EVERYTHING IS ILLUMINATED

SAMUEL LELIÈVRE, THIERRY MONTEIL, AND BARAK WEISS

Abstract. We study geometrical properties of translation surfaces: the finite blocking property, bounded blocking property, and illumination properties. These are elementary properties which can be fruitfully studied using the dynamical behavior of the $\mathrm{SL}(2,\mathbb{R})$-action on the moduli space of translation surfaces. We characterize surfaces with the finite blocking property and bounded blocking property, completing work of the second-named author [Mon05]. Concerning the illumination problem, we also extend results of Hubert-Schmoll-Troubetzkoy [HST08], removing the hypothesis that the surface in question is a lattice surface, thus settling a conjecture of [HST08]. Our results crucially rely on the recent breakthrough results of Eskin-Mirzakhani [EM] and Eskin-Mirzakhani-Mohammadi [EMM], and on related results of Wright [Wria].

1. Introduction

A *translation surface* is a finite union of polygons, glued along parallel edges by translations, up to a cut and paste equivalence. These structures arise in the study of billiards, interval exchange transformations, and various problems in group theory and geometry. See [MT02, Zor06] for comprehensive introductions and detailed definitions. The purpose of this paper is to apply recent breakthrough results of Eskin-Mirzakhani [EM] and Eskin-Mirzakhani-Mohammadi [EMM], on the dynamics of a group action on the moduli space of translation surfaces, to some elementary geometrical questions concerning translation surfaces. We begin with some definitions.

A pair of points $(x, y) \in M \times M$ is *finitely blocked* if there exists a finite set $B \subset M$ which does not contain x or y and intersects every straight-line trajectory connecting x and y. A set B with this property is called a *blocking set* for (x, y), and the minimal cardinality of a blocking set is called the *blocking cardinality of* (x, y) and is denoted by $\mathrm{bc}(x, y)$. A translation surface M has the *blocking property* if any pair $(x, y) \in M \times M$ is finitely blocked, and the *bounded blocking property* if there is a number n such that any pair $(x, y) \in M \times M$ is finitely blocked with blocking cardinality at most n. If x and y are finitely blocked with blocking cardinality zero, that is, if there is no straightline path on M from x to y, then we say that x and y *do not illuminate each other*. A translation surface M is a *torus cover* if there is a surjective translation map from M to a torus (the singularities of M may project to one or several points on the torus). Equivalently (see e.g. [Mon05]), the subgroup of $\mathbb{R}^2$ generated by holonomies of absolute periods on M is discrete.

Our first result settles a question of the second-named author, see [Mon05, Mon09].

Theorem 1. *For a translation surface M, the following are equivalent:*

(1) *M is a torus cover.*

(2) *M has the blocking property.*
(3) *There is an open set $U \subset M \times M$ such that any pair of points in U is finitely blocked.*
(4) *M has the bounded blocking property.*

Hubert, Schmoll and Troubetzkoy [HST08] have constructed an example of a translation surface M which is not a torus cover, and in which there are infinitely many pairs of points which do not illuminate each other. In fact, there is an involution $\tau : M \to M$ such that for any $x \in M$, there is no straight line between x and $\tau(x)$. See §6.3 for similar examples. This shows that in (3), it is not enough to suppose that U is infinite.

Our second result concerns questions of illumination. The classical illumination problem was first posed in the 1950's, when it was asked whether there exists a polygonal room with a pair of points which do not illuminate each other. First examples were found by Tokarsky [Tok95] and Boshernitzan (unpublished), and this raised the question of classification and possible cardinality of pairs of points which do not illuminate one another on translation surfaces. We refer to [HST08] or the Wikipedia page `http://en.wikipedia.org/wiki/Illumination_problem` for a brief history. We show:

Theorem 2. *For any translation surface M, and any point $x \in M$, the set of points y which are not illuminated by x is finite.*

Moreover, the set

$$\{(x, y) : x \text{ and } y \text{ do not illuminate each other}\}$$

is the union of a finite set, and of finitely many translations surfaces M' embedded in $M \times M$, such that the projections $p_i|M' : M' \to M$ are both finite-degree covers.

Here $p_i : M \times M \to M,\ i = 1, 2$ are the natural projections onto the first and second factors respectively.

Theorem 2 strengthens results of [HST08], which deal with surfaces which have a large group of translation automorphisms. Namely, Theorem 2 was proved in [HST08] under the additional hypothesis that M is a lattice surface, and when M is a pre-lattice surface, the first assertion of the theorem was shown, with 'countable' in place of 'finite' (for the definitions see §2.3). The first assertion of Theorem 2 settles [HST08, Conjecture 1]. In §5 we deduce Theorem 2 from the more general Theorem 11. In §6 we give examples which elaborate on related examples given in [HST08].

A standard 'unfolding' technique (see [MT02, Zor06]) leads to the following result, justifies the title of this paper. It is a special case of [OP01, Conjecture 1].

Corollary 3. *Let P be a rational polygon. Then for any $x \in P$ there are at most finitely many points y for which there is no geodesic trajectory between x and y.*

There is a moduli space $\mathcal{H}$ parameterizing all translation surfaces sharing some topological data, and this space is equipped with an action of the group $G \stackrel{\text{def}}{=} \mathrm{SL}(2, \mathbb{R})$. The breakthrough work [EM, EMM] has made it possible to analyze the dynamics of this action in great detail. Our analysis depends crucially on this work, as well as on additional work of Wright [Wria].

We note that the crucial feature which make our analysis possible is that the geometric properties we consider give rise to subsets of $\mathcal{H}$ which are closed and

G-invariant. It has long been known that a detailed understanding of the G-action would shed light on the illumination problem, as well as on many similar 'elementary' problems. For more papers applying the dynamics of the G-action to the analysis of closed and G-invariant geometrical properties of translation surfaces, see [Vee95, Vor96, Mon05, Mon09, HST08, SW10, SW07, LW].

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2. Preliminaries

We begin by briefly recalling the definitions of translation surfaces and strata, and refer to [MT02, Zor06] for more details. Fix a topological orientable surface S of genus g, a finite subset $\Sigma = \{x_1, \ldots, x_k\}$ of S, and non-negative integers $\alpha_1, \ldots, \alpha_k$ so that $\sum_i \alpha_i = 2g - 2$. We allow some of the α_i to be zero and require $k \neq 0$. A translation surface M of type $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is a surface M homeomorphic to S, with k labelled singular points $\{\xi_1, \ldots, \xi_k\}$, equipped with an equivalence class of atlases of *planar charts*, i.e. maps from open subsets of $M \smallsetminus \{\xi_1, \ldots, \xi_k\}$ to $\mathbb{C}$, such that:

- Transition maps for the charts are translations.
- At each ξ_i the charts give rise to a cone type singularity of angle $2\pi(\alpha_i + 1)$.

As usual two atlases are considered equivalent if their union is also an atlas of the same type, and two translation surfaces are considered equivalent if there is a homeomorphism from one to the others, which is a translation in charts, and maps the distinguished finite set $\{\xi_i\}$ of one translation surface bijectively to the other in a way which respects the numbering. Note that an atlas of planar charts on $M \smallsetminus \Sigma$ naturally induces a translation structure on $(M \smallsetminus \Sigma) \times (M \smallsetminus \Sigma)$, with charts taking values in $\mathbb{C}^2$ and for which transition maps are translations. We will call this the *Cartesian product translation structure on* M^2.

The points ξ_i are called *singularities*. Note that we have allowed singularities with cone angle 2π (as happens when $\alpha_i = 0$). Such singularities are sometimes referred to as *marked points*. Note also that in contrast to the convention used by some authors, our convention is that singularities are labeled.

