HOROSPHERICAL DYNAMICS IN INVARIANT SUBVARIETIES

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Abstract. We consider the horospherical foliation on any invariant subvariety in the moduli space of translation surfaces. This foliation can be described dynamically as the strong unstable foliation for the geodesic flow on the invariant subvariety, and geometrically, it is induced by the canonical splitting of $\mathbb{C}$-valued cohomology into its real and imaginary parts. We define a natural volume form on the leaves of this foliation, and define horospherical measures as those measures whose conditional measures on leaves are given by the volume form. We show that the natural measures on invariant subvarieties, and in particular, the Masur-Veech measures on strata, are horospherical. We show that these measures are the unique horospherical measures giving zero mass to the set of surfaces with no horizontal saddle connections, extending work of Lindenstrauss-Mirzakhani and Hamenstädt for principal strata. We describe all the leaf closures for the horospherical foliation.

1. Introduction

It is an interesting fact that geometric questions about rational polygonal billiards can be addressed by studying the dynamics on moduli spaces of translation surfaces. This is one of many reasons to study the dynamics on moduli spaces of translation surfaces — see the surveys [MT02, Zor06, FM14, Wri15b] for other motivation and a survey of results. We remind the reader that this moduli space is partitioned into *strata*, which correspond to translation surfaces of a fixed topological type. The group $G \stackrel{\mathrm{def}}{=} \mathrm{SL}_2(\mathbb{R})$ acts on each stratum.

The *horocycle flow* is given by

$$U \stackrel{\mathrm{def}}{=} \{u_s : s \in \mathbb{R}\} \subset G, \quad \text{where} \quad u_s \stackrel{\mathrm{def}}{=} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

The analogy between dynamics on strata and homogeneous dynamics has been fruitful. In the setting of homogeneous dynamics U-actions and G-actions were analyzed in work of Ratner which showed that orbit closures and ergodic invariant probability measures are surprisingly well-behaved. The dynamics of G-actions (and moreover the dynamics of its subgroup P of upper triangular matrices) on strata were analyzed in two papers [EM18, EMM15] where it was shown that orbit closures and ergodic invariant measures have nice descriptions (see Section 2.2 for a precise statement).

The G-orbit closures are endowed with a wealth of geometrical structures, among which is the *horospherical foliation* which plays the role of the strong unstable manifold foliation for the one parameter diagonal subgroup which is called the geodesic flow (see §3.2). In §3 we will define *horospherical measures*. Loosely speaking, the horospherical leaves are endowed with affine structures and the horospherical measures are those for which the conditional measures on theses leaves are translation invariant with respect to these affine structures.

1.1. **Statement of results.** All measures considered in this paper are Borel regular Radon measures on strata of translation surfaces. Any G-orbit closure $\mathcal{M}^{(1)} \subset \mathcal{H}^{(1)}$ supports a unique ergodic G-invariant finite smooth measure; we will refer to this measure as the *special flat* measure on $\mathcal{M}^{(1)}$. The following are the main results of this paper.

Theorem 1.1. *The special flat measure on any G-orbit closure is horospherical.*

We will say that a measure μ is *saddle connection free* if μ-a.e. surface has no horizontal saddle connections.

Theorem 1.2. *Up to scaling, the only saddle connection free horospherical measure on a G-orbit closure is the special flat measure.*

We emphasize that horospherical measures are a *a priori* not assumed to be finite. It is thus a consequence of Theorem 1.2 that horospherical measures are finite under the saddle connection free assumption; it seems likely, but we were not able to prove, that all horospherical ergodic measures are finite. Theorem 1.2 was announced without proof in [BSW22, Claim 1, §9]. The saddle connection free assumption cannot be removed; for example, the length measure on a periodic horocycle trajectory in a closed G-orbit. In §5 we will give more interesting examples of invariant subvarieties and horospherical measures on them, which are not the special flat measure. We will also classify (see §5.1) all the horospherical measures on the simplest nontrivial invariant subvarieties, namely the eigenform loci in $\mathcal{H}(1, 1)$.

If a surface has a horizontal cylinder then so does any surface on its horospherical leaf. We will say that a leaf of the horospherical foliation is *cylinder-free* if all surfaces on the leaf have no horizontal cylinders. We say that a measure μ on $\mathcal{M}$ is cylinder-free if μ-a.e. surface has no horizontal cylinders. In §5 we give examples of horospherical measures which are not special flat and for which almost every point has a horizontal saddle connection. For these measures it is also the case that almost every point has a cylinder. It seems likely that this is always the case; or in other words, that in Theorem 1.2 the condition 'saddle connection free' can be weakened to 'cylinder-free'. The analogous assertion about orbit closures is true:

Theorem 1.3. *Any cylinder-free leaf for the horospherical foliation is dense.*

The proof of Theorem 1.3 uses a statement of independent interest (Theorem 7.2), about extending horizontal saddle connections while staying inside invariant suborbifolds. This result was explained to us by Paul Apisa and Alex Wright, and its proof is given in Appendix A.

The *geodesic flow* is the restriction of the G-action to the subgroup

$$A \stackrel{\mathrm{def}}{=} \{g_t : t \in \mathbb{R}\} \subset G, \quad \text{where } g_t \stackrel{\mathrm{def}}{=} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{1}$$

From a dynamical perspective, the horospherical foliation is the strong unstable foliation for the geodesic flow. Our arguments yield a simpler proof of the following theorem.

Theorem 1.4 ([EM18, EMM15])**.** *The special flat measure is the unique A-invariant horospherical measure on any G-orbit closure. Any leaf for the weak-unstable foliation on any G-orbit closure is dense.*

1.2. **Further motivation, prior work, and some ideas from the proofs.** The work of Eskin, Mirzakhani and Mohammadi gives a very detailed understanding of invariant measures and sets for the G-action and the P-action on strata of translation surfaces. A central remaining open problem is to understand horocycle invariant ergodic measures. As we will see in §3, horospherical measures are horocycle-invariant; thus understanding horospherical measures can be seen as a contribution to the problem of understanding general horocycle-invariant measures.

A previous measure rigidity result for horospherical measures was obtained in 2008, independently by Lindenstrauss and Mirzakhani [LM08] and by Hamenstädt [Ham09]. They were interested in understanding mapping class group invariant measures on the space of measured laminations. By a 'duality principle' (see [LM08, §5]) this question is very closely related to the problem of classifying horospherical measures on the principal stratum.

Our argument for Theorem 1.2 follows [LM08], which in turn is inspired by ideas of Dani [Dan78] and Margulis [Mar04] The main ingredients are the mixing of the A-action, the use of dynamical boxes and how they transform under the A-action, and nondivergence results for the U-action (which in the present context were obtained in [MW02]). After the requisite preparations, this argument is given in §4. In order to carry out the details of this argument, we give a precise description of horospherical measures and special flat measures, and their decomposition into conditional measures in flow boxes in §3.

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2. Preliminaries

In this section we introduce our objects of study and set up our notation. There are many approaches to these definitions. In our approach, the linear orbifold structure (or affine orbifold structure) given by period coordinates will be important and we will stress this point of view in what follows. A suitable reference for the theory utilizing this point of view is [BSW22, §2], and unless stated otherwise, our notation, terminology and assumptions are as in [BSW22]. See also [MT02, Zor06, FM14, Wri15b]. See [Gol] for a general discussion of affine manifolds.

2.1. **Strata and period coordinates.** Let S be a connected, compact orientable surface of genus g, $\Sigma = \{\xi_1, \ldots, \xi_k\} \subset S$ a finite set, $a_1, \ldots, a_k$ non-negative integers with $\sum a_i = 2g-2$, and $\mathcal{H} = \mathcal{H}(a_1, \ldots, a_k)$ the corresponding stratum of translation surfaces. We let $\mathcal{H}_{\mathrm{m}} = \mathcal{H}_{\mathrm{m}}(a_1, \ldots, a_k)$ denote the stratum of marked translation surfaces and $\pi : \mathcal{H}_{\mathrm{m}} \to \mathcal{H}$ the forgetful mapping. It will be useful to assume that singular points are labeled, or equivalently, $\mathcal{H} = \mathcal{H}_{\mathrm{m}}/\mathrm{Mod}(S, \Sigma)$, where $\mathrm{Mod}(S, \Sigma)$ is the group of isotopy classes of orientation-preserving homeomorphisms of S fixing Σ, up to an isotopy fixing Σ. We will typically denote elements of $\mathcal{H}$ by the letter q when we want to consider them as points of $\mathcal{H}$, and by the letter M or M_q when we want to consider their underlying topological or geometrical properties as spaces in their own right. To make this distinction clear we will write M_q for the underlying surface associated with a point $q \in \mathcal{H}$. Points in $\mathcal{H}_{\mathrm{m}}$ will be typically denoted by boldface letters such as $\boldsymbol{q}$.

We recall the definition of the map $\mathrm{dev} : \mathcal{H}_{\mathrm{m}} \to H^1\left(S, \Sigma; \mathbb{R}^2\right)$. For an oriented path γ in M_q which is either closed or has endpoints at singularities, let $\mathrm{hol}(M_q, \gamma) \stackrel{\mathrm{def}}{=} \left(\int_\gamma dx_q, \int_\gamma dy_q\right)$, where dx_q and dy_q are the 1-forms on M_q inherited from the the forms dx and dy on the plane. Given $\boldsymbol{q} \in \mathcal{H}_{\mathrm{m}}$ represented by $f : S \to M_q$, where M_q is a translation surface, we define $\mathrm{dev}(\boldsymbol{q}) \stackrel{\mathrm{def}}{=} f^*(\mathrm{hol}(M_q, \cdot))$. The map dev is also known in the literature as the *period map*. There is an open cover $\{\mathcal{U}_\tau\}$ of $\mathcal{H}_{\mathrm{m}}$, indexed by triangulations τ of S with triangles whose vertices are in Σ, such that the restricted maps

$$\varphi_\tau \stackrel{\mathrm{def}}{=} \mathrm{dev}|_{\mathcal{U}_\tau}, \quad \varphi_\tau : \mathcal{U}_\tau \to H^1\left(S, \Sigma; \mathbb{R}^2\right)$$

are homeomorphisms onto their image. The charts φ_τ give an atlas with affine overlap maps and endow $\mathcal{H}_{\mathrm{m}}$ with a structure of affine manifold. This atlas of charts $\{(\mathcal{U}_\tau, \varphi_\tau)\}$ is known as the *period coordinate atlas.*

The $\mathrm{Mod}(S, \Sigma)$-action on $\mathcal{H}_{\mathrm{m}}$ is properly discontinuous and affine, and hence $\mathcal{H}$ inherits the structure of affine orbifold, and the map $\pi : \mathcal{H}_{\mathrm{m}} \to \mathcal{H}$ is an orbifold covering map. We can associate to any affine manifold a *holonomy cover* and a *developing map*. In this case $\mathcal{H}_{\mathrm{m}}$ is the holonomy cover and dev is the developing map of $\mathcal{H}$ (see [Gol]).

The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on translation surfaces in $\mathcal{H}$ and $\mathcal{H}_{\mathrm{m}}$ by modifying planar charts. It acts on $H^1\left(S, \Sigma; \mathbb{R}^2\right)$ via its action on the coefficients $\mathbb{R}^2$. The $\mathrm{GL}_2^+(\mathbb{R})$-action commutes with the $\mathrm{Mod}(S, \Sigma)$-action, and thus the map π is $\mathrm{GL}_2^+(\mathbb{R})$-equivariant for these actions. The $\mathrm{GL}_2^+(\mathbb{R})$-action on $\mathcal{H}_{\mathrm{m}}$ is free, since $\mathrm{dev}(g\boldsymbol{q}) \neq \mathrm{dev}(\boldsymbol{q})$ for any nontrivial $g \in \mathrm{GL}_2^+(\mathbb{R})$.

We have a coordinate splitting of $\mathbb{R}^2$ and we write $\mathbb{R}^2 = \mathbb{R}_{\mathrm{x}} \oplus \mathbb{R}_{\mathrm{y}}$ to distinguish the two summands in this splitting. There is a corresponding splitting of cohomology

$$H^1\left(S, \Sigma; \mathbb{R}^2\right) = H^1(S, \Sigma; \mathbb{R}_{\mathrm{x}}) \oplus H^1\left(S, \Sigma; \mathbb{R}_{\mathrm{y}}\right). \tag{2}$$

We refer to the summands in this splitting as the *horizontal space* and *vertical space* respectively.

It can also be useful to identify the coefficients with $\mathbb{C}$ and consider $H^1(S, \Sigma; \mathbb{C})$. This is the most natural choice when we are considering Abelian differentials. An $\mathbb{R}$-*structure* on a complex vector space V is given by a choice of a real subspace $W \subset V$ so that $V = W \oplus \mathbf{i}W$. If V is equipped with an $\mathbb{R}$-structure we say that a complex subspace $V' \subset V$ is *defined over* $\mathbb{R}$ if $V' = W' \oplus \mathbf{i}W'$ for some real subspace $W' \subset W$. We give $H^1(S, \Sigma; \mathbb{C})$ the $\mathbb{R}$-structure corresponding to the real subspace $H^1(S, \Sigma; \mathbb{R}) = H^1(S, \Sigma; \mathbb{R}_{\mathrm{x}}) \subset H^1(S, \Sigma; \mathbb{C})$. In this language $\mathbf{i}H^1(S, \Sigma; \mathbb{R}) = H^1(S, \Sigma; \mathbb{R}_{\mathrm{y}})$.

If V has an $\mathbb{R}$-structure then an $\mathbb{R}$-*basis* for V is a basis for V as a complex vector space consisting of elements of W. We say that a linear equation on V has *real coefficients* if it is represented by a matrix with real entries when expressed with respect to an $\mathbb{R}$-basis.

Lemma 2.1. *Let $V \subset H^1(S, \Sigma; \mathbb{C})$ be a complex vector space. The following are equivalent:*

1. *V is invariant under the action of $\mathrm{GL}_2^+(\mathbb{R})$.*
2. *$V \subset H^1(S, \Sigma; \mathbb{C})$ is defined over $\mathbb{R}$.*
3. *V is the set of zeros of a collection of linear equations with real coefficients.*

Proof. The universal coefficients theorem identifies $H^1\left(S,\Sigma;\mathbb{R}^2\right)$ with

$$\mathrm{Hom}\left(H_1(S,\Sigma;\mathbb{R}),\mathbb{R}^2\right).$$

With respect to this identification $\mathrm{GL}_2^+(\mathbb{R})$ acts on the left by post-composition. Let

$$a \stackrel{\mathrm{def}}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b \stackrel{\mathrm{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and } c \stackrel{\mathrm{def}}{=} aba^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that multiplication by a corresponds to multiplication by $\mathbf{i}$, and multiplication by b and c correspond respectively to projections onto the two summands in (2).

We first prove the implication (2) $\implies$ (1). If V is a complex vector space then it is invariant under a, b and c. Since a, b, c generate the ring of 2×2 real matrices, V is $\mathrm{GL}_2^+(\mathbb{R})$-invariant.

It is clear that (3) implies (2), and we now prove that (1) implies (3). If $h : H_1(S,\Sigma;\mathbb{R}) \to \mathbb{R}^2$ is an element of $\mathrm{Hom}(H_1(S,\Sigma;\mathbb{R}),\mathbb{R}^2)$ and $L \in \mathrm{GL}_2^+(\mathbb{R})$, then h maps to $L \circ h$. In particular h and h' differ by an element of $\mathrm{GL}_2^+(\mathbb{R})$ exactly when they have the same kernel. Thus if V is $\mathrm{GL}_2^+(\mathbb{R})$ invariant it corresponds to a subspace $K \subset H_1(S,\Sigma;\mathbb{R}) \to \mathbb{R}^2$. If k_j is a basis for K then V is determined by the equations $v \cdot k_j = 0$ where the dot represents the Kronecker pairing. These equations have real coefficients. □

We have a restriction map $\mathrm{Res} : H^1(S,\Sigma;\mathbb{R}^2) \to H^1(S;\mathbb{R}^2)$ (given by restricting a cochain to closed paths). Since Res is topologically defined, its kernel ker(Res) is $\mathrm{Mod}(S,\Sigma)$-invariant. Moreover our convention that singular points are marked implies that the $\mathrm{Mod}(S,\Sigma)$-action on ker(Res) is trivial.

Define

$$Z = \ker(\mathrm{Res}) \cap H^1(S,\Sigma;\mathbb{R}_{\mathrm{x}}). \tag{3}$$

Since monodromy acts trivially on Z, for any $v \in Z$ the constant vector field on $H^1(S,\Sigma;\mathbb{R}^2)$ in direction v pulls back to a well-defined vector field on $\mathcal{H}_{\mathrm{m}}$ on $\mathcal{H}$ via the local diffeomorphism dev. Integrating this vector field gives a locally defined *real REL flow (corresponding to v)* $(t,q) \mapsto \mathrm{Rel}_{tv}(q)$. For every $q \in \mathcal{H}$ a trajectory is defined for $t \in I_q$, where the *domain of definition* $I_q = I_q(v)$ is an open interval of $\mathbb{R}$ which contains 0. This interval is all of $\mathbb{R}$ if the underlying surface M_q has no horizontal saddle connections. If $q \in \mathcal{H}$, $s \in \mathbb{R}$ and $t \in I_q$ then $t \in I_{u_s q}$, and $\mathrm{Rel}_{tv}(u_s q) = u_s \mathrm{Rel}_{tv}(q)$. The set

$$Z^{(q)} \stackrel{\mathrm{def}}{=} \{v \in Z : \mathrm{Rel}_v(q) \text{ is defined}\} = \{v \in Z : 1 \in I_q(v)\}, \tag{4}$$

as well as the sets $I_q(v)$, are explicitly described in [BSW22, Thm. 6.1].

