decreases by a factor of at most β. Our choice of j_ℓ ensures that (3.4) holds for $i = \ell$. Now we let $B'_\ell = B\big(f(\mathbf{x}_\ell), \rho'_\ell\big)$. We claim that B'_ℓ is a valid move in Game 2. By (3.4) and the definition of ρ'_i,

$$\beta' = \beta^n \le \frac{\rho_\ell}{\rho_{\ell-1}} = \frac{\rho'_\ell}{\rho'_{\ell-1}} < \beta^{n-1} = \frac{\beta'}{\beta}. \tag{3.5}$$

Since $B_{j_\ell} \subset B_{j_{\ell-1}}$, we must have $\operatorname{dist}(\mathbf{x}_{\ell-1}, \mathbf{x}_\ell) \le \rho_{\ell-1} - \rho_\ell$ and thus

$$\operatorname{dist}\big(f(\mathbf{x}_{\ell-1}), f(\mathbf{x}_\ell)\big) \le C_1 \operatorname{dist}(\mathbf{x}_{\ell-1}, \mathbf{x}_\ell) \le C_1(\rho_{\ell-1} - \rho_\ell) = \rho'_{\ell-1} - \rho'_\ell\,.$$

Hence $B'_\ell \subset B'_{\ell-1}$. Since $B_{j_\ell} \cap A_{j_{\ell-1}} = \varnothing$ and $A_{j_{\ell-1}} \supset f^{-1}\left(\mathcal{L}'^{(\eta\rho'_{\ell-1})}_{\ell-1}\right)$, it follows via (3.5) that

$$\operatorname{dist}(f(\mathbf{x}_\ell), \mathcal{L}'_{\ell-1}) \ge \eta\rho'_{\ell-1} = \left(1 + \frac{1}{\beta}\right)\beta'\rho'_{\ell-1} > \beta'\rho'_{\ell-1} + \rho'_\ell.$$

That is, $B'_\ell \cap \mathcal{L}'^{(\beta'\rho'_{\ell-1})}_{\ell-1} = \varnothing$, hence $B'_\ell \subset B'_{\ell-1} \smallsetminus A'_{\ell-1}$, proving the claim.

Now choose A'_ℓ according to Alice's strategy in Game 2, say $A'_\ell = \mathcal{L}'^{(\varepsilon'_\ell)}_\ell$ for some $\varepsilon'_\ell \le \beta'\rho'_\ell$. We will show that f^{-1} does not move A'_ℓ too much from a hyperplane neighborhood in B_{j_ℓ}. To this end, fix $\mathbf{y}' \in \mathcal{L}'_\ell$, and define $\mathcal{L}_\ell \stackrel{\text{def}}{=} \bar{L}_{\mathbf{y}'}(\mathcal{L}'_\ell)$, which is also a k-dimensional subspace. For any $\mathbf{y} \in B'_\ell$ with $\operatorname{dist}(\mathbf{y}, \mathcal{L}'_\ell) < \eta\rho'_\ell$, let $\mathbf{y}_0$ be the projection of $\mathbf{y}$ onto $\mathcal{L}'_\ell$. Then, by the definition of C_2,

$$\|f^{-1}(\mathbf{y}) - f^{-1}(\mathbf{y}_0)\| \le C_2\|\mathbf{y} - \mathbf{y}_0\| \le C_2\eta\rho'_\ell.$$

Since we've reindexed so that the diameter of B_1 is small enough to force the quantity in (3.1) to be less than $C_2\eta$, we have

$$\|f^{-1}(\mathbf{y}) - \bar{L}_{\mathbf{y}'}(\mathbf{y}_0)\| \le \|f^{-1}(\mathbf{y}) - f^{-1}(\mathbf{y}_0)\| + \|\bar{L}_{\mathbf{y}'}(\mathbf{y}_0) - f^{-1}(\mathbf{y}_0)\| \le 2C_2\eta\rho'_\ell.$$

It follows that for

$$\varepsilon_\ell \stackrel{\text{def}}{=} 2C_2\eta\rho'_\ell = C\eta\rho_\ell$$

we have

$$A_{j_\ell} \stackrel{\text{def}}{=} \mathcal{L}^{(\varepsilon_\ell)}_\ell \supset f^{-1}\big(\mathcal{L}'^{(\eta\rho'_\ell)}_\ell\big).$$

But, by (3.2), $C\eta < \beta$, so that $\varepsilon_\ell < \beta\rho_\ell$, i.e. A_{j_ℓ} is a valid move for Alice in Game 1 satisfying (iii). This concludes the inductive step and completes the proof. ☐

For the proof of Theorem 2.5 we need the so-called 'simplex lemma', whose proof goes back to Schmidt and Davenport:

Lemma 3.1. [21, Lemma 4] *For every $\beta \in (0,1)$ and for every $k \in \mathbb{N}$ let*

$$U_k \stackrel{\text{def}}{=} \left\{\tfrac{\mathbf{p}}{q} : q \in \mathbb{N},\ \mathbf{p} \in \mathbb{Z}^d \text{ and } \beta^{\frac{-d}{d+1}(k-1)} \le q < \beta^{\frac{-d}{d+1}k}\right\}.$$

Denote by V_d the volume of the d-dimensional unit ball. Then for every

$$0 < r < \beta(d!\,V_d)^{-1/d} \tag{3.6}$$

and for every $\mathbf{x} \in \mathbb{R}^d$ there exists an affine hyperplane $\mathcal{L}$ such that

$$U_k \cap B(\mathbf{x}, \beta^{k-1}r) \subset \mathcal{L}.$$

Proof of Theorem 2.5. Let $\beta < 1/3$. When the hyperplane β-absolute game begins, Alice makes dummy moves until the radius ρ is small enough to satisfy (3.6) with $r \le \rho$. Then one sets $c = \beta^2\rho$. After this, let B_{j_k} be the subsequence of moves where the radius ρ_{j_k} first satisfies $\beta^{k-1}\rho \ge \rho_{j_k} > \beta^k\rho$. On turns not in this subsequence Alice makes dummy moves; on the turns j_k, consider the rational points in U_k. $U_k \cap B_{j_k}$ is contained in a hyperplane $\mathcal{L}_k$ by Lemma 3.1, so Alice chooses

$$A_{j_k+1} \stackrel{\text{def}}{=} \mathcal{L}_k^{(\beta^{k+1}\rho)}\,.$$

But note that

$$\beta^{k+1}\rho = c\beta^{k-1} \ge cq^{-(1+1/d)}$$

whenever q is a denominator of one of the rational points from U_k. Thus any $\mathbf{x} \in B_{j_k+1}$ is at least $cq^{-(1+1/d)}$ away from $\mathbf{p}/q \in U_k$. Satisfying this for all k and comparing with (1.3) shows $\cap_j B_j \in \mathbf{BA}_d$. ☐

Proof of Theorem 2.6. Let λ be the spectral radius of R. If $\lambda = 1$, then obviously every eigenvalue of R must have modulus 1. By a theorem of Kronecker [22], they must be roots of unity, so there exists an $N \in \mathbb{N}$ such that the only eigenvalue of R^N is 1. Thus $R^N = I$. Hence, for any $y \in \mathbb{T}^d$,

$$\tilde{E}(R^N, y) \supset \mathbb{R}^d \smallsetminus (\mathbf{y} + \mathbb{Z}^d)\,,$$

where $\mathbf{y}$ is an arbitrary vector in $\pi^{-1}(y)$. Thus $\tilde{E}(R^N, y)$ is HAW, since $\mathbf{y}+\mathbb{Z}^d$ is countable. Hence $\tilde{E}(R^N, z)$ is HAW whenever $z \in f_R^{-i}(y)$, where $0 \le i < N$. Thus the intersection

$$\tilde{E}(R, y) = \bigcap_{i=0}^{N-1} \bigcap_{z \in f_R^{-i}(y)} \tilde{E}(R^N, x)$$

is also HAW.

