EVERYTHING IS ILLUMINATED

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ABSTRACT. We study geometrical properties of translation surfaces: the finite blocking property, bounded blocking property, and illumination properties. These are elementary properties which can be fruitfully studied using the dynamical behavior of the $\text{SL}(2,\mathbb{R})$-action on the moduli space of translation surfaces. We characterize surfaces with the finite blocking property and bounded blocking property, completing work of the second-named author [Mon05]. Concerning the illumination problem, we also extend results of Hubert-Schmoll-Troubetzkoy [HST08], removing the hypothesis that the surface in question is a lattice surface, thus settling a conjecture of [HST08]. Our results crucially rely on the recent breakthrough results of Eskin-Mirzakhani [EM] and Eskin-Mirzakhani-Mohammadi [EMM], and on related results of Wright [Wria].

1. Introduction

A translation surface is a finite union of polygons, glued along parallel edges by translations, up to a cut and paste equivalence. These structures arise in the study of billiards, interval exchange transformations, and various problems in group theory and geometry. See [MT02, Zor06] for comprehensive introductions and detailed definitions. The purpose of this paper is to apply recent breakthrough results of Eskin-Mirzakhani [EM] and Eskin-Mirzakhani-Mohammadi [EMM], on the dynamics of a group action on the moduli space of translation surfaces, to some elementary geometrical questions concerning translation surfaces. We begin with some definitions.

A pair of points $(x,y) \in M \times M$ is finitely blocked if there exists a finite set $B \subset M$ which does not contain $x$ or $y$ and intersects every straight-line trajectory connecting $x$ and $y$. A set $B$ with this property is called a blocking set for $(x,y)$, and the minimal cardinality of a blocking set is called the blocking cardinality of $(x,y)$ and is denoted by $bc(x,y)$. A translation surface $M$ has the blocking property if any pair $(x,y) \in M \times M$ is finitely blocked, and the bounded blocking property if there is a number $n$ such that any pair $(x,y) \in M \times M$ is finitely blocked with blocking cardinality at most $n$. If $x$ and $y$ are finitely blocked with blocking cardinality zero, that is, if there is no straightline path on $M$ from $x$ to $y$, then we say that $x$ and $y$ do not illuminate each other. A translation surface $M$ is a torus cover if there is a surjective translation map from $M$ to a torus (the singularities of $M$ may project to one or several points on the torus). Equivalently (see e.g. [Mon05]), the subgroup of $\mathbb{R}^2$ generated by holonomies of absolute periods on $M$ is discrete.

Our first result settles a question of the second-named author, see [Mon05, Mon09].

**Theorem 1.** For a translation surface $M$, the following are equivalent:

1. $M$ is a torus cover.
(2) $M$ has the blocking property.

(3) There is an open set $U \subset M \times M$ such that any pair of points in $U$ is finitely blocked.

(4) $M$ has the bounded blocking property.

Hubert, Schmoll and Troubetzkoy [HST08] have constructed an example of a translation surface $M$ which is not a torus cover, and in which there are infinitely many pairs of points which do not illuminate each other. In fact, there is an involution $\tau : M \to M$ such that for any $x \in M$, there is no straight line between $x$ and $\tau(x)$. See §6.3 for similar examples. This shows that in (3), it is not enough to suppose that $U$ is infinite.

Our second result concerns questions of illumination. The classical illumination problem was first posed in the 1950’s, when it was asked whether there exists a polygonal room with a pair of points which do not illuminate each other. First examples were found by Tokarsky [Tok95] and Boshernitzan (unpublished), and this raised the question of classification and possible cardinality of pairs of points which do not illuminate one another on translation surfaces. We refer to [HST08] or the Wikipedia page http://en.wikipedia.org/wiki/Illumination_problem for a brief history. We show:

**Theorem 2.** For any translation surface $M$, and any point $x \in M$, the set of points $y$ which are not illuminated by $x$ is finite.

Moreover, the set

$$\{(x, y) : x \text{ and } y \text{ do not illuminate each other}\}$$

is the union of a finite set, and of finitely many translations surfaces $M'$ embedded in $M \times M$, such that the projections $p_i|M' : M' \to M$ are both finite-degree covers.

Here $p_i : M \times M \to M$, $i = 1, 2$ are the natural projections onto the first and second factors respectively.

Theorem 2 strengthens results of [HST08], which deal with surfaces which have a large group of translation automorphisms. Namely, Theorem 2 was proved in [HST08] under the additional hypothesis that $M$ is a lattice surface, and when $M$ is a pre-lattice surface, the first assertion of the theorem was shown, with ‘countable’ in place of ‘finite’ (for the definitions see §2.3). The first assertion of Theorem 2 settles [HST08, Conjecture 1]. In §5 we deduce Theorem 2 from the more general Theorem 11. In §6 we give examples which elaborate on related examples given in [HST08].

A standard ‘unfolding’ technique (see [MT02, Zor06]) leads to the following result, justifies the title of this paper. It is a special case of [OP01, Conjecture 1].

**Corollary 3.** Let $P$ be a rational polygon. Then for any $x \in P$ there are at most finitely many points $y$ for which there is no geodesic trajectory between $x$ and $y$.

There is a moduli space $\mathcal{H}$ parameterizing all translation surfaces sharing some topological data, and this space is equipped with an action of the group $G \defeq \text{SL}(2, \mathbb{R})$. The breakthrough work [EM, EMM] has made it possible to analyze the dynamics of this action in great detail. Our analysis depends crucially on this work, as well as on additional work of Wright [Wria].

We note that the crucial feature which make our analysis possible is that the geometric properties we consider give rise to subsets of $\mathcal{H}$ which are closed and
G-invariant. It has long been known that a detailed understanding of the G-action would shed light on the illumination problem, as well as on many similar ‘elementary’ problems. For more papers applying the dynamics of the G-action to the analysis of closed and G-invariant geometrical properties of translation surfaces, see [Vee95, Vor96, Mon05, Mon09, HST08, SW10, SW07, LW].

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2. **Preliminaries**

We begin by briefly recalling the definitions of translation surfaces and strata, and refer to [MT02, Zor06] for more details. Fix a topological orientable surface $S$ of genus $g$, a finite subset $\Sigma = \{x_1, \ldots, x_k\}$ of $S$, and non-negative integers $\alpha_1, \ldots, \alpha_k$ so that $\sum \alpha_i = 2g - 2$. We allow some of the $\alpha_i$ to be zero and require $k \neq 0$. A translation surface $M$ of type $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is a surface $M$ homeomorphic to $S$, with $k$ labelled singular points $\{\xi_1, \ldots, \xi_k\}$, equipped with an equivalence class of atlases of planar charts, i.e. maps from open subsets of $M \setminus \{\xi_1, \ldots, \xi_k\}$ to $\mathbb{C}$, such that:

- Transition maps for the charts are translations.
- At each $\xi_i$ the charts give rise to a cone type singularity of angle $2\pi(\alpha_i + 1)$.

As usual two atlases are considered equivalent if their union is also an atlas of the same type, and two translation surfaces are considered equivalent if there is a homeomorphism from one to the others, which is a translation in charts, and maps the distinguished finite set $\{\xi_i\}$ of one translation surface bijectively to the other in a way which respects the numbering. Note that an atlas of planar charts on $M \setminus \Sigma$ naturally induces a translation structure on $(M \setminus \Sigma) \times (M \setminus \Sigma)$, with charts taking values in $\mathbb{C}^2$ and for which transition maps are translations. We will call this the *Cartesian product translation structure on $M^2$*.