A homeomorphism $S \to M$ which maps each x_i to ξ_i is called a *marking*. We can use a marking and the planar charts of M to evaluate the integrals of directed paths on S beginning and ending in Σ. Such an integral is a complex number whose real and imaginary components measure respectively the total horizontal and vertical distance travelled when moving in M along the image of the path. Denote by $\mathcal{H}(\vec{\alpha})$ the set of translation surfaces of type $\vec{\alpha}$. It is called a *stratum* and is equipped with a natural topology defined as follows. The discussion above shows that the marking gives rise to a map

$$\mathcal{H}(\vec{\alpha}) \to H^1(S, \Sigma; \mathbb{C}).$$

It is known that the maps above constitute an atlas of charts which endow $\mathcal{H}(\vec{\alpha})$ with the structure of a linear orbifold. We will call these coordinates *period coordinates*. With respect to period coordinates, the change of a marking constitutes a change of coordinates via a unimodular integral matrix, so $\mathcal{H}(\vec{\alpha})$ is naturally endowed with a Lebesgue measure and a $\mathbb{Q}$-structure. It is known that each stratum has finitely many connected components.

The group G acts on each stratum component $\mathcal{H}$ by postcomposition of planar charts. That is, identifying the field of complex numbers with the plane $\mathbb{R}^2$ in the usual way, each $g \in G$ is a linear map of $\mathbb{R}^2$ and we use it to replace each chart $M \supset U \xrightarrow{\varphi} \mathbb{C} \cong \mathbb{R}^2$ with the chart $g \circ \varphi : U \to \mathbb{R}^2$. For each stratum component $\mathcal{H}$, the subset $\mathcal{H}^{(1)}$ consisting of area one surfaces is a sub-orbifold which in period coordinates is cut out by a quadratic condition. It is preserved by the G-action, and G acts ergodically preserving a natural smooth finite measure obtained from the Lebesgue measure by a cone construction. Given a translation surface M and a positive real number t, we denote by tM the translation surface obtained by multiplying all planar charts of M by the scalar t.

2.1. **Adding marked points.** We will need some notation for the operation of covering a stratum by a corresponding stratum with one or two additional marked points.

Given a stratum component $\mathcal{H}$, we denote by $\mathcal{H}'$ the corresponding stratum component of surfaces with one additional marked point, and by $\mathcal{H}''$ the corresponding stratum component of surfaces with two additional marked points. More formally this is defined as follows. Suppose $\mathcal{H}$ is a component of $\mathcal{H}(\vec{\alpha})$ where $\vec{\alpha} \stackrel{\text{def}}{=} (\alpha_1, \ldots, \alpha_k)$ and $\Sigma \stackrel{\text{def}}{=} \{x_1, \ldots, x_k\}$ is a finite subset of cardinality k in the topological surface S. Let x_{k+1}, x_{k+2} denote two distinct points on $S \smallsetminus \Sigma$, set $\alpha_{k+1} = \alpha_{k+2} = 0$, set

$$\Sigma' \stackrel{\text{def}}{=} \Sigma \cup \{x_{k+1}\}, \quad \Sigma'' \stackrel{\text{def}}{=} \Sigma' \cup \{x_{k+2}\}, \quad \vec{\alpha}' \stackrel{\text{def}}{=} (\alpha_1, \ldots, \alpha_{k+1}), \quad \vec{\alpha}'' \stackrel{\text{def}}{=} (\alpha_1, \ldots, \alpha_{k+2}),$$

and let $\varphi' : \mathcal{H}(\vec{\alpha}') \to \mathcal{H}(\vec{\alpha})$, $\varphi'' : \mathcal{H}(\vec{\alpha}'') \to \mathcal{H}(\vec{\alpha}')$ be the *forgetful maps* obtained by deleting the points corresponding to x_{k+1}, x_{k+2} from the domain of any planar chart. Let $\varphi \stackrel{\text{def}}{=} \varphi' \circ \varphi''$. The three maps $\varphi', \varphi'', \varphi$ are bundle maps for the respective bundles $\mathcal{H}(\vec{\alpha}'), \mathcal{H}(\vec{\alpha}''), \mathcal{H}(\vec{\alpha}'')$ with bases $\mathcal{H}(\vec{\alpha}), \mathcal{H}(\vec{\alpha}'), \mathcal{H}(\vec{\alpha})$ and fibers $S \smallsetminus \Sigma, S \smallsetminus \Sigma', (S \smallsetminus \Sigma)^2 \smallsetminus \Delta$ respectively (Δ is the diagonal). Finally we let $\mathcal{H}', \mathcal{H}''$ be the connected components of $\mathcal{H}(\vec{\alpha}')$ and $\mathcal{H}(\vec{\alpha}'')$ covering the component $\mathcal{H}$.

One easily checks from the definitions that the maps $\varphi, \varphi', \varphi''$ are G-equivariant, and that the fibers are linear manifolds in period coordinates. Moreover note that the linear structure on a fiber $\varphi'^{-1}(M) \cong S \smallsetminus \Sigma$ coincides with the translation structure afforded by the translation charts on M, and similarly, the linear structure on a fiber $\varphi^{-1}(M) \cong (S \smallsetminus \Sigma)^2 \smallsetminus \Delta$ coincides with the Cartesian product translation structure on M^2. In the sequel we will refer to x_{k+1}, x_{k+2} as the first and second marked points for the covers $\mathcal{H}'' \to \mathcal{H}' \to \mathcal{H}$. Note that we allow $\mathcal{H}$ to contain additional marked points.

2.2. **Recent dynamical breakthroughs.** We now state the results of [EM, EMM, Wria] mentioned in the introduction. This requires some terminology. We say that a subset $\mathcal{L}_0 \subset \mathcal{H}$ is a *complex linear manifold defined over* $\mathbb{R}$ if for each of the charts $\mathcal{H} \to H^1(S, \Sigma; \mathbb{C}) \cong \mathbb{C}$ obtained by fixing a marking, the image of $\mathcal{L}_0$ is the intersection of an open set with an affine subspace whose linear part is a $\mathbb{C}$-linear vector space defined over $\mathbb{R}$. Note that the real dimension of a complex linear manifold is even. Given $\mathcal{L} \subset \mathcal{H}^{(1)}$, we denote

$$\widehat{\mathcal{L}} \stackrel{\text{def}}{=} \{tM' : t > 0, M' \in \mathcal{L}\}.$$

If ν is a measure on $\mathcal{H}$ then $\mu(A) = \nu(\{tx : x \in A, t \in (0,1]\}$ is a measure on $\mathcal{H}^{(1)}$ and we say that μ *is obtained by coning off* ν. We say that $\mathcal{L} \subset \mathcal{H}^{(1)}$ is an *affine invariant manifold* if it is G-invariant, is the support of an ergodic G-invariant measures μ, $\widehat{\mathcal{L}}$ is a complex linear manifold defined over $\mathbb{R}$, and μ is obtained by coning off Lebesgue measure on $\widehat{\mathcal{L}}$.

Theorem 4 (Eskin-Mirzakhani-Mohammadi)**.** *For each stratum component $\mathcal{H}$ and each $M \in \mathcal{H}^{(1)}$, the orbit closure $\mathcal{L} \stackrel{\mathrm{def}}{=} \overline{GM}$ is an affine invariant manifold. The collection of affine invariant manifolds of $\mathcal{H}$ obtained as orbit-closures for the G-action is countable. If $\mathcal{L}_n$, $n \geq 1$ is a sequence of distinct affine invariant manifolds of some dimension k contained in $\mathcal{H}$, then after passing to a subsequence, the set of accumulation points*

$$\{M \in \mathcal{H} : \exists M_n \in \mathcal{L}_n \text{ such that } M_n \to M\}$$

is an affine invariant manifold $\mathcal{L}_\infty$ with $\dim \mathcal{L}_\infty > k$ and $\{M_n\} \subset \mathcal{L}_\infty$.