Otherwise let $\ell \in \mathbb{N}$ be the smallest integer such that $\lambda^{-\ell} < \beta$, and let $a = |\det(R)^{-\ell}|$. Then $R^{-j}(\mathbb{Z}^d) \subset a\mathbb{Z}^d$ for $j \in \{0, 1, \ldots, \ell\}$. Let $b > 0$ be such that $R^{-j}\big(B(0,1)\big) \subset B(0, b)$ for $0 \le j \le \ell$.

Let V and W be the largest R-invariant subspaces on which all eigenvalues have absolute value equal to, and less than, λ respectively. Then, since R is semisimple, $\mathbb{R}^d = V \oplus W$. Since the eigenvalues of $R|_V$ are of absolute value λ and R is semisimple, there exists $\delta_1 > 0$ such that, for all $\mathbf{v} \in V$ and $j \in \mathbb{N}$,

$$\|R^{-j}\mathbf{v}\| \le \delta_1 \lambda^{-j} \|\mathbf{v}\|. \tag{3.7}$$

Similarly, since all eigenvalues of R^{-1} have absolute value at least λ^{-1} and R is semisimple, there exists $\delta_2 > 0$ such that for all $\mathbf{u} \in \mathbb{R}^d$ and $j \in \mathbb{N}$

$$\|R^{-j}\mathbf{u}\| \ge \delta_2 \lambda^{-j} \|\mathbf{u}\|. \tag{3.8}$$

Again, choose an arbitrary vector $\mathbf{y} \in \pi^{-1}(y)$. Let t_0 be the minimum positive value of $\frac{1}{3b} \operatorname{dist}\big(\mathbf{y} - R^{-j}(\mathbf{y}), a\mathbf{z}\big)$, ranging over $j \in \{0, 1, \ldots, \ell\}$ and $\mathbf{z} \in \mathbb{Z}^d$. Then since $b \ge 1$, by the triangle inequality we have that, for any $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{Z}^d$ and $0 \le j \le \ell$ such that $\mathbf{y} + \mathbf{m}_1 \ne \mathbb{R}^{-j}(\mathbf{y} + \mathbf{m}_2)$,

(3.9)
$$\operatorname{dist}\big(B(\mathbf{y}+\mathbf{m}_1, t_0), R^{-j}(B(\mathbf{y}+\mathbf{m}_2, t_0)\big) \ge \operatorname{dist}\big(B(\mathbf{y}+\mathbf{m}_1, bt_0), B(R^{-j}(\mathbf{y}+\mathbf{m}_2), bt_0)\big) \ge t_0$$

Let $k, j_1, j_2 \in \mathbb{Z}_+$ be such that $j_1 \le j_2$ and $\beta^{-k} \le \lambda^{j_i} < \beta^{-(k+1)}$. Note that, by our choice of l, $0 \le j_2 - j_1 \le \ell$. By (3.8) and (3.9), for any $0 < t < t_0$ and $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{Z}^d$ such that $\mathbf{y} + \mathbf{m}_1 \ne \mathbb{R}^{-j_2+j_1}(\mathbf{y} + \mathbf{m}_2)$,

$$\operatorname{dist}\big(R^{-j_1}\big(B(\mathbf{y}+\mathbf{m}_1, t)\big), R^{-j_2}\big(B(\mathbf{y}+\mathbf{m}_2, t)\big)\big) \ge \delta_2 t_0 \beta^{k+1}. \tag{3.10}$$

Let $p : \mathbb{R}^d \to V$ be the projection onto V parallel to W, and let $M = \|p\|_{op}$ (with respect to the Euclidean norm). Alice will play arbitrarily until Bob chooses a ball of radius $\rho \le \frac{\beta}{2}\delta_2 t_0$. Reindexing, call this ball B_1. Choose a subsequence of moves B_{jk} such that B_{jk} has radius smaller than $\frac{1}{2}\delta_2 t_0 \beta^{k+1}$, so by (3.10) if it intersects two sets of the form $R^{-j_i}\big(B(\mathbf{y}+\mathbf{m}_i, t)\big)$ with $\beta^{-k} \le \lambda^{j_i} < \beta^{-(k+1)}$, $j_1 \le j_2$, and $\mathbf{m}_i \in \mathbb{Z}^d$, then we must have $\mathbf{y} + \mathbf{m}_1 = R^{-j_2+j_1}(\mathbf{y}+\mathbf{m}_2)$. Then by our choice of b,

$$R^{-j_2}\big(B(\mathbf{y}+\mathbf{m}_2, t)\big) \subset R^{-j_1}\left(B\big(R^{-j_2+j_1}(\mathbf{y}+\mathbf{m}_2), bt\big)\right) = R^{-j_1}\big(B(\mathbf{y}+\mathbf{m}_1, bt)\big)\,.$$

Thus all sets of the above form that intersect B_{jk} must be contained in a single set of the form $R^{-j}\big(B(\mathbf{y}+\mathbf{m}, bt)\big)$. Since $\operatorname{diam}\big(p(B(\mathbf{y}, bt))\big) \le 2Mbt$, we have by (3.7) that for $t = \min\left(t_0, \frac{\beta\rho}{2Mb\delta_1}\right)$,

$$\operatorname{diam}\left(p\big(R^{-j}(B(\mathbf{y}+\mathbf{m}, t))\big)\right) \le 2\delta_1 Mbt\lambda^{-j} \le 2\delta_1 Mbt\beta^k \le \beta^{k+1}\rho,$$

so each $R^{-j}\big(B(\mathbf{y}+\mathbf{m}, t)\big)$ intersecting B_{jk} is contained in $\mathcal{L}^{(\varepsilon)}$, where $\mathcal{L}$ contains a translate of W and $\varepsilon \le \beta^{k+1}\rho$.

On her $(k+1)$-st move, Alice will remove $\mathcal{L}^{(\varepsilon)}$. By induction, Alice will play in such a way that $\mathbf{x} \in \cap B_{jk}$ satisfies $\|R^j(\mathbf{x}) - \mathbf{m} - \mathbf{y}\| \ge t$ for all $\mathbf{m} \in \mathbb{Z}^d$ and $j \in \mathbb{Z}_+$. Hence, $\mathbf{x} \in \tilde{E}(R, y)$, and Alice wins. □

4. Games played on proper subsets

An application of Schmidt games to proving abundance of badly approximable vectors on certain fractal subsets of $\mathbb{R}^d$ first appeared in [12], and was motivated by earlier related results [33, 17, 18, 21]. The main idea is simple: if K is a closed subset of $\mathbb{R}^d$ and $0 < \alpha, \beta < 1$, one lets Bob and Alice successively choose nested closed balls in $\mathbb{R}^d$ as in (2.1) and (2.2), but with an additional constraint that the centers of all the balls belong to K. (An equivalent approach is to view K as a metric space with the metric induced from $\mathbb{R}^d$ and play the (α, β)-game there with $S \cap K$ being the target set, but replacing the usual containment of balls with the stronger one, namely the containment of the corresponding balls in $\mathbb{R}^d$.) One says that $S \subset \mathbb{R}^d$ is (α, β)-*winning on* K if Alice has a strategy guaranteeing that $\cap_i B_i \in S$ regardless of the way Bob chooses to play; α-*winning on* K and *winning on* K are defined accordingly. Note that $\cap_i B_i$ is also automatically in K since the latter is closed and the centers of balls are chosen to be in K.