The points $\xi_i$ are called **singularities**. Note that we have allowed singularities with cone angle $2\pi$ (as happens when $\alpha_i = 0$). Such singularities are sometimes referred to as **marked points**. Note also that in contrast to the convention used by some authors, our convention is that singularities are labeled.

A homeomorphism $S \to M$ which maps each $x_i$ to $\xi_i$ is called a **marking**. We can use a marking and the planar charts of $M$ to evaluate the integrals of directed paths on $S$ beginning and ending in $\Sigma$. Such an integral is a complex number whose real and imaginary components measure respectively the total horizontal and vertical distance travelled when moving in $M$ along the image of the path. Denote by $\mathcal{H}(\vec{\alpha})$ the set of translation surfaces of type $\vec{\alpha}$. It is called a **stratum** and is equipped with a natural topology defined as follows. The discussion above shows that the marking gives rise to a map

$$\mathcal{H}(\vec{\alpha}) \to H^1(S, \Sigma; \mathbb{C}).$$

It is known that the maps above constitute an atlas of charts which endow $\mathcal{H}(\vec{\alpha})$ with the structure of a linear orbifold. We will call these coordinates **period coordinates**. With respect to period coordinates, the change of a marking constitutes
a change of coordinates via a unimodular integral matrix, so $\mathcal{H}(\vec{\alpha})$ is naturally endowed with a Lebesgue measure and a $\mathbb{Q}$-structure. It is known that each stratum has finitely many connected components.

The group $G$ acts on each stratum component $\mathcal{H}$ by postcomposition of planar charts. That is, identifying the field of complex numbers with the plane $\mathbb{R}^2$ in the usual way, each $g \in G$ is a linear map of $\mathbb{R}^2$ and we use it to replace each chart $M \supset U \xrightarrow{\varphi} \mathbb{C} \cong \mathbb{R}^2$ with the chart $g \circ \varphi : U \to \mathbb{R}^2$. For each stratum component $\mathcal{H}$, the subset $\mathcal{H}^{(1)}$ consisting of area one surfaces is a sub-orbifold which in period coordinates is cut out by a quadratic condition. It is preserved by the $G$-action, and $G$ acts ergodically preserving a natural smooth finite measure obtained from the Lebesgue measure by a cone construction. Given a translation surface $M$ and a positive real number $t$, we denote by $tM$ the translation surface obtained by multiplying all planar charts of $M$ by the scalar $t$.

### 2.1. Adding marked points.

We will need some notation for the operation of adding a stratum by a corresponding stratum with one or two additional marked points.

Given a stratum component $\mathcal{H}$, we denote by $\mathcal{H}'$ the corresponding stratum component of surfaces with one additional marked point, and by $\mathcal{H}''$ the corresponding stratum component of surfaces with two additional marked points. More formally this is defined as follows. Suppose $\mathcal{H}$ is a component of $\mathcal{H}(\vec{\alpha})$ where $\vec{\alpha} \equiv (\alpha_1, \ldots, \alpha_k)$ and $\Sigma \equiv \{x_1, \ldots, x_k\}$ is a finite subset of cardinality $k$ in the topological surface $S$. Let $x_{k+1}, x_{k+2}$ denote two distinct points on $S \setminus \Sigma$, set $\alpha_{k+1} = \alpha_{k+2} = 0$, set $\Sigma' \equiv \Sigma \cup \{x_{k+1}\}$, $\Sigma'' \equiv \Sigma' \cup \{x_{k+2}\}$, $\vec{\alpha}' \equiv (\alpha_1, \ldots, \alpha_{k+1})$, $\vec{\alpha}'' \equiv (\alpha_1, \ldots, \alpha_{k+2})$, and let $\varphi' : \mathcal{H}(\vec{\alpha}') \to \mathcal{H}(\vec{\alpha})$, $\varphi'' : \mathcal{H}(\vec{\alpha}'') \to \mathcal{H}(\vec{\alpha})$ be the forgetful maps obtained by deleting the points corresponding to $x_{k+1}, x_{k+2}$ from the domain of any planar chart. Let $\varphi \equiv \varphi' \circ \varphi''$. The three maps $\varphi', \varphi'', \varphi$ are bundle maps for the respective bundles $\mathcal{H}(\vec{\alpha})$, $\mathcal{H}(\vec{\alpha}')$, $\mathcal{H}(\vec{\alpha}'')$ with bases $\mathcal{H}(\vec{\alpha})$, $\mathcal{H}(\vec{\alpha}')$, $\mathcal{H}(\vec{\alpha}'')$ and fibers $S \setminus \Sigma, S \setminus \Sigma'$, $(S \setminus \Sigma)^2 \setminus \Delta$ respectively ($\Delta$ is the diagonal). Finally we let $\mathcal{H}', \mathcal{H}''$ be the connected components of $\mathcal{H}(\vec{\alpha}')$ and $\mathcal{H}(\vec{\alpha}'')$ covering the component $\mathcal{H}$.

One easily checks from the definitions that the maps $\varphi, \varphi', \varphi''$ are $G$-equivariant, and that the fibers are linear manifolds in period coordinates. Moreover note that the linear structure on a fiber $\varphi'^{-1}(M) \cong S \setminus \Sigma$ coincides with the translation structure afforded by the translation charts on $M$, and similarly, the linear structure on a fiber $\varphi^{-1}(M) \cong (S \setminus \Sigma)^2 \setminus \Delta$ coincides with the Cartesian product translation structure on $M^2$. In the sequel we will refer to $x_{k+1}, x_{k+2}$ as the first and second marked points for the covers $\mathcal{H}'' \to \mathcal{H}' \to \mathcal{H}$. Note that we allow $\mathcal{H}$ to contain additional marked points.

### 2.2. Recent dynamical breakthroughs.

We now state the results of [EM, EMM, Wria] mentioned in the introduction. This requires some terminology. We say that a subset $\mathcal{L}_0 \subset \mathcal{H}$ is a complex linear manifold defined over $\mathbb{R}$ if for each of the charts $\mathcal{H} \to H^1(S, \Sigma; \mathbb{C}) \cong \mathbb{C}$ obtained by fixing a marking, the image of $\mathcal{L}_0$ is the intersection of an open set with an affine subspace whose linear part is a $\mathbb{C}$-linear vector space defined over $\mathbb{R}$. Note that the real dimension of a complex linear manifold is even. Given $\mathcal{L} \subset \mathcal{H}^{(1)}$, we denote

$$\hat{\mathcal{L}} \equiv \{tM' : t > 0, M' \in \mathcal{L}\}.$$
If \( \nu \) is a measure on \( \mathcal{H} \) then \( \mu(A) = \nu(\{tx : x \in A, t \in (0, 1]\}) \) is a measure on \( \mathcal{H}^{(1)} \) and we say that \( \mu \) is obtained by coning off \( \nu \). We say that \( \mathcal{L} \subset \mathcal{H}^{(1)} \) is an affine invariant manifold if it is \( G \)-invariant, is the support of an ergodic \( G \)-invariant measures \( \mu, \hat{L} \) is a complex linear manifold defined over \( \mathbb{R} \), and \( \mu \) is obtained by coning off Lebesgue measure on \( \hat{L} \).