2.2. **Invariant subvarieties.** In this subsection, we introduce our notion of *invariant subvarieties* and *irreducible invariant subvarieties*. It will be shown in [SY], using the work of Eskin-Mirzakhani [EM18] and Eskin-Mirzakhani-Mohammadi [EMM15], that an irreducible invariant subvariety is exactly a $\mathrm{GL}_2^+(\mathbb{R})$-orbit closure while an invariant subvariety is a finite union of such $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures.

Definition 2.2. *A d-*dimensional linear manifold *is a submanifold L of $\mathcal{H}_{\mathrm{m}}$ which is a connected component of* $\mathrm{dev}^{-1}(V)$ *where V is a d-dimensional complex subspace of $H^1\left(S,\Sigma;\mathbb{R}^2\right)$ defined over $\mathbb{R}$.*

Since the developing map is equivariant and $\mathrm{Mod}(S,\Sigma)$ acts linearly on the space $H^1\left(S,\Sigma;\mathbb{R}^2\right)$, it follows that $\mathrm{Mod}(S,\Sigma)$ takes a d-dimensional linear manifold to a d-dimensional linear manifold. If L is a linear manifold corresponding to $V_L \subset H^1\left(S,\Sigma;\mathbb{R}^2\right)$, we denote by Γ_L be the subgroup of $\mathrm{Mod}(S,\Sigma)$ that preserves L. Since the developing map dev is $\mathrm{Mod}(S,\Sigma)$-equivariant, we get an induced action of Γ_L on V_L. We say that L is an *equilinear manifold* if furthermore we have $\det\left(\gamma|_{V_L}\right) = \pm 1$ for every $\gamma \in \Gamma_L$.

Definition 2.3. *A d-dimensional invariant subvariety is a subset $\mathcal{M} \subset \mathcal{H}$ such that $\pi^{-1}(\mathcal{M})$ is a locally finite union of d-dimensional equilinear manifolds.*

We will write $d = \dim(\mathcal{M})$; in some texts this is referred to as the complex dimension of $\mathcal{M}$. The term "invariant" in the definition of invariant subvariety is justified by the following:

Proposition 2.4. *An invariant subvariety is closed and $\mathrm{GL}_2^+(\mathbb{R})$-invariant.*

Proof. Since V is a closed subset of $H^1\left(S,\Sigma;\mathbb{R}^2\right)$ it follows that $\mathrm{dev}^{-1}(V)$ is a closed subset of $\mathcal{H}_{\mathrm{m}}$. It follows that a linear manifold is a closed subset of $\mathcal{H}_{\mathrm{m}}$. The set $\pi^{-1}(\mathcal{M})$ is closed because it is a locally finite union of closed sets, and this implies that $\mathcal{M}$ is closed.

Since π is $\mathrm{GL}_2^+(\mathbb{R})$-equivariant, it is enough to prove that $\pi^{-1}(\mathcal{M})$ is $\mathrm{GL}_2^+(\mathbb{R})$-invariant. Let L be a linear submanifold contained in $\pi^{-1}(\mathcal{M})$ which maps to V_L under dev. By definition, V_L is defined over $\mathbb{R}$ and by Lemma 2.1 it is invariant under the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $H^1\left(S,\Sigma;\mathbb{R}^2\right)$. Since dev is $\mathrm{GL}_2^+(\mathbb{R})$-equivariant the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{H}_{\mathrm{m}}$ preserves $\mathrm{dev}^{-1}(V_L)$. Since $\mathrm{GL}_2^+(\mathbb{R})$ is connected, the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{H}_{\mathrm{m}}$ preserves L. Since $\pi^{-1}(\mathcal{M})$ is a union of linear submanifolds it follows that it is invariant under $\mathrm{GL}_2^+(\mathbb{R})$. □

Definition 2.5. *A d-dimensional invariant subvariety is said to be* irreducible *if it cannot be written as a union of two proper distinct d-dimensional invariant subvarieties.*

We have the following equivalent characterization:

Proposition 2.6. *Let $\mathcal{M}$ be a d-dimensional invariant subvariety. Then $\mathcal{M}$ is irreducible if and only if for any d-dimensional equilinear manifold $L \subset \pi^{-1}(\mathcal{M})$, we have*

$$\bigcup_{\gamma\in\mathrm{Mod}(S,\Sigma)} L\cdot\gamma = \pi^{-1}(\mathcal{M}). \tag{5}$$

For the proof of Proposition 2.6 we will need the following:

Lemma 2.7. *If L and L' are distinct d-dimensional linear submanifolds, then $\pi(L)\cap\pi(L')$ is a meager subset of $\pi(L)$ and of $\pi(L')$.*

Proof. We first show that $\pi^{-1}\left(\pi(L)\cap\pi(L')\right)$ is a countable union of sets of dimension less than d. We have:

$$\pi^{-1}(\pi(L)) = \bigcup_{\gamma\in\mathrm{Mod}(S,\Sigma)} L\cdot\gamma \quad\text{and}\quad \pi^{-1}(\pi(L')) = \bigcup_{\gamma\in\mathrm{Mod}(S,\Sigma)} L'\cdot\gamma.$$

Now consider an intersection $(L\cdot\gamma)\cap(L'\cdot\gamma')$. We have $\mathrm{dev}(L\cdot\gamma)\subset V$ and $\mathrm{dev}(L'\cdot\gamma)\subset V'$ for d-dimensional linear subspaces of $H^1(S,\Sigma;\mathbb{R}^2)$. If $V = V'$ then $L\cdot\gamma$ and $L'\cdot\gamma'$ are topological components of $\mathrm{dev}^{-1}(V)$ so they are either disjoint or equal. If $L \cdot \gamma = L' \cdot \gamma'$ then $L = L' \cdot \gamma' \gamma^{-1}$ so $\pi^{-1}(\pi(L)) = \pi^{-1}(\pi(L'))$ and $\pi(L) = \pi(L')$ contrary to assumption.

If $V \neq V'$ then $(L \cdot \gamma) \cap (L' \cdot \gamma') \subset \mathrm{dev}^{-1}(V \cap V')$. This is a complex subspace of positive codimension so its inverse image is a nowhere dense subset of the d-dimensional manifolds $L \cdot \gamma$ and $L' \cdot \gamma'$. Thus $\pi^{-1}(\pi(L) \cap \pi(L'))$ is a meager subset of $\pi^{-1}(\pi(L))$ and $\pi^{-1}(\pi(L'))$. Since π is open and the intersections $(L \cdot \gamma) \cap (L' \cdot \gamma')$ are closed, the projection $\pi(L) \cap \pi(L')$ is a meager subset of $\pi(L)$ and $\pi(L')$. □

Proof of Proposition 2.6. Say that $\mathcal{M}$ is irreducible and let L be a d-dimensional equilinear manifold in $\pi^{-1}(\mathcal{M})$. If (5) does not hold, we can write $\pi^{-1}(\mathcal{M})$ as a countable union of orbits of distinct linear submanifolds $L_1, L_2, \ldots$, as

$$\pi^{-1}(\mathcal{M}) = \bigcup_{\ell} \bigcup_{\gamma \in \mathrm{Mod}(S,\Sigma)} L_\ell \cdot \gamma,$$

where $L = L_1$ and the list $\{L_i\}$ contains more than one element. We have

$$\mathcal{M} = \bigcup_{\ell} \pi(L_\ell).$$

We define

$$A \stackrel{\mathrm{def}}{=} \pi(L_1) \text{ and } B \stackrel{\mathrm{def}}{=} \bigcup_{1<\ell} \pi(L_\ell).$$

Since $\mathcal{M}$ is irreducible, and since we have assumed that (5) fails, we have $\mathcal{M} = B$. This implies $A \subset B$, and hence $\pi(L_1) = \bigcup_\ell \pi(L_1) \cap \pi(L_\ell)$. According to Lemma 2.7 $\pi(L_1) \cap \pi(L_\ell)$ is a meager subset of $\pi(L_1)$ so our decomposition of $\pi(L_1)$ expresses $\pi(L_1)$ as a meager set and violates the Baire category theorem. We conclude that $\mathcal{M} = A$ which is what we wanted to show.

Now assume that for any d-dimensional equilinear manifold $L \subset \pi^{-1}(\mathcal{M})$ we have (5). Suppose we have a decomposition $\mathcal{M} = A \cup B$ where

$$A = \bigcup_j \pi(L_j) \text{ and } B = \bigcup_k \pi(L'_k),$$

where both collections $\{L_i\}$, $\{L'_k\}$ are $\mathrm{Mod}(S,\Sigma)$-invariant and comprised of d-dimensional equilinear manifolds. By (5), $\pi^{-1}(A)$ and $\pi^{-1}(B)$ are either empty or equal to $\pi^{-1}(\mathcal{M})$. Thus A and B are not proper subsets of $\mathcal{M}$. □

It follows from Proposition 2.6 that if $\mathcal{M}$ is a d-dimensional irreducible invariant subvariety and L is a d-dimensional equilinear manifold contained in $\pi^{-1}(\mathcal{M})$, then $\pi(L) = \mathcal{M}$. This motivates the following definition that will be used throughout the text

Definition 2.8. *Let $\mathcal{M}$ be a d-dimensional irreducible invariant subvariety. A* lift *of $\mathcal{M}$ is a d-dimensional equilinear manifold $L \subset \pi^{-1}(\mathcal{M})$.*

The following result establishes the link between $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures and invariant subvarieties. In the forthcoming [SY], it will be deduced from the results of [EMM15, EM18].

Theorem 2.9. *Irreducible invariant subvarieties and $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures coincide. Furthermore, any invariant subvariety is a finite union of irreducible invariant subvarieties.*

Standing assumption: all the invariant subvarieties we will consider from now on will be assumed to be irreducible.

Let $\mathcal{M}$ be a d-dimensional invariant subvariety. We conclude this section by constructing a Radon measure supported on $\mathcal{M}$ which will be uniquely defined up to a multiplicative constant. This will require some constructions which are summarized in Appendix B. Let L be a lift of $\mathcal{M}$, let $V_L = \mathrm{dev}(L)$ and let Γ_L be the stabilizer in $\mathrm{Mod}(S,\Sigma)$ of L. Let α be a volume form on L that is obtained as the pullback by dev of an element of the top degree exterior power of V_L. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts smoothly on $\mathcal{H}_{\mathrm{m}}$. Denoting by g^* the pull-back operator on differential forms corresponding to the action of $g \in \mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{H}_{\mathrm{m}}$, we have

$$\forall g \in \mathrm{GL}_2^+(\mathbb{R}), \quad g^*\alpha = (\det g)^d \ \alpha. \tag{6}$$

The volume form α defines a measure on L that we denote by μ_L. Since L is an equilinear manifold, the measure μ_L is Γ_L-invariant. Furthermore, since $\mathrm{Mod}(S,\Sigma)$ acts transitively on the set of irreducible components, it can be arranged that for any $\gamma \in \mathrm{Mod}(S,\Sigma)$, $\gamma_* m_L = m_{L\cdot\gamma}$. This means that the sum

$$\tilde{\mu}_{\mathcal{M}} = \sum_{L \subset \pi^{-1}(\mathcal{M})} \mu_L,$$

where the sum ranges over the lifts of $\mathcal{M}$, is a $\mathrm{Mod}(S,\Sigma)$-invariant measure on $\mathcal{H}_{\mathrm{m}}$. The measure $\tilde{\mu}_{\mathcal{M}}$ is a Radon measure, which follows from the fact that the collection of irreducible components is locally finite. Using Proposition B.3, there is a unique Radon measure $\mu_{\mathcal{M}}$ on $\mathcal{H}_{\mathrm{m}}$ such that for any $f \in C_c(\mathcal{H}_{\mathrm{m}})$, we have

$$\int_{\mathcal{H}_{\mathrm{m}}} f \ d\tilde{\mu}_{\mathcal{M}} = \int_{\mathcal{H}} \left(\int_{\mathcal{H}_{\mathrm{m}}} f \ d\theta_q \right) \ d\mu_{\mathcal{M}}(q),$$

where

$$\theta_q \stackrel{\mathrm{def}}{=} \sum_{\mathbf{q} \in \pi^{-1}(q)} N(\mathbf{q}) \cdot \delta_{\mathbf{q}}, \quad N(\mathbf{q}) \stackrel{\mathrm{def}}{=} |\{\gamma \in \mathrm{Mod}(S,\Sigma) : \mathbf{q} = \mathbf{q} \cdot \gamma\}|$$

(as in equation (45)). The measure $\mu_{\mathcal{M}}$ is supported on $\mathcal{M}$. It follows from Lemma B.2 that it is $\mathrm{SL}_2(\mathbb{R})$-invariant. We call it the *linear measure on* $\mathcal{M}$. Notice that this is a slight abuse of language as $\mu_{\mathcal{M}}$ is only determined up to a multiplicative constant.

2.3. **Area one locus, cone construction, and special linear measures.** Let $\boldsymbol{q} \in \mathcal{H}_{\mathrm{m}}$, let $q = \pi(\boldsymbol{q})$, and let $M = M_q$ be the underlying translation surface. The area of M can be expressed using period coordinates as follows. We define a Hermitian form on $H^1(S,\Sigma;\mathbb{C})$ by

$$(\alpha, \beta) = \frac{1}{2\mathbf{i}} \int_S \alpha \wedge \bar{\beta}. \tag{7}$$

(See [BSW22, §2.5] for a topological interpretation of equation (7).) The area of M is then given by $(\mathrm{dev}(\boldsymbol{q}), \mathrm{dev}(\boldsymbol{q}))$. This is thus a quadratic formula in period coordinates.

For the purposes of this paper we will use a related *real valued* bracket $\langle \alpha, \beta \rangle$ involving the pairing of horizontal and vertical classes. Say that on a marked surface M_q we have a 1-form α corresponding to an element of $H^1(S,\Sigma;\mathbb{R}_{\mathrm{x}})$ (a horizontal form) and a 1-form β corresponding to an element of $H^1(S,\Sigma;\mathbb{R}_{\mathrm{y}})$ (a vertical form). Then

$$\langle \alpha, \beta\rangle = \int_S \alpha\wedge\beta$$

and this gives

$$\mathrm{area}(M_q) = \langle dx_q, dy_q\rangle = \int_S dx_q\wedge dy_q. \tag{8}$$

We denote the subset of surfaces in $\mathcal{H}_{\mathrm{m}}$ and $\mathcal{H}$ of area one by $\mathcal{H}_{\mathrm{m}}^{(1)}$ and $\mathcal{H}^{(1)}$. More generally, when $\mathcal{M}$ is an invariant subvariety and L is a lift of $\mathcal{M}$, we also denote by $\mathcal{M}^{(1)}$ and $L^{(1)}$ their intersection with the area-one locus. The latter are G-invariant and invariant under real REL flows (where defined).

We recall that there is a *rescaling action* of $\mathbb{R}_+^*$ on $\mathcal{H}$ that corresponds to the action of the subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ of scalar matrices with positive coefficients. We consider the *cone measure* $m_{\mathcal{M}}$ on $\mathcal{M}^{(1)}$ defined for any Borel subset $A\subset\mathcal{M}^{(1)}$ by

$$m_{\mathcal{M}}(A) \stackrel{\mathrm{def}}{=} \mu_{\mathcal{M}}(\mathrm{cone}(A)), \quad \text{where } \mathrm{cone}(A) \stackrel{\mathrm{def}}{=} \{t\cdot a : t\in(0,1],\ a\in A\}. \tag{9}$$

When $\mathcal{M}$ is the whole stratum $\mathcal{H}$, the measure $m_{\mathcal{H}}$ is called the *Masur-Veech measure*. More generally, we shall call the measure $m_{\mathcal{M}}$ the *special flat measure* on $\mathcal{M}$. If L is a lift of $\mathcal{M}$, we can perform the same cone construction with the measure μ_L and we denote by m_L the corresponding measure. Let $\tilde{m}_{\mathcal{M}}$ be the pre-image of $m_{\mathcal{M}}$ under π, that is the unique measure on $\mathcal{H}_{\mathrm{m}}$ such that for any $f\in C_c(\mathcal{H}_{\mathrm{m}})$,

$$\int_{\pi^{-1}(\mathcal{M})} f\ d\tilde{m}_{\mathcal{M}} = \int_{\mathcal{M}}\left(\int_{\tilde{X}} f\ d\theta_q\right) dm_{\mathcal{M}}(q) \tag{10}$$

(see Definition B.1). It is easily verified that

$$\tilde{m}_{\mathcal{M}} = \sum_{L\subset\pi^{-1}(\mathcal{M})} m_L. \tag{11}$$

2.4. **The sup-norm Finsler metric.** We now recall the sup-norm Finsler metric on $\mathcal{H}_{\mathrm{m}}$. This structure was studied by Avila, Gouëzel and Yoccoz, for proofs and more details see [AGY06] and [AG10]. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$. For a translation surface q, denote by Λ_q the collection of saddle connections on M_q and let $\ell_q(\sigma) = \|\mathrm{hol}_q(\sigma)\|$ be the length of $\sigma\in\Lambda_q$. For $\beta\in H^1(M_q,\Sigma_q;\mathbb{R}^2)$ we set

$$\|\beta\|_q \stackrel{\mathrm{def}}{=} \sup_{\sigma\in\Lambda_q}\frac{\|\beta(\sigma)\|}{\ell_q(\sigma)}. \tag{12}$$

We now define a Finsler metric for $\mathcal{H}_{\mathrm{m}}$. Let $f : S\to M_q$ be a marking map representing a marked surface $\boldsymbol{q}\in\mathcal{H}_{\mathrm{m}}$. Using period coordinates we can identify the tangent space to $\mathcal{H}_{\mathrm{m}}$ at $\boldsymbol{q}$ with $H^1(S,\Sigma;\mathbb{R}^2)$. Then

$$\|\beta\|_{\boldsymbol{q}} \stackrel{\mathrm{def}}{=} \sup_{\tau\in\Lambda_{\boldsymbol{q}}}\frac{\|\beta(f(\tau))\|}{\ell_q(f(\tau))} \tag{13}$$

is a norm on $H^1(S,\Sigma;\mathbb{R}^2)$. It satisfies the equivariance property

$$\forall h\in\mathrm{Mod}(S,\Sigma), \quad \|\beta\|_{\boldsymbol{q}} = \|h^*\beta\|_{\boldsymbol{q}.h}, \tag{14}$$

where $\boldsymbol{q}.h$ is represented by the marking map $f\circ h$. The map

$$T(\mathcal{H}_{\mathrm{m}})\to\mathbb{R}, \qquad (\boldsymbol{q},\beta)\mapsto\|\beta\|_{\boldsymbol{q}}$$

is continuous. The Finsler metric defines a distance function[1] on $\mathcal{H}_{\mathrm{m}}$ which we call the *sup-norm distance* and define as follows:

$$\mathrm{dist}(\boldsymbol{q}_0, \boldsymbol{q}_1) \stackrel{\mathrm{def}}{=} \inf_{\gamma} \int_0^1 \|\gamma'(\tau)\|_{\gamma(\tau)} d\tau, \tag{15}$$

where γ ranges over smooth paths $\gamma : [0,1] \to \mathcal{H}$ with $\gamma(0) = \boldsymbol{q}_0$ and $\gamma(1) = \boldsymbol{q}_1$. The topology induced by the sup norm distance on $\mathcal{H}_{\mathrm{m}}$ is the one induced by period coordinates, and the resulting metric space is proper and complete. We can use the distance function on $\mathcal{H}_{\mathrm{m}}$ to define a distance function on $\mathcal{H}$ by

$$\mathrm{dist}(q_0, q_1) = \inf\{\mathrm{dist}(\boldsymbol{q}_0, \boldsymbol{q}_1) : \boldsymbol{q}_i \in \pi^{-1}(q_i),\ i = 0, 1\}.$$

3. Horospherical measures

Let $\mathcal{M}$ be an invariant subvariety of dimension n. The goal of this section is to define the horospherical foliation on $\mathcal{M}$ and the related horospherical measures, which are our object of study in this paper. These objects will be defined via their counterparts for the irreducible components of $\pi^{-1}(\mathcal{M})$.