The following lemma, proved in [12] and adapted from Schmidt [26], gives a condition on a set K allowing one to estimate from below the Hausdorff dimension of $S \cap K$ for every S which is winning on K. We will use the following notation: for $K \subset \mathbb{R}^d$, $\mathbf{x} \in K$, $\rho > 0$ and $0 < \beta < 1$, let $N_K(\beta, \mathbf{x}, \rho)$ denote the maximum number of disjoint balls of radius $\beta\rho$ centered on K and contained in $B(\mathbf{x}, \rho)$.

Lemma 4.1. [12, Theorem 3.1] *Suppose there exists positive M, δ, ρ_0 and β_0 such that*

$$N_K(\beta, \mathbf{x}, \rho) \geq M\beta^{-\delta} \tag{4.1}$$

whenever $\mathbf{x} \in K$, $\rho < \rho_0$ and $\beta < \beta_0$. Then $\dim(S \cap K \cap U) \geq \delta$ whenever U is an open set with $U \cap K \neq \varnothing$ and S is winning on K.

We remark that in [12] the lemma is stated for sets K supporting absolutely friendly measures and with $\beta_0 = 1$ and $U = \mathbb{R}^d$, but the proof does not require all this. In particular, whenever (4.1) can be satisfied with $\delta = \dim(K)$, such as when K supports an Ahlfors regular measure (see the next section for more detail), this gives the full Hausdorff dimension of the intersection of K and S at any point of K.

Our next goal is to define the k-dimensional β-absolute game played on K. In this new version Bob and Alice will choose sets B_i and A_i as in the game played on $\mathbb{R}^d$, with Bob's choices subjected to the additional constraint that they are centered on K. That is, Bob initially chooses $\mathbf{x}_1 \in K$ and $\rho_1 > 0$, thus defining a closed ball $B_1 = B(\mathbf{x}_1, \rho_1)$; then at each stage, after Bob chooses $\mathbf{x}_i \in K$ and $\rho_i > 0$, Alice chooses an affine subspace $\mathcal{L}$ of dimension k and $0 < \varepsilon \leq \beta$, and then Bob chooses $\mathbf{x}_{i+1} \in K$ and $\rho_{i+1} \geq \beta\rho_i$ such that

$$B_{i+1} \stackrel{\text{def}}{=} B(\mathbf{x}_{i+1}, \rho_{i+1}) \subset B_i \smallsetminus A_i \,.$$

As before, we declare Alice the winner if (2.3) holds, and say that $S \subset \mathbb{R}^d$ is *k-dimensionally β-absolute winning on K* if Alice has a winning strategy regardless of how Bob chooses to play. Observe however that it might sometimes happen that Alice removes all or most of K after finitely many steps[1], leaving no valid moves for Bob. For example, if K is a singleton, Alice can remove it on her first turn, forcing the game to stop. In this case, we will declare Bob the winner (and so it would not be beneficial for Alice to play in this fashion).

Let us say that $S \subset \mathbb{R}^d$ is *k-dimensionally absolute winning on K* if there exists β_0 such that Alice has a strategy to win the k-dimensional β-absolute game game on K for all $\beta \leq \beta_0$. As before, $(d-1)$-dimensional absolute winning on K will be referred to as *hyperplane absolute winning* (*HAW*) on K.

Because Alice loses when she leaves Bob with no valid moves, it will be useful to have a condition on K which guarantees that Bob will have an available move even when Alice decides to choose a subspace which meets K. The following geometric condition ensures exactly this for small enough initial radius, which as we will see can be assumed without loss of generality.

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[1]Note that the same scenario could happen in the game on $\mathbb{R}^d$ if β was chosen to be bigger than $1/3$, hence this restriction on β in the rule introduced by McMullen.

Definition 4.2. *A closed set $K \subset \mathbb{R}^d$ is said to be k-dimensionally β-diffuse (here $0 \le k < d$, $0 < \beta < 1$) if there exists $\rho_K > 0$ such that for any $0 < \rho \le \rho_K$, $\mathbf{x} \in K$, and any k-dimensional affine subspace $\mathcal{L}$, there exists $\mathbf{x}' \in K$ such that*

$$\mathbf{x}' \in B(\mathbf{x}, \rho) \smallsetminus \mathcal{L}^{(\beta\rho)}.$$

We say that K is k-dimensionally diffuse if it is k-dimensionally β_0-diffuse for some $\beta_0 < 1$ (and hence for all $\beta \le \beta_0$). When $k = d-1$, this property will be referred to as hyperplane diffuseness; clearly it implies k-dimensional diffuseness for all k.

For a class of trivial examples of sets satisfying those conditions, it is clear that $\mathbb{R}^d$ itself is hyperplane β-diffuse for all $\beta < 1$, and so is the closure of any bounded open subset of $\mathbb{R}^d$ with smooth boundary; more generally, any m-dimensional compact smooth submanifold of $\mathbb{R}^d$ is k-dimensionally diffuse whenever $m > k$. Moreover, many proper subsets of $\mathbb{R}^d$ can also be shown to have the diffuseness property, such as, most notably, limit sets of irreducible family of self-similar or self-conformal contractions of $\mathbb{R}^d$ [16, 17, 32]. More examples will be given in §5.

As we will be using this definition to choose balls in the absolute game, it will be useful to have the following equivalent version of the definition.

Lemma 4.3. *Let K be k-dimensionally β'-diffuse and let $\beta \le \frac{\beta'}{2+\beta'}$. Then for any $0 < \rho \le \rho_K$, $\mathbf{x} \in K$, and any k-dimensional affine subspace $\mathcal{L}$, there exists $\mathbf{x}' \in K$ such that*

$$B(\mathbf{x}', \beta\rho) \subset B(\mathbf{x}, \rho) \smallsetminus \mathcal{L}^{(\beta\rho)}\,. \tag{4.2}$$

Proof. Clearly it suffices to prove the result for $\beta = \frac{\beta'}{2+\beta'}$. Let $\mathbf{x} \in K$, $0 < \rho \le \rho_K$, and $\mathcal{L}$ a k-dimensional affine subspace. Since $\beta' = \frac{2\beta}{1-\beta}$, we can use diffuseness to find a point $\mathbf{x}' \in K$ such that

$$\mathbf{x}' \in B\big(\mathbf{x}, (1-\beta)\rho\big) \smallsetminus \mathcal{L}^{(\beta'(1-\beta)\rho)} = B(\mathbf{x}, (1-\beta)\rho) \smallsetminus \mathcal{L}^{(2\beta\rho)}.$$

This $\mathbf{x}'$ satisfies (4.2). □

The diffuseness condition can also be stated in terms of microsets, a notion introduced by H. Fursternberg in [14]. Let B_1 be the unit ball in $\mathbb{R}^d$ and given a ball B, let T_B be the homothety sending B to B_1. Let K be a closed subset of $\mathbb{R}^d$. Any Hausdorff-metric limit point of a sequence of sets $T_{B_i}(B_i \cap K)$, with each B_i centered on K and $\mathrm{diam}(B_i) \to 0$, is called a *microset* of K.