**Theorem 4** (Eskin-Mirzakhani-Mohammadi). For each stratum component \( \mathcal{H} \) and each \( M \in \mathcal{H}^{(1)} \), the orbit closure \( \mathcal{L} \overset{\text{def}}{=} GM \) is an affine invariant manifold. The collection of affine invariant manifolds of \( \mathcal{H} \) obtained as orbit-closures for the \( G \)-action is countable. If \( \mathcal{L}_n, n \geq 1 \) is a sequence of distinct affine invariant manifolds of some dimension \( k \) contained in \( \mathcal{H} \), then after passing to a subsequence, the set of accumulation points

\[
\{ M \in \mathcal{H} : \exists M_n \in \mathcal{L}_n \text{ such that } M_n \to M \}
\]

is an affine invariant manifold \( \mathcal{L}_\infty \) with \( \dim \mathcal{L}_\infty > k \) and \( \{ M_n \} \subset \mathcal{L}_\infty \).

Note that the results of [EMM] work for strata with marked points, i.e. they allow \( \alpha_i = 0 \).

Suppose that the number of singularities \( k \) is at least 2. Let \( H_1(S) \) and \( H_1(S, \Sigma) \) denote respectively the absolute and relative homology groups. Then we have \( H_1(S) \subset H_1(S, \Sigma) \) and we can restrict each 1-cocycle in \( H^1(S, \Sigma; \mathbb{C}) \) to the subspace \( H_1(S) \); that is we get a natural restriction map \( H^1(S, \Sigma; \mathbb{C}) \to H^1(S; \mathbb{C}) \). The kernel REL of this map is a subspace of \( H^1(S, \Sigma; \mathbb{C}) \) of real dimension \( 2(k-1) \), and we have a foliation of \( H^1(S, \Sigma; \mathbb{C}) \) by cosets of REL. Since the restriction map \( H^1(S, \Sigma; \mathbb{C}) \to H^1(S; \mathbb{C}) \) is topological, the space REL is independent of a marking, that is can be used to unequivocally define a linear foliation of \( \mathcal{H}(\alpha) \) using period coordinates. This foliation of \( \mathcal{H}(\alpha) \) is called the REL foliation. The \( G \)-action respects the REL foliation and hence we have a linear foliation of \( \mathcal{H} \) by leaves tangent to \( g \oplus \text{REL} \), where we use \( g \) to denote the tangent to the foliation by \( G \)-orbits. We denote this foliation by \( G \oplus \text{REL} \). Following [Wria], if a closed \( G \)-invariant and \( G \)-ergodic linear manifold \( \mathcal{L} \) is contained in a single leaf of the foliation \( G \oplus \text{REL} \), we say that it is of cylinder rank one. A translation surface \( M \) is completely periodic if in any cylinder direction on \( M \) there is a complete cylinder decomposition.

**Theorem 5** (Wright [Wria], Theorems 1.5 and 1.6). A linear manifold \( \mathcal{L} \) as above is of cylinder rank one if and only if any surface in \( \mathcal{L} \) is completely periodic.

We will need the following Lemma. Note that its assertion would be trivial if the fiber of \( \varphi \) were compact.

**Lemma 6.** Let \( M \in \mathcal{H} \) and \( M'' \in \varphi^{-1}(M) \subset \mathcal{H}'' \). Let \( \mathcal{L} \overset{\text{def}}{=} GM \) and \( \mathcal{L}'' \overset{\text{def}}{=} GM'' \). Then \( \varphi|\mathcal{L}'' \) is an open mapping and hence \( \dim \varphi(\mathcal{L}'') = \dim \mathcal{L} \).

**Proof.** According to [EMM], there are Borel probability measures \( \mu, \mu'' \) on \( \mathcal{H}, \mathcal{H}'' \) respectively such that \( \mathcal{L} = \text{supp} \mu, \mathcal{L}'' = \text{supp} \mu'' \). We first claim that \( \mu = \varphi_* \mu'' \). To this end note that Theorems 2.6 and 2.10 in [EMM] provide an averaging method converging to \( \mu, \mu'' \); that is, in both of these theorems, one finds probability measures \( \nu_T \) on \( G \), such that for any continuous compactly supported functions \( f, f'' \) on \( \mathcal{H} \) and \( \mathcal{H}'' \) respectively,

\[
\int_G f(gM)dg_T(g) \to_{T \to \infty} \int_{\mathcal{H}} f\,d\mu \quad \text{(resp.,} \quad \int_G f(gM'')dg_T(g) \to_{T \to \infty} \int_{\mathcal{H}''} f''\,d\mu'' \quad).\]
By a standard argument we may assume that this is also true if \( f'' \) is continuous and has a finite limit at infinity; in particular, for \( f \in C_c(\mathcal{H}) \) we may take \( f'' = f \circ \varphi \). Thus by equivariance we have
\[
\int_{\mathcal{H}} f d\mu \leftarrow \int_{G} f(gM)d\nu_T(g) = \int_{G} f''(gM')d\nu_T(g) \rightarrow \int_{\mathcal{H}} f \circ \varphi d\mu'',
\]
and this implies that \( \mu = \varphi_*\mu'' \).

The map \( \varphi|_{\mathcal{L}''} : \mathcal{L}'' \to \mathcal{L} \) is an affine map of affine manifolds. In order to show that it is open it suffices to show that its derivative is surjective at every point \( x \in \mathcal{L}'' \). If not, then there is a neighborhood \( U \) of \( x \) in \( \mathcal{L}'' \) such that \( \varphi(U) \) is contained in a proper affine submanifold of \( \mathcal{L} \). Such a proper affine submanifold must have zero measure for the flat measure class on \( \mathcal{L} \), i.e. \( \mu(\varphi(U)) = 0 \). By the preceding paragraph this implies \( \mu''(U) = 0 \) which is impossible. \( \square \)

2.3. The Veech group, lattice surfaces, and periodic points. An affine automorphism of a translation surface \( M \) is a homeomorphism \( \varphi : M \to M \) which is affine in charts. In this case, by connectedness, its derivative \( D\varphi \) is a constant \( 2 \times 2 \) matrix of determinant \( \pm 1 \). We denote by \( \text{Aff}^+(M) \) the group of orientation-preserving affine automorphisms, i.e. those for which \( D\varphi \in G \). We say that \( \varphi \) is a parabolic automorphism if \( D\varphi \) is a parabolic matrix, i.e. is not the identity but has both eigenvalues equal to 1. The Veech group of \( M \) is the image under the homomorphism \( D : \text{Aff}^+(M) \to G \) of the group of orientation-preserving affine automorphisms. We say that \( M \) is a lattice surface if its Veech group is a lattice in \( G \). Equivalently, by a theorem of Smillie (see [Vee95, SW07]), the orbit \( GM \) is closed. Following [HST08] we say that \( M \) is a pre-lattice surface if \( \text{Aff}^+(M) \) contains two non-commuting parabolic automorphisms. Veech [Vee89] showed that a lattice surface is a pre-lattice surface, justifying the terminology. A point \( x \in M \) is called periodic if its orbit under \( \text{Aff}^+(M) \) is finite.