3.1. **Boxes.** We now define a notion of *boxes.* They will be used throughout the text and will play two roles: boxes give local coordinates on invariant subvarieties (more precisely, on the irreducible components of their pre-image by π) that are convenient for the study of horospherical measures; additionally, they will be used in a mixing argument in the proof of Theorem 1.2.

From now on, we identify $H^1(S, \Sigma, \mathbb{C})$ with $H^1(S, \Sigma, \mathbb{R}^2)$ as in §2.1. Let $V \subset H^1(S, \Sigma, \mathbb{C})$ be a complex linear subspace defined over $\mathbb{R}$. We have

$$V = V_{\mathrm{x}} \oplus V_{\mathrm{y}}, \tag{16}$$

where

$$V_{\mathrm{x}} \stackrel{\mathrm{def}}{=} V \cap H^1(S, \Sigma; \mathbb{R}_{\mathrm{x}}) \quad \text{and} \quad V_{\mathrm{y}} \stackrel{\mathrm{def}}{=} V \cap H^1(S, \Sigma; \mathbb{R}_{\mathrm{y}})$$

are identified by the isomorphism $V_{\mathrm{x}} \ni v \mapsto \mathbf{i}v \in V_{\mathrm{y}}$. We define

$$V^{(1)} \stackrel{\mathrm{def}}{=} \{(x, y) \in V : x \in V_{\mathrm{x}},\ y \in V_{\mathrm{y}},\ \langle x, y\rangle = 1\},$$

and denote by

$$\pi_{\mathrm{x}} : V \to V_{\mathrm{x}}, \qquad \pi_{\mathrm{y}} : V \to V_{\mathrm{y}} \tag{17}$$

the projections corresponding to the direct sum decomposition (16), and by π'_{x} the projection from $\pi_{\mathrm{x}}^{-1}(V_{\mathrm{x}} \smallsetminus \{0\})$ to the projective space $\mathbf{P}(V_{\mathrm{x}})$. Finally let

$$\Psi : V^{(1)} \to \mathbf{P}(V_{\mathrm{x}}) \times V_{\mathrm{y}}, \qquad \Psi(q) = (\pi'_{\mathrm{x}}(q), \pi_{\mathrm{y}}(q)). \tag{18}$$

Lemma 3.1. *The map Ψ is a local diffeomorphism.*

Proof. Say that $(x_0, y_0) \in V^{(1)}$ is mapped by Ψ to $(\bar{x}_0, y_0)$ in $\mathbf{P}(V_{\mathrm{x}}) \times V_{\mathrm{y}}$. We will construct a local inverse. Since $\langle x_0, y_0\rangle = 1$ we can find neighborhoods U_{x} of x_0 in V_{x} and U_{y} of y_0 in V_{y} so that $\langle x, y\rangle > 0$ for $x \in U_{\mathrm{x}}$ and $y \in U_{\mathrm{y}}$. We define maps

$$\tilde{\psi} : U_{\mathrm{x}} \times U_{\mathrm{y}} \to V^{(1)}, \qquad \tilde{\psi}(x, y) = \left(\frac{x}{\langle x, y\rangle}, y\right) \tag{19}$$

and

$$\psi : U'_{\mathrm{x}} \times U_{\mathrm{y}} \to V^{(1)}, \qquad \psi([x], y) \stackrel{\mathrm{def}}{=} \tilde{\psi}(x, y), \qquad \text{where } U'_{\mathrm{x}} \stackrel{\mathrm{def}}{=} \pi'_{\mathrm{x}}(U_{\mathrm{x}}). \tag{20}$$

[1] In order to avoid confusion we use 'distance function' to refer to what is often called a metric.

The map $\tilde{\psi}$ is smooth and descends in a well-defined way to define ψ. We see that $\Psi \circ \psi$ is the identity map, i.e., ψ is a local inverse of Ψ. □

Definition 3.2 (Boxes)**.** *Let L be a lift of $\mathcal{M}$ and let $V = \mathrm{dev}(L)$. A* box in L *is a relatively compact subset $\boldsymbol{B} \subset L^{(1)}$ together with a diffeomorphism $\varphi : U'_{\mathrm{x}} \times U_{\mathrm{y}} \to \boldsymbol{B}$ such that, in the notations above,*

- *U'_{x} and U_{y} are open sets in V_{y} and $\mathbf{P}(V_{\mathrm{x}})$ respectively.*
- *$\Psi \circ \mathrm{dev} \circ \varphi = \mathrm{Id}$.*

For $y \in U_{\mathrm{y}}$, the plaque of y in $\boldsymbol{B}$ *is the set $\boldsymbol{L}_y \stackrel{\mathrm{def}}{=} \varphi(U'_{\mathrm{x}} \times \{y\})$.*

The composition in the second item in Definition 3.2 makes sense since $\mathrm{dev}(L^{(1)}) \subset V^{(1)}$, in light of equation (8). It should be understood as a choice of a suitable parameterization for boxes. Note that the data $\varphi, U'_{\mathrm{x}} \times U_{\mathrm{y}}$ are implicit in the notion of a box, but in order to avoid excessive notation we simply write $\boldsymbol{B}$.

More generally, a *box in* $\pi^{-1}(\mathcal{M})$ is a box in one of the irreducible components of $\pi^{-1}(\mathcal{M})$. Such a box $\boldsymbol{B}$ will be called *regular* if for any $\gamma \in \mathrm{Mod}(S, \Sigma)$ either $\boldsymbol{B} \cdot \gamma \cap \boldsymbol{B} = \emptyset$ or $\gamma \in \Gamma$, where Γ is the stabilizer in $\mathrm{Mod}(S, \Sigma)$ of $\boldsymbol{B}$ (*i.e.* the set of $\gamma \in \mathrm{Mod}(S, \Sigma)$ such that $\boldsymbol{B} \cdot \gamma = \boldsymbol{B}$). When $\boldsymbol{B}$ is regular, the map π induces a homeomorphism $\boldsymbol{B}/\Gamma \to \pi(\boldsymbol{B})$. In particular the image of a regular box by π is an open subset of $\mathcal{M}$. Since $\mathrm{Mod}(S, \Sigma)$ acts diagonally on $\mathbf{P}(H^1(S, \Sigma, \mathbb{R}_{\mathrm{x}})) \times H^1(S, \Sigma, \mathbb{R}_{\mathrm{y}})$, the set of boxes is preserved by the action of $\mathrm{Mod}(S, \Sigma)$. Furthermore, a finite intersection of boxes is a box. Thus, by Lemma 3.1, for every $\boldsymbol{q} \in \pi^{-1}(\mathcal{M})$, there is a regular box in $\pi^{-1}(\mathcal{M})$ containing $\boldsymbol{q}$.

Remark 3.3. *There is an asymmetry in the definition of a box; we could equally well define a box using V_{x} and $\mathbf{P}(V_{\mathrm{y}})$, but we will make no use of that kind of box.*

3.2. **Definition of the horospherical foliation.** Recall that a smooth map of manifolds is a submersion if its derivative is of full rank at every point. The implicit function theorem implies that the connected components of the fibers of a submersion are the leaves of a foliation.

Definition 3.4. *Let L be a lift of $\mathcal{M}$ and let V be the linear space on which L is modeled. The foliations on $L^{(1)}$ induced by the submersions*

$$\pi'_{\mathrm{x}} \circ \mathrm{dev} : L^{(1)} \to \mathbf{P}(V_{\mathrm{x}}) \quad \text{and} \quad \pi_{\mathrm{y}} \circ \mathrm{dev} : L^{(1)} \to V_{\mathrm{y}},$$

are called the weak stable *and* strong unstable *foliations. They are denoted respectively by W^s_L and W^{uu}_L. The leaf of $\boldsymbol{q} \in L^{(1)}$ for the weak stable foliation is denoted by $W^s_L(\boldsymbol{q})$ and the leaf of $\boldsymbol{q}$ for the strong unstable foliation is denoted by $W^{uu}_L(\boldsymbol{q})$.*

It follows from Lemma 3.1 that these foliations are well-defined, and the leaves of these foliations are everywhere transverse.

Lemma 3.5. *The action of* $\mathrm{Mod}(S, \Sigma)$ *permutes the leaves of W^{uu}_L. For any leaf F, the restriction* $\mathrm{dev}|_F$ *is a local homeomorphism to an affine subspace of V and with respect to this affine structure, the subgroup $\Gamma_L \stackrel{\mathrm{def}}{=} \{\gamma \in \mathrm{Mod}(S, \Sigma) : L \cdot \gamma = L\}$ acts on the leaves of W^{uu}_L by affine maps.*

Proof. The monodromy preserves the product splitting $V = V_{\mathrm{x}} \oplus V_{\mathrm{y}}$ and acts linearly on each factor. Thus the monodromy acts projectively on $\mathbf{P}(V_{\mathrm{x}})$. Since dev is monodromy equivariant, the leaves of the foliations W^s_L and W^{uu}_L are permuted by the action of $\mathrm{Mod}(S, \Sigma)$.

For the second assertion, it is clear from the definitions that dev maps the leaf F to a set of the form $\{(x, y_0) \in V : x \in V_{\mathrm{x}} \text{ and } \langle x, y_0 \rangle = 1\}$ for some fixed $y_0 \in V_{\mathrm{y}}$, and by Lemma 3.1, the map $\mathrm{dev}|_F$ is a local diffeomorphism. The last assertion follows from the $\mathrm{Mod}(S, \Sigma)$-equivariance of dev and the fact that $\mathrm{Mod}(S, \Sigma)$ preserves the bracket $\langle \cdot, \cdot \rangle$. □

Remark 3.6. *Lemma 3.5 equips the leaves of the foliation W^{uu} with an affine manifold structure. This structure need not be geodesically complete. Using real Rel deformations, one easily constructs affine geodesics in a leaf $W^{uu}_L(\boldsymbol{q})$ which contain a surface with a horizontal saddle connection whose length goes to zero as one moves along the leaf. There are additional sources of non-completeness involving surfaces whose horizontal foliation is minimal but not uniquely ergodic, see* [MW14]*. Furthermore, using* [MW14, Thm. 1.2]*, one can show that each leaf $W^{uu}_L(\boldsymbol{q})$ is mapped by the developing map homeomorphically to an explicitly described convex domain in $H^1(S, \Sigma; \mathbb{R}_{\mathrm{x}})$, defined by finitely many linear inequalities.*

It follows from Lemma 3.5 that the partition of $L^{(1)}$ given by the leaves W^{uu}_L induces a partition of $\mathcal{M}^{(1)}$. We denote it by W^{uu} and if $q \in \mathcal{M}$, we denote by $W^{uu}(q)$ the element of the partition that contains q. We emphasize that W^{uu} does not depend on the choice of a particular irreducible component used to define it. This is a consequence of the fact that $\mathrm{Mod}(S, \Sigma)$ acts on $H^1(S, \Sigma, \mathbb{C})$ by real endomorphisms and thus preserves the splitting into real and imaginary parts of cohomology classes.

Definition 3.7. *A* horosphere *is an element of the partition of W^{uu}.*

Remark 3.8. *Occasionally, we may call the partition W^{uu} the* horospherical foliation *of $\mathcal{M}$, even though $\mathcal{M}$ is generally not a manifold. Even if this will play no role in the rest of the paper, we justify this choice of terminology for the sake of completeness: the invariant subvariety $\mathcal{M}$ can be seen to have the structure of a properly immersed manifold $\mathcal{M}$, i.e., is the image of a manifold $\mathcal{N}$ under a proper orbifold immersion $f : \mathcal{N} \to \mathcal{H}$ and there is a foliation on $\mathcal{N}$ whose leaves are sent to horospheres by f. We can choose $\mathcal{N}$ to be the quotient of L by a finite-index torsion-free normal subgroup Γ_0 of $\mathrm{Mod}(S, \Sigma)$ and $f : L/\Gamma_0 \to \mathcal{H}$, $\boldsymbol{q}\Gamma_0 \mapsto \pi(\boldsymbol{q})$. By Lemma 3.5, the horospherical foliation on L descends to a foliation on the manifold L/Γ_0. The leaves of this foliation are indeed mapped to horospheres and f is an orbifold immersion. The fact that it is proper follows from the fact that the collection of irreducible components of $\pi^{-1}(\mathcal{M})$ is locally finite.*

Reversing the roles of π_{x} and π_{y}, and defining π'_{y} in an analogous fashion, we also define the *strong stable* and *unstable* foliations W^{ss}_L and W^u_L as those induced by the submersion $\pi_{\mathrm{x}} \circ \mathrm{dev}$, $\pi'_{\mathrm{y}} \circ \mathrm{dev}$ respectively. Lemma 3.5 holds for these foliations as well, with obvious modifications. Summarizing: for every $\boldsymbol{q} \in L$ we have

$$W^{ss}_L(\boldsymbol{q}) \subset W^s_L(\boldsymbol{q}), \quad W^{uu}_L(\boldsymbol{q}) \subset W^u_L(\boldsymbol{q}),$$

the leaves $W^{ss}_L(\boldsymbol{q})$ and $W^{uu}_L(\boldsymbol{q})$ have a natural affine structure and, for $n = \dim(\mathcal{M})$, we have

$$\dim W^{ss}_L(\boldsymbol{q}) = \dim W^{uu}_L(\boldsymbol{q}) = n - 1, \qquad \dim W^s_L(\boldsymbol{q}) = \dim W^u_L(q) = n.$$

As we saw in §2.4, the sup-norm Finsler metric induces a distance function on $\mathcal{H}_{\mathrm{m}}$ as a path metric. We will induce distance functions on leaves of the stable and strong stable foliations using the same approach. For $\boldsymbol{q}_0, \boldsymbol{q}_1 \in \mathcal{H}_{\mathrm{m}}$ belonging to the same stable (respectively, strong stable) leaf, we define $\mathrm{dist}^{(s)}(\boldsymbol{q}_0, \boldsymbol{q}_1)$ (respectively, $\mathrm{dist}^{(ss)}(\boldsymbol{q}_0, \boldsymbol{q}_1)$) by the formula in equation (15), but making the additional requirement that the entire path γ is contained in the stable (respectively strong stable) leaf of the $\boldsymbol{q}_i$.

We similarly define $\mathrm{dist}^{(s)}(q_0, q_1)$ and $\mathrm{dist}^{(ss)}(q_0, q_1)$ for $q_0, q_1 \in \mathcal{H}$ belonging to the same stable (respectively, strong stable) leaf. We will call the distance functions $\mathrm{dist}^{(s)}$, $\mathrm{dist}^{(ss)}$ the *stable (resp. strong stable) sup-norm distance function.*

These distance functions have the following properties:

Proposition 3.9. *Let L a lift of $\mathcal{M}$ and let $\boldsymbol{q}_0, \boldsymbol{q}_1 \in L$.*

(1) If $\boldsymbol{q}_0, \boldsymbol{q}_1$ are in the same stable (resp., strong stable leaf) leaf then $\mathrm{dist}(\boldsymbol{q}_0, \boldsymbol{q}_1) \leq \mathrm{dist}^{(s)}(\boldsymbol{q}_0, \boldsymbol{q}_1)$ (resp., $\mathrm{dist}(\boldsymbol{q}_0, \boldsymbol{q}_1) \leq \mathrm{dist}^{(ss)}(\boldsymbol{q}_0, \boldsymbol{q}_1)$).

(2) If $\boldsymbol{q}_0, \boldsymbol{q}_1$ are in the same strong stable leaf then for all $t \geq 0$,

$$\mathrm{dist}^{(ss)}(g_t\boldsymbol{q}_0, g_t\boldsymbol{q}_1) \leq \mathrm{dist}^{(ss)}(\boldsymbol{q}_0, \boldsymbol{q}_1).$$

And the same holds for the strong unstable leaf.

(3) If $\boldsymbol{q}_1 = g_t\boldsymbol{q}_0$ for some $t \in \mathbb{R}$ then $\mathrm{dist}^{(s)}(\boldsymbol{q}_0, \boldsymbol{q}_1) \leq |t|$.