Note that the results of [EMM] work for strata with marked points, i.e. they allow $\alpha_i = 0$.

Suppose that the number of singularities k is at least 2. Let $H_1(S)$ and $H_1(S, \Sigma)$ denote respectively the absolute and relative homology groups. Then we have $H_1(S) \subset H_1(S, \Sigma)$ and we can restrict each 1-cocycle in $H^1(S, \Sigma; \mathbb{C})$ to the subspace $H_1(S)$; that is we get a natural restriction map $H^1(S, \Sigma; \mathbb{C}) \to H^1(S; \mathbb{C})$. The kernel REL of this map is a subspace of $H^1(S, \Sigma; \mathbb{C})$ of real dimension $2(k-1)$, and we have a foliation of $H^1(S, \Sigma; \mathbb{C})$ by cosets of REL. Since the restriction map $H^1(S, \Sigma; \mathbb{C}) \to H^1(S; \mathbb{C})$ is topological, the space REL is independent of a marking, that is can be used to unequivocally define a linear foliation of $\mathcal{H}(\vec{\alpha})$ using period coordinates. This foliation of $\mathcal{H}(\vec{\alpha})$ is called the *REL foliation.* The G-action respects the REL foliation and hence we have a linear foliation of $\mathcal{H}$ by leaves tangent to $\mathfrak{g} \oplus \mathrm{REL}$, where we use $\mathfrak{g}$ to denote the tangent to the foliation by G-orbits. We denote this foliation by $G \oplus \mathrm{REL}$. Following [Wria], if a closed G-invariant and G-ergodic linear manifold $\mathcal{L}$ is contained in a single leaf of the foliation $G \oplus \mathrm{REL}$, we say that it is *of cylinder rank one.* A translation surface M is *completely periodic* if in any cylinder direction on M there is a complete cylinder decomposition.

Theorem 5 (Wright [Wria], Theorems 1.5 and 1.6)**.** *A linear manifold $\mathcal{L}$ as above is of cylinder rank one if and only if any surface in $\mathcal{L}$ is completely periodic.*

We will need the following Lemma. Note that its assertion would be trivial if the fiber of φ were compact.

Lemma 6. *Let $M \in \mathcal{H}$ and $M'' \in \varphi^{-1}(M) \subset \mathcal{H}''$. Let $\mathcal{L} \stackrel{\mathrm{def}}{=} \overline{GM}$ and $\mathcal{L}'' \stackrel{\mathrm{def}}{=} \overline{GM''}$. Then $\varphi|_{\mathcal{L}''}$ is an open mapping and hence $\dim \varphi(\mathcal{L}'') = \dim \mathcal{L}$.*

Proof. According to [EMM], there are Borel probability measures μ, μ'' on $\mathcal{H}$, $\mathcal{H}''$ respectively such that $\mathcal{L} = \mathrm{supp}\mu$, $\mathcal{L}'' = \mathrm{supp}\mu''$. We first claim that $\mu = \varphi_* \mu''$. To this end note that Theorems 2.6 and 2.10 in [EMM] provide an averaging method converging to μ, μ''; that is, in both of these theorems, one finds probability measures ν_T on G, such that for any continuous compactly supported functions f, f'' on $\mathcal{H}$ and $\mathcal{H}''$ respectively,

$$\int_G f(gM) d\nu_T(g) \to_{T \to \infty} \int_{\mathcal{H}} f d\mu \text{ (resp., } \int_G f(gM'') d\nu_T(g) \to_{T \to \infty} \int_{\mathcal{H}''} f'' d\mu'' \text{).}$$

By a standard argument we may assume that this is also true if f'' is continuous and has a finite limit at infinity; in particular, for $f \in C_c(\mathcal{H})$ we may take $f'' = f \circ \varphi$. Thus by equivariance we have

$$\int_{\mathcal{H}} f d\mu \longleftarrow \int_G f(gM) d\nu_T(g) = \int_G f''(gM'') d\nu_T(g) \longrightarrow \int_{\mathcal{H}''} f \circ \varphi \, d\mu'',$$

and this implies that $\mu = \varphi_* \mu''$.

The map $\varphi|_{\mathcal{L}''} : \mathcal{L}'' \to \mathcal{L}$ is an affine map of affine manifolds. In order to show that it is open it suffices to show that its derivative is surjective at every point $x \in \mathcal{L}''$. If not, then there is a neighborhood $\mathcal{U}$ of x in $\mathcal{L}''$ such that $\varphi(\mathcal{U})$ is contained in a proper affine submanifold of $\mathcal{L}$. Such a proper affine submanifold must have zero measure for the flat measure class on $\mathcal{L}$, i.e. $\mu(\varphi(\mathcal{U})) = 0$. By the preceding paragraph this implies $\mu''(\mathcal{U}) = 0$ which is impossible. □

2.3. **The Veech group, lattice surfaces, and periodic points.** An *affine automorphism* of a translation surface M is a homeomorphism $\varphi : M \to M$ which is affine in charts. In this case, by connectedness, its derivative $D\varphi$ is a constant 2×2 matrix of determinant ± 1. We denote by $\mathrm{Aff}^+(M)$ the group of orientation-preserving affine automorphisms, i.e. those for which $D\varphi \in G$. We say that φ is a *parabolic automorphism* if $D\varphi$ is a parabolic matrix, i.e., is not the identity but has both eigenvalues equal to 1. The *Veech group* of M is the image under the homomorphism $D : \mathrm{Aff}^+(M) \to G$ of the group of orientation-preserving affine automorphisms. We say that M is a *lattice surface* if its Veech group is a lattice in G. Equivalently, by a theorem of Smillie (see [Vee95, SW07]), the orbit GM is closed. Following [HST08] we say that M is a *pre-lattice surface* if $\mathrm{Aff}^+(M)$ contains two non-commuting parabolic automorphisms. Veech [Vee89] showed that a lattice surface is a pre-lattice surface, justifying the terminology. A point $x \in M$ is called *periodic* if its orbit under $\mathrm{Aff}^+(M)$ is finite.

2.3.1. *Example.* In Lemma 6 we showed that $\varphi''|_{\mathcal{L}''} : \mathcal{L}'' \to \mathcal{L}$ is an open map. Given that $\mathcal{L}$ is connected, this leads to the question of whether $\varphi|_{\mathcal{L}''}$ is surjective. The following example of Alex Wright shows that an open affine map of orbit-closures need not be surjective. Let $M \in \mathcal{H}$ be a lattice surface which admits an involution τ (e.g. M could be a surface of genus 2 and τ could be the hyper-elliptic involution). Let $\mathcal{L} = GM$ be the orbit of M (which in this case coincides with the orbit closure), let $x \in M$ be a non-periodic point, and let $M' \stackrel{\mathrm{def}}{=} (M, x)$ be the surface in $\mathcal{H}'$ obtained by marking the point x. It was proved in [HST08], and follows easily from Theorem 4, that $\mathcal{L}' \stackrel{\mathrm{def}}{=} \overline{GM'}$ coincides with $\varphi'^{-1}(GM)$ (i.e. all surfaces in GM marked at all nonsingular points). Now let $y \stackrel{\mathrm{def}}{=} \tau(x) \neq x$, let $M'' \stackrel{\mathrm{def}}{=} M(x, y)$ be the surface in $\mathcal{H}''$ obtained by marking M at the two points x, y, let $\mathcal{L}'' \stackrel{\mathrm{def}}{=} \overline{GM''}$, and let $\varphi'' : \mathcal{H}'' \to \mathcal{H}'$ be the affine map which forgets the second marked point. We have

$$\mathcal{L}'' \subset \{(M_0, x_0, y_0) \in \mathcal{H}'' : M_0 \in \mathcal{L}, \, \tau(x_0) = y_0 \neq x_0\},$$

since the set on the right-hand side is closed and G-invariant. This implies that $\varphi''(\mathcal{L}'') \subset \{(M_0, x_0) : M_0 \in GM, \tau(x_0) \neq x_0\}$, and in particular $\varphi''|_{\mathcal{L}''}$ is not surjective. However the proof of Lemma 6 shows that $\varphi''|_{\mathcal{L}''}$ is open.