2.3.1. Example. In Lemma 6 we showed that \( \varphi''|_{\mathcal{L}''} : \mathcal{L}'' \to \mathcal{L} \) is an open map. Given that \( \mathcal{L} \) is connected, this leads to the question of whether \( \varphi|_{\mathcal{L}''} \) is surjective. The following example of Alex Wright shows that an open affine map of orbit-closures need not be surjective. Let \( M \in \mathcal{H} \) be a lattice surface which admits an involution \( \tau \) (e.g. \( M \) could be a surface of genus 2 and \( \tau \) could be the hyper-elliptic involution). Let \( \mathcal{L} = GM \) be the orbit of \( M \) (which in this case coincides with the orbit closure), let \( x \in M \) be a non-periodic point, and let \( M'' \overset{\text{def}}{=} (M, x) \) be the surface in \( \mathcal{H}' \) obtained by marking the point \( x \). It was proved in [HST08], and follows easily from Theorem 4, that \( \mathcal{L}' \overset{\text{def}}{=} GM' \) coincides with \( \varphi^{-1}(GM) \) (i.e. all surfaces in \( GM \) marked at all nonsingular points). Now let \( y \overset{\text{def}}{=} \tau(x) \neq x \), let \( M'' \overset{\text{def}}{=} M(x, y) \) be the surface in \( \mathcal{H}'' \) obtained by marking \( M \) at the two points \( x, y \), let \( \mathcal{L}'' \overset{\text{def}}{=} GM'' \), and let \( \varphi'' : \mathcal{H}' \to \mathcal{H}'' \) be the affine map which forgets the second marked point. We have
\[
\mathcal{L}'' \subset \{(M_0, x_0, y_0) \in \mathcal{H}'' : M_0 \in \mathcal{L}, \tau(x_0) = y_0 \neq x_0\},
\]
since the set on the right-hand side is closed and \( G \)-invariant. This implies that \( \varphi''|_{\mathcal{L}''} \subset \{(M_0, x_0) : M_0 \in GM, \tau(x_0) \neq x_0\} \), and in particular \( \varphi''|_{\mathcal{L}''} \) is not surjective. However the proof of Lemma 6 shows that \( \varphi''|_{\mathcal{L}''} \) is open.

Using one additional marked point one can find similar examples that show that in general, in Lemma 6, one need not have \( \varphi(\mathcal{L}'') = \mathcal{L} \).
3. Bounded blocking defines closed sets

Let $M$ be a translation surface with singularity set $\Sigma$, and let $\hat{M}^2 = \{(x, y) \in (M \setminus \Sigma)^2 : x \neq y\}$. If $Z$ is a topological space and $A \subset B$ are subsets of $Z$, when we say that $A$ is closed as a subset of $B$, we mean that $A$ is closed in the relative topology, i.e. $A = B \cap \overline{A}$.

**Lemma 7.** For any fixed integer $n \geq 0$, the following hold:

(I) For a fixed translation surface $M$, the set

$$F_n(M) \overset{\text{def}}{=} \{(x, y) \in \hat{M}^2 : \text{bc}(x, y) \leq n\}$$

is closed as a subset of $\hat{M}^2$.

(II) For a fixed translation surface $M$, and a fixed nonsingular $x \in M$, the set

$$F_n(M, x) \overset{\text{def}}{=} \{y \in M \setminus (\Sigma \cup \{x\}) : \text{bc}(x, y) \leq n\}$$

is closed as a subset of $M \setminus (\Sigma \cup \{x\})$.

(III) The set $F_n \subset \mathcal{H}''$ consisting of all surfaces on which the first and second marked points are finitely blocked of blocking cardinality at most $n$, is closed in $\mathcal{H}''$.

(IV) For a fixed stratum $\mathcal{H}$, the set of $M_0 \in \mathcal{H}$ for which any pair $(x, y) \in \hat{M}^2_0$ satisfies $\text{bc}(x, y) \leq n$ is closed in $\mathcal{H}$.

Moreover, there is $\ell$ such that if the set

$$\{(x, y) \in M^2 : \text{bc}(x, y) \leq n\}$$

is dense in $M^2$, then $M$ has the bounded blocking property with blocking cardinality at most $\ell$.

**Proof.** We will denote a surface in $\Sigma \cup \{x\}$ by $(M, x, y)$, where $x$ and $y$ are respectively the first and second marked points on $M$. The topology on $\mathcal{H}''$ is such that when $(M_k, x_k, y_k) \to (M, x, y)$, for any parametrized line segment $\{\sigma(t) : t \in [0, 1]\}$ on $M$ between $x$ and $y$, for any large enough $k$ there are parametrized line segments $\{\sigma_k(t) : t \in [0, 1]\}$ such that $\sigma_k(t) \to \sigma(t)$ for all $t$ see [MT02, Zor06] for details. Here a parameterized line segment is a constant speed straight line in each chart and does not contain singular points in its interior.

We begin with the proof of (III). Let $(M_k, x_k, y_k)$ be a sequence that converges to $(M, x, y)$ in $\mathcal{H}''$, where $(x_k, y_k)$ belongs to $F_n(M_k)$ for all $k$. Let $\{b_k^{(1)}, \ldots, b_k^{(n)}\} \subset M_k$ be a blocking set for $(x_k, y_k)$. Passing to a subsequence, we may assume that $b_k^{(i)}$ converges to a point $b^{(i)} \in M$ for each $i$. By the above description of the topology of $\mathcal{H}''$, if $\{b^{(1)}, \ldots, b^{(n)}\}$ does not contain $x$ or $y$ then it is a blocking set for $(x, y)$ in $M$ and we are done.

We now discuss the case that some of the $b^{(i)}$ are equal to $x$ or $y$. We modify the set $\{b^{(1)}, \ldots, b^{(n)}\}$ as follows. For any $i$ for which $b^{(i)}$ is different from both $x$ and $y$, we set $B^{(i)} = b^{(i)}$. Suppose $i$ is such that $b^{(i)} = x$. Let $r > 0$ be smaller than half the length of the shortest saddle connection on $M$. This implies that $r$ is smaller than half the distance between $x$ and $y$, and that there is no singularity in the ball $B(x, r)$ with center $x$ and radius $r$.

For $k$ large enough, $B(x_k, r)$ is an embedded flat disk in $M_k$ that contains $b_k^{(i)}$, and there is a unique trajectory $\delta_k^{(i)}$ from $x_k$ to $b_k^{(i)}$ that stays within this disk. Let
be the point on $\delta_k^{(i)}$ at distance $r$ from $x_k$. Passing again to a subsequence we assume that $B_k^{(i)}$ converges to a point $B^{(i)}$ in $M$. Note that this point is distinct from $x$ and $y$ for each such $i$. We repeat this procedure for each $i$ for which $b_k^{(i)}$ is equal to either $x$ or $y$, passing at each stage to a further subsequence.

Let us prove that $\{B^{(1)}, \ldots, B^{(n)}\}$ is a blocking set for $(x,y)$ in $M$. Let $\sigma$ be a trajectory from $x$ to $y$. We can assume without loss of generality that $\sigma$ is simple, i.e. does not intersect itself. Let $\sigma_k$ be the segment between $x_k$ and $y_k$ that converges pointwise to $\sigma$. If $\sigma_k$ meets one of the $B_k^{(i)}$ for infinitely many $k$, $B^{(i)}$ belongs to $\sigma$ and we are done.

Assume by contradiction that there is an index $i$ such that, for infinitely many $k$, $\sigma_k$ meets $b_k^{(i)}$ but not any $B_k^{(i)}$. In particular, $b_k^{(i)}$ converges to either $x$ or $y$.

Suppose for concreteness that it converges to $x$. Since $B_k^{(i)}$ does not belong to $\sigma_k$, the subsegment $\sigma'_k$ of $\sigma_k$ between $x$ and $b_k^{(i)}$ is not equal to the segment $\delta_k^{(i)}$ defined above. In particular the length of this subsegment is bounded below and it converges to a nontrivial subsegment $\sigma'$ of $\sigma$, which is a loop from $x$ to $x$. This contradicts the simplicity of $\sigma$, completing the proof of (III).