(4) Statements (1), (2) *and* (3) *also hold in $\mathcal{H}$, for q_0, q_1 in place of $\boldsymbol{q}_0, \boldsymbol{q}_1$.*

Proof. Assertion (1) is obvious from definitions, and assertions (2) and (3) are proved in [AG10, §5] (where what we call the strong stable foliation is referred to as the stable foliation). The assertions for $\mathcal{H}$ follows from the corresponding ones for $\mathcal{H}_{\mathrm{m}}$. $\square$

Remark 3.10. *Almost everywhere, the horospheres $W^{uu}(q)$ and $W^{ss}(q)$ are actually the unstable and stable manifolds of the geodesic flow. That is, for any q, and almost every (with respect to the measure class induced by the affine structure on leaves) $q_1 \in W^{uu}(q)$, $q_2 \in W^{ss}(q)$, we have*

$$\mathrm{dist}(g_tq, g_tq_1) \underset{t\to-\infty}{\longrightarrow} 0 \quad \textit{and} \quad \mathrm{dist}(g_tq, g_tq_2) \underset{t\to\infty}{\longrightarrow} 0.$$

This is proved in [Vee86] *(see also* [EMM22]*) for $\mathcal{M} = \mathcal{H}^{(1)}$. The same result for general invariant subvarieties can be proved by adapting the arguments used in* [EMM22]*.*

3.3. **Definition of horospherical measures.** Let L be a lift of $\mathcal{M}$ as in Subsection 2.2 and let $V \subset H^1(S, \Sigma, \mathbb{C})$ be the subspace on which L is modeled. We write $V = V_{\mathrm{x}} \oplus V_{\mathrm{y}}$ as in equation (2). Let η_{x} and η_{y} be the translation invariant volume forms on V_{x} and V_{y} determined by a choice of an element of the top degree wedge power of V_{x} and V_{y}. Define

$$\alpha_{\mathrm{x}} \stackrel{\mathrm{def}}{=} (\pi_{\mathrm{x}} \circ \mathrm{dev})^*(\eta_{\mathrm{x}}), \qquad \alpha_{\mathrm{y}} \stackrel{\mathrm{def}}{=} (\pi_{\mathrm{y}} \circ \mathrm{dev})^*(\eta_{\mathrm{y}}). \tag{21}$$

We recall that the measure μ_L on L was defined in Section 2.2 as the integral of a volume form α. From now on, this form will be chosen so that $\alpha = \alpha_{\mathrm{x}} \wedge \alpha_{\mathrm{y}}$. We define the *Euler vector field* E on $\mathcal{H}_{\mathrm{m}}$ such that for any $\boldsymbol{q} \in \mathcal{H}_{\mathrm{m}}$,

$$E(q) = E_x(\boldsymbol{q}) \stackrel{\mathrm{def}}{=} \frac{\partial}{\partial t}\Big|_{t=0} \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \cdot \boldsymbol{q}.$$

This vector field can be thought of as the tangent vector to the rescaling action, which justifies our choice of terminology. Notice furthermore that the image of E by dev is the usual Euler vector field $e(v) = v$ on $H^1(S, \Sigma; \mathbb{C})$. This is due to the fact that dev is $\mathrm{GL}_2^+(\mathbb{R})$-equivariant. Since L is a linear manifold, the vector field E is tangent to it. We use this to define the form

$$\beta_{\mathrm{x}} \stackrel{\mathrm{def}}{=} \iota_E \alpha_{\mathrm{x}},$$

i.e., the contraction of α_{x} by the Euler field E. The restriction of β_{x} to the leaves of W_L^{uu} induces a volume form. We denote by $\nu_{\beta_{\mathrm{x}}}$ the induced measures. We emphasize that this defines a system of measures, one on each leaf $W_L^{uu}(\boldsymbol{q})$, so one should write $\nu^L_{\beta_{\mathrm{x}},\boldsymbol{q}}$ instead of $\nu_{\beta_{\mathrm{x}}}$; we omit this in our notation. We say that a measure m on L is *horospherical* if it is supported on $L^{(1)}$ and its conditional measures on the leaves of W_L^{uu} are given by the measures $\nu_{\beta_{\mathrm{x}}}$. More precisely, this means that for any box $\boldsymbol{B}$ in L, there is a measure λ on U_{y} such that for any compactly supported continuous function $f : L \to \mathbb{R}$,

$$\int_{\boldsymbol{B}} f \; d\nu = \int_{U_{\mathrm{y}}} \left(\int_{\boldsymbol{L}_y} f \, d\nu_{\beta_{\mathrm{x}}} \right) \, d\lambda(y). \tag{22}$$

Remark 3.11. *The measure λ is a so-called 'transverse measure' for the horospherical foliation. This means it is a system of measures on sets tranverse to the foliation which is invariant under holonomy along leaves, see* [CC03, Vol. 1, 10.1.13 & 11.5.2]. *According to the theory of transverse measures equation* (22) *yields as a bijection between horospherical measures and transverse measures. We will not be using this point of view in this paper.*

Remark 3.12. *Let ϕ_t be a smooth flow acting on L. A measure ν is said to be invariant if for any $t \in \mathbb{R}$ we have $(\phi_t)_*\nu = \nu$. This definition is equivalent to requiring that the conditional measures of ν on the orbits of ϕ_t be multiples of the Lebesgue measure dt, i.e., invariant under the maps $\phi_s x \mapsto \phi_{t+s} x$ for any fixed t. The equivalence can be shown by disintegrating ν on flow boxes, i.e., boxes whose horizontal plaques are pieces of ϕ_t-orbits. By Lemma 3.5, leaves of W_L^{uu} are modeled on linear subspaces, and thus one could try and define horospherical measures as those that are invariant under translation along the leaves. However, these translations are not part of a globally defined group action; for instance trajectories might escape to infinity in finite time. Our definition of horospherical measures is inspired by the second characterization of invariant measures, where the foliation by orbits of ϕ_t is replaced by the strong unstable foliation and the translation invariant measure dt is replaced by $\nu_{\beta_{\mathrm{x}}}$.*

In order to define a notion of horospherical measures on $\mathcal{M}$, we first need some terminology: let ν be a Radon measure on $\mathcal{M}$ and let $\tilde{\nu}$ be its pre-image by π as in equation (10) (see also Appendix B). By construction, the measure $\tilde{\nu}$ is supported on $\pi^{-1}(\mathcal{M})$. If L is a lift of $\mathcal{M}$, then the restriction of the measure $\tilde{\nu}$ to L is called *the lift of ν* corresponding to L. More generally, a *lift* of ν is a measure of the form $\tilde{\nu}|_L$ where L is any lift of $\mathcal{M}$. For instance, the measures m_L in equation (11) are the lifts of $m_{\mathcal{M}}$.

Definition 3.13 (Horospherical measure)**.** *A Radon measure ν on $\mathcal{M}$ is* horospherical *if its lifts are horospherical.*

By Proposition 2.6, it is enough that one of the lifts is horospherical, as the action of $\mathrm{Mod}(S, \Sigma)$ preserves the set of horospherical measures on $\mathcal{H}_{\mathrm{m}}$. By definition, a horospherical measure is supported on $L^{(1)}$. We have the following useful local disintegration formula:

Proposition 3.14. *Let ν be a horospherical measure on $\mathcal{M}$. For any regular box $\varphi : U'_{\mathrm{x}} \times U_{\mathrm{y}} \to \boldsymbol{B}$ in $\pi^{-1}(\mathcal{M})$, there is a measure λ on U_{y} such that for any compactly supported continuous function $f : \mathcal{M} \to \mathbb{R}$, denoting $B = \pi(\boldsymbol{B})$ we have*

$$\int_B f\, d\nu = \int_{U_{\mathrm{y}}} \left(\int_{\boldsymbol{L}_y} f \circ \pi\, d\nu_{\beta_{\mathrm{x}}} \right) d\lambda(y). \tag{23}$$

Proof. Let L be a lift of $\mathcal{M}$ in which $\boldsymbol{B}$ is contained. We denote by Γ the stabilizer in $\mathrm{Mod}(S, \Sigma)$ of $\boldsymbol{B}$, and by $\tilde{\nu}$ the pre-image of ν under π. By definition, the measure $\tilde{\nu}|_L$ is horospherical, and we let λ_0 be a measure on $\mathcal{U}_{\mathrm{y}}$ as in equation (22). We set $\lambda \overset{\mathrm{def}}{=} \frac{1}{|\Gamma|}\lambda_0$, and claim that λ satisfies equation (23). Indeed, let $f \in C_c(\mathcal{H})$ and assume for now that the support of f is contained in $\bar{B}$. Let h be the function that is equal to $f \circ \pi$ on $\bar{\boldsymbol{B}}$ and 0 elsewhere. This function is continuous and its support is contained in $\bar{\boldsymbol{B}}$ by construction. Using that the stabilizer of $\bar{\boldsymbol{B}}$ in $\mathrm{Mod}(S, \Sigma)$ is also Γ, we calculate that for any $q \in \mathcal{H}$, we have $\int_{\mathcal{H}_{\mathrm{m}}} h\, d\theta_q = |\Gamma| f(q)$. We have

$$\int_B f\, d\nu = \frac{1}{|\Gamma|} \int_{\mathcal{H}} h\, d\tilde{\nu} = \frac{1}{|\Gamma|} \int_{U_{\mathrm{y}}} \left(\int_{\boldsymbol{L}_y} f \circ \pi\, d\nu_{\beta_{\mathrm{x}}} \right) d\lambda_0(y),$$

which is what we wanted. In case the support of f is arbitrary, we pick a sequence ψ_n of uniformly bounded smooth functions with support contained in $\bar{B}$ and that converge pointwise to 1_B, the indicator function of B, and we apply the previous computation to $\psi_n f$ in place of f. We have

$$\int_B \psi_n f\, d\nu = \int_{U_{\mathrm{y}}} \left(\int_{\boldsymbol{L}_y} \psi_n f \circ \pi\, d\nu_{\beta_{\mathrm{x}}} \right) d\lambda(y).$$

Passing to the limit using Lebesgue's dominated convergence, we obtain equation (23). □

3.4. **The special flat measures are horospherical.** In this subsection we prove Theorem 1.1, which gives us our first examples of horospherical measures. Namely we will show that the Masur-Veech measures on strata, and more generally, the special flat measures defined in equation (9), are horospherical.

Let $\mathcal{M}$ be an invariant subvariety and let L be a lift of $\mathcal{M}$. In order to establish Theorem 1.1, we shall first establish that the measure m_L as in equation (11) is horospherical. This will be achieved in Proposition 3.17. We need some preparatory results. We recall that the measure m_L is obtained by the cone construction applied to μ_L, i.e., for any Borel set $A \subset L^{(1)}$,

$$m_L(A) = \mu_L(\mathrm{cone}(A)),$$

and the measure μ_L is itself obtained by integration of $\alpha = \alpha_{\mathrm{x}} \wedge \alpha_{\mathrm{y}}$, where α_{x} and α_{y} are as in equation (21). Let $\beta \overset{\mathrm{def}}{=} \iota_E \alpha$. By construction, β induces a volume form on $L^{(1)}$ and we denote by μ_β the measure obtained by integration of β. The following relates the measure μ_β and the cone measure m_L.

Lemma 3.15. *We have*

$$\mu_\beta = \dim(\mathcal{M}) \cdot m_L.$$

Proof. The proof is an application of Stokes' theorem. It follows from equation (6) that the Lie derivative of α with respect to the Euler vector field satisfies $\mathcal{L}_E(\alpha) = \dim(\mathcal{M}) \cdot \alpha$. Let U be an open set in $L^{(1)}$ contained in one chart for the manifold structure on L. Notice that the only part of the boundary $\partial\text{cone}(U)$ of $\text{cone}(U)$ to which E is not tangent is U itself. In particular, the only part of $\partial\text{cone}(U)$ on which $\iota_E\alpha$ does not vanish is U. We have from Stokes' formula (for manifolds with corners, see e.g. [LL12]) that

$$\int_{\text{cone}(U)} d(\iota_E\alpha) = \int_{\partial\text{cone}(U)} \iota_E\alpha = \int_U \iota_E\alpha = \mu_\beta(U).$$

It follows from the Cartan formula that $\mathcal{L}_E(\alpha) = d\iota_E\alpha + \iota_E d\alpha$. Since α is closed, we deduce that $d\iota_E(\alpha) = \mathcal{L}_E(\alpha)$. Gathering everything, we obtain

$$\dim(\mathcal{M}) \cdot m_L(U) = \mu_\beta(U).$$

This is true for all U as above, and these open sets generate the Borel σ-algebra on $L^{(1)}$. ◻

We introduce two new vector fields E_x and E_y on $\mathcal{H}_\mathrm{m}$:

$$E_x(\boldsymbol{q}) \overset{\text{def}}{=} \frac{\partial}{\partial t}\Big|_{t=0} \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} \cdot \boldsymbol{q} \quad \text{and} \quad E_y(\boldsymbol{q}) \overset{\text{def}}{=} \frac{\partial}{\partial t}\Big|_{t=0} \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \cdot \boldsymbol{q}. \tag{24}$$

By definition we have $E = E_x + E_y$, and for any $\boldsymbol{q} \in \mathcal{H}_\mathrm{m}$, we have

$$\frac{\partial}{\partial t}\Big|_{t=0} g_t \cdot \boldsymbol{q} = E_x(\boldsymbol{q}) - E_y(\boldsymbol{q}). \tag{25}$$

For the proof of Theorem 1.1, we will also need the following calculation:

Lemma 3.16. *The restrictions of the forms α_x and α_y to $L^{(1)}$ satisfy*

$$\iota_E(\alpha_\mathrm{x} \wedge \alpha_\mathrm{y}) = 2(\iota_E\alpha_\mathrm{x}) \wedge \alpha_\mathrm{y}.$$

Proof. Let $n = \dim(\mathcal{M})$. We begin by observing that on restriction to $L^{(1)}$, we have

$$\iota_{(E_\mathrm{x} - E_\mathrm{y})}(\alpha_\mathrm{x} \wedge \alpha_\mathrm{y}) = 0. \tag{26}$$

Indeed, we deduce from equation (25) that $E_x - E_y$ is tangent to $L^{(1)}$. In particular, since $L^{(1)}$ has dimension $2n-1$ (as a real vector space), any family of $2n-1$ linearly independent vector fields that are tangent to $L^{(1)}$ contain $E_\mathrm{x} - E_\mathrm{x}$ in their span. This implies equation (26).

Now we calculate:

$$\begin{aligned}\iota_E(\alpha_\mathrm{x} \wedge \alpha_\mathrm{y}) &= \iota_{2E_\mathrm{x} - (E_\mathrm{x} - E_\mathrm{y})}(\alpha_\mathrm{x} \wedge \alpha_\mathrm{y}) \\ &\overset{(26)}{=} 2\iota_{E_\mathrm{x}}(\alpha_\mathrm{x} \wedge \alpha_\mathrm{y}) = 2\iota_{E_\mathrm{x}}\alpha_\mathrm{x} \wedge \alpha_\mathrm{y} + (-1)^n \alpha_\mathrm{x} \wedge \iota_{E_x}\alpha_\mathrm{y}.\end{aligned}$$

The last equality follows from the Leibniz formula for contractions

$$\iota_V(\alpha \wedge \beta) = (\iota_V\alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge \iota_V\beta.$$

Now, notice that E_x is tangent to the fibers of $\pi_\mathrm{y} \circ \text{dev}$. Since $\alpha_\mathrm{y} = (\pi_\mathrm{y} \circ \text{dev})^* \eta_\mathrm{y}$, we deduce that $\iota_{E_\mathrm{x}}\alpha_\mathrm{y} = 0$. Similarly, we prove that $\iota_{E_\mathrm{y}}\alpha_\mathrm{x} = 0$ and thus

$$\iota_E(\alpha_\mathrm{x} \wedge \alpha_\mathrm{y}) = 2\iota_{E_\mathrm{x}}\alpha_\mathrm{x} = 2\iota_E\alpha_\mathrm{x}.$$

◻

Proposition 3.17. *The measure m_L is horospherical.*

Proof. It follows from Lemma 3.15 that m_L is given, up to a multiplicative constant, by integration of the differential form $\beta = \iota_E(\alpha_{\mathrm{x}} \wedge \alpha_{\mathrm{y}})$. Lemma 3.16 implies that $\beta = 2\beta_{\mathrm{x}} \wedge \alpha_{\mathrm{y}}$. Notice that both the forms β_{x} and α_{y} are *basic*, i.e., they are obtained by pullback of forms on $\mathbf{P}(V_{\mathrm{x}})$ and V_{y} by the projections $\mathrm{dev} \circ \pi'_{\mathrm{x}}$ and $\mathrm{dev} \circ \pi_{\mathrm{y}}$. Indeed, we have $\alpha_{\mathrm{y}} = (\mathrm{dev} \circ \pi_{\mathrm{y}})^* \eta_y$ and using Lemma 3.1, we can build a differential form β'_{x} on $\mathbf{P}(V_{\mathrm{x}})$ such that $(\mathrm{dev} \circ \pi'_{\mathrm{x}})^* \beta'_{\mathrm{x}} = \beta_x$.