Using one additional marked point one can find similar examples that show that in general, in Lemma 6, one need not have $\varphi(\mathcal{L}'') = \mathcal{L}$.

3. Bounded blocking defines closed sets

Let M be a translation surface with singularity set Σ, and let $\widehat{M^2} = \{(x, y) \in (M \smallsetminus \Sigma)^2 : x \neq y\}$. If Z is a topological space and $A \subset B$ are subsets of Z, when we say that A *is closed as a subset of* B, we mean that A is closed in the relative topology, i.e. $A = B \cap \overline{A}$.

Lemma 7. *For any fixed integer $n \geq 0$, the following hold:*

(I) *For a fixed translation surface M, the set*

$$F_n(M) \stackrel{\text{def}}{=} \{(x, y) \in \widehat{M^2} : \mathrm{bc}(x, y) \leq n\}$$

is closed as a subset of $\widehat{M^2}$.

(II) *For a fixed translation surface M, and a fixed nonsingular $x \in M$, the set*

$$F_n(M, x) \stackrel{\text{def}}{=} \{y \in M \smallsetminus (\Sigma \cup \{x\}) : \mathrm{bc}(x, y) \leq n\}$$

is closed as a subset of $M \smallsetminus (\Sigma \cup \{x\})$.

(III) *The set $\mathcal{F}_n \subset \mathcal{H}''$ consisting of all surfaces on which the first and second marked points are finitely blocked of blocking cardinality at most n, is closed in $\mathcal{H}''$.*

(IV) *For a fixed stratum $\mathcal{H}$, the set of $M_0 \in \mathcal{H}$ for which any pair $(x, y) \in \widehat{M_0^2}$ satisfies $\mathrm{bc}(x, y) \leq n$ is closed in $\mathcal{H}$.*

Moreover, there is ℓ such that if the set

$$\left\{(x, y) \in M^2 : \mathrm{bc}(x, y) \leq n\right\} \tag{1}$$

is dense in M^2, then M has the bounded blocking property with blocking cardinality at most ℓ.

Proof. We will denote a surface in $\mathcal{H}''$ by (M, x, y), where x and y are respectively the first and second marked points on M. The topology on $\mathcal{H}''$ is such that when $(M_k, x_k, y_k) \to (M, x, y)$, for any parametrized line segment $\{\sigma(t) : t \in [0, 1]\}$ on M between x and y, for any large enough k there are parametrized line segments $\{\sigma_k(t) : t \in [0, 1]\}$ such that $\sigma_k(t) \to \sigma(t)$ for all t – see [MT02, Zor06] for details. Here a parameterized line segment is a constant speed straight line in each chart and does not contain singular points in its interior.

We begin with the proof of (III). Let (M_k, x_k, y_k) be a sequence that converges to (M, x, y) in $\mathcal{H}''$, where (x_k, y_k) belongs to $F_n(M_k)$ for all k. Let $\left\{b_k^{(1)}, \ldots, b_k^{(n)}\right\} \subset M_k$ be a blocking set for (x_k, y_k). Passing to a subsequence, we may assume that $b_k^{(i)}$ converges to a point $b^{(i)} \in M$ for each i. By the above description of the topology of $\mathcal{H}''$, if $\left\{b^{(1)}, \ldots, b^{(n)}\right\}$ does not contain x or y then it is a blocking set for (x, y) in M and we are done.

We now discuss the case that some of the $b^{(i)}$ are equal to x or y. We modify the set $\left\{b^{(1)}, \ldots, b^{(n)}\right\}$ as follows. For any i for which $b^{(i)}$ is different from both x and y, we set $B^{(i)} = b^{(i)}$. Suppose i is such that $b^{(i)} = x$. Let $r > 0$ be smaller than half the length of the shortest saddle connection on M. This implies that r is smaller than half the distance between x and y, and that there is no singularity in the ball $B(x, r)$ with center x and radius r.

For k large enough, $B(x_k, r)$ is an embedded flat disk in M_k that contains $b_k^{(i)}$, and there is a unique trajectory $\delta_k^{(i)}$ from x_k to $b_k^{(i)}$ that stays within this disk. Let

$B_k^{(i)}$ be the point on $\delta_k^{(i)}$ at distance r from x_k. Passing again to a subsequence we assume that $B_k^{(i)}$ converges to a point $B^{(i)}$ in M. Note that this point is distinct from x and y for each such i. We repeat this procedure for each i for which $b^{(i)}$ is equal to either x or y, passing at each stage to a further subsequence.

Let us prove that $\{B^{(1)}, \ldots, B^{(n)}\}$ is a blocking set for (x, y) in M. Let σ be a trajectory from x to y. We can assume without loss of generality that σ is simple, i.e. does not intersect itself. Let σ_k be the segment between x_k and y_k that converges pointwise to σ. If σ_k meets one of the $B_k^{(i)}$ for infinitely many k, $B^{(i)}$ belongs to σ and we are done.

Assume by contradiction that there is an index i such that, for infinitely many k, σ_k meets $b_k^{(i)}$ but not any $B_k^{(j)}$. In particular, $b_k^{(i)}$ converges to either x or y. Suppose for concreteness that it converges to x. Since $B_k^{(i)}$ does not belong to σ_k, the subsegment σ_k' of σ_k between x and $b_k^{(i)}$ is not equal to the segment $\delta_k^{(i)}$ defined above. In particular the length of this subsegment is bounded below and it converges to a nontrivial subsegment σ' of σ, which is a loop from x to x. This contradicts the simplicity of σ, completing the proof of (III).

Clearly (III) $\implies$ (I) $\implies$ (II) and (III) $\implies$ (IV). It remains to prove the final assertion. Suppose $x, y \in M$ and there are $x_k \to x, y_k \to y$ such that $\mathrm{bc}(x_k, y_k) \leq n$. We need to prove that $\mathrm{bc}(x, y) \leq \ell$, for some ℓ which depends only on M and n. If x, y are distinct nonsingular points, then for large enough k the points x_k, y_k are also distinct and nonsingular, and the claim follows from (I). We will consider three cases, adapting the proof of (III) to each one:

Case 1. $x = y$ is a nonsingular point. We will show that in this case the previous proof applies and we can take $\ell = 2n$.

We construct the points $B^{(i)}$ as follows. Passing to subsequences we assume the existence of each of the limits $b_i = \lim_{k\to\infty} b_k^{(i)}$, and when $b_i \neq x$, we set $B_i = b_i$ as before. When $b_i = x$, in place of the short segments $\delta_k^{(i)}$ appearing in the previous argument, we consider two segments — one from x_k to $b_k^{(i)}$ and one from y_k to $b_k^{(i)}$. We denote these by $\delta_{k,1}^{(i)}, \delta_{k,2}^{(i)}$ and construct points $B_{k,1}^{(i)}, B_{k,2}^{(i)}$ by 'sliding' $b_k^{(i)}$ along these segments as in the preceding argument. Taking limits, in each case in which $b_i = x$ we get two limit points, so re-indexing we get a total of at most $2n$ points.