Clearly (III) $\Rightarrow$ (I) $\Rightarrow$ (II) and (III) $\Rightarrow$ (IV). It remains to prove the final assertion. Suppose $x,y \in M$ and there are $x_k \to x, y_k \to y$ such that $bc(x_k, y_k) \leq n$. We need to prove that $bc(x,y) \leq \ell$, for some $\ell$ which depends only on $M$ and $n$. If $x,y$ are distinct nonsingular points, then for large enough $k$ the points $x_k, y_k$ are also distinct and nonsingular, and the claim follows from (I). We will consider three cases, adapting the proof of (III) to each one:

**Case 1.** $x = y$ is a nonsingular point. We will show that in this case the previous proof applies and we can take $\ell = 2n$.

We construct the points $B^{(i)}$ as follows. Passing to subsequences we assume the existence of each of the limits $b_i = \lim_{k \to \infty} b_k^{(i)}$, and when $b_i \neq x$, we set $B_i = b_i$ as before. When $b_i = x$, in place of the short segments $\delta_k^{(i)}$ appearing in the previous argument, we consider two segments — one from $x_k$ to $b_k^{(i)}$ and one from $y_k$ to $b_k^{(i)}$. We denote these by $\delta_k^{(i)}', \delta_k^{(i)}''$ and construct points $B_{k,1}^{(i)}, B_{k,2}^{(i)}$ by ‘sliding’ $b_k^{(i)}$ along these segments as in the preceding argument. Taking limits, in each case in which $b_i = x$ we get two limit points, so re-indexing we get a total of at most $2n$ points.

We now show that this is a blocking set. Let $\sigma$ be a segment from $x$ to $x$, which does not contain any of the $B_i$. It is not contained in the ball $B = B(x, r)$ appearing in the previous proof. Let $\sigma_k$ be a sequence of parameterized line segments converging to $\sigma$. We can assume that none of these segments contains any of the $B_{k,i}, i = 1, \ldots, n, j = 1, 2$. The only place in the proof of (III) in which we used that $x \neq y$ is that we needed to know that the subsegment $\sigma'$ of $\sigma$ constructed in the proof is a proper subsegment of $\sigma$. In the case $x = y$ there are two subsegments $\sigma'_k, \sigma''_k$ between $x$ and $b_k^{(i)}$, neither of which is equal to $\delta_k^{(i)}$, since $\sigma_k$ does not contain any of the $B_{k,i}^{(i)}$. In particular, each of them leaves the disk $B(x_k, r)$ and hence has length at least $r$. So in the limit they both converge to nontrivial loops $\sigma', \sigma''$ from $x$ to itself, whose concatenation is $\sigma$. This gives the desired contradiction to the simplicity of $\sigma$.

**Case 2.** $x \neq y$ and at least one of them is a singularity. We will show that in this case we can take $\ell = n(\tau + 1)^2$, where $\tau\pi$ is the maximal cone angle of a singularity on $M$. 

Assume that $x$ is a singularity, let $r$ be as in the preceding proof, and let $\mathcal{U}_1, \ldots, \mathcal{U}_{r+1}$ be open convex subsets of $M$ of diameter less than $r$, such that $\bigcup \mathcal{U}_k = B(x, r) \setminus \{x\}$ and $x$ is in the closure of each $\mathcal{U}_k$. Such sets exist by our choice of $\tau$ and $r$, e.g. we may take them to be open half-disks centered at $x$. If $y$ is also a singularity, we similarly choose $\mathcal{U}_1', \ldots, \mathcal{U}_{r+1}'$ covering $B(y, r) \setminus \{y\}$.

We now choose sequences $x_k^{(s)}$ such that $x_k^{(s)} \in \mathcal{U}_k$ and $x_k^{(s)} \to_{k \to \infty} x$. If $y$ is also a singularity we similarly choose sequences $y_k^{(t)}$ which approach $y$ from within $\mathcal{U}_k'$. We also require that $bc(x_k^{(s)}, y_k^{(t)}) \leq n$ for each $k, s, t$. Such sequences exist since (1) is dense. For each choice of $(s, t) \in \{1, \ldots, \tau + 1\}^2$ we perform the procedure explained in the proof of (III). Namely we take blocking sets $\{b_k^{(i, s, t)} : i = 1, \ldots, n\}$ which block all segments between $x_k^{(s)}$ and $y_k^{(t)}$. Pass to subsequences to assume that $\lim_k b_k^{(i, s, t)}$ exists for each $i, s, t$, and define $B^{(i, s, t)}$ to be this limit if it is distinct from $x$ and $y$. If the limit is $x$ we modify $b_k^{(i, s, t)}$ by letting $B_k^{(i, s, t)}$ be the unique point of distance $r$ from $x$ along the continuation of the unique segment $s_k^{(i, s, t)}$ which connects $x$ and $b_k^{(i, s, t)}$ and which passes through $\mathcal{U}_k$. Then we take $B^{(i, s, t)}$ to be the limit $\lim_k B_k^{(i, s, t)}$ (passing to subsequences if necessary). We perform a similar modification if $\lim_k b_k^{(i, s, t)} = y$. This procedure gives us a set

$$\left\{B^{(i, s, t)} : i \in \{1, \ldots, n\}, (s, t) \in \{1, \ldots, \tau + 1\}^2\right\},$$

which we claim is a blocking set for $x, y$.

Indeed for each segment $\sigma$ from $x$ to $y$, we can assume that it approaches $x$ from within $\mathcal{U}_s$ and $y$ from within $\mathcal{U}_t$. Then for large enough $k$ there are segments $s_k$ from $x_k^{(s)}$ to $y_k^{(t)}$ which approach $\sigma$ pointwise. Working with these segments as in the proof of (III), we see that $\sigma$ is blocked by $B^{(i, s, t)}$ for some $i$.

Case 3. $x = y$ is a singular point. In this case we use both of the arguments used in Cases 1 and 2. We leave the details to the reader. \qed

**Corollary 8.** Let $BB_n$ denote the set of surfaces which have the bounded blocking property, with blocking cardinality at most $n$. Then there is $\ell \in \mathbb{N}$ such that if $M \in BB_n$ then $\overline{GM} \subset BB_\ell$.

**Proof.** Let us say that $M_\ell$ is $n$-blocking for distinct nonsingular points if for any $(x, y) \in M_\ell^2$, $bc(x, y) \leq n$. Then the set of such surfaces is closed by Lemma 7(IV). Also, if $M'' \in H''$ has $x, y$ as the first and second marked points, and $x(g), y(g)$ are the first and second marked points on $gM''$, then $bc(x, y) \leq n$ implies $bc(x(g), y(g)) \leq n$. This implies that the property of being $n$-blocking for distinct nonsingular points is $G$-invariant. Since $M \in BB_n$, $M$ is $n$-blocking for distinct nonsingular points. Thus any surface in $\overline{GM}$ is also $n$-blocking for distinct nonsingular points, and the claim follows from the last assertion in Lemma 7. \qed

A similar argument also shows:

**Proposition 9.** Let $M$ be a translation surface, $\xi$ a singular point on $M$ and $n \geq 0$ an integer. Recalling our convention that singularities on translation surfaces are labeled, we can use the notation $\xi$ for a singular point of any other surface in $H$. Let $F' \subset H'$ denote the set of surfaces on which the marked point $y$ satisfies
bc(ξ, y) ≤ n. Then \( F'_n \) is closed in \( \mathcal{H}' \). In particular, \( \{ y \in M \setminus \Sigma : bc(\xi, y) \leq n \} \) is closed as a subset of \( M \setminus \Sigma \).

**Proof.** We repeat the proof of Lemma 7(III), replacing everywhere \( x \) with \( \xi \) and also \( x_k \) with \( \xi \).