Now, let $\varphi : U'_{\mathrm{x}} \times U_{\mathrm{y}} \to \boldsymbol{B}$ be a box in $L^{(1)}$ and let $f \in C_c(L)$. Notice that $\varphi^* \alpha_{\mathrm{y}} = \eta_{\mathrm{y}}$ and $\varphi^* \beta_{\mathrm{x}} = \beta'_{\mathrm{x}}$. We have:

$$\begin{aligned}
\int_{\boldsymbol{B}} f \cdot \beta_{\mathrm{x}} \wedge \alpha_{\mathrm{y}} &= \int_{U'_{\mathrm{x}} \times U_{\mathrm{y}}} f \circ \varphi \cdot \beta'_{\mathrm{x}} \wedge \eta_{\mathrm{y}} \\
&= \int_{U_{\mathrm{y}}} \Big(\int_{U'_{\mathrm{x}} \times \{y\}} f \circ \varphi \cdot \beta'_{\mathrm{x}} \Big) \cdot \eta_{\mathrm{y}} \\
&= \int_{U_{\mathrm{y}}} \Big(\int_{\boldsymbol{L}_y} f \cdot \beta_{\mathrm{x}} \Big) \cdot \eta_{\mathrm{y}}.
\end{aligned}$$

If we let λ be the measure on U_{y} given by integration of the form $\frac{2}{\dim(\mathcal{M})} \cdot \eta_y$, we obtain

$$\int_{\boldsymbol{B}} f \; dm_L = \int_{U_{\mathrm{y}}} \Big(\int_{\boldsymbol{L}_y} f \; d\nu_{\beta_{\mathrm{x}}} \Big) \, d\lambda(y).$$

□

Proof of Theorem 1.1. By definition, in order to prove that $m_{\mathcal{M}}$ is horospherical, we need to show that the lifts of $m_{\mathcal{M}}$ are horospherical. We recall from equation (11) that the lifts of $m_{\mathcal{M}}$ are given by the m_L. Theorem 1.1 is then a consequence of Proposition 3.17. □

3.5. **The horocycle flow, real REL, and horospherical measures.** In this subsection we will show that the horocycle flow and the real Rel deformations move surfaces points in their horospherical leaf, and preserve horospherical measures.

Proposition 3.18. *Horospheres and horospherical measures are horocycle flow-invariant.*

Proof. For $\boldsymbol{q} \in \mathcal{H}_{\mathrm{m}}$ with $\mathrm{dev}(\boldsymbol{q}) = (x, y)$ and $s \in \mathbb{R}$, we have

$$\mathrm{dev}(u_s \boldsymbol{q}) = (x + sy, y). \tag{27}$$

This implies that the horocycle flow maps $W^{uu}_L(\boldsymbol{q})$ to itself and since π is $\mathrm{GL}_2^+(\mathbb{R})$-equivariant, we deduce that horospheres are preserved by the horocycle flow.

We also deduce from equation (27) that the horocycle flow preserves the form α_{x}. Since the horocycle flow commutes with the rescaling action, it also preserves the Euler vector field, from which we deduce that for any $s \in \mathbb{R}$ we have ${u_s}^* \beta_{\mathrm{x}} = \beta_{\mathrm{x}}$. In particular, the horocycle flow preserves the measures $\nu_{\beta_{\mathrm{x}}}$.

Let ν be a horospherical measure and let $s_0 > 0$. We claim that for any $q \in \mathcal{M}$ such that the orbit segment $\{u_s q : s \in \mathbb{R}, \ |s| \leq s_0\}$ is embedded, i.e., Uq is not a periodic horocycle orbit with period of length smaller than $2s_0$, there is an open set $\mathcal{U} \subset \mathcal{M}$ containing q such that for any compactly supported continuous function f with support contained in $\mathcal{U}$ and any $s \in \mathbb{R}$ with $|s| \leq s_0$, we have:

$$\int_{\mathcal{M}} f \circ u_s \; d\nu = \int_{\mathcal{M}} f \; d\nu. \tag{28}$$

To see this, let $\boldsymbol{q} \in \mathcal{H}_{\mathrm{m}}$ be such that $\pi(\boldsymbol{q}) = q$ and define

$$\boldsymbol{\sigma} \stackrel{\mathrm{def}}{=} \{u_s \boldsymbol{q} : s \in \mathbb{R},\ |s| \leq s_0\}.$$

Let Γ be the stabilizer in $\mathrm{Mod}(S, \Sigma)$ of $\boldsymbol{q}$. Since the $\mathrm{GL}_2^+(\mathbb{R})$-action on $\mathcal{H}_{\mathrm{m}}$ commutes with $\mathrm{Mod}(S, \Sigma)$, the group Γ acts trivially on $\boldsymbol{\sigma}$ and since Uq is not a periodic orbit with period smaller than $2s_0$, we have that that $\boldsymbol{\sigma} \cdot \gamma \cap \boldsymbol{\sigma} = \emptyset$ for any $\gamma \in \mathrm{Mod}(S, \Sigma)$, unless $\gamma \in \Gamma$. By thickening $\boldsymbol{\sigma}$, we can find a box $\boldsymbol{B} \subset \pi^{-1}(\mathcal{M})$ containing $\boldsymbol{\sigma}$ and up to replacing $\boldsymbol{B}$ with $\cap_{\gamma \in \Gamma} \boldsymbol{B} \cdot \gamma$, we can assume that $\boldsymbol{B}$ is regular. By construction, for any $s \in \mathbb{R}$ with $|s| \leq s_0$, the surface $u_s q$ belongs to $B = \pi(\boldsymbol{B})$ and lies on the plaque of q. Since the horocycle flow acts continuously, there is a neighborhood $\mathcal{U} \subset B$ around q such that for any $\boldsymbol{q}' \in \pi^{-1}(\mathcal{U}) \cap \boldsymbol{B}$ and $|s| \leq s_0$, we have $u_s \boldsymbol{q}' \in \boldsymbol{B}$. Let λ be a transverse measure on U_{y}, i.e., a measure as in equation (23), and let f be a continuous function with support contained in $\mathcal{U}$. It follows from equation (23) that

$$\begin{aligned}
\int_{\mathcal{M}} f \circ u_s \, d\nu &= \int_{U_{\mathrm{y}}} \left(\int_{\boldsymbol{L}_y} f \circ u_s \circ \pi \, d\nu_{\beta_{\mathrm{x}}} \right) \, d\lambda(y) \\
&= \int_{U_{\mathrm{y}}} \left(\int_{\boldsymbol{L}_y} f \circ \pi \, du_{s*}\nu_{\beta_{\mathrm{x}}} \right) \, d\lambda(y) \\
&= \int_{U_{\mathrm{y}}} \left(\int_{\boldsymbol{L}_y} f \circ \pi \, d\nu_{\beta_{\mathrm{x}}} \right) \, d\lambda(y) = \int_{\mathcal{M}} f \, d\nu.
\end{aligned}$$

The second equality follows from the fact that π is $\mathrm{GL}_2^+(\mathbb{R})$-equivariant and the fact that for any $y \in U_{\mathrm{y}}$, the action of u_s maps $\mathrm{supp}(f \circ \pi) \cap \boldsymbol{L}_y$ inside $\boldsymbol{L}_y$, which in turns is implied by our choice of $\mathcal{U}$ and the fact that the horocycle flow maps the leaves of W_L^{uu} into themselves.

For any $f \in C_c(\mathcal{H})$, define

$$s_f \stackrel{\mathrm{def}}{=} \inf\{s > 0 : \mathrm{supp}(f) \text{ contains a periodic surface of period } s\}.$$

It is easy to see that s_f is always positive, and using the first part of the proof together with a partition of unity argument, we can show that equation (28) holds for f and $s \in \mathbb{R}$ with $|s| \leq s_f$. Furthermore, notice that for any $s \in \mathbb{R}$, we have $s_f = s_{f \circ u_s}$. Writing $s = ks'$ with $k \in \mathbb{N}$ and $|s'| \leq s_f$, we obtain

$$\int_{\mathcal{M}} f \circ u_s \, d\nu = \int_{\mathcal{M}} f \circ u_{(k-1)s'} \, d\nu = \cdots = \int_{\mathcal{M}} f \, d\nu.$$

This proves that ν is horocycle flow-invariant. □

For any irreducible invariant subvariety $\mathcal{M}$, we let

$$Z_{\mathcal{M}} \stackrel{\mathrm{def}}{=} V \cap Z, \tag{29}$$

where V is the model space of any lift of $\pi^{-1}(\mathcal{M})$. Notice that the space $Z_{\mathcal{M}}$ actually does not depend on the choice of particular lift. This is a consequence of Proposition 2.6 together with with the fact that $\mathrm{Mod}(S, \Sigma)$ acts trivially on $\ker(\mathrm{Res})$.

Proposition 3.19. *Let $v \in Z_{\mathcal{M}}$, $q \in \mathcal{M}$, and suppose $\mathrm{Rel}_v(q)$ is defined. Then $\mathrm{Rel}_v(q) \in W^{uu}(q)$. If ν is a horospherical measure and $\mathrm{Rel}_v(q)$ is defined for ν-a.e. q, then ν is invariant under the (almost everywhere defined) map $q \mapsto \mathrm{Rel}_v(q)$.*

Proof. Let $Z^{(q)}$ be as in equation (4). Since $Z_{\mathcal{M}} \subset V$, where V is the subspace that $\mathcal{M}$ is modeled on, we have $\mathrm{Rel}_v(q) \in \mathcal{M}$ if $q \in \mathcal{M}$ and $v \in Z_{\mathcal{M}} \cap Z^{(q)}$. The only properties of the horocycle flow which were used in the proof of Proposition 3.18 are that u_{s_0} preserves the horospheres, and acts on them by translations. The same properties are valid for the action of Rel_v for $v \in Z_{\mathcal{M}}$. Indeed, Rel_v sends surfaces of area one to surfaces of area one, and if $\mathrm{dev}(\boldsymbol{q}) = (x, y)$ then $\mathrm{dev}(\mathrm{Rel}_v \boldsymbol{q}) = (x + v, y)$. □

If a measure μ on $\mathcal{M}$ is saddle-connection free, then for μ-a.e. $q \in \mathcal{M}$, $\mathrm{Rel}_v(q)$ is defined for every $v \in Z_{\mathcal{M}}$, and satisfies the 'group law' property

$$\forall v_1, v_2 \in Z_{\mathcal{M}}, \quad \mathrm{Rel}_{v_1}\left(\mathrm{Rel}_{v_2}(q)\right) = \mathrm{Rel}_{v_1+v_2}(q).$$

Following [Wri15a], we say that an irreducible invariant subvariety $\mathcal{M}$ *is of rank one* if $\dim(\mathrm{Res}(V)) = 2$, where V is the model space of any lift of $\pi^{-1}(\mathcal{M})$. In the rank one case we have the following converse to Propositions 3.18 and 3.19:

Proposition 3.20. *If $\mathcal{M}$ is an invariant subvariety of rank one and μ is a saddle connection-free measure, then μ is horospherical if and only if it is invariant under the horocycle and the real Rel flows.*

Proof. By a dimension count, we see that when $\mathcal{M}$ has rank one, the dimension of horospheres is the same as $\dim(Z_{\mathcal{M}}) + 1$. This means that the horosphere $W^{uu}(q)$ satisfies

$$W^{uu}(q) = \{\mathrm{Rel}_v(u_s q) : s \in \mathbb{R},\ v \in Z_{\mathcal{M}}\},$$

that is the group action generated by the horocycle flow and real Rel acts transitively on the horospheres. As we saw in the proofs of Propositions 3.18 and 3.19, this action is by translations, with respect to the affine structure on $W^{uu}(q)$ afforded by Lemma 3.5. Since the measures $\nu_{\beta_{\mathrm{x}}}$ are the unique (up to scaling) translation-invariant measures on the affine manifolds $L^{(1)}$, the invariance of μ under the horocycle and real Rel flows implies that the conditional measures on the plaques in a box are given by $\nu_{\beta_{\mathrm{x}}}$. □

3.6. **Further properties.** Let X be a manifold with a foliation, and a Borel measure μ. We say that μ is *ergodic for the foliation* if any Borel subset A which is a union of leaves satisfies either $\mu(A) = 0$ or $\mu(X \smallsetminus A) = 0$. For instance we have:

Proposition 3.21. *The special flat measure on an invariant subvariety is ergodic for the horospherical foliation.*

Proof. This follows from Proposition 3.18 and the ergodicity of the special flat measure with respect to the U-action. □

Denote by $\mathcal{P}^{(uu)}(\mathcal{M})$ the collection of horospherical measures on $\mathcal{M}$ with total mass at most one. The following standard results in ergodic theory are valid in the context of horospherical measures:

Proposition 3.22. *For the horospherical foliation on any invariant subvariety $\mathcal{M}^{(1)}$, we have:*

1. *The space $\mathcal{P}^{(uu)}(\mathcal{M})$, with the weak-$*$ topology, is a compact convex set.*
2. *A horospherical probability measure is ergodic if and only if it is an extreme point of $\mathcal{P}^{(uu)}(\mathcal{M})$.*

(3) *For any probability measure* $\mu \in \mathcal{P}^{(uu)}(\mathcal{M})$ *there is a probability space* (Θ, η) *and a measurable map* $\Theta \to \mathcal{P}^{(uu)}(\mathcal{M})$, $\theta \mapsto \nu_\theta$, *such that* ν_θ *is ergodic and a probability measure for* η*-a.e.* θ, *and* $\mu = \int_\Theta \nu_\theta \, d\eta(\theta)$.

(4) *If* $\mu_1, \mu_2 \in \mathcal{P}^{(uu)}(\mathcal{M})$ *such that* $\mu_1 \ll \mu_2$, *and* μ_2 *is ergodic, then* $\mu_1 = c\mu_2$ *for some* $c \geq 0$.

Proof. It is clear from definitions that if $\mu_1, \mu_2 \in \mathcal{P}^{(uu)}(\mathcal{M})$, and $\alpha \in (0,1)$, then $\alpha\mu_1 + (1-\alpha)\mu_2 \in \mathcal{P}^{(uu)}(\mathcal{M})$. It is also clear that condition (23), and the condition $\mu(\mathcal{M}) \leq 1$, are both closed conditions. This proves the first assertion. The remaining assertions follow from Choquet's theorem by standard arguments, see e.g. [CC03, Chap. 2.6]. $\square$

We say that two surfaces $q, q' \in \mathcal{H}$ are *horizontally equivalent* if there is a homeomorphism $M_q \to M_{q'}$ of the underlying surfaces, that preserves the labels of singularities and maps the horizontal saddle connections of M_q bijectively to those of $M_{q'}$. Note that a horizontal equivalence only preserves *certain* horizontal structure. It preserves saddle connections but need not preserve the horizontal foliation.

Proposition 3.23. *Any two surfaces in the same horospherical leaf are horizontally equivalent.*

Proof. It suffices to show this upstairs; that is, we let $\boldsymbol{q}, \boldsymbol{q}' \in \mathcal{H}_{\mathrm{m}}$ with $\mathbf{q}' \in W^{uu}_L(\boldsymbol{q})$, let $f : S \to M_q$ and $f' : S \to M_{q'}$ be marking maps representing $\boldsymbol{q}, \boldsymbol{q}'$ and show that $f' \circ f^{-1}$ gives a bijection of horizontal saddle connections. We first discuss $\boldsymbol{q}' \in W^{uu}_L(\boldsymbol{q})$ which are sufficiently close to $\boldsymbol{q}$. Let $f : S \to M_q$ be a marking map representing $\boldsymbol{q}$ and let $\sigma_1, \ldots, \sigma_r$ be the horizontal saddle connections on M_q. Using a neighborhood $\mathcal{U}_\tau$ corresponding to a fixed triangulation τ of S, we see that there is a neighborhood $\mathcal{U}$ of $\boldsymbol{q}$ such that for any surface $\boldsymbol{q}' \in \mathcal{U}$, represented by $f' : S \to M_{q'}$, the paths $f' \circ f^{-1}(\sigma_i)$ are represented by saddle connections on $M_{q'}$. Furthermore, if $\boldsymbol{q}' \in \mathcal{U} \cap W^{uu}_L(\boldsymbol{q})$ then these paths are horizontal saddle connections, so that $f' \circ f^{-1} : M_q \to M_{q'}$ is a homeomorphism mapping the horizontal saddle connections of M_q injectively to horizontal saddle connections on $M_{q'}$.

Now choose $\boldsymbol{q}_{\max} \in W^{uu}_L(\boldsymbol{q})$ so that it has the maximal number of horizontal saddle connections. We will show that the set $\mathcal{V}$ of surfaces in $W^{uu}_L(\boldsymbol{q})$ which are horizontally equivalent to $\boldsymbol{q}_{\max}$ is open and closed, and this will conclude the proof. By the preceding discussion, $\mathcal{V}$ is open in $W^{uu}_L(\boldsymbol{q})$. Furthermore, if $\boldsymbol{q}_n \to \boldsymbol{q}_\infty$ is a convergent sequence of surfaces in $W^{uu}_L(\boldsymbol{q})$, with $\boldsymbol{q}_n \in \mathcal{V}$, then the horizontal saddle connections on the surfaces M_{q_n} have length bounded uniformly from above and below, and so converge to paths on M_{q_∞} which are represented by horizontal saddle connections or by finite concatenations of horizontal saddle connections. Thus $\boldsymbol{q}_\infty$ has at least the same number of saddle connections as $\boldsymbol{q}_{\max}$, and so, by maximality, $\boldsymbol{q}_\infty \in \mathcal{V}$. This completes the proof. $\square$

From Proposition 3.23 we deduce:

Corollary 3.24. *If* ν *is an ergodic horospherical measure then there is a subset* $\mathcal{M}' \subset \mathcal{M}$ *of full* ν*-measure such any two surfaces in* $\mathcal{M}'$ *are horizontally equivalent.*

Remark 3.25. *In* [BSW22, Def. 5.1]*, using boundary marked surfaces,* topological horizontal equivalence *is introduced. In this definition the homeomorphism* $M_q \to M_{q'}$ *is required to preserve additional structure, e.g. the angular differences between saddle connections at each singular point. Proposition 3.23 and Corollary 3.24 hold for this finer notion of equivalence as well.*

4. Saddle connection free horospherical measures

In this section we will prove Theorem 1.2. We first state and prove some auxiliary statements.