We now show that this is a blocking set. Let σ be a segment from x to x, which does not contain any of the B_i. It is not contained in the ball $B = B(x, r)$ appearing in the previoius proof. Let σ_k be a sequence of parameterized line segments converging to σ. We can assume that none of these segments contains any of the $B_{k,j}^{(i)}, i = 1, \ldots, n, j = 1, 2$. The only place in the proof of (III) in which we used that $x \neq y$ is that we needed to know that the subsegment σ' of σ constructed in the proof is a proper subsegment of σ. In the case $x = y$ there are two subsegments σ_k', σ_k'' between x and $b_k^{(i)}$, neither of which is equal to $\delta_k^{(i)}$, since σ_k does not contain any of the $B_{k,j}^{(i)}$. In particular each of them leaves the disk $B(x_k, r)$ and hence has length at least r. So in the limit they both converge to nontrivial loops σ', σ'' from x to itself, whose concatenation is σ. This gives the desired contradiction to the simplicity of σ.

Case 2. $x \neq y$ and at least one of them is a singularity. We will show that in this case we can take $\ell = n(\tau + 1)^2$, where $\tau\pi$ is the maximal cone angle of a singularity on M.

Assume that x is a singularity, let r be as in the preceding proof, and let $\mathcal{U}_1, \ldots, \mathcal{U}_{\tau+1}$ be open convex subsets of M of diameter less than r, such that $\bigcup \mathcal{U}_s = B(x,r) \smallsetminus \{x\}$ and x is in the closure of each $\mathcal{U}_s$. Such sets exist by our choice of τ and r, e.g. we may take them to be open half-disks centered at x. If y is also a singularity, we similarly choose $\mathcal{U}'_1, \ldots, \mathcal{U}'_{\tau+1}$ covering $B(y,r) \smallsetminus \{y\}$.

We now choose sequences $x_k^{(s)}$ such that $x_k^{(s)} \in \mathcal{U}_s$ and $x_k^{(s)} \to_{k\to\infty} x$. If y is also a singularity we similarly choose sequences $y_k^{(t)}$ which approach y from within $\mathcal{U}'_t$. We also require that $\mathrm{bc}\left(x_k^{(s)}, y_k^{(t)}\right) \leq n$ for each k, s, t. Such sequences exist since (1) is dense. For each choice of $(s,t) \in \{1, \ldots, \tau+1\}^2$ we perform the procedure explained in the proof of (III). Namely we take blocking sets $\left\{b_k^{(i,s,t)} : i = 1, \ldots, n\right\}$ which block all segments between $x_k^{(s)}$ and $y_k^{(t)}$, pass to subsequences to assume that $\lim_k b_k^{(i,s,t)}$ exists for each i, s, t, and define $B^{(i,s,t)}$ to be this limit if it is distinct from x and y. If the limit is x we modify $b_k^{(i,s,t)}$ by letting $B_k^{(i,s,t)}$ be the unique point of distance r from x along the continuation of the unique segment $\delta_k^{(i,s,t)}$ which connects x and $b_k^{(i,s,t)}$ and which passes through $\mathcal{U}_s$. Then we take $B^{(i,s,t)}$ to be the limit $\lim_k B_k^{(i,s,t)}$ (passing to subsequences if necessary). We perform a similar modification if $\lim_k b_k^{(i,s,t)} = y$. This procedure gives us a set

$$\left\{B^{(i,s,t)} : i \in \{1, \ldots, n\}, (s,t) \in \{1, \ldots, \tau+1\}^2\right\},$$

which we claim is a blocking set for x, y.

Indeed for each segment σ from x to y, we can assume that it approaches x from within $\mathcal{U}_s$ and y from within $\mathcal{U}'_t$. Then for large enough k there are segments σ_k from $x_k^{(s)}$ to $y_k^{(t)}$ which approach σ pointwise. Working with these segments as in the proof of (III), we see that σ is blocked by $B^{(i,s,t)}$ for some i.

Case 3. $x = y$ is a singular point. In this case we use both of the arguments used in Cases 1 and 2. We leave the details to the reader. □

Corollary 8. *Let* BB_n *denote the set of surfaces which have the bounded blocking property, with blocking cardinality at most* n*. Then there is* $\ell \in \mathbb{N}$ *such that if* $M \in \mathrm{BB}_n$ *then* $\overline{GM} \subset \mathrm{BB}_\ell$*.*

Proof. Let us say that M_0 is *n-blocking for distinct nonsingular points* if for any $(x,y) \in \widehat{M_0^2}$, $\mathrm{bc}(x,y) \leq n$. Then the set of such surfaces is closed by Lemma 7(IV). Also, if $M'' \in \mathcal{H}''$ has x, y as the first and second marked points, and $x(g), y(g)$ are the first and second marked points on gM'', then $\mathrm{bc}(x,y) \leq n$ implies $\mathrm{bc}(x(g), y(g)) \leq n$. This implies that the property of being n-blocking for distinct nonsingular points is G-invariant. Since $M \in \mathrm{BB}_n$, M is n-blocking for distinct nonsingular points. Thus any surface in $\overline{GM}$ is also n-blocking for distinct nonsingular points, and the claim follows from the last assertion in Lemma 7. □

A similar argument also shows:

Proposition 9. *Let* M *be a translation surface,* ξ *a singular point on* M *and* $n \geq 0$ *an integer. Recalling our convention that singularities on translation surfaces are labeled, we can use the notation* ξ *for a singular point of any other surface in* $\mathcal{H}$*. Let* $\mathcal{F}'_n \subset \mathcal{H}'$ *denote the set of surfaces on which the marked point* y *satisfies*

$\mathrm{bc}(\xi, y) \le n$. Then $\mathcal{F}'_n$ is closed in $\mathcal{H}'$. In particular, $\{y \in M \smallsetminus \Sigma : \mathrm{bc}(\xi, y) \le n\}$ is closed as a subset of $M \smallsetminus \Sigma$.

Proof. We repeat the proof of Lemma 7(III), replacing everywhere x with ξ and also x_k with ξ.

In this case the set $B = B(\xi, r)$ is a topological disk which is metrically a finite cover of a flat disk, branched over its center point ξ. Then B is star-shaped with respect to its center point ξ and it is still the case that there is a unique straight segment from ξ to any point in B which is contained in B. We can thus define the segment $\delta_k^{(i)}$ as in the proof of (III), and the same argument applies. □

4. Characterization of the finite blocking property

In this section we will prove Theorem 1. A translation surface is *purely periodic* if it is completely periodic and all cylinders in such a decomposition have commensurable circumferences. The following was proved in [Mon09]:

Proposition 10 (Monteil)**.** *If M has the blocking property then M is purely periodic.*

Proof of Theorem 1. The implication (1) $\implies$ (2) is proved in [Mon05], and it is immediate that (2) $\implies$ (3). We first show (4) $\implies$ (1), that is we assume that M has the bounded blocking property and we show that it is a torus cover.

Let $\mathcal{L} \stackrel{\mathrm{def}}{=} \overline{GM}$. By assumption there is n such that $M \in \mathrm{BB}_n$, and by Corollary 8 this means $\mathcal{L}$ is contained in BB_ℓ for some ℓ. By Proposition 10 this means that every surface in $\mathcal{L}$ is completely periodic and by Proposition 5, $\mathcal{L}$ is of cylinder rank one.