Lemma 4.4. *K is k-dimensionally diffuse if and only if no microset of K is contained in a k-dimensional affine subspace.*

Proof. We will define the *k-dimensional width* of a set A to be

$$\inf\{\varepsilon : \mathcal{L}^{(\varepsilon)} \supset A \text{ for some } k\text{-dimensional affine subspace } \mathcal{L}\}.$$

Notice that k-dimensional width is continuous with respect to the Hausdorff metric, and that A is contained in a k-dimensional affine subspace if and only if its k-dimensional width is zero.

First suppose K is k-dimensionally β-diffuse for some β, and let B_i be a sequence of balls centered on K with radii $\rho_i \to 0$. Then for sufficiently large i, the radius of B_i is less than ρ_K, so the diffuseness assumption guarantees that, for every k-dimensional affine subspace $\mathcal{L}$, $(B_i \smallsetminus \mathcal{L}^{(\beta\rho_i)}) \cap K \ne \varnothing$. Thus, $T_{B_i}(B_i \cap K)$ has k-dimensional width at least β for all sufficiently large i, so any Hausdorff-metric limit point of this sequence does as well.

Conversely, suppose K is not k-dimensionally β-diffuse. Then there exist balls B_i centered on K with radii $\rho_i \to 0$ and k-dimensional affine subspaces $\mathcal{L}_i$ such that $B_i \smallsetminus \mathcal{L}_i^{(\beta\rho_i)}$ is disjoint from K, so that each $T_B(B_i \cap K)$ has k-dimensional width at most β. By the continuity of the k-dimensional width, there exists a microset of K with k-dimensional width at most β. Thus, if K is not k-dimensionally diffuse, there exist microsets of K with arbitrarily small k-dimensional width, and by compactness, there are microsets with zero k-dimensional width. □

One advantage of playing on a diffuse set is the following lemma, which is a generalization of (2.5):

Lemma 4.5. *If K is k-dimensionally β-diffuse and S is k-dimensionally absolute winning on K, then S is k-dimensionally β'-absolute winning on K whenever $\beta' \le \frac{\beta}{2+\beta}$.*

Proof. There is no loss of generality in assuming that the diameters of the balls chosen by Bob tend to zero. Thus when the β'-game begins, Alice can choose A_i disjoint from B_i until Bob has chosen a ball of radius less than ρ_K. Reindexing, call this B_1. We have assumed S is k-dimensionally β''-absolute winning for all small enough positive β''; in particular for some $\beta'' \le \beta'$. Note that at any stage of the game, playing with the parameter β' instead of β'' affords Alice more possible moves but eliminates some of Bob's possible moves. By Lemma 4.3, in view of $\beta' \le \frac{\beta}{2+\beta}$, Bob will never win by having *all* of his potential moves removed, so a β''-strategy for the initial ball B_1 is also a β'-strategy for this initial choice. $\square$

Our next result asserts that diffuseness of K is sufficient for a lower estimate on the dimension of sets winning on K:

Theorem 4.6. *Let $K \subset \mathbb{R}^d$ be k-dimensionally β-diffuse set, let $S \subset \mathbb{R}^d$ be winning on K, and let U be an open set with $U \cap K \ne \varnothing$. Then*

$$\dim(S \cap K \cap U) \ge \frac{-\log(k+2)}{\log\beta - \log(2+\beta)}\,. \tag{4.3}$$

Proof. It is easy to see that, by Lemma 4.3, $N_K(\beta', \mathbf{x}, \rho) \ge k+2$ for $\beta' \le \frac{\beta}{2+\beta}$, $\rho < \rho_K$ and $\mathbf{x} \in K$; thus $N_K\left(\left(\frac{\beta}{2+\beta}\right)^n, \mathbf{x}, \rho\right) \ge (k+2)^n$, so we get the hypothesis of Lemma 4.1 with $\delta = \frac{-\log(k+2)}{\log\beta - \log(2+\beta)}$ and $M = \frac{1}{k+2}$. $\square$

Proving a set to be k-dimensionally absolute winning on K has several useful implications. First of all, it implies the winning property discussed in the beginning of the section:

Proposition 4.7. *Let $K \subset \mathbb{R}^d$ be k-dimensionally β-diffuse, and let $S \subset \mathbb{R}^d$ be k-dimensionally absolute winning on K. Then S is $\frac{\beta}{2+\beta}$-winning[2] on K.*

Proof. By Lemma 4.5, we know that S is k-dimensionally β'-absolute winning on K for any $\beta' \le \frac{\beta}{2+\beta}$. Let $\alpha = \frac{\beta}{2+\beta}$, $0 < \beta'' < 1$, and $\beta' = \alpha\beta'' < \frac{\beta}{2+\beta}$. We want to win the (α, β'')-game on K using the strategy we have in the k-dimensional β'-absolute game. When the game begins, Alice will choose A_i to be concentric with B_i until, reindexing, Bob chooses $B_1 = B(x, \rho)$ with $\rho < \rho_K$. Now at the i-th stage, suppose Alice's strategy is to remove $\mathcal{L}^{(\varepsilon\rho)}$ where $\varepsilon \le \beta' < \alpha$. Since K is k-dimensionally β-diffuse, by Lemma 4.3 there exists $\mathbf{x}' \in B(\mathbf{x}, \rho) \cap K$ such that $B(\mathbf{x}', \alpha\rho) \subset B(\mathbf{x}, \rho) \smallsetminus \mathcal{L}^{(\varepsilon\rho)}$, so she chooses $\mathbf{x}'$ as the center of her ball. Bob's next move B_i is a ball contained in Alice's of radius at least $\alpha\beta''\rho = \beta'\rho$, so this is a valid move for Bob in the k-dimensional β'-absolute game. Since B_i is the same ball in the two games, we have $\cap B_i \in S$, so S is α-winning. $\square$

In particular, in view of Theorem 4.6, (4.3) holds whenever $K \subset \mathbb{R}^d$ is k-dimensionally β-diffuse and $S \subset \mathbb{R}^d$ is k-dimensionally absolute winning on K. One also has

Proposition 4.8. *Let K be any closed subset of $\mathbb{R}^d$, $\beta > 0$, and for each $i \in \mathbb{N}$ let S_i be k-dimensionally β'-absolute winning on K for any $\beta' \le \beta$. Then the countable intersection $\cap_i S_i$ is also k-dimensionally β'-absolute winning on K for any $\beta' \le \beta$.*

Proof. The proof is exactly the same as Schmidt's original proof [26, Theorem 2] for α-winning sets. $\square$

Now that we have described consequences of being able to win an absolute game on a diffuse set K, the next natural question is how to one can verify this kind of winning property. Remarkably, it turns out that this property can be simply extracted from the corresponding one for $K = \mathbb{R}^d$. More generally, the following holds:

[2] In fact the argument shows that S is $\frac{\beta}{2+\beta}$-strong winning on K, where the latter is defined as in [23].