In this case the set \( B = B(\xi, r) \) is a topological disk which is metrically a finite cover of a flat disk, branched over its center point \( \xi \). Then \( B \) is star-shaped with respect to its center point \( \xi \) and it is still the case that there is a unique straight segment from \( \xi \) to any point in \( B \) which is contained in \( B \). We can thus define the segment \( \delta^{(i)}_k \) as in the proof of (III), and the same argument applies. □

4. Characterization of the finite blocking property

In this section we will prove Theorem 1. A translation surface is purely periodic if it is completely periodic and all cylinders in such a decomposition have commensurable circumferences. The following was proved in [Mon09]:

**Proposition 10** (Monteil). If \( M \) has the blocking property then \( M \) is purely periodic.

**Proof of Theorem 1.** The implication \( (1) \Rightarrow (2) \) is proved in [Mon05], and it is immediate that \( (2) \Rightarrow (3) \). We first show \( (4) \Rightarrow (1) \), that is we assume that \( M \) has the bounded blocking property and we show that it is a torus cover.

Let \( \mathcal{L} \equiv \mathcal{G}M \). By assumption there is \( n \) such that \( M \in \mathcal{B}B_n \), and by Corollary 8 this means \( \mathcal{L} \) is contained in \( \mathcal{B}B_\ell \) for some \( \ell \). By Proposition 10 this means that every surface in \( \mathcal{L} \) is completely periodic and by Proposition 5, \( \mathcal{L} \) is of cylinder rank one.

Recall that the field of definition of \( \mathcal{L} \) is the smallest field such that in any coordinate chart \( U \) on \( \mathcal{H} \) given by period coordinates, the connected components of \( U \cap \mathcal{L} \) are cut out by linear equations with coefficients in \( k \) (see [Wrib]). By [Wria, Theorem 1.9], for any completely periodic surface \( M' \in \mathcal{L} \), and any cylinder decomposition on \( M' \) with circumferences \( c_1, \ldots, c_r \) the field of definition \( k \) of \( \mathcal{L} \) satisfies

\[
\mathbb{Q} \left[ \left\{ \frac{c_i}{c_j} : i, j = 1, \ldots, r \right\} \right].
\]

By Proposition 10, any surface in \( \mathcal{L} \) is purely periodic, so \( k = \mathbb{Q} \). Therefore \( \mathcal{L} \) contains a surface with rational holonomies, i.e. a square-tiled surface \( M' \). Since \( M' \) is square-tiled the holonomy of absolute periods on \( M' \) is a discrete subset of \( \mathbb{C} \). Motion in the \( G \oplus \text{REL} \) leaf only changes the holonomy of absolute periods by a linear map, and therefore for any \( M \in \mathcal{L} \), the holonomy of absolute periods is discrete, i.e., any \( M \in \mathcal{L} \) is a torus cover. This proves \( (4) \Rightarrow (1) \).

Now we prove \( (3) \Rightarrow (4) \). We have an open set \( U_1 \) in \( M \times M \) consisting of pairs of points on \( M \) blocked from each other by finitely many points, that is, \( U_1 \cap M^2 \subset \bigcup_n F_n(M) \). Each \( F_n(M) \) is closed as a subset of \( M^2 \) by Lemma 7(I), so by Baire category, there is \( n \) such that \( F_n(M) \) contains an open set \( U_2 \). Each pair of points \( (x, y) \) in \( U_2 \) defines a surface in \( \mathcal{H}'' \), namely \( M'' = (M, x, y) \). Let \( \mathcal{L}(M'') \equiv \mathcal{G}M'' \subset \mathcal{H}'' \). By Theorem 4, \( \hat{\mathcal{L}}(M'') \) is a linear manifold of even dimension contained in \( \mathcal{F}_n \) and the collection of such linear submanifolds is countable. By Lemma 7(III), \( \hat{\mathcal{L}}(M'') \subset \mathcal{F}_n \).
The fiber $\varphi^{-1}(M)$ is a linear submanifold of $\mathcal{H}''$ identified with $\hat{M}^2$. Therefore for any $M'', \Omega(M'') \overset{\text{def}}{=} \varphi^{-1}(M) \cap \hat{L}(M'')$ is also a linear submanifold, and its dimension is 0, 2 or 4. We have covered $U_2$, an open subset of a four-dimensional manifold, by countably many linear manifolds of dimensions at most four. By Baire category, there is $M''$ for which $\Omega(M'')$ is a linear manifold of dimension four. Since $\varphi^{-1}(M)$ is connected, it coincides with $\Omega(M'')$.

We have proved that

$$\varphi^{-1}(M) = \Omega(M'') \subset \hat{L}(M'') \subset \mathcal{F}_n;$$

that is, any two distinct nonsingular points in $M$ are of blocking cardinality at most $n$. Applying the last assertion of Lemma 7, we see that $M$ has the bounded blocking property. \hfill \Box

5. Illumination

In this section we will study some illumination problems. Recall that two points $x, y$ on a translation surface $M$ do not illuminate each other if and only if they are finitely blocked with blocking cardinality zero. Also recall that $p_1, p_2$ denote the projections onto the first and second factors of $M \times M$. The following result is the main result of this section.

**Theorem 11.** Let $M$ be a translation surface, let $n$ be a non-negative integer. Then:

(i) For any $x \in M$, the set \(\{y \in M : \text{bc}(x, y) \leq n\}\) is either finite or contains $M \setminus (\Sigma \cup \{x\})$.

(ii) The set \(\{(x, y) \in M^2 : \text{bc}(x, y) \leq n\}\) either contains $M^2$, or is a finite union of 0 and 2 dimensional linear submanifolds of $M \times M$. The 2-dimensional linear submanifolds are of one of the following forms: $F \times M, M \times F$, where $F \subset M$ is finite, or a translation surface $S$ embedded affinely in $M \times M$, where for $i = 1, 2, \tau_i = p_i|_S : S \to M$ is a finite-degree covering map such that $\tau_2 \circ \tau_1^{-1}$ is a multiplication by a scalar $\lambda$ satisfying $\lambda^2 \in \mathbb{Q}$.

**Theorem 11 implies Theorem 2.** We apply Theorem 11 with $n = 0$. It is clear that the second alternative in (i) cannot hold, since for any $x$, all nearby points illuminate $x$. Also, in (ii), the cases $F \times M$ and $M \times F$ do not arise, since any point illuminates some other point. \hfill \Box

**Proof of Theorem 11.** Keep the notation of §2.1 and Lemma 7. We will first prove (i) in case $x$ is a regular point of $M$. Let $M' \in \varphi^{-1}(M) \subset \mathcal{H}'$ denote the surface with first marked point at $x$. We need to show that

$$A \overset{\text{def}}{=} \{y \in M \setminus (\Sigma \cup \{x\}) : \text{bc}(x, y) \leq n\},$$

which we may identify with $\mathcal{F}_n \cap \varphi''^{-1}(M')$, is either finite or coincides with $\varphi''^{-1}(M')$. Let us assume $A \not\subset \varphi''^{-1}(M')$. Since $\mathcal{F}_n$ is closed and $G$-invariant, $A$ is a union of at most countably many linear manifolds, which are of the form $\mathcal{L}(M''_0) \overset{\text{def}}{=} GM''_0$ for $M''_0 \in A$. For each $M''_0$, $\mathcal{L}(M''_0) \cap \varphi''^{-1}(M')$ is a linear manifold of dimension 0 or 2 by Theorem 4. If the dimension were 2, $A$ would coincide with the fiber $\varphi''^{-1}(M)$ by connectedness, and hence each $\mathcal{L}(M''_0)$ is finite. To conclude the proof of (i) we need to show that in fact there are only finitely many distinct sets $\mathcal{L}(M''_0)$. If there were infinitely many this would mean that $\mathcal{F}_n$ contains infinitely
Applying Theorem 4 as in the preceding paragraph we see that the details to the reader. This follows from arguments of [HST08], but we give an independent argument for completeness.