Recall that the *field of definition* of $\mathcal{L}$ is the smallest field such that in any coordinate chart U on $\mathcal{H}$ given by period coordinates, the connected components of $U \cap \mathcal{L}$ are cut out by linear equations with coefficients in k (see [Wrib]). By [Wria, Theorem 1.9], for any completely periodic surface $M' \in \mathcal{L}$, and any cylinder decomposition on M' with circumferences $c_1, \ldots, c_r$ the field of definition k of $\mathcal{L}$ satisfies

$$k \subset \mathbb{Q}\left[\left\{\frac{c_i}{c_j} : i, j = 1, \ldots, r\right\}\right].$$

By Proposition 10, any surface in $\mathcal{L}$ is purely periodic, so $k = \mathbb{Q}$. Therefore $\mathcal{L}$ contains a surface with rational holonomies, i.e. a square-tiled surface M'. Since M' is square-tiled the holonomy of absolute periods on M' is a discrete subset of $\mathbb{C}$. Motion in the $G \oplus \mathrm{REL}$ leaf only changes the holonomy of absolute periods by a linear map, and therefore for any M in $\mathcal{L}$, the holonomy of absolute periods is discrete, i.e., any $M \in \mathcal{L}$ is a torus cover. This proves (4) $\implies$ (1).

Now we prove (3) $\implies$ (4). We have an open set U_1 in $M \times M$ consisting of pairs of points on M blocked from each other by finitely many points, that is, $U_1 \cap \widehat{M^2} \subset \bigcup_n F_n(M)$. Each $F_n(M)$ is closed as a subset of $\widehat{M^2}$ by Lemma 7(I), so by Baire category, there is n such that $F_n(M)$ contains an open set U_2. Each pair of points (x, y) in U_2 defines a surface in $\mathcal{H}''$, namely $M'' = (M, x, y)$. Let $\mathcal{L}(M'') \stackrel{\mathrm{def}}{=} \overline{GM''} \subset \mathcal{H}''$. By Theorem 4, $\widehat{\mathcal{L}}(M'')$ is a linear manifold of even dimension contained in $\mathcal{F}_n$ and the collection of such linear submanifolds is countable. By Lemma 7(III), $\widehat{\mathcal{L}}(M'') \subset \mathcal{F}_n$.

The fiber $\varphi^{-1}(M)$ is a linear submanifold of $\mathcal{H}''$ identified with $\widehat{M^2}$. Therefore for any M'', $\Omega(M'') \stackrel{\text{def}}{=} \varphi^{-1}(M) \cap \widehat{\mathcal{L}}(M'')$ is also a linear submanifold, and its dimension is $0, 2$ or 4. We have covered U_2, an open subset of a four-dimensional manifold, by countably many linear manifolds of dimensions at most four. By Baire category, there is M'' for which $\Omega(M'')$ is a linear manifold of dimension four. Since $\varphi^{-1}(M)$ is connected, it coincides with $\Omega(M'')$.

We have proved that

$$\varphi^{-1}(M) = \Omega(M'') \subset \widehat{\mathcal{L}}(M'') \subset \mathcal{F}_n;$$

that is, any two distinct nonsingular points in M are of blocking cardinality at most n. Applying the last assertion of Lemma 7, we see that M has the bounded blocking property. $\square$

5. Illumination

In this section we will study some illumination problems. Recall that two points x, y on a translation surface M do not illuminate each other if and only if they are finitely blocked with blocking cardinality zero. Also recall that p_1, p_2 denote the projections onto the first and second factors of $M \times M$. The following result is the main result of this section.

Theorem 11. *Let M be a translation surface, let n be a non-negative integer. Then:*

(i) *For any $x \in M$, the set $\{y \in M : \mathrm{bc}(x, y) \leq n\}$ is either finite or contains $M \smallsetminus (\Sigma \cup \{x\})$.*

(ii) *The set $\{(x, y) \in M^2 : \mathrm{bc}(x, y) \leq n\}$ either contains $\widehat{M^2}$, or is a finite union of 0 and 2 dimensional linear submanifolds of $M \times M$. The 2-dimensional linear submanifolds are of one of the following forms: $F \times M$, $M \times F$, where $F \subset M$ is finite, or a translation surface S embedded affinely in $M \times M$, where for $i = 1, 2$, $\tau_i = p_i|_S : S \to M$ is a finite-degree covering map such that $\tau_2 \circ \tau_1^{-1}$ is a multiplication by a scalar λ satisfying $\lambda^2 \in \mathbb{Q}$.*

Theorem 11 implies Theorem 2. We apply Theorem 11 with $n = 0$. It is clear that the second alternative in (i) cannot hold, since for any x, all nearby points illuminate x. Also, in (ii), the cases $F \times M$ and $M \times F$ do not arise, since any point illuminates some other point. $\square$

Proof of Theorem 11. Keep the notation of §2.1 and Lemma 7. We will first prove (i) in case x is a regular point of M. Let $M' \in \varphi'^{-1}(M) \subset \mathcal{H}'$ denote the surface with first marked point at x. We need to show that

$$A \stackrel{\text{def}}{=} \{y \in M \smallsetminus (\Sigma \cup \{x\}) : \mathrm{bc}(x, y) \leq n\},$$

which we may identify with $\mathcal{F}_n \cap \varphi''^{-1}(M')$, is either finite or coincides with $\varphi''^{-1}(M')$. Let us assume $A \subsetneq \varphi''^{-1}(M')$. Since $\mathcal{F}_n$ is closed and G-invariant, A is a union of at most countably many linear manifolds, which are of the form $\mathcal{L}(M''_0) \stackrel{\text{def}}{=} \overline{GM''_0}$ for $M''_0 \in A$. For each M''_0, $\mathcal{L}(M''_0) \cap \varphi''^{-1}(M')$ is a linear manifold of dimension 0 or 2 by Theorem 4. If the dimension were 2, A would coincide with the fiber $\varphi''^{-1}(M)$ by connectedness, and hence each $\mathcal{L}(M''_0)$ is finite. To conclude the proof of (i) we need to show that in fact there are only finitely many distinct sets $\mathcal{L}(M''_0)$. If there were infinitely many this would mean that $\mathcal{F}_n$ contains infinitely many G-orbit-closures, so by Theorem 4 they would accumulate on an orbit-closure of greater dimension, also contained in $\mathcal{F}_n$. By Lemma 6, each $\mathcal{L}(M_0'')$ projects onto $\mathcal{L}$, so the only way for the dimension to increase would be in the direction of the fiber $\varphi''^{-1}(M')$; that is, once again we would have that $A = \varphi''^{-1}(M')$, contrary to assumption.

In case $x = \xi$ is a singularity we repeat the argument, using Proposition 9 instead of Lemma 7, $\mathcal{F}_n'$ instead of $\mathcal{F}_n$, φ' instead of φ'' and $\mathcal{H}'$ instead of $\mathcal{H}''$. We leave the details to the reader.

The proof of (ii) is similar. Suppose that

$$\widehat{M^2} \not\subset A \stackrel{\text{def}}{=} \{(x,y) \in M^2 : \text{bc}(x,y) \leq n\}.$$

Applying Theorem 4 as in the preceding paragraph we see that A is the union of finitely many 0-dimensional and finitely many 2-dimensional linear manifolds. We need to show that all of the 2-dimensional manifolds have the stated form. This follows from arguments of [HST08], but we give an independent argument for completeness.