Proposition 4.9. *If $L \subset K$ are both k-dimensionally diffuse and $S \subset \mathbb{R}^d$ is k-dimensionally absolute winning on K, then it is also k-dimensionally absolute winning on L. In particular, every set which is k-dimensionally absolute winning on $\mathbb{R}^d$ is k-dimensionally absolute winning on every k-dimensionally diffuse set.*

Proof. We assume S is k-dimensionally absolute winning on K, and show that it is k-dimensionally absolute winning on L. Let β be small enough that L and K are both k-dimensionally β-diffuse, so that by Lemma 4.5 S is k-dimensionally $\frac{\beta}{2+\beta}$-absolute winning on K. Then consider the $\frac{\beta}{2+\beta}$-game played on L. Alice will choose A_i disjoint from B_i until Bob has chosen a ball of radius less than $\min(\rho_L, \rho_K)$. Reindexing, call this B_1. Since playing on L instead of K restricts Bob's choices but not Alice's, and because Lemma 4.5 guarantees that Bob will never win by having no valid moves available to him, Alice may use her strategy for the game played on K. □

Combining the above proposition with Theorem 2.4 we obtain the following statement, proving Theorem 1.2:

Theorem 4.10. *Let $S \subset \mathbb{R}^d$ be k-dimensionally absolute winning, $U \subset \mathbb{R}^d$ open, and K a k-dimensionally diffuse set. Then for any C^1 nonsingular map $f : U \to \mathbb{R}^d$, the set $f^{-1}(S) \cup U^c$ is k-dimensionally absolute winning on K. Consequently, for any sequence $\{f_i\}$ of C^1 diffeomorphisms of U onto (possibly different) open subsets of $\mathbb{R}^d$, the set* (1.5) *has positive Hausdorff dimension.*

Proof. For the first part, in view of Proposition 4.9, it suffices to show that $f^{-1}(S) \cup U^c$ is k-dimensionally absolute winning on $\mathbb{R}^d$, which is exactly the conclusion of Theorem 2.4. As for the second assertion, the union of the set (1.5) with U^c is k-dimensionally absolute winning on K by the first part, Lemma 4.5 and Proposition 4.8, hence winning on K by Proposition 4.7, hence its intersection with U, that is, the set (1.5) itself, has positive Hausdorff dimension by Theorem 4.6. □

In particular, by Theorems 2.5 and 2.6, the sets $\mathbf{BA}_d$ and $\tilde{E}(R, y)$, as well as their countable intersections and differmorphic images, always intersect hyperplane diffuse subsets of $\mathbb{R}^d$, and the intersection has positive Hausdorff dimension. This conclusion not only generalizes many known results on intersection of sets of these types with fractals, see [2, 3], but provides a more conceptual 'two-step' proof: to find uncountably many points of $K \cap S$, one has to check separately the diffuseness of K and the absolute winning property of S.

It is instructive to point out another simple consequence of Proposition 4.9. Let Γ be a one-cusp discrete group of isometries of the hyperbolic space $\mathbb{H}^n$, and let $D(\Gamma)$ be the set of lifts of endpoints of bounded geodesics in $\mathbb{H}^n/\Gamma$. Dani [4] showed that $D(\Gamma)$ is winning, and McMullen [23] strengthened this result, showing $D(\Gamma)$ to be absolute winning. Aravinda [1] proved that the intersection of the $D(\Gamma)$ with any C^1 curve is winning (and hence has Hausdorff dimension 1). This can now be seen to be an immediate corollary of Proposition 4.9 and the 0-dimensional diffuseness property of smooth curves.

Our next goal is to study the concept of the incompressibility on K, which calls for showing that the set (1.5) has full Hausdorff dimension. However for that we need additional assumptions on K, phrased in terms of properties of a measure whose support is equal to K. This is discussed in the next section.

5. Measures and the proof of Theorem 1.1

We start with a definition introduced in [17]: if μ is a locally finite Borel measure on $\mathbb{R}^d$ and $C, \gamma > 0$, one says that μ is (C, γ)-*absolutely decaying*[3] if there exists $\rho_0 > 0$ such that, for all $0 < \rho < \rho_0$, all $\mathbf{x} \in \operatorname{supp} \mu$, all affine hyperplanes $\mathcal{L} \subset \mathbb{R}^d$ and all $\varepsilon > 0$, one has

$$\mu\left(B(\mathbf{x}, \rho) \cap \mathcal{L}^{(\varepsilon)}\right) < C \left(\frac{\varepsilon}{\rho}\right)^{\gamma} \mu\big(B(\mathbf{x}, \rho)\big) . \tag{5.1}$$

[3] This terminology differs slightly from the one in [17], where a less uniform version was considered.

We will say that μ is *absolutely decaying* if it is (C,γ)-absolutely decaying for some positive C,γ.

Another useful property is the so-called Federer (doubling) condition. One says that μ is *D-Federer* if there exists $\rho_0 > 0$ such that

$$\mu\big(B(\mathbf{x},2\rho)\big) < D\mu\big(B(\mathbf{x},\rho)\big) \quad \forall\, \mathbf{x}\in \text{supp}\ \mu,\ \forall\, 0<\rho<\rho_0\,, \tag{5.2}$$

and *Federer* if it is D-Federer for some $D>0$. Measures which are both absolutely decaying and Federer are called *absolutely friendly*, a term coined in [25].

Many examples of absolutely friendly measures can be found in [17, 18, 32, 29]. The Federer condition is very well studied; it obviously holds when μ is *Ahlfors regular*, i.e. when there exist positive δ, c_1, c_2, ρ_0 such that

$$c_1\rho^\delta \le \mu\big(B(\mathbf{x},\rho)\big) \le c_2\rho^\delta \quad \forall\, \mathbf{x}\in \text{supp}\ \mu,\ \forall\, 0<\rho<\rho_0\,. \tag{5.3}$$

The above property for a fixed δ will be referred to as *δ-Ahlfors regularity*. It is easy to see that the Hausdorff dimension of the support of a δ-Ahlfors regular measure is equal to δ. An important class of examples of absolutely decaying and Ahlfors regular measures is provided by limit measures of irreducible families of contracting self-similar [17] or self-conformal [32] transformations of $\mathbb{R}^d$ satisfying the open set condition, as defined by Hutchinson [16]. See however [18] for an example of an absolutely friendly measure which is not Ahlfors regular.

Our next result shows that supports of absolutely decaying measures have the diffuseness property:

Proposition 5.1. *Let μ be absolutely decaying; then $K = \text{supp}\ \mu$ is hyperplane diffuse (and hence also k-dimensionally diffuse for all $1\le k<d$).*

Proof. Let $K = \text{supp}\,\mu$. If $B = B(\mathbf{x},\rho)$, where we assume $\rho<\rho_0$, and if $\mathcal{L}$ is an affine hyperplane, then

$$\mu\left(B\big(\mathbf{x},\rho\big)\cap \mathcal{L}^{(\beta\rho)}\right) < C\beta^\gamma \mu\left(B\big(\mathbf{x},\rho\big)\right).$$

If $\beta = \left(\frac{1}{C}\right)^{1/\gamma}$, then $C\beta^\gamma < 1$, hence the intersection of K with $B\big(\mathbf{x},\rho\big)\smallsetminus \mathcal{L}^{(\beta\rho)}$ is not empty. □

In particular, in view of Theorem 4.6, if K is the support of μ as above, U an open set with $K\cap U\neq\varnothing$ and S is winning on K, then $S\cap K\cap U$ has positive Hausdorff dimension. Note though that we can directly obtain a dimension bound for sets winning on supports of absolutely decaying measures using Lemma 4.1:

Lemma 5.2. *Let μ be (C,γ)-absolutely decaying, let $K = \text{supp}\ \mu$, and let $S\subset\mathbb{R}^d$ be winning on K. Then $\dim(S\cap K\cap U)\ge\gamma$ for any open U with $K\cap U\neq\varnothing$.*

Proof. For any $\beta<1$, let k be such that for sufficiently small ρ, $\inf_{\mathbf{x}\in K} N_K(\beta,\mathbf{x},\rho)\ge k$. For any $\mathbf{x}\in K$, take k hyperplane neighborhoods of width 2β contained in the ball around $\mathbf{x}$ of radius $(1-\beta)\rho$. Their total measure is at most $kC\frac{(2\beta)^\gamma}{(1-\beta)^\gamma}$ of the measure of the ball $B\big(\mathbf{x},(1-\beta)\rho\big)$. Thus if $k<\frac{(1-\beta)^\gamma}{C(2\beta)^\gamma}$, there is a point of K outside the union of these hyperplane neighborhoods and in $B\big(\mathbf{x},(1-\beta)\rho\big)$. This point can be the center of a new $\beta\rho$-ball in $B(\mathbf{x},\rho)$, and now we have $k+1$ disjoint balls (since each hyperplane neighborhood contains a ball). Thus $N_K(\beta,\mathbf{x},\rho)\ge\frac{(1-\beta)^\gamma}{C(2\beta)^\gamma}$, so for sufficiently small β we get $N_K(\beta,\mathbf{x},\rho)\ge M\beta^{-\gamma}$ with some constant M. Now we apply Lemma 4.1 which yields the desired dimension estimate. □

We remark that the statement of the above lemma has been known, in view of [19, Proposition 5.2], for absolutely friendly measures, that is, under the additional Federer condition, and the lemma weakens the assumption to being just absolutely decaying. Note also that a similar argument was used in [12] to prove

Lemma 5.3. [12, Theorem 5.1] *Let μ be δ-Ahlfors regular, let $K = \text{supp}\ \mu$, and let $S\subset\mathbb{R}^d$ be winning on K. Then $\dim(S\cap K\cap U)=\delta=\dim(K\cap U)$ for every open set $U\subset\mathbb{R}^d$ with $V\cap K\neq\varnothing$.*

In particular, the conclusion of the lemma holds whenever S is k-dimensionally absolute winning on K as above. Combining with Theorem 2.4, we obtain

Corollary 5.4. *Let μ be absolutely decaying and Ahlfors regular, and let $S \subset \mathbb{R}^d$ be k-dimensionally absolute winning; then S is strongly C^1 incompressible on* $\operatorname{supp} \mu$.

Note that Theorem 1.1 follows immediately from Corollary 5.4 and Theorem 2.5.

The following proposition, together with Proposition 5.1, shows that the notions of hyperplane diffuse sets and supports of absolutely decaying measures are closely related.

Proposition 5.5. *Let K be a hyperplane diffuse subset of $\mathbb{R}^d$ and $U \subset \mathbb{R}^d$ open with $U \cap K \neq \varnothing$. Then there exists an absolutely decaying measure μ such that* $\operatorname{supp} \mu \subset K \cap U$.

Whether or not it is possible to construct an absolutely decaying measure μ with $\operatorname{supp} \mu = K$ is an open question.

In order to prove the proposition, we will need the following

Lemma 5.6. *Given $\beta_0 > 0$ there exists a positive $\beta' < \beta_0$ such that, for every $\mathbf{x} \in \mathbb{R}^d$, $\rho > 0$, and $\mathbf{y}_1, \ldots, \mathbf{y}_d \in B(\mathbf{x}, \rho)$ such that the balls $B(\mathbf{y}_i, \beta_0\rho)$ are contained in $B(\mathbf{x}, \rho)$ and are pairwise disjoint, if a hyperplane $\mathcal{L}$ intersects each ball $B(\mathbf{y}_i, \beta'\rho)$, then*

$$B(\mathbf{x}, \rho) \cap \mathcal{L}^{(\beta'\rho)} \subset B(\mathbf{x}, \rho) \cap \mathcal{L}(\mathbf{y}_1, \ldots, \mathbf{y}_d)^{(\beta_0\rho)},$$

where $\mathcal{L}(\mathbf{y}_1, \ldots, \mathbf{y}_d)$ is the hyperplane passing through the points $\mathbf{y}_i$.

Proof. By rescaling, it suffices to prove the case that $\mathbf{x} = 0$ and $\rho = 1$. If the statement fails then there exists a sequence of d-tuples $(\mathbf{y}_{1,n}, \ldots, \mathbf{y}_{d,n})$ as above and hyperplanes $\mathcal{L}_n$ such that $\mathcal{L}_n$ intersects each $B(\mathbf{y}_{i,n}, \frac{1}{n})$ but

$$B(0,1) \cap \mathcal{L}_n^{(\frac{1}{n})} \not\subset B(0,1) \cap \mathcal{L}(\mathbf{y}_{1,n}, \ldots, \mathbf{y}_{d,n})^{(\beta_0)}.$$

By the compactness of $B(0,1)$ there is a limit point $(\mathbf{y}_1, \ldots, \mathbf{y}_d)$ of that sequence of d-tuples, and the corresponding subsequence of hyperplanes $B(0,1) \cap \mathcal{L}_{n_j}$ converges to $B(0,1) \cap \mathcal{L}(\mathbf{y}_1, \ldots, \mathbf{y}_d)$ in the Hausdorff metric, as does $B(0,1) \cap \mathcal{L}(\mathbf{y}_{1,n_j}, \ldots, \mathbf{y}_{d,n_j})$. Hence, for each $\varepsilon > 0$, there is some $m > 1/\varepsilon$ such that $\operatorname{dist}(\mathbf{y}_{i,m}, \mathbf{y}_i) < \varepsilon$ for each $1 \le i \le d$ and

$$B(0,1) \cap \mathcal{L}_m^{(\varepsilon)} \subset B(0,1) \cap \mathcal{L}(\mathbf{y}_1, \ldots, \mathbf{y}_d)^{(2\varepsilon)} \subset B(0,1) \cap \mathcal{L}(\mathbf{y}_{1,m}, \ldots, \mathbf{y}_{d,m})^{(3\varepsilon)}. \tag{5.4}$$

But, by the definition of $\mathcal{L}_m$,

$$B(0,1) \cap \mathcal{L}_m^{(\varepsilon)} \not\subset B(0,1) \cap \mathcal{L}(\mathbf{y}_{1,m}, \ldots, \mathbf{y}_{d,m})^{(\beta_0)}.$$

This contradicts (5.4) as long as $3\varepsilon < \beta_0$. □

Proof of Proposition 5.5. We will construct μ by iteratively distributing mass. Let ρ_K be as in Definition 4.2 and let $\beta_0 < 1/3$ be small enough to satisfy (4.2) with $\beta = \beta_0$. Let $\beta \le \frac{1}{3}\beta_0$ be small enough so that $\beta' = 2\beta$ is as in Lemma 5.6. Let $\mathbf{x}_0 \in U \cap K$ and $0 < \rho_0 < \rho_K$ small enough to guarantee $B(\mathbf{x}_0, \rho_0) \subset U$. Take $B(\mathbf{x}_0, \rho_0) = A_0$ and assign it mass 1. Now, given any $1 \le \ell \le d$ and points $\mathbf{y}_1, \ldots, \mathbf{y}_\ell \in A_0$, there is a hyperplane $\mathcal{L}$ passing through these points, and Lemma 4.3 guarantees the existence of a point $\mathbf{y}_{\ell+1} \in A_0 \cap K$ such that $B(\mathbf{y}_{\ell+1}, \beta_0\rho_0) \subset A_0$ is disjoint from $\mathcal{L}^{(\beta_0\rho_0)}$. Hence we can choose $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,d+1} \in K$ with $B(\mathbf{x}_{1,i}, \beta_0\rho_0) \subset A_0$ disjoint such that $B(\mathbf{x}_{1,d+1}, \beta_0\rho_0)$ is disjoint from $\mathcal{L}(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,d})^{(\beta_0\rho_0)}$.