4.1. **The Jacobian distortion in a box.** The different plaques in a box can be compared to each other using the structure of a box. Namely, let $\varphi : U'_{\mathrm{x}} \times U_{\mathrm{y}} \to \boldsymbol{B} \subset L^{(1)}$ be a box. For any point $y \in U_{\mathrm{y}}$ we define

$$\varphi_y : U'_{\mathrm{x}} \to \boldsymbol{L}_y, \qquad \varphi_y([x]) \stackrel{\mathrm{def}}{=} \varphi([x], y),$$

where $\boldsymbol{L}_y$ is the plaque of y in $\boldsymbol{B}$ (see Definition 3.2).

For any two points y_0 and y_1 in U_{y}, the map $\varphi_{y_0,y_1} \stackrel{\mathrm{def}}{=} \varphi_{y_1} \circ \varphi_{y_0}^{-1}$ is a diffeomorphism between the plaques $\boldsymbol{L}_{y_0}$ and $\boldsymbol{L}_{y_1}$ in B, identifying points parameterized by the same point in U'_{x}. Define

$$\delta_{y_1} : \boldsymbol{L}_{y_0} \to \mathbb{R}, \quad \delta_{y_1}(\boldsymbol{q}) \stackrel{\mathrm{def}}{=} \langle x_{\boldsymbol{q}}, y_1 \rangle^{-\dim(\mathcal{M})},$$

where $x_{\boldsymbol{q}} = \pi_{\mathrm{x}} \circ \mathrm{dev}(\boldsymbol{q})$. The diffeomorphism φ_{y_0,y_1} is not measure preserving. Instead, we have the following:

Proposition 4.1. *(Jacobian calculation) For any two points $y_0, y_1 \in U_{\mathrm{y}}$ we have*

$$(\varphi_{y_0,y_1})^* \left(\beta_{\mathrm{x}}|_{\boldsymbol{L}_{y_1}}\right) = \delta_{y_1} \cdot \left(\beta_{\mathrm{x}}|_{\boldsymbol{L}_{y_0}}\right). \tag{30}$$

Proof. For any $y \in U_{\mathrm{y}}$, write

$$\bar{L}_y \stackrel{\mathrm{def}}{=} \pi_{\mathrm{x}} \circ \mathrm{dev}(\boldsymbol{L}_y),$$

where π_{x} is the projection in equation (17). Then $\bar{L}_y$ is an open subset of an affine hyperplane in V_{x}, which is a translate of

$$\{x \in V_{\mathrm{x}} : \langle x, y \rangle = 0\}. \tag{31}$$

By Definition 3.2 the map

$$F : \bar{L}_y \to \boldsymbol{L}_y, \qquad F(x) \stackrel{\mathrm{def}}{=} \varphi([x], y)$$

is a diffeomorphism with inverse $\pi_{\mathrm{x}} \circ \mathrm{dev}$. We denote by e_{x} the Euler vector field on V_{x}. Notice that $F^*\beta_{\mathrm{x}}$ is the restriction to $\bar{L}_y$ of $\iota_{e_{\mathrm{x}}}\eta_{\mathrm{x}}$. Indeed, we calculate

$$F^*\beta_{\mathrm{x}} = F^*\iota_E(\pi_{\mathrm{x}} \circ \mathrm{dev}^*\eta_{\mathrm{x}}) = F^*(\pi_{\mathrm{x}} \circ \mathrm{dev})^* \left(\iota_{e_{\mathrm{x}}}\eta_{\mathrm{x}}\right) = \iota_{E_{\mathrm{x}}}(\eta_{\mathrm{x}}).$$

The map F gives a chart of $\boldsymbol{L}_y$ in which β_{x} is $\iota_{e_{\mathrm{x}}}\eta_{\mathrm{x}}$. We shall perform our calculation in these charts and verify equation (30) in $\bar{L}_y$ instead of L_y. Let $y_0, y_1 \in U_{\mathrm{y}}$, and set

$$h : \bar{L}_{y_1} \to \mathbb{R}, \qquad h(x) \stackrel{\mathrm{def}}{=} \frac{1}{\langle x, y_1 \rangle}.$$

The map $\varphi_{y_0,y_1} : L_{y_0} \to L_{y_1}$ is expressed in charts simply as the map

$$\bar{\varphi}_{y_0,y_1} : \bar{L}_{y_0} \to \bar{L}_{y_1}, \quad \bar{\varphi}_{y_0,y_1}(x) = h(x)\, x.$$

This implies by the multivariable product rule that

$$(D\bar{\varphi}_{y_0,y_1})_x(v) = h(x)\, v + (Dh)_x(v)\, x.$$

Hence, for $v_1, \ldots, v_{d-1}$ in the tangent space to $\bar{L}_{y_0}$ at x, denoting $n = \dim(\mathcal{M})$ we have:

$$\begin{aligned}
&\left((\bar{\varphi}_{y_0,y_1})^* \iota_{e_{\mathrm{x}}} \eta_x\right)_x (v_1, \ldots, v_{d-1}) \\
=&(\iota_{e_{\mathrm{x}}} \eta_x)_{\bar{\varphi}_{y_0,y_1}(x)} \Big(D_x \bar{\varphi}_{y_0,y_1}(v_1), \ldots, D_x \bar{\varphi}_{y_0,y_1}(v_{d-1}) \Big) \\
=&(\eta_{\mathrm{x}})_{h(x)x} \Big(h(x)x, h(x)v_1 + D_x h(v_1)\, x, \ldots, h(x)v_{d-1} + D_x h(v_{d-1})\, x \Big) \\
=&(\eta_{\mathrm{x}})_{h(x)x} \Big(h(x)x, h(x)v_1, \ldots, h(x)v_{d-1} \Big) = h(x)^n (\iota_{e_{\mathrm{x}}} \eta_{\mathrm{x}})_x (v_1, \ldots, v_{d-1}).
\end{aligned}$$

This is Formula (30). $\square$

Notice that for any $y_0, y_1 \in U_{\mathrm{y}}$ and $[x] \in U'_{\mathrm{x}}$, we have

$$\delta_{y_1} \circ \varphi_{y_0}([x]) = \left(\frac{\langle x, y_0 \rangle}{\langle x, y_1 \rangle} \right)^{\dim(\mathcal{M})}.$$

This leads us to define the *distortion of* $\boldsymbol{B}$ as follows:

$$\delta_{\boldsymbol{B}} \stackrel{\text{def}}{=} \sup \left\{ \left| 1 - \left(\frac{\langle x, y_0 \rangle}{\langle x, y_1 \rangle} \right)^d \right| : [x] \in U'_{\mathrm{x}},\ y_0, y_1 \in U_{\mathrm{y}} \right\}.$$

Remark 4.2. *The quantity* $\frac{\langle x, y_0 \rangle}{\langle x, y_1 \rangle}$ *has the following geometric interpretation. The points* $\varphi([x], y_0)$ *and* $\varphi([x], y_1)$ *are in the same weak stable leaf, and this leaf is further foliated by strong stable leaves. The geodesic flow maps a given weak stable leaf to itself, permuting the strong stable leaves inside it. The choice* $t = \log\left(\frac{\langle x, y_0 \rangle}{\langle x, y_1 \rangle}\right)$ *is the value of* $t \in \mathbb{R}$ *for which* g_t *maps* $\varphi([x], y_0)$ *to the strong stable leaf of* $\varphi([x], y_1)$.

The distortion can be used to bound the variation of the mass of the horospherical plaques of $\boldsymbol{B}$ with respect to the measures $\nu_{\beta_{\mathrm{x}}}$. Indeed, by an easy change of variables, using φ_{y_0,y_1} we have

$$|\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_{y_1}) - \nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_{y_0})| \le \delta_{\boldsymbol{B}}\, \nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_{y_0}).$$

From this it follows that

$$\left| \frac{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_{y_1})}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_{y_0})} - 1 \right| \le \delta_{\boldsymbol{B}}. \tag{32}$$

The distortion of a box is well-behaved with respect to the geodesic flow. For $t \in \mathbb{R}$ and any $\boldsymbol{B}$ in $\pi^{-1}(\mathcal{M})$, we write

$$\boldsymbol{B}_t \stackrel{\text{def}}{=} g_t(\boldsymbol{B}) \quad \text{and} \quad B_t \stackrel{\text{def}}{=} \pi(\boldsymbol{B}_t).$$

Proposition 4.3. *Let* $\boldsymbol{B}$ *be a box in* $\pi^{-1}(\mathcal{M})$. *Then* $\boldsymbol{B}_t$ *is a box with* $\delta_{\boldsymbol{B}_t} = \delta_{\boldsymbol{B}}$ *and it is regular whenever* $\boldsymbol{B}$ *is.*

Proof. Let $\varphi : U'_{\mathrm{x}} \times U_{\mathrm{y}} \to L^{(1)}$ be the parametrization of $\boldsymbol{B}$, where L is an irreducible component of $\pi^{-1}(\mathcal{M})$ and let $\bar{U}_{\mathrm{y}}$ be the image of U_{y} under multiplication by e^{-t}, and let $\bar{\varphi}([x], y) \stackrel{\text{def}}{=} g_t \circ \varphi([x], e^t y)$. Using the fact that the geodesic flow preserves the splitting into stable and horospherical foliation, and acts on V_{y} by multiplication by e^{-t}, we see that $\bar{\varphi} : U'_{\mathrm{x}} \times \bar{U}_{\mathrm{y}} \to L_1^{(1)}$ is a parameterization of $\boldsymbol{B}_t$ as in Definition 3.2. Also, for $i = 0, 1$, if $([x], \bar{y}_i) \in U'_{\mathrm{x}} \times \bar{U}_{\mathrm{y}}$, where $\bar{y}_i = e^{-t} y_i$, then

$$\frac{\langle x, \bar{y}_0 \rangle}{\langle x, \bar{y}_1 \rangle} = \frac{\langle x, y_0 \rangle}{\langle x, y_1 \rangle}.$$

This implies that the distortion of $\boldsymbol{B}$ is the same as the distortion of $\boldsymbol{B}_t$.

The last statement follows from the fact that the actions of $\mathrm{GL}_2^+(\mathbb{R})$ and $\mathrm{Mod}(S,\Sigma)$ commute. $\square$

4.2. **Thickness of a box.** We now introduce the notion of the thickness of a box. To define this quantity we use the sup-norm Finsler metric of §2.4 to induce a distance function on leaves of the stable foliation. We rely on work of Avila and Gouezel [AG10, §5], who defined a similar distance function on the leaves of the strong stable foliation.

For a subset of a stable leaf, we denote by $\mathrm{diam}^{(s)}$ its diameter with respect to the distance function $\mathrm{dist}^{(s)}$. We define the *thickness* of the box $\boldsymbol{B}$ as

$$\tau_{\boldsymbol{B}} \stackrel{\text{def}}{=} \sup_{[x]\in U'_{\mathrm{x}}} \mathrm{diam}^{(s)}\ \varphi(\{[x]\}\times U_{\mathrm{y}});$$

that is, the maximal diameter of a plaque for the stable foliation.

We will need boxes whose thickness is also well-behaved under the geodesic flow. Similarly to Proposition 4.3, we have:

Proposition 4.4. *For any $\varepsilon>0$ and any $\boldsymbol{q}\in\pi^{-1}(\mathcal{M}^{(1)})$, there is a regular box $\boldsymbol{B}$ in $\pi^{-1}(\mathcal{M})$ containing $\boldsymbol{q}$ such that for any $t\geq 0$, $\tau_{\boldsymbol{B}_t}\leq\varepsilon$.*

Proof. Let L be a lift of $\mathcal{M}$ that contains $\boldsymbol{q}$ and let Γ be the stabilizer in $\mathrm{Mod}(S,\Sigma)$ of $\boldsymbol{q}$. Since $\mathrm{Mod}(S,\Sigma)$ acts properly discontinuously on $\mathcal{H}_{\mathrm{m}}$, there is a neighborhood $\mathcal{V}$ containing $\boldsymbol{q}$ such that for any $\gamma\in\mathrm{Mod}(S,\Sigma)$, either $\mathcal{V}\cdot\Gamma\cap\mathcal{V}=\emptyset$ or $\gamma\in\Gamma$. By Lemma 3.1, let $\bar{\boldsymbol{B}}\subset\mathcal{V}$ be a box containing $\boldsymbol{q}$ and let $\bar\varphi:\bar U'_{\mathrm{x}}\times\bar U_{\mathrm{y}}\to\bar{\boldsymbol{B}}$ be the parametrization of $\bar{\boldsymbol{B}}$. Let $\mathrm{dev}(\mathbf{q})=(x_0,y_0)$, let $\hat U'_{\mathrm{x}}$ be a neighborhood of $[x_0]$ whose closure is contained in $\bar U'_{\mathrm{x}}$, and let

$$\mathbf{C}\stackrel{\text{def}}{=}\varphi_1\left(\hat U'_{\mathrm{x}}\times\{y_0\}\right).$$

That is, $\mathbf{C}$ is a bounded subset of a horospherical leaf, contained in a plaque of $\bar{\boldsymbol{B}}$, and with closure in the interior of $\bar{\boldsymbol{B}}$. Let $\varepsilon_1\in\left(0,\frac{\varepsilon}{4}\right)$ be small enough so that

$$\mathbf{C}_1\stackrel{\text{def}}{=}\bigcup_{|t|\leq\varepsilon_1} g_t(\mathbf{C})\subset\bar{\boldsymbol{B}},$$

and let $\varepsilon_2\in\left(0,\frac{\varepsilon}{4}\right)$ such that

$$\mathbf{C}_2\stackrel{\text{def}}{=}\bigcup_{\boldsymbol{q}_1\in\mathbf{C}_1}\left\{\boldsymbol{q}_2\in W^{ss}_L(\boldsymbol{q}_1):\mathrm{dist}^{(ss)}(\boldsymbol{q}_1,\boldsymbol{q}_2)<\varepsilon_2\right\}$$

is contained in $\bar{\boldsymbol{B}}$. Such numbers $\varepsilon_1,\varepsilon_2$ exist because $\mathbf{C}$ is bounded, and $\mathbf{C}_2$ contains a neighborhood of $\boldsymbol{q}$. We can therefore let $U'_{\mathrm{x}}\subset\hat U'_{\mathrm{x}}$ and $U_{\mathrm{y}}\subset\bar U_{\mathrm{y}}$ be small enough open sets so that $\boldsymbol{B}=\bar\varphi(U'_{\mathrm{x}}\times U_{\mathrm{y}})$ contains $\boldsymbol{q}$ and is contained in $\mathbf{C}_2$. Since $\boldsymbol{B}$ is contained in $\mathcal{V}$, we may replace $\boldsymbol{B}$ by $\cap_{\gamma\in\Gamma}\boldsymbol{B}\cdot\gamma$ and we can assume that B is regular, with stabilizer Γ.

For $\boldsymbol{q}\in\boldsymbol{B}$, let $\boldsymbol{L}^s(\boldsymbol{q})$ be the plaque through $\boldsymbol{q}$ for the weak stable foliation, that is, the connected component of $\boldsymbol{q}$ in $\boldsymbol{B}\cap W^s_L(\boldsymbol{q})$. For each $\boldsymbol{q}_2\in\boldsymbol{B}$ there is a point $\boldsymbol{q}_0$, which is the unique point in the intersection $\mathbf{C}\cap\boldsymbol{L}^s(\boldsymbol{q}_2)$, and a path from $\boldsymbol{q}_0$ to $\boldsymbol{q}_2$ which is a concatenation of two paths γ_1 and γ_2. The path $\gamma_1=\{g_t\boldsymbol{q}_0:t\in I\}$ from $\boldsymbol{q}_0$ to $\boldsymbol{q}_1$ goes along a geodesic arc, where I is an interval of length at most ε_1. The path γ_2 from $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$ has sup-norm length at most ε_2 and is contained in $W^{ss}_L(\boldsymbol{q}_2)$. Since $\varepsilon_1,\varepsilon_2<\frac{\varepsilon}{4}$, each point in any stable plaque in $\boldsymbol{B}$ is within distance at most $\frac{\varepsilon}{2}$ from the unique point at the intersection of this plaque with $\mathbf{C}$, where the distance is measured using the distance function $\mathrm{dist}^{(s)}$. Concatenating such paths we see that the diameter of any stable plaque in $\boldsymbol{B}$ is at most ε, and this implies the same bound for stable plaques in B. That is, the thickness of B is less than ε. By Proposition 3.9, the lengths of geodesic paths and of paths in strong stable leaves, do not increase when pushed by g_t for $t \geq 0$. Thus the same argument (using the pushes of γ_1 and γ_2 by g_t) give the required upper bound on the thickness of $\boldsymbol{B}_t$. $\square$

For a compactly supported continuous function f on $\mathcal{M}$, we denote by ω_f its continuity modulus with respect to the sup-norm distance function. In particular, $\omega_f(t) \to 0$ as $t \to 0+$ and

$$|f(q_1) - f(q_2)| \leq \omega_f(\mathrm{dist}(q_1, q_2)) \quad \text{ for any } q_1, q_2 \in \mathcal{M}.$$

The following key lemma says that for any horospherical measure ν, any regular box $\boldsymbol{B}$ and any test function f, the integral of f with respect to $\nu|_B$ can be approximated by the integral of $f \circ \pi$ with respect to $\nu_{\beta_{\mathrm{x}}}$ on any one horospherical plaque of $\boldsymbol{B}$, provided that $\boldsymbol{B}$ has small distortion and small thickness. We recall that $B \subset \mathcal{M}$ is defined as the image of $\boldsymbol{B}$ by π.