Let $N \subset (M \smallsetminus \Sigma)^2$ be a 2-dimensional linear manifold in A. By Theorem 4, N is $\mathbb{C}$-linear, i.e., in the translation charts of $M \times M$, a neighborhood of N is the set of solutions of an equation of the form

$$az_1 + bz_2 = 0 \tag{2}$$

(up to a translation). Moreover N is defined over $\mathbb{R}$ so we can take $a, b \in \mathbb{R}$. If $a = 0$ then any connected component of N is of the form $M \times \{x\}$ for some $x \in M$, i.e. N is of the form $M \times F$. Similarly if $b = 0$ then N has the form $F \times N$. Now we consider the case when a, b are both nonzero.

Since the transition maps for the translation atlas are translations, a and b can actually be taken to be independent of the neighborhood, and the Cartesian product translation structure on M^2, restricted to N, endows N with a natural structure of a translation surface (see [HST08, §3] for more details), where N is locally modelled on the plane (2). Since a and b are both nonzero, each of the projections $\tau_i = p_i|_N$ has a nonsingular derivative, so by connectedness, each τ_i is a finite covering map.

The plane (2) can be identified with $\mathbb{C}$ in many ways and thus the translation surface structure on N is only naturally defined up to a scalar multiple. However, for any fixed choice of translation structure on N, each of the maps τ_i is the composition of a dilation and a translation covering. Let k_i be the degree of the covering map τ_i, and let λ_i be the associated dilation. The choice of the λ_i depends on a choice of the translation structure on N, but since the derivative of $\tau_2 \circ \tau_1^{-1}$ is the map $z_1 \mapsto -\frac{a}{b} z_1$, we have $\lambda \stackrel{\text{def}}{=} \frac{\lambda_2}{\lambda_1} = -\frac{a}{b}$. We can compute the area of N using each of the maps τ_i, to obtain

$$\text{area}(N) = \frac{k_i}{\lambda_i^2} \text{area}(M).$$

Comparing these formulae for $i = 1, 2$ we see that $\lambda^2 = \left(\frac{a}{b}\right)^2 = \frac{k_2}{k_1} \in \mathbb{Q}$. $\square$

6. Examples and questions

Let T be the standard torus, obtained from the unit square $[0,1]^2$ by gluing opposite sides to each other by translations. Denote by π the projection from $\mathbb{R}^2$ to T. For any nonzero integer n, notice that the map $\mathbb{R}^2 \to \mathbb{R}^2$, $x \mapsto nx$ descends to a map $m_n : T \to T$ which multiplies both components by n in $\mathbb{R}/\mathbb{Z}$, and is therefore

n^2 to 1. We describe blocking cardinalities of pairs of points in T and blocking sets realizing them.

Lemma 12. (a) *If x and y are distinct points on T, their blocking cardinality is* $\mathrm{bc}(x,y) = 4$.

(b) *It is realized by the blocking set $B(x,y) = m_2^{-1}(x+y)$, which contains the midpoint of any geodesic from x to y.*

(c) *This is the unique blocking set of size* 4.

Proof. Let $\widetilde{x}$, $\widetilde{y}$ denote points in $\mathbb{R}^2$ which project to x, y on T. Let $u = (1,0)$, $v = (0,1)$, $w = (1,1)$. The four segments from $\widetilde{y}$ to the four points $\widetilde{x}$, $\widetilde{x}+u$, $\widetilde{x}+v$, $\widetilde{x}+w$ (four corners of a unit square) project to segments with disjoint interiors on T, so at least 4 points are required to block the pair (x,y). On the other hand, any line segment in T from x to y is the projection of a line segment in $\mathbb{R}^2$ from $\widetilde{x}$ to $\widetilde{y}+au+bv$ with a and b in $\mathbb{Z}$. Such a segment has midpoint $\frac{1}{2}(\widetilde{x}+\widetilde{y}+au+bv)$. This midpoint in $\mathbb{R}^2$ projects to one of the points $\frac{1}{2}(x+y)$, $\frac{1}{2}(x+y+u)$, $\frac{1}{2}(x+y+v)$, $\frac{1}{2}(x+y+w)$, which are the four points in T comprising $m_2^{-1}(x+y)$. This proves that the set $B(x,y)$ is a blocking set and that $\mathrm{bc}(x,y) \leq 4$. So (a) and (b) are proved.

We now prove (c). We saw that the four segments from $\widetilde{y}$ to $\widetilde{x}$, $\widetilde{x}+u$, $\widetilde{x}+v$, $\widetilde{x}+w$ project to segments on T with disjoint interiors, so a blocking set for (x,y) must contain at least a point in each of them. Consider the segment from $\widetilde{y}+v$ to $\widetilde{x}+u$. The only intersection of its projection to T with the interiors of our four segments is its midpoint m which is also the midpoint of the segment from y to $y+w$. So a blocking set not containing m would need to contain at least 5 points. Similar reasoning proves the other three points in the proposed set $B(x,y)$ have to be in a blocking set of cardinality 4. □

The following two lemmas extend this description to configurations blocking a point from itself, and describe larger blocking sets on T. They are proved by similar arguments and we leave the details to the reader.

Lemma 13. (a) *If $x = y$, then the blocking cardinality is* $\mathrm{bc}(x,x) = 3$.

(b) *It is realized by the blocking set $B(x,x) = m_2^{-1}(2x) \setminus \{x\}$, which is the set of midpoints of all primitive geodesics from x to x. This blocking set can also be described as $B(x,x) = x + B_0$ where $B_0 = B(0,0) = m_2^{-1}(0) \setminus \{0\}$.*

(c) *This is the unique blocking set of size* 3.

Lemma 14. (a) *Let n and a be relatively prime integers with $1 \leq a < n$. For any pair of points (x,y) with $x \neq y$, the set $B = m_n^{-1}(ax + (n-a)y)$ is a blocking set of cardinality n^2 for the pair (x,y). It contains the point located a/n of the way along each line segment from x to y on T.*

(b) *Let $n \geq 2$ be an integer. For the pair of points (x,x) with $x = 0$, the set*

$$B_0 = m_n^{-1}(0) \setminus \{0\} = \{(a/n, b/n) : 0 \leq a < n, \quad 0 \leq b < n, \quad (a,b) \neq (0,0)\}$$

is a blocking set of cardinality $n^2 - 1$.

For the pair of points (x,x) with $x \neq 0$, the set $B = x + B_0$ is a blocking set of cardinality $n^2 - 1$, also equal to $m_n^{-1}(nx)$.

We will use these computations to compute blocking configurations on brached covers of T. Recall that if $M \to T$ is a branched translation cover, a singularity of

M corresponds to a ramification point of the cover, and if the angle at a singularity x is $2\pi k$ then k is called the *ramification index* of x.

Lemma 15. *Suppose M is a torus cover of degree d, with arbitrary branch locus and ramification type, and let $p : M \to T$ denote the covering map.*

(a) *For a pair (x, y) of points of M such that $p(x) \neq p(y)$, if B' is a blocking set for $(p(x), p(y))$ on T, then $B = p^{-1}(B')$ is a blocking set for (x, y), of cardinality at most d times that of B', with equality when B contains no zero of M, i.e. no ramification point of p.*

(b) *In particular,*
   - *for almost every pair (x, y) of points of M, $\mathrm{bc}(x, y) \leq 4d$.*
   - *for pairs (x, y) of points of M, such that the set $B(p(x), p(y))$ contains branch points of p, the bound above is decreased by the sum of the ramification indices of the ramification points above these branch points.*

(c) *For a pair of points (x, y) on M such that $p(x) = p(y)$ (whether $x = y$ or not), $p^{-1}(B(p(x), p(x)))$ is a blocking set, so that $\mathrm{bc}(x, y) \leq 3d$. As above, when $B(p(x), p(y))$ contains branch points of p, the bound is decreased by the sum of the ramification indices of the ramification points above these branch points.*

Proof. Both (a) and (b) are easy, and (c) follows from the following observation. When $p(x) = p(y)$, any geodesic path γ from x to y projects to a geodesic γ' from $p(x)$ to itself, possibly non primitive. Considering the restriction of the geodesic γ, if γ' is not primitive, to its initial part until it first reaches a point projecting to $p(x)$, we see that (c) holds. □

6.1. **Example 1.** The following example shows that quite general maps τ_1, τ_2 may arise in Theorem 11.