By Lemma 5.6, this implies that for any hyperplane $\mathcal{L}$, its neighborhood $\mathcal{L}^{(\beta\rho_0)}$ can intersect at most d of the balls $B(\mathbf{x}_{1,i}, \beta\rho_0)$. Indeed, if it intersects the first d of these balls, then $\mathcal{L}$ intersects the corresponding balls $B(\mathbf{x}_{1,i}, \beta'\rho_0)$, which implies $\mathcal{L}^{(\beta\rho_0)} \subset \mathcal{L}(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,d})^{(\beta_0\rho_0)}$ which is disjoint from $B(\mathbf{x}_{1,d+1}, \beta_0\rho)$. We will take $A_1^{(i)} = B(\mathbf{x}_{1,i}, \beta\rho_0)$ and distribute the mass uniformly. Note that these children of A_0 are separated by $2\beta\rho_0$ from each other. Now suppose $A_{j,i} = B(\mathbf{x}_{j,i}, \beta^j\rho_0)$ comprise the jth stage of the construction, with each $\mathbf{x}_{j,i} \in K$. For each i, we again use the diffuseness condition and the lemma to find $\mathbf{x}_{j+1,\ell} \in K$, for $(i-1)(d+1) < \ell \le i(d+1)$, such that $A_{j+1,\ell} = B(\mathbf{x}_{j+1,\ell}, \beta^{j+1}\rho_0) \subset A_{j,i}$ are separated by at least $2\beta^{j+1}\rho_0$ and $\mathcal{L}^{(\beta^{j+1}\rho_0)}$ intersects at most d of these balls for any hyperplane $\mathcal{L}$, and distribute the mass uniformly among them. It is clear that the support of the limit measure μ is contained in K. We claim it is absolutely decaying.

Let $\rho < \rho_0$, $\mathbf{x} \in \operatorname{supp}\mu$, $0 < \varepsilon < \frac{1}{2}\beta\rho$, and $\mathcal{L}$ a hyperplane. Take $i, j \in \mathbb{N}$ such that

$$2\beta^i \rho_0 \le \rho < 2\beta^{i-1}\rho_0 \tag{5.5}$$

and

$$\frac{1}{2}\beta^{j+1}\rho \le \varepsilon < \frac{1}{2}\beta^j \rho. \tag{5.6}$$

Since $\mathbf{x} \in \operatorname{supp}\mu$, it is contained in some stage-i ball A_i and some stage-$(i-1)$ ball A_{i-1}. By (5.5), $B(\mathbf{x},\rho) \supset A_i$, and since A_{i-1} is separated by $2\beta^{i-1}\rho_0$ from every other stage-$(i-1)$ ball, $B(\mathbf{x},\rho) \cap \operatorname{supp}\mu \subset A_{i-1}$. Since $\varepsilon < \beta^i\rho_0$, the ε-neighborhood of $\mathcal{L}$ intersects at most d of the children of A_{i-1}. Since the measure is distributed uniformly among them, these children carry $\frac{d}{d+1}$ of the measure of A_{i-1}. Similarly, by (5.6), for each $\ell \le j$, $\mathcal{L}^{(\varepsilon)}$ intersects at most d of the children of each stage-$(i+j-2)$ ball. It thus follows that

$$\begin{aligned}\mu\big(B(\mathbf{x},\rho) \cap \mathcal{L}^{(\varepsilon)}\big) &\le \left(\frac{d}{d+1}\right)^j \mu(A_{i-1}) \\ &= \left(\frac{d}{d+1}\right)^j (d+1)\mu(A_i) \le \left(\frac{d}{d+1}\right)^j (d+1)\mu\big(B(\mathbf{x},\rho)\big).\end{aligned}$$

But, taking $\gamma = \frac{\log[d/(d+1)]}{\log\beta} > 0$, we have by (5.6) that

$$\left(\frac{d}{d+1}\right)^j = \beta^{j\gamma} \le 2^\gamma\beta^{-\gamma}\left(\frac{\varepsilon}{\rho}\right)^\gamma.$$

Hence, (5.1) holds for $C = 2^\gamma\beta^{-\gamma}(d+1)$ whenever $\varepsilon < \frac{1}{2}\beta\rho$. But if $\varepsilon \ge \frac{1}{2}\beta\rho$, then $C\left(\frac{\varepsilon}{\rho}\right)^\gamma$ is not less than 1, therefore (5.1) holds trivially. Thus μ is absolutely decaying. □

6. Concluding remarks

6.1. **Winning sets and strong incompressibility.** The main result of the paper specifies a strengthening of the winning property which enables one to deduce strong C^1 incompressibility, and, in particular, which is invariant under C^1 diffeomorphisms. We remark here that the class of sets winning in Schmidt's original version of the game does not have such strong invariance properties. In other words, it is possible to exhibit an example of a winning subset of $\mathbb{R}^2$ whose diffeomorphic image is not winning. Here is one construction, motivated by [23, §4]. Given $a \in \mathbb{N}$ and $\theta > 1$, we denote by $P_{a,\theta}$ the *almost arithmetic progression* which starts with a and has difference θ, that is,

$$P_{a,\theta} \stackrel{\text{def}}{=} \{\lceil a + j\theta\rceil : j = 0, 1, \dots\}.$$

Then consider

$$S = \{(x,y) \in \mathbb{R}^2 : \exists\, a \in \mathbb{N} \text{ and } \theta > 1 \text{ such that } x_i = y_i = 0 \text{ whenever } i \in P_{a,\theta}\}, \tag{6.1}$$

where x_i, y_i are the digits of the base 3 expansions of x, y. We also let

$$\tilde{S} = \{(x,y) \in \mathbb{R}^2 : x_i = y_i = 0 \text{ for some } i \in \mathbb{N}\}.$$

Clearly S is a subset of $\tilde{S}$; it is not hard to see that S has Lebesgue measure zero, and $\tilde{S}$ has full Lebesgue measure.

Proposition 6.1. (a) *S (and therefore $\tilde{S}$) is winning.*

(b) *Define $f_n(x,y) \stackrel{\text{def}}{=} (3^{-n}x, y)$. Then $f_n(\tilde{S})$ is not $\frac{4}{3^n}$-winning, and hence neither is $f_n(S)$.*

Proof. For (a), take $\alpha = 1/108$ and, without loss of generality, assume the radius of B_1 is $\rho < \frac{1}{6\alpha}$. Then let a be the integer satisfying

$$\frac{1}{6 \cdot 3^{a+1}} \leq \alpha\rho < \frac{1}{6 \cdot 3^a}.$$

This implies $\rho \geq 18 \cdot 3^{-(a+1)} = 2 \cdot 3^{1-a}$. Hence, B_1 contains a square of sidelength 3^{-a} consisting of pairs (x, y) with $x_a = y_a = 0$. Since $\alpha\rho < \frac{1}{2 \cdot 3^a}$, Alice can choose A_1 to be contained in this square.