Lemma 4.5. *Let ν be a horospherical measure, let $f \in C_c\left(\mathcal{M}^{(1)}\right)$ and let $\boldsymbol{B}$ be a regular box such that $\nu(B) > 0$. Then for any $y \in U_{\mathrm{y}}$,*

$$\left| \frac{1}{\nu(B)} \int_B f \, d\nu - \frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_y)} \int_{\boldsymbol{L}_y} f \circ \pi \; d\nu_{\beta_{\mathrm{x}}} \right| \leq \omega_f(\tau_{\boldsymbol{B}}) + 2\|f\|_\infty \delta_{\boldsymbol{B}}.$$

Proof. For $y, y' \in U_{\mathrm{y}}$, let $\varphi_{y,y'} : \boldsymbol{L}_y \to \boldsymbol{L}_{y'}$ be as in §4.1. On the one hand, for any $y' \in U_{\mathrm{y}}$ we have

$$\begin{aligned} \left| \int_{\boldsymbol{L}_{y'}} f \circ \pi \, d\nu_{\beta_{\mathrm{x}}} - \int_{\boldsymbol{L}_{y'}} f \circ \pi \circ \varphi_{y',y} \, d\nu_{\beta_{\mathrm{x}}} \right| &\leq \int_{\boldsymbol{L}_{y'}} |f \circ \pi - f \circ \pi \circ \varphi_{y',y}| \; d\nu_{\beta_{\mathrm{x}}} \\ &\leq \omega_f(\tau_{\boldsymbol{B}}) \nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_{y'}). \end{aligned}$$

The second inequality follows from the fact that, by definition of the thickness, for any $[x] \in U'_{\mathrm{x}}$, the distance between the points $\varphi([x])$ and $\varphi(\varphi_{y',y}([x]))$, with respect to the distance function $\mathrm{dist}^{(s)}$, is at most $\tau_{\boldsymbol{B}}$ and thus also with respect to the distance function dist, together with the fact that π is a contraction.

On the other hand, by the definition of $\delta_{\boldsymbol{B}}$, we have:

$$\begin{aligned} \left| \int_{\boldsymbol{L}_{y'}} f \circ \pi \circ \varphi_{y',y} \, d\nu_{\beta_{\mathrm{x}}} - \int_{\boldsymbol{L}_y} f \circ \pi \, d\nu_{\beta_{\mathrm{x}}} \right| &= \left| \int_{\boldsymbol{L}_y} f \circ \pi \, d\varphi^*_{y,y'} \nu_{\beta_{\mathrm{x}}} - \int_{\boldsymbol{L}_y} f \circ \pi \, d\nu_{\beta_{\mathrm{x}}} \right| \\ &\leq \|f\|_\infty \delta_{\boldsymbol{B}} \nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_y). \end{aligned}$$

The last inequality follows from Proposition 4.1 and the definition of δ_B. Using equation (32) we deduce that for any $y, y' \in V_{\mathrm{y}}$,

$$\left| \frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_{y'})} \int_{\boldsymbol{L}_{y'}} f \circ \pi \, d\nu_{\beta_{\mathrm{x}}} - \frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_y)} \int_{\boldsymbol{L}_y} f \circ \pi \, d\nu_{\beta_{\mathrm{x}}} \right| \leq \omega_f(\tau_{\boldsymbol{B}}) + 2\|f\|_\infty \delta_{\boldsymbol{B}}.$$

Let $y_0 \in U_{\mathrm{y}}$ and let λ be a measure on U_{y} as in equation (23). Notice that $\nu(B) = \int_{U_{\mathrm{y}}} \nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_y)\, d\lambda(y)$. Therefore

$$\begin{aligned}
&\left| \frac{1}{\nu(B)} \int_B f\, d\nu - \frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_{y_0})} \int_{\boldsymbol{L}_{y_0}} f \circ \pi\, d\nu_{\beta_{\mathrm{x}}} \right| \\
\leq & \left| \frac{1}{\nu(B)} \int_{U_{\mathrm{y}}} \left(\int_{L_y} f \circ \pi\, d\nu_{\beta_{\mathrm{x}}} \right) d\lambda(y) - \frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_{y_0})} \int_{\boldsymbol{L}_{y_0}} f \circ \pi\, d\nu_{\beta_{\mathrm{x}}} \right| \\
\leq & \frac{1}{\nu(B)} \int_{U_{\mathrm{y}}} \nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_y) \left| \frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_y)} \int_{\boldsymbol{L}_y} f \circ \pi\, d\nu_{\beta_{\mathrm{x}}} - \frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_{y_0})} \int_{\boldsymbol{L}_{y_0}} f \circ \pi\, d\nu_{\beta_{\mathrm{x}}} \right| \, d\lambda(y) \\
\leq & \omega_f(\tau_{\boldsymbol{B}}) + 2\|f\|_\infty \delta_{\boldsymbol{B}}.
\end{aligned}$$

□

4.3. Mixing of geodesics, nondivergence of horocycles.

We recall the following useful results:

Lemma 4.6 (Nondivergence of the horocycle flow [MW02])**.** *For any $\varepsilon > 0$ and $c > 0$ there is a compact $K \subset \mathcal{M}^{(1)}$ such that for any $q \in \mathcal{M}^{(1)}$, one of the following holds:*

- $\liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_K(u_s q)\, ds > 1 - \varepsilon$ *(where $\mathbf{1}_K$ is the indicator of K).*
- *The surface q has a horizontal saddle connection of length smaller than c.*

Lemma 4.6 will be used at several places in this text. The first fact we deduce from it is the following:

Lemma 4.7. *Let ν be a saddle connection free horospherical measure and let $\delta > 0$. Then there is a regular box $\boldsymbol{B} \subset \pi^{-1}(\mathcal{M})$, a constant $c > 0$ and an unbounded increasing sequence of times t_i such that:*

(a) For all $i \geq 0$, $\nu(B_{t_i}) > c\nu(\mathcal{M})$, where $B_{t_i} = \pi(\boldsymbol{B}_{t_i})$.
(b) Both the thickness and distortion of each $\boldsymbol{B}_{t_i}$ are smaller than δ.

In particular it follows from (a) that ν is finite.

Proof. Let K be a compact subset as in Lemma 4.6 for $\varepsilon = \frac{1}{2}, c = 1$, and denote $\nu_t \stackrel{\text{def}}{=} (g_{-t})_*\nu$. By Proposition 3.18, ν is U-invariant, and since g_t normalizes U, the same holds for ν_t. An application of a generalisation of the Birkhoff ergodic theorem for locally finite measures (see [Kre85, Thm. 2.3] for a general formulation) to the function $\mathbf{1}_K$ shows that there is a function $f \in L^1(\nu_t)$ such that $\|f\|_{L^1(\nu_t)} \leq \|\mathbf{1}_K\|_{L^1(\nu_t)} = \nu_t(K)$ and for ν_t-almost every $q \in \mathcal{M}^{(1)}$,

$$\frac{1}{T}\left|\{s \in [0,T] :\ u_s q \in K\}\right| \underset{T\to\infty}{\longrightarrow} f(q).$$

Since the $\{g_t\}$-action preserves the property of having horizontal saddle connections, ν_t is also saddle connection free and thus by Lemma 4.6, we have almost surely $f(q) > \frac{1}{2}$. As a consequence,

$$\nu_t(K) \geq \int_{\mathcal{M}} f\, d\nu_t > \frac{\nu_t(\mathcal{M})}{2}.$$

For every $\delta > 0$, using Proposition 4.4, $K \cap \mathcal{M}^{(1)}$ can be covered by the image by π of regular boxes $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_N$ whose distortion is smaller than δ, and for which the thickness of $g_t(\boldsymbol{B}_j)$ is smaller than δ, for each j and each $t \geq 0$. By Lemma 4.3, the distortion of $g_t(\boldsymbol{B}_j)$ is also less than δ for each j and each $t \geq 0$. Let $c \stackrel{\text{def}}{=} \frac{1}{2N}$. For each t, there is $j = j(t) \in \{1, \dots, N\}$ such that

$$\nu_t(B_j) \geq \frac{\nu_t(K)}{N} > c\,\nu_t(\mathcal{M}).$$

Let $t_i \to \infty$ be a sequence along which $j = j(t_i)$ is constant. Then (a) and (b) hold for $\boldsymbol{B} = \boldsymbol{B}_j$. □

Lemma 4.8 (Mixing of the geodesic flow)**.** *For any invariant subvariety $\mathcal{M}$, the geodesic flow is mixing with respect to the special flat measure on $\mathcal{M}^{(1)}$.*

For a proof and detailed discussion of this result and its quantitative strengthenings, see [FM14, Chap. 4] or [EMM22].

4.4. **Putting it all together.** We have gathered all the ingredients needed to give the proof of our main result.

Proof of Theorem 1.2. Let ν be a horospherical measure. We assume first that ν is ergodic for the horospherical foliation. We will show that the special flat measure $m_{\mathcal{M}}$ is absolutely continuous with respect to ν. To see this, let A be a Borel set of positive measure for $m_{\mathcal{M}}$. Since $m_{\mathcal{M}}$ is a Radon measure, in particular inner regular, there is a compact K contained in A such that $m_{\mathcal{M}}(K) > 0$. Let U be an open set that contains A and let $f : \mathcal{M}^{(1)} \to [0,1]$ be a continuous function whose support is contained in U and that evaluates to 1 on K. Such a function exists by Urysohn's Lemma. Let $\varepsilon > 0$, and choose $\delta > 0$ so that

$$\omega_f(\delta) + 2\|f\|_\infty \delta < \varepsilon.$$

By Lemma 4.7, there is $c > 0$, a regular box $\boldsymbol{B}$ and $t_i \to \infty$ such that for each i, $\tau_{\boldsymbol{B}_{t_i}} < \delta$ and $\delta_{\boldsymbol{B}} < \delta$, and $\nu(B_{t_i}) \geq c\nu(\mathcal{M})$. Applying Lemma 4.5 to both ν and $m_{\mathcal{M}}$ we obtain

$$\left| \frac{1}{\nu(B_{t_i})} \int_{B_{t_i}} f\, d\nu - \frac{1}{m_{\mathcal{M}}(B_{t_i})} \int_{B_{t_i}} f\, dm_{\mathcal{M}} \right| < 2\varepsilon.$$

By mixing of the geodesic flow with respect to $m_{\mathcal{M}}$, there is $i > 0$ large enough such that $m_{\mathcal{M}}(B_{t_i} \cap K) > m_{\mathcal{M}}(B)(m_{\mathcal{M}}(K) - \varepsilon)$. Therefore:

$$\begin{aligned} \frac{\nu(U)}{c\,\nu(\mathcal{M}^{(1)})} \geq & \frac{\nu(U)}{\nu(B_{t_i})} \geq \frac{1}{\nu(B_{t_i})} \int_{B_{t_i}} f\, d\nu \\ > & \frac{1}{m_{\mathcal{M}}(B_{t_i})} \int_{B_{t_i}} f\, dm_{\mathcal{M}} - 2\varepsilon \\ \geq & \frac{m_{\mathcal{M}}(B_{t_i} \cap K)}{m_{\mathcal{M}}(B_{t_i})} - 2\varepsilon > m_{\mathcal{M}}(K) - 3\varepsilon. \end{aligned}$$

Since ε was chosen arbitrarily, we have proven $\nu(U) \geq c\,\nu(\mathcal{M})\, m_{\mathcal{M}}(K)$. We deduce by inner regularity of the measure ν that $\nu(A)$ is positive. This completes the proof that $m_{\mathcal{M}} \ll \nu$.

It follows from Proposition 3.22 that $m_{\mathcal{M}} = c\nu$ for some $c \geq 0$, and since $m_{\mathcal{M}}$ is nonzero, $c > 0$ and $\nu = \frac{1}{c} m_{\mathcal{M}}$. For general ν, we obtain from the case just discussed

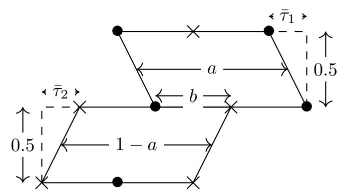

FIGURE 1. A completely periodic surface in $\mathcal{H}(1,1)$. The two singularities are marked with $\bullet$ and $\times$.

that all the ergodic components of the measure ν are proportional to the special flat measure and thus ν itself is proportional to the special flat measure. $\square$

5. Examples of horospherical measures

The simplest example of a horospherical measure which is not the special flat measure occurs when $\mathcal{M}$ is a closed $\mathrm{GL}_2^+(\mathbb{R})$-orbit. In this case the leaves of the horospherical foliation are the U-orbits, and the length measure on a closed periodic U-orbit is a horospherical measure; indeed, in this case, the transverse measure λ in equation (23) is atomic.

In order to obtain more complicated examples, we use the following:

Proposition 5.1. *Let $W^{uu}(q)$ be a closed horosphere in $\mathcal{M}$. Then $W^{uu}(q)$ is the support of a horospherical measure ν whose lifts are the measures $\nu^L_{\beta_x,\boldsymbol{q}}$ where L is a lift of $\mathcal{M}$ and $\pi(\boldsymbol{q}) = q$.*

Proof. The horosphere $W^{uu}(q)$ is closed if and only the collection $\left\{W^{uu}_L(\boldsymbol{q})\right\}$ is locally finite, where $\boldsymbol{q}$ ranges over $\pi^{-1}(q)$ and L ranges over the lifts of $\mathcal{M}$ that contain $\boldsymbol{q}$. Each of the $W^{uu}_L(\boldsymbol{q})$ carries the Radon measure $\nu^L_{\beta_x,\boldsymbol{q}}$ and the measure

$$\tilde{\nu} \stackrel{\text{def}}{=} \sum \nu^L_{\beta_x,\boldsymbol{q}}$$

is a $\mathrm{Mod}(S,\Sigma)$-invariant Radon measure on $\pi^{-1}(\mathcal{M})$. Let ν be the Radon measure on $\mathcal{M}$ whose lift is $\tilde{\nu}$ (see Proposition B.3). The measure ν is horospherical by construction. $\square$

To construct an example of a closed horosphere, we use *horizontally periodic* surfaces, i.e., surfaces which can be represented as a finite union of horizontal cylinders. Let $\mathcal{M} = \mathcal{H}(1,1)$. This stratum is an invariant subvariety of dimension 5 (see Definition 2.3), and thus its horospherical leaves have real dimension 4. Let a, b be real numbers with $a, b \in (0,1)$ and $0 < b < \min(a, 1-a)$, let $\tau_1, \tau_2 \in \mathbb{S}^1 \stackrel{\text{def}}{=} \mathbb{R}/\mathbb{Z}$, and set $\bar{\tau}_1 \stackrel{\text{def}}{=} a\tau_1$ and $\bar{\tau}_2 \stackrel{\text{def}}{=} (1-a)\tau_2$, so that $\bar{\tau}_1, \bar{\tau}_2$ take values in circles of circumference $a, 1-a$ respectively. Define the surface $q = q_{a,b,\tau_1,\tau_2} \in \mathcal{M}$ by the polygonal representation shown in Figure 1. In the horizontal direction it is comprised of two cylinders, each of height 0.5, and of areas $0.5a$ and $0.5(1-a)$. The parameters τ_1, τ_2 are called *twist parameters.* Changing them by adding an integer amounts to performing the corresponding number of Dehn twists in the two cylinders, and thus does not change the surface q.

It is clear that varying the parameters a, b, τ_1, τ_2 results in surfaces that belong to the horospherical leaf of q, and thus, by a dimension count, they locally parameterize the leaf of q. In either of the cases $b \to 0$ or $b \to \min(a, 1-a)$, the surface q_{a,b,τ_1,τ_2} has shorter and shorter horizontal saddle connections on the boundaries of the cylinders, and thus exits compact subsets of $\mathcal{M}^{(1)}$. This means that the horosphere $W^{uu}(q)$ is closed and that the map

$$\mathbb{S}^1 \times \mathbb{S}^1 \times \{(a,b) \in (0,1)^2 : 0 < b < \min(a, 1-a)\} \to \mathcal{M}^{(1)} \quad (a, b, \tau_1, \tau_2) \mapsto q_{a,b,\tau_1,\tau_2}$$

is a proper homeomorphism onto $W^{uu}(q)$.

It can be checked that in this case the map $(a, b, \bar{\tau}_1, \bar{\tau}_2) \mapsto \mathrm{dev}(q_{a,b,\tau_1,\tau_2})$ is affine in charts. Thus the horospherical measure can be written explicitly (up to scaling) as $d\nu(q_{a,b,\tau_1,\tau_2}) = da\, db\, d\bar{\tau}_1\, d\bar{\tau}_2$.

Remark 5.2. *For the horospherical measure constructed in the preceding example, the space $Z_{\mathcal{M}}$ (defined in equation (29)) is one dimensional, and for every surface q in the support of this measure, $Z^{(q)}$ (defined in equation (4)) is a bounded interval. Moreover, for any $v \in Z_{\mathcal{M}}$ there is a positive measure set of surfaces q (with small values of a) for which $\mathrm{Rel}_v(q)$ is not defined. This shows that the hypothesis in Proposition 3.19, that $\mathrm{Rel}_v(q)$ is defined, is not always satisfied.*

It is no coincidence that the closed horospheres in the two preceding examples consist of horizontally periodic surfaces.

Proposition 5.3. *For any $\mathcal{M}$ and any $q \in \mathcal{M}$, the surface M_q is horizontally periodic if and only if $W^{uu}(q)$ is closed. In this case every surface in $W^{uu}(q)$ is horizontally periodic, and the horospherical measure on $W^{uu}(q)$ constructed in Proposition 5.1 is finite.*

Proof. Suppose first that M_q is horizontally periodic, and let $f : S \to M_q$ be a marking map representing $\boldsymbol{q} \in \pi^{-1}(q)$. Let $C_1, \ldots, C_s$ be the horizontal cylinders on M_q, and let c_j, h_j denote respectively the circumference and height of C_j. Since the area of M_q is one,

$$\sum_{j=1}^{s} c_j h_j = 1. \tag{33}$$

Let $\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_s \in H_1(S, \Sigma)$ be a generating set satisfying the following:

- The collection $\{f(\alpha_i) : i = 1, \ldots, r+s\}$ consists of saddle connections.
- $\{f(\alpha_i) : i = 1, \ldots, r\}$, are all the horizontal saddle connections on cylinder boundaries, and are oriented from left to right.
- For $j = 1, \ldots, s$, $f(\alpha_{r+j})$ is contained in C_j, connects one boundary component of C_j to the other, and is oriented from bottom to top.