Proposition 16. *Let a, b be positive integers with $\gcd(a, b) = 1$, let $n = a + b$, and let*

$$X = \{(-ax, bx) : x \in T\}.$$

Also let $p : M \to T$ be a translation cover with branching locus $m_n^{-1}(0)$, and non-trivial ramification at each pre-image of each branch point, and let

$$Y = (p \times p)^{-1}(X).$$

Then any pair of points in Y do not illuminate each other.

Proof. For $x \in \mathbb{R}^2$, the point 0 is a/n along the geodesic in $\mathbb{R}^2$ from $-ax$ to bx. Thus, according to Lemma 14, the set $B = m_n^{-1}(0)$ is a common blocking set, of cardinality n^2, for all pairs of points in X. Thus the statement follows from Lemma 15. □

6.2. **Example 2.** The following examples show that the map $\tau_2 \circ \tau_1^{-1}$ could be a translation. Let $M = T$ be the torus, and consider

$$N = \{(x, y) \in M^2 : \mathrm{bc}(x, y) \leq 3\}.$$

Then according to Lemma 13, N contains the diagonal $\{(x, x) : x \in M\}$ but according to Lemma 12, $N \neq M^2$. Therefore the diagonal is one of the linear submanifolds appearing in Theorem 11, and we can have $\tau_2 \circ \tau_1^{-1} = \mathrm{Id}$.

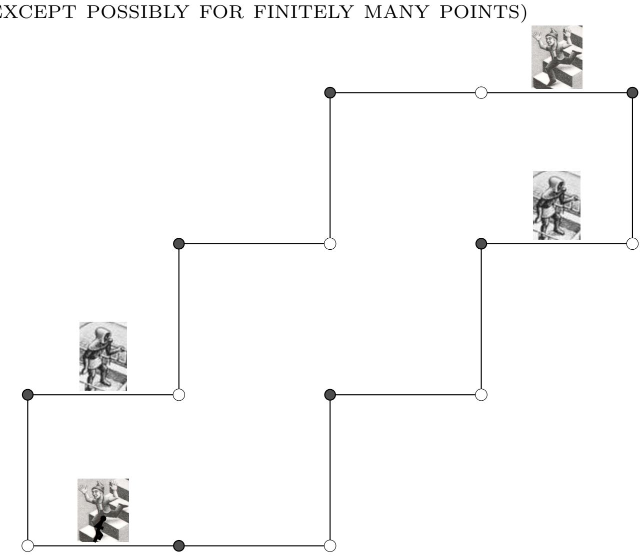

FIGURE 1. The Escher double staircase. Sides marked with identical stair-climbers are identified; unmarked sides are identified with the corresponding opposite sides.

Similar examples in which $\tau_2 \circ \tau_1^{-1}$ is a nontrivial translation can be obtained by taking M to be a cyclic cover of T, for example the Escher staircase (see Figure 1). This surface admits a degree 3 cover $p : M \to T$ and it has a nontrivial translation automorphism $D : M \to M$ moving one step up the ladder. Let x and y be any two points such that $D(x) = y$. Then $p(x) = p(y)$, and according to Lemma 15(c), $\mathrm{bc}(x, y) \leq 9$. It is not hard to find an explicit pair of points x, y for which $\mathrm{bc}(x, y) > 9$. This shows that if we take this surface M and $n = 9$, then we can have a subsurface N for which $D = \tau_2 \circ \tau_1^{-1}$ is a translation automorphism.

6.3. **Example 3.** Using the torus and Lemmas 12 and 13 we easily find sequences $x_k \to x, y_k \to y$ for which $\mathrm{bc}(x, y) < \lim_k \mathrm{bc}(x_k, y_k)$, i.e. the blocking cardinality is not continuous. The following example shows that it is not even lower semi-continuous, i.e. it may increase when taking limits. It also shows that in Lemma 7(I), we cannot replace $\widehat{M^2}$ with M^2.

Let M be a surface in $\mathcal{H}(2)$. Then M admits a hyper-elliptic involution h, whose set of fixed points consists of the unique singularity ξ, and 5 non-singular Weierstrass points. We claim that whenever $h(x) = y, x \neq y$, we have $\mathrm{bc}(x, y) \leq 5$. Indeed, in this case, the action of h swaps x and $h(x)$, and acts by rotation by π. So h maps any segment between σ between x and y to another segment from x to y, of the same length and in the same direction. Since x and y are distinct regular points there is only one such segment, i.e. h maps σ to itself, reversing the orientation on it. So its midpoint must be fixed by h, that is, the Weierstrass points form a blocking set for the pair (x, y).

On the other hand by constructing explicit disjoint segments, it is not hard to show that $\mathrm{bc}(\xi, \xi) \geq 9$. For example we can present M as the union of three parallelograms (an L-shaped presentation), and the diagonal and edges of these parallelograms contain 9 disjoint segments from ξ to ξ. Now taking $x_k \to \xi$, we have $y_k = h(x_k) \to \xi$, and

$$5 \geq \lim_k \mathrm{bc}(x_k, y_k), \quad \mathrm{bc}(\lim_k x_k, \lim_k y_k) = \mathrm{bc}(\xi, \xi) \geq 9.$$

6.4. **Questions.** 1. Let $N \subset M \times M$ be a 2-dimensional linear submanifold as in Theorem 11(ii), and let λ_1, λ_2 be the derivatives of the translation maps τ_1, τ_2. The quotient $\lambda = \lambda_1/\lambda_2$ is called the *slope* of N. In Example 2 the slope is 1, and in Example 1 the slope can be an arbitrary negative rational number. It would be interesting to know whether other slopes are possible. In particular, do the cases $\lambda = 0$, $\lambda = \infty$ actually arise in connection with blocking configurations? Do positive rational slopes arise, except for $\lambda = 1$?

2. More generally, suppose $N \subset M \times M$ is an embedded translation surface for which the maps $\tau_i : N \to M$ are the composition of a dilation and a translation, and let λ be the derivative of the composition $\tau_2 \circ \tau_1^{-1}$. In the proof of Theorem 11 we showed that $\lambda^2 \in \mathbb{Q}$. Is it possible that λ is irrational?

3. In the last assertion of Lemma 7, can we take $\ell = n$? Example 3 shows that this does not follow from a simple continuity argument. Also in connection with Example 3, does there exist a similar example in which the point ξ is nonsingular? That is, an example of a surface M with a regular point ξ and two sequences x_k, y_k converging to ξ, such that $\mathrm{bc}(\xi, \xi) > \lim_k \mathrm{bc}(x_k, y_k)$?

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Laboratoire de mathématique d'Orsay, Umr 8628 Cnrs/Université Paris-Sud, Bât. 425, campus Orsay, 91405 Orsay cedex, France, samuel.lelievre@math.u-psud.fr

LIRMM - CNRS (UMR 5506), 161 rue Ada 34392 Montpellier cedex 5 France thierry.monteil@lirmm.fr

School of Math. Sciences, Tel Aviv University, Israel, barakw@post.tau.ac.il