Now take $\theta = -\log_3(\alpha\beta)$, and assume A_k has been chosen so that for all $(x, y) \in A_k$ and $0 \leq j < k$ we have $x_i = y_i = 0$ for $i = \lceil a + \theta j \rceil$. Then the radius of B_{k+1} is

$$(\alpha\beta)^k \rho = 3^{-\theta k} \rho \geq 2 \cdot 3^{1-a-k\theta}.$$

Let $i = \lceil a + k\theta \rceil$. Then $i \geq a + k\theta$, so

$$2 \cdot 3^{1-i} \leq 2 \cdot 3^{1-a-k\theta} \leq (\alpha\beta)^k \rho.$$

Thus, B_{k+1} must contain a square of sidelength 3^{-i} consisting of pairs (x, y) with $x_i = y_i = 0$. Furthermore, $i < a + k\theta + 1$, so

$$(\alpha\beta)^k \alpha\rho < \frac{1}{6 \cdot 3^{a+k\theta}} \leq \frac{1}{2 \cdot 3^i}\,.$$

Therefore Alice can choose A_{k+1} to be contained in this square. This implies that $(x, y) \in \cap A_k$ must satisfy $x_i = y_i = 0$ for all $i \in P_{a,\theta}$.

Next, we give a strategy for Bob with target set $f_n(\tilde{S})$, with $\alpha = 4 \cdot 3^{-n}$ and $\beta = \frac{1}{4}3^{-n}$. We will show that Bob can play the game in such a way that, regardless of Alice's play, we will have for each $i \in \mathbb{N}$ either $x_{i+n} = 1$ or $y_i = 1$. Bob begins by choosing B_0 to be a ball of radius $1/2$ consisting of pairs (x, y) with $x_0 = y_0 = 1$. Note that the radius of A_k is $\frac{\alpha(\alpha\beta)^{k-1}}{2} = \frac{2}{3^{2n(k-1)+n}}$, so it contains a square of sidelength $3^{-2nk+n-1}$ consisting of points (x, y) satisfying $x_{2nk-n+1} = y_{2nk-n+1} = 1$. Also, the radius of B_{k+1} is $\frac{(\alpha\beta)^k}{2} = \frac{1}{2 \cdot 3^{2nk}}$, so B_{k+1} can be chosen so that $x_i = y_i = 1$ for $(2k-1)n < i \leq 2kn$. Let $(x, y) \in \cap B_k$ and $i \in \mathbb{N}$. Let $m = 0, 1, \ldots$ be such that $mn < i \leq (m+1)n$. If m is odd, then $m = 2k - 1$ for some $k \in \mathbb{N}$ and $y_i = 1$. If m is even, then $m = 2(k-1)$ for some $k \in \mathbb{N}$, therefore $(2k-1)n < i + n \leq 2kn$ and $x_{i+n} = 1$. Hence, there does not exist an $i \in \mathbb{N}$ for which $x_{i+n} = y_i = 0$, so $(x, y) \notin f_n(S)$. □

The above argument also shows that there exists a C^∞ diffeomorphism f of $\mathbb{R}^2$ (constructed so that for each m there exists a strip $n \leq x \leq n+1$ on which $f^{-1} = f_m^{-1}$) such that $f(\tilde{S})$ is not winning. However, it is clear that $\tilde{S}$ is of full Lebesgue measure, and hence so are all its diffeomorphic images considered above and their countable intersections. Therefore it cannot serve as a counterexample to the strong affine incompressibility of winning sets. Whether or not such a counterexample is furnished by S as in (6.1), that is, whether or not the intersection (1.1) for this S has Hausdorff dimension less than 2, is not clear to the authors. In fact it would be interesting to find out whether or not there exists a winning subset S of $\mathbb{R}^d$ which is not strongly (strongly affinely, strongly C^1) incompressible. Or maybe even for which the intersection (1.1), with no uniform bound on bi-Lipschitz norms of f_i, can be empty (this is impossible for $d = 1$, as proved by Schmidt in [26]).

6.2. **VWA is strongly incompressible.** A straightforward application of the Baire category theorem shows that $\bigcap_i f_i^{-1}(S)$ is nonempty whenever S is residual and the f_i are homeomorphisms; however this does not imply that it has full Hausdorff dimension, and thus does not imply incompressibility. K. Falconer [9] introduced a theory which implies lower bounds on the dimension of $\bigcap_i f_i^{-1}(S)$ for certain residual sets S. These ideas were developed further by A. Durand in [7], see also references therein. We illustrate these results by exhibiting another incompressible set arising in Diophantine approximation. Namely, for $\tau > 0$ denote by $J_{d,\tau}$ the set of τ-*approximable* vectors in $\mathbb{R}^d$, that is,

$$J_{d,\tau} \stackrel{\text{def}}{=} \left\{ \mathbf{x} \in \mathbb{R}^d : \text{ there are infinitely many } \mathbf{p} \in \mathbb{Z}^d,\, q \in \mathbb{N} \text{ with } \left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| < \frac{1}{q^\tau} \right\}.$$

It is well known that when $\tau > \frac{d+1}{d}$, the sets $J_{d,\tau}$ have Lebesgue measure zero, and $\dim(J_{d,\tau})$ tends to d as $\tau \to \frac{d+1}{d}$. It was shown by Durand, see [7, Thm. 1 and Prop. 3], that for any nonempty open $U \subset \mathbb{R}^d$ and a sequence $f_1, f_2, \ldots$ of bi-Lipschitz maps $U \to \mathbb{R}^d$, the Hausdorff dimension of the intersection $\bigcap_i f_i^{-1}(J_{d,\tau})$ is equal to $\dim(J_{d,\tau})$. Thus, if one denotes by $\mathbf{VWA}_d \stackrel{\text{def}}{=} \bigcup_{\tau > \frac{d+1}{d}} J_{d,\tau}$ the set of *very well approximable* vectors in $\mathbb{R}^d$, it follows that

$$\dim\left(\bigcap_i f_i^{-1}(J_{d,\tau})\right) = d\,;$$

that is, $\mathbf{VWA}_d$ is strongly incompressible.

Note however that $\mathbf{VWA}_d$ is not a winning set, since it is contained in $\mathbb{R}^d \smallsetminus \mathbf{BA}_d$. Furthermore, the analogue of Theorem 1.1 does not hold for $\mathbf{VWA}_d$: indeed, using Lemma 3.1 it is not hard to find a closed set K, supporting an absolutely decaying and Ahlfors regular measure, which is contained in $\mathbf{BA}_d$ (see [18] where such constructions are explained), so that $K \cap \mathbf{VWA}_d = \varnothing$.

The example described above brings up a natural open question: is the set $\mathbf{BA}_d$ (together with its cousins defined in terms of toral endomorphisms) strongly incompressible? That is, can one weaken the C^1 assumption on the maps f_i to just bi-Lipschitz? The methods of the present paper do not seem to be enough to answer this question.

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