We need to show that all of the 2-dimensional manifolds have the stated form. Of finitely many 0-dimensional and finitely many 2-dimensional linear manifolds.

Let \( F \) be the set of solutions of an equation of the form \( az_1 + b z_2 = 0 \) (up to a translation). Moreover \( N \) is defined over \( \mathbb{R} \) so we can take \( a, b \in \mathbb{R} \). If \( a = 0 \) then any connected component of \( N \) is of the form \( M \times \{ x \} \) for some \( x \in M \), i.e. \( N \) is of the form \( M \times F \). Similarly if \( b = 0 \) then \( N \) has the form \( F \times N \). Now we consider the case when \( a, b \) are both nonzero.

Since the transition maps for the translation atlas are translations, \( a \) and \( b \) can actually be taken to be independent of the neighborhood, and the Cartesian product translation structure on \( M^2 \), restricted to \( N \), endows \( N \) with a natural structure of a translation surface (see [HST08, §3] for more details), where \( N \) is locally modelled on the plane \( (2) \). Since \( a \) and \( b \) are both nonzero, each of the projections \( \tau_i = p_i|_N \) has a nonsingular derivative, so by connectedness, each \( \tau_i \) is a finite covering map.

The plane \( (2) \) can be identified with \( \mathbb{C} \) in many ways and thus the translation surface structure on \( N \) is only naturally defined up to a scalar multiple. However, for any fixed choice of translation structure on \( N \), each of the maps \( \tau_i \) is the composition of a dilation and a translation covering. Let \( k_i \) be the degree of the covering map \( \tau_i \), and let \( \lambda_i \) be the associated dilation. The choice of the \( \lambda_i \) depends on a choice of the translation structure on \( N \), but since the derivative of \( \tau_2 \circ \tau_1^{-1} \) is the map \( z_1 \mapsto -\frac{a}{b} z_1 \), we have \( \lambda \stackrel{\text{def}}{=} \lambda_2 \lambda_1^{-1} = -\frac{a}{b} \). We can compute the area of \( N \) using each of the maps \( \tau_i \), to obtain

\[
\text{area}(N) = \frac{k_1}{\lambda_i^2} \text{area}(M).
\]

Comparing these formulae for \( i = 1, 2 \) we see that \( \lambda^2 = \left(\frac{a}{b}\right)^2 = \frac{k_2}{k_1} \in \mathbb{Q}. \)

6. Examples and questions

Let \( T \) be the standard torus, obtained from the unit square \([0, 1]^2\) by gluing opposite sides to each other by translations. Denote by \( \pi \) the projection from \( \mathbb{R}^2 \) to \( T \). For any nonzero integer \( n \), notice that the map \( \mathbb{R}^2 \to \mathbb{R}^2, x \mapsto nx \) descends to a map \( m_n : T \to T \) which multiplies both components by \( n \) in \( \mathbb{R}/\mathbb{Z} \), and is therefore
that the set midpoint in $\mathbb{R}$ from itself, and describe larger blocking sets on (a) arguments and we leave the details to the reader.

Lemma 12.  
(a) If $x$ and $y$ are distinct points on $T$, their blocking cardinality is $bc(x, y) = 4$.

(b) It is realized by the blocking set $B(x, y) = m_2^{-1}(x + y)$, which contains the midpoint of any geodesic from $x$ to $y$.

(c) This is the unique blocking set of size 4.

Proof. Let $\bar{x}$, $y$ denote points in $\mathbb{R}^2$ which project to $x$, $y$ on $T$. Let $u = (1, 0)$, $v = (0, 1)$, $w = (1, 1)$. The four segments from $y$ to the four points $\bar{x}$, $\bar{x} + u$, $\bar{x} + v$, $\bar{x} + w$ (four corners of a unit square) project to segments with disjoint interiors on $T$, so at least 4 points are required to block the pair $(x, y)$. On the other hand, any line segment in $T$ from $x$ to $y$ is the projection of a line segment in $\mathbb{R}^2$ from $\bar{x}$ to $\bar{y} + au + bv$ with $a$ and $b$ in $\mathbb{Z}$. Such a segment has midpoint $\frac{1}{2}(\bar{x} + \bar{y} + au + bv)$. This midpoint in $\mathbb{R}^2$ projects to one of the points $\frac{1}{2}(x + y)$, $\frac{1}{2}(x + y + u)$, $\frac{1}{2}(x + y + v)$, $\frac{1}{2}(x + y + w)$, which are the four points in $T$ comprising $m_2^{-1}(x + y)$. This proves that the set $B(x, y)$ is a blocking set and that $bc(x, y) \leq 4$. So (a) and (b) are proved.

We now prove (c). We saw that the four segments from $\bar{y}$ to $\bar{x}$, $\bar{x} + u$, $\bar{x} + v$, $\bar{x} + w$ project to segments on $T$ with disjoint interiors, so a blocking set for $(x, y)$ must contain at least a point in each of them. Consider the segment from $\bar{y} + v$ to $\bar{x} + u$. The only intersection of its projection to $T$ with the interiors of our four segments is its midpoint $m$ which is also the midpoint of the segment from $y$ to $y + w$. So a blocking set not containing $m$ would need to contain at least 5 points. Similar reasoning proves the other three points in the proposed set $B(x, y)$ have to be in a blocking set of cardinality 4.

The following two lemmas extend this description to configurations blocking a point from itself, and describe larger blocking sets on $T$. They are proved by similar arguments and we leave the details to the reader.

Lemma 13.  
(a) If $x = y$, then the blocking cardinality is $bc(x, x) = 3$.

(b) It is realized by the blocking set $B(x, x) = m_2^{-1}(2x) \setminus \{x\}$, which is the set of midpoints of all primitive geodesics from $x$ to $x$. This blocking set can also be described as $B(x, x) = x + B_0$ where $B_0 = B(0, 0) = m_2^{-1}(0) \setminus \{0\}$.

(c) This is the unique blocking set of size 3.

Lemma 14.  
(a) Let $n$ and $a$ be relatively prime integers with $1 \leq a < n$. For any pair of points $(x, y)$ with $x \neq y$, the set $B = m_2^{-1}(ax + (n - a)y)$ is a blocking set of cardinality $n^2$ for the pair $(x, y)$. It contains the point located $a/n$ of the way along each line segment from $x$ to $y$ on $T$.

(b) Let $n \geq 2$ be an integer. For the pair of points $(x, x)$ with $x = 0$, the set $B_0 = m_2^{-1}(0) \setminus \{0\} = \{(a/n, b/n) : 0 \leq a < n, \ 0 \leq b < n, \ (a, b) \neq (0, 0)\}$ is a blocking set of cardinality $n^2 - 1$.

For the pair of points $(x, x)$ with $x \neq 0$, the set $B = x + B_0$ is a blocking set of cardinality $n^2 - 1$, also equal to $m_2^{-1}(nx)$.

We will use these computations to compute blocking configurations on branched covers of $T$. Recall that if $M \to T$ is a branched translation cover, a singularity of
$M$ corresponds to a ramification point of the cover, and if the angle at a singularity
$x$ is $2\pi k$ then $k$ is called the ramification index of $x$.

**Lemma 15.** Suppose $M$ is a torus cover of degree $d$, with arbitrary branch locus and ramification type, and let $p : M \to T$ denote the covering map.

(a) For a pair $(x, y)$ of points of $M$ such that $p(x) \neq p(y)$, if $B'$ is a blocking set for $(p(x), p(y))$ on $T$, then $B = p^{-1}(B')$ is a blocking set for $(x, y)$, of cardinality at most $d$ times that of $B'$, with equality when $B$ contains no zero of $M$, i.e. no ramification point of $p$.

(b) In particular,
- for almost every pair $(x, y)$ of points of $M$, $bc(x, y) \leq 4d$.
- for pairs $(x, y)$ of points of $M$, such that the set $B(p(x), p(y))$ contains branch points of $p$, the bound above is decreased by the sum of the ramification indices of the ramification points above these branch points.

(c) For a pair of points $(x, y)$ on $M$ such that $p(x) = p(y)$ (whether $x = y$ or not), $p^{-1}(B(p(x), p(x)))$ is a blocking set, so that $bc(x, y) \leq 3d$. As above, when $B(p(x), p(y))$ contains branch points of $p$, the bound is decreased by the sum of the ramification indices of the ramification points above these branch points.

**Proof.** Both (a) and (b) are easy, and (c) follows from the following observation. When $p(x) = p(y)$, any geodesic path $\gamma$ from $x$ to $y$ projects to a geodesic $\gamma'$ from $p(x)$ to itself, possibly non primitive. Considering the restriction of the geodesic $\gamma$, if $\gamma'$ is not primitive, to its initial part until it first reaches a point projecting to $p(x)$, we see that (c) holds. \hfill $\Box$

6.1. **Example 1.** The following example shows that quite general maps $\tau_1, \tau_2$ may arise in Theorem 11.

**Proposition 16.** Let $a, b$ be positive integers with $\gcd(a, b) = 1$, let $n = a+b$, and let

$$X = \{(ax, bx) : x \in T\}.$$ 

Also let $p : M \to T$ be a translation cover with branching locus $m_n^{-1}(0)$, and non-trivial ramification at each pre-image of each branch point, and let

$$Y = (p \times p)^{-1}(X).$$

Then any pair of points in $Y$ do not illuminate each other.

**Proof.** For $x \in \mathbb{R}^2$, the point $0$ is $a/n$ along the geodesic in $\mathbb{R}^2$ from $-ax$ to $bx$. Thus, according to Lemma 14, the set $B = m_n^{-1}(0)$ is a common blocking set, of cardinality $n^2$, for all pairs of points in $X$. Thus the statement follows from Lemma 15. \hfill $\Box$

6.2. **Example 2.** The following examples show that the map $\tau_2 \circ \tau_1^{-1}$ could be a translation. Let $M = T$ be the torus, and consider

$$N = \{(x, y) \in M^2 : bc(x, y) \leq 3\}.$$ 

Then according to Lemma 13, $N$ contains the diagonal $\{(x, x) : x \in M\}$ but according to Lemma 12, $N \neq M^2$. Therefore the diagonal is one of the linear submanifolds appearing in Theorem 11, and we can have $\tau_2 \circ \tau_1^{-1} = \text{Id}$. 

Figure 1. The Escher double staircase. Sides marked with identical stair-climbers are identified; unmarked sides are identified with the corresponding opposite sides.

Similar examples in which \( \tau_2 \circ \tau_1^{-1} \) is a nontrivial translation can be obtained by taking \( M \) to be a cyclic cover of \( T \), for example the Escher staircase (see Figure 1). This surface admits a degree 3 cover \( p : M \to T \) and it has a nontrivial translation automorphism \( D : M \to M \) moving one step up the ladder. Let \( x \) and \( y \) be any two points such that \( D(x) = y \). Then \( p(x) = p(y) \), and according to Lemma 15(c), \( bc(x, y) \leq 9 \). It is not hard to find an explicit pair of points \( x, y \) for which \( bc(x, y) > 9 \). This shows that if we take this surface \( M \) and \( n = 9 \), then we can have a subsurface \( N \) for which \( D = \tau_2 \circ \tau_1^{-1} \) is a translation automorphism.

6.3. Example 3. Using the torus and Lemmas 12 and 13 we easily find sequences \( x_k \to x, y_k \to y \) for which \( bc(x, y) < \lim_k bc(x_k, y_k) \), i.e. the blocking cardinality is not continuous. The following example shows that it is not even lower semi-continuous, i.e. it may increase when taking limits. It also shows that in Lemma 7(I), we cannot replace \( \hat{M}^2 \) with \( M^2 \).

Let \( M \) be a surface in \( \mathcal{H}(2) \). Then \( M \) admits a hyper-elliptic involution \( h \), whose set of fixed points consists of the unique singularity \( \xi \), and 5 non-singular Weierstrass points. We claim that whenever \( h(x) = y, x \neq y \), we have \( bc(x, y) \leq 5 \). Indeed, in this case, the action of \( h \) swaps \( x \) and \( h(x) \), and acts by rotation by \( \pi \). So \( h \) maps any segment between \( \sigma \) between \( x \) and \( y \) to another segment from \( x \) to \( y \), of the same length and in the same direction. Since \( x \) and \( y \) are distinct regular points there is only one such segment, i.e. \( h \) maps \( \sigma \) to itself, reversing the orientation on it. So its midpoint must be fixed by \( h \), that is, the Weierstrass points form a blocking set for the pair \( (x, y) \).

On the other hand by constructing explicit disjoint segments, it is not hard to show that \( bc(\xi, \xi) \geq 9 \). For example we can present \( M \) as the union of three parallelograms (an \( L \)-shaped presentation), and the diagonal and edges of these parallelograms contain 9 disjoint segments from \( \xi \) to \( \xi \). Now taking \( x_k \to \xi \), we
have $y_k = h(x_k) \to \xi$, and

$$5 \geq \lim_{k} \text{bc}(x_k, y_k), \quad \text{bc}(\lim_{k} x_k, \lim_{k} y_k) = \text{bc}(\xi, \xi) \geq 9.$$

### 6.4. Questions

1. Let $N \subset M \times M$ be a 2-dimensional linear submanifold as in Theorem 11(ii), and let $\lambda_1, \lambda_2$ be the derivatives of the translation maps $\tau_1, \tau_2$. The quotient $\lambda = \lambda_1/\lambda_2$ is called the slope of $N$. In Example 2 the slope is 1, and in Example 1 the slope can be an arbitrary negative rational number. It would be interesting to know whether other slopes are possible. In particular, do the cases $\lambda = 0, \lambda = \infty$ actually arise in connection with blocking configurations? Do positive rational slopes arise, except for $\lambda = 1$?

2. More generally, suppose $N \subset M \times M$ is an embedded translation surface for which the maps $\tau_i : N \to M$ are the composition of a dilation and a translation, and let $\lambda$ be the derivative of the composition $\tau_2 \circ \tau_1^{-1}$. In the proof of Theorem 11 we showed that $\lambda^2 \in \mathbb{Q}$. Is it possible that $\lambda$ is irrational?

3. In the last assertion of Lemma 7, can we take $\ell = n$? Example 3 shows that this does not follow from a simple continuity argument. Also in connection with Example 3, does there exist a similar example in which the point $\xi$ is nonsingular? That is, an example of a surface $M$ with a regular point $\xi$ and two sequences $x_k, y_k$ converging to $\xi$, such that $\text{bc}(\xi, \xi) > \lim_{k} \text{bc}(x_k, y_k)$?

### References


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