Write the holonomies $\mathrm{hol}(M_{\boldsymbol{q}}, \alpha_i)$ as

$$\begin{aligned} \mathrm{hol}(M_{\boldsymbol{q}}, \alpha_i) = &(t_i, 0) \quad i = 1, \ldots, r \\ \mathrm{hol}(M_{\boldsymbol{q}}, \alpha_j) = &(\tau_j, h_j) \quad j = 1, \ldots, s. \end{aligned} \tag{34}$$

For each j and each boundary component of C_j, we have

$$\sum_{i \in \mathcal{I}} t_i = c_j, \tag{35}$$

where $\mathcal{I}$ is a subset of $\{1, \ldots, r\}$ containing the saddle connections comprising the boundary component. The numbers t_i, h_j, τ_j also satisfy some linear equations $L_1, \ldots, L_t$, which describe the space that L is modeled on, in a neighborhood of $\boldsymbol{q}$.

Let $\boldsymbol{q}' \in W^{uu}_L(\boldsymbol{q})$. We first show that the underlying surface $M_{q'}$ is horizontally periodic. There is a continuous path $\sigma \mapsto \boldsymbol{q}(\sigma)$, with $\boldsymbol{q}(0) = \boldsymbol{q}$ and $\boldsymbol{q}(1) = \boldsymbol{q}'$, such that $\boldsymbol{q}(\sigma) \in W^{uu}_L(\boldsymbol{q})$ for every $\sigma \in [0, 1]$. By definition of the horospherical foliation, for any $\sigma \in [0, 1]$, $\mathrm{dev}(\boldsymbol{q}(\sigma)) - \mathrm{dev}(\boldsymbol{q}) \in H^1(S, \Sigma; \mathbb{R}_{\mathrm{x}})$. That is to say, there are $t_1(\sigma), \ldots, t_r(\sigma), \tau_1(\sigma), \ldots, \tau_s(\sigma) \in \mathbb{R}$ such that equation (34) holds for $\boldsymbol{q}(\sigma)$. Note that h_j is independent of σ, that the numbers $t_i(\sigma), h_j, \tau_j(\sigma)$ also satisfy the equations $L_1, \ldots, L_t$, and that the numbers $c_j(\sigma)$ defined by equation (35) also satisfy equation (33).

Assume first that

$$t_i(\sigma) > 0 \quad \text{for all } \sigma \in [0, 1] \ \text{ and } i \in \{1, \ldots, r\}. \tag{36}$$

For each $j \in \{1, \ldots, s\}$, the set

$$\{\sigma \in [0, 1] : \text{ the curve } \alpha_{r+j} \text{ crosses a horizontal cylinder on } \boldsymbol{q}(\sigma)\}$$

is open (this is a general property of cylinders, see e.g. [MT02, §4.1]) and closed (since the heights h_j are fixed). Therefore, by a connectedness argument, $\boldsymbol{q}'$ is also made of s horizontal cylinders. By equation (33), these cylinders occupy the entire area of $\boldsymbol{q}(\sigma)$, and thus $\boldsymbol{q}(\sigma)$ is horizontally periodic.

Now if equation (36) fails, let $\sigma_{\min}$ be the smallest value of σ for which it fails. When σ increases to $\sigma_{\min}$ from below, the surfaces $\boldsymbol{q}(\sigma)$ have shorter and shorter horizontal saddle connections on the boundaries of cylinders, and this means that the surfaces $\boldsymbol{q}(\sigma)$ cannot converge to $\boldsymbol{q}(\sigma_{\min})$. This shows that equation (36) holds and proves that all surfaces in $W^{uu}_L(\boldsymbol{q})$ are horizontally periodic.

The set of parameters t_i, τ_j giving surfaces in $W^{uu}(q)$ is bounded. Indeed, the c_i defined by equation (35) are bounded by equation (33), and this implies that the numbers $t_i \in (0, \max_j c_j)$ are bounded. Changing the τ_j by adding an integer multiple of c_j amounts to performing Dehn twists in the cylinder C_j and does not change the projection of the surface to $\mathcal{M}$. That is, the numbers τ_j can be taken to lie in the bounded set $[0, c_j)$. Also, as the parameters t_i leave compact subsets of the bounded domain described above, at least one of the horizontal saddle connections on the corresponding surface has length going to zero. This implies that the bounded set of surfaces we have just described by varying the parameters t_i, τ_j projects to the entire leaf $W^{uu}(q)$, that this leaf is properly embedded, and that all surfaces in this leaf are horizontally periodic.

Furthermore, we can use equation (33) to express c_1 as a function of $c_2, \ldots, c_s$ (a constant function when $s = 1$), and using the linear equations defining L, we can write some of the variables c_j, τ_j, t_i as linear combinations of a linearly independent set of variables. We can then write the horospherical measure up to scaling as $d\nu(q) = \prod_{j=\mathcal{J}_1} dc_j \prod_{j \in \mathcal{J}_2} d\tau_j \prod_{i \in \mathcal{J}_3} dt_i$, for some subsets of indices, and thus the preceding discussion shows that the total measure of the leaf is bounded.

Now suppose that M_q is not horizontally periodic. According to [SW04], the horocycle orbit Uq consists of surfaces that are not horizontally periodic, but there is $q' \in \overline{Uq}$ such that $M_{q'}$ is horizontally periodic. By Proposition 3.18, $Uq \subset W^{uu}(q)$, and thus $q' \in \overline{W^{uu}(q)}$. Since M_q is not horizontally periodic, according to the first part of the proof, $q' \notin W^{uu}(q)$. This shows that the leaf $W^{uu}(q)$ has an accumulation point that is not contained in the leaf, which is to say that $W^{uu}(q)$ is not closed. □

5.1. **Classification of horospherical measures in the eigenform loci in $\mathcal{H}(1,1)$.** The stratum $\mathcal{H}(1,1)$ contains a countable collection of complex 3-dimensional invariant subvarieties known as *eigenform loci*. This terminology is due to McMullen, who gave a complete classification of these invariant subvarieties in a sequence of papers (see [McM07] and references therein), following the first such examples discovered by Calta [Cal04]. The horocycle invariant measures and orbit-closures for the U-action on an eigenform locus, were classified in [BSW22] (these classification results require Theorem 1.2 of the present work). We can classify the horospherical measures inside eigenform loci as follows:

Theorem 5.4. *Let $\mathcal{M}$ be an eigenform locus in $\mathcal{H}(1,1)$, and let ν be an ergodic horospherical measure on $\mathcal{M}$. Then either ν is the special flat measure $m_{\mathcal{M}}$ or ν is the measure given by Proposition 5.1 on a closed horosphere $W^{uu}(q)$ of a horizontally periodic surface $q \in \mathcal{M}$.*

Proof. If ν is saddle connection free then it is the special flat measure by Theorem 1.2, and by Proposition 5.3, ν is supported on a properly embedded leaf if and only if ν is supported on horizontally periodic surfaces. The only remaining possibilities are that for a set of positive measure, surfaces have one horizontal saddle connection or a horizontal slit, corresponding to cases (3), (4) or (6) of [BSW22, Thm. 9.1]. We assume that such a horospherical measure ν exists, and we will reach a contradiction. By Corollary 3.24, each of the conditions (3), (4), (6) is invariant under the horospherical foliation and so by ergodicity, one of these cases holds for ν-a.e. surface.

Let $f : \operatorname{supp}\nu \to \mathbb{R}_+$ be the function that assigns to a surface the length of a horizontal saddle connection (note that in case (3) there is a unique horizontal saddle connection and in cases (4) and (6) there are two of equal length). Let

$$\mathcal{M}_t \stackrel{\text{def}}{=} f^{-1}(t), \qquad \mathcal{M}_{c\le c'} \stackrel{\text{def}}{=} f^{-1}([c,c']),$$

and let $c' > c > 0$ such that $\eta \stackrel{\text{def}}{=} \nu(\mathcal{M}_{c\le c'}) > 0$. Let $K \subset \mathcal{H}(1,1)$ be the compact set given by Lemma 4.6, corresponding to c and to $\varepsilon \stackrel{\text{def}}{=} \frac{1}{2}$. The U-action preserves the lengths of horizontal saddle connections, and thus preserves $\mathcal{M}_{c\le c'}$. The U-action also preserves ν, by Proposition 3.18. We claim as in the proof of Lemma 4.7 that

$$\nu\left(K \cap \mathcal{M}_{c\le c'}\right) \ge \frac{\eta}{2} \tag{37}$$

(and in particular η is finite). Indeed, defining $\nu' \stackrel{\text{def}}{=} \nu|_{\mathcal{M}_{c\le c'}}$, we have by the Birkhoff ergodic theorem that for ν'-a.e. x,

$$\frac{1}{T}\int_0^T \mathbf{1}_K(u_s x)\, ds \longrightarrow_{T\to\infty} f(x), \tag{38}$$

where f is a function satisfying

$$\|f\|_{L^1(\nu')} \le \|\mathbf{1}_K\|_{L^1(\nu')} = \nu(K \cap \mathcal{M}_{c\le c'}).$$

By Lemma 4.6 we have $f(x) \ge \frac{1}{2}$ whenever the limit in equation (38) exists, and this gives equation (37).

Now note that by Proposition 3.19, ν is invariant under Rel deformations. In our context there are Rel deformations which increase the lengths of saddle connections by an arbitrary fixed number. This implies that for every $s > 0$,

$$\nu(\mathcal{M}_{c \le c'}) = \nu(\mathcal{M}_{c+s \le c'+s}).$$

Repeating the arguments by which we obtained equation (37) we obtain for every $s > 0$,

$$\nu\left(K \cap \mathcal{M}_{c+s \le c'+s}\right) \ge \frac{\eta}{2}.$$

For $s > c' - c$, and $j \in \mathcal{N}$, the sets $\mathcal{M}_{c+js \le c'+js}$ are disjoint. This implies that

$$\nu(K) \ge \sum_{j=1}^{\infty} \nu\left(K \cap \mathcal{M}_{c+js \le c'+js}\right) = \infty,$$

contradicing the fact that ν is a Radon measure. □

5.2. **A further example in $\mathcal{H}(2)$.** Since there is currently no classification of horospherical measures in $\mathcal{H}(2)$, it is of interest to give examples. In this subsection we construct an ergodic horospherical measure which is not the special flat measure and is not supported on one properly embedded horospherical leaf. Its support is contained in the four-dimensional invariant subvariety $\mathcal{M} = \mathcal{H}(2)$, the genus two stratum consisting of surfaces with one singular point of order two.

Recall from Corollary 3.24 that for a given ergodic horospherical measure, almost all surfaces are horizontally equivalent. In Figure 2 we show a typical surface q for our horospherical measure, and a typical topological picture of its horizontal saddle connections. These saddle connections will be denoted by δ and δ'. They disconnect the surface into a horizontal cylinder C, shaded gray in Figure 2, and a torus T.

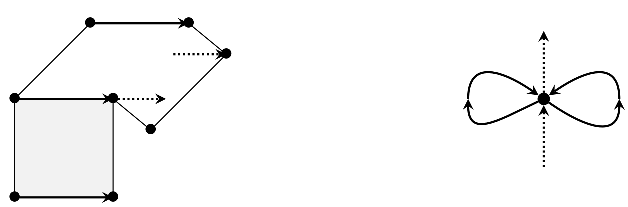

FIGURE 2. A surface in $\mathcal{H}(2)$ with two horizontal saddle connections, bounding a horizontal cylinder. On the right, the corresponding horizontal saddle connection diagram.

Let x be the length of δ and δ', let η be a saddle connection passing from top to bottom of the cylinder C, and let its holonomy be (a, τ). Fix $\boldsymbol{q} \in \pi^{-1}(q)$. The height of C is constant and equal to a in a neighborhood of $\boldsymbol{q}$ in $W^{uu}_L(\boldsymbol{q})$. The area of C is ax, and hence

$$0 < x < \frac{1}{a}. \tag{39}$$

Moreover, changing τ by an integer multiple of x amounts to performing a Dehn twist in C so does not change the surface M_q. Thus we may take

$$\tau \in [0, x). \tag{40}$$

When varying surfaces within their horospherical leaves, we change horizontal components of all saddle connections, and thus changing τ and x we stay in the horospherical leaf. Similarly, by Proposition 3.18, $u_{s'}q \in W^{uu}(q)$ for every s'. Moreover, if $M_q = C \cup T$ as above, the surface $u_s^{(T)} M_q$ obtained by performing the horocycle flow on T and leaving C unchanged is also in $W^{uu}(q)$. It is easy to check that changing the three parameters x, τ, s gives a linear mapping in period coordinates, and that the three corresponding tangent directions in directions in $T_q(\mathcal{M})$ are linearly independent. Since $\dim(\mathcal{M}) = 4$, the dimension of the horospherical leaves in $\mathcal{M}$ is three, so the variables x, τ, s give an affine parameterization of a neighborhood of q in $W^{uu}(q)$.

Since the height a of C remains constant in $W^{uu}(q)$, by equations (39) and (40), the variables x, τ take values in the bounded domain

$$\Delta \stackrel{\mathrm{def}}{=} \left\{ (x, \tau) : 0 \le \tau < x < \frac{1}{a} \right\}.$$

We construct a bundle $\mathcal{B}$ with base Δ, and a homogeneous space fiber, as follows. Let $\mathrm{Tor} \stackrel{\mathrm{def}}{=} G/\mathrm{SL}_2(\mathbb{R})_2(\mathbb{Z})$, the space of tori of some fixed area. This area is usually taken to be one, but by rescaling, can be taken to be any fixed number. For each $x \in \left(0, \frac{1}{a}\right)$, let $\mathrm{Tor}(x)$ denote the space of tori of area $1 - ax$ and with an embedded horizontal segment of length x. This is the complement in Tor of a closed set with empty interior (consisting of periodic horocycles of period at most x). Define $\mathcal{B}$ to be the bundle with base Δ and such that the fiber over $(x, \tau) \in \Delta$ is $\mathrm{Tor}(x)$.

Let μ be the G-invariant probability measure on Tor. Since the set of surfaces which do not admit an embedded horizontal segment of some length is of μ-measure zero, we can also think of μ as a probability measure μ_x on $\mathrm{Tor}(x)$. For $(x, \tau) \in \Delta$ let $C = C(x, \tau)$ be a cylinder of height a, circumference x and twist τ. We have a map

$$\Psi : \mathcal{B} \to \mathcal{H}(2)$$

defined by gluing the torus T from $\mathrm{Tor}(x)$, with a slit of length x, to the cylinder $C(x, \tau)$. Let

$$\nu \stackrel{\mathrm{def}}{=} \int_0^{1/a} \int_0^x \Psi_*(\mu_x)\, d\tau\, dx.$$

The image $\Psi(\mathcal{B})$ is a five-dimensional properly embedded submanifold of $\mathcal{M}$, consisting of all surfaces that can be presented as in Figure 2 for some fixed choice of $a > 0$. Along any sequence of elements $(x, \tau) \in \Delta$ leaving compact subsets, we have either $x \to 0$ or the area $1 - ax$ of T goes to zero, and in both cases the surfaces in the image of Ψ have short saddle connections. This shows that $\Psi(\mathcal{B})$ is properly embedded. Since ν is invariant under translations using the affine coordinates x, τ, s, it is a finite horospherical measure supported on $\Psi(\mathcal{B})$.

6. The geodesic flow and weak unstable foliation

Proof of Theorem 1.4. Let ν be a horospherical measure that is invariant by the geodesic flow. We will assume that ν is ergodic and is not equal to the special flat measure, and prove that $\nu = 0$. By Theorem 1.2, ν-a.e surface in $\mathcal{M}^{(1)}$ has a horizontal saddle connection. Denote by f the map that maps a surface to the length of its shortest horizontal saddle connection. Using Lemma 4.6 as in the proof of Theorem 5.4, we have that for any $c > 0$,

$$\nu\left(\mathcal{M}_{\geq c}\right) < \infty, \quad \text{ where } \mathcal{M}_{\geq c} \stackrel{\text{def}}{=} f^{-1}([c,\infty)).$$

Since $\nu\left(\bigcap_{c'\geq c}\mathcal{M}_{\geq c'}\right) = 0$, this implies

$$\nu\left(\mathcal{M}_{\geq c'}\right) \longrightarrow_{c'\to\infty} 0.$$

The geodesic flow expands the horizontal direction, and hence

$$g_t\left(\mathcal{M}_{\geq c}\right) = \mathcal{M}_{\geq ce^t}.$$

By invariance of the measure ν under the A-action, we obtain that $\nu\left(\mathcal{M}_{\geq c}\right) = \nu\left(\mathcal{M}_{\geq ce^t}\right) \to 0$, and hence $\nu\left(\mathcal{M}_{\geq c}\right) = 0$ for any $c > 0$. This implies $\nu = 0$.

We now show that any leaf for the weak-unstable foliation is dense. Let $q \in \mathcal{M}_1^{(1)}$, let U be an open set contained in $\mathcal{M}^{(1)}$ and let f be a nonzero non negative compactly supported function whose support is contained in U. In order to show $U \cap W^u(q) \neq \emptyset$ we will show that there is $p \in W^u(q)$ such that $f(p) > 0$. Let $\varepsilon \stackrel{\text{def}}{=} \int_{\mathcal{M}} f \, d\mu_{\mathcal{M}} > 0$, let ω_f denote the continuity modulus of f with respect to the sup-norm distance function, and let $\boldsymbol{q} \in \pi^{-1}(q)$. Using Propositions 4.3 and 4.4, let $\boldsymbol{B}$ be a regular box containing $\boldsymbol{q}$ such that for any $t \geq 0$, the box $\boldsymbol{B}_t \stackrel{\text{def}}{=} g_t(\boldsymbol{B})$ satisfies $\omega_f(\tau_{\boldsymbol{B}_t}) + 2\|f\|_\infty \delta_{\boldsymbol{B}_t} < \frac{\varepsilon}{4}$. Let $m_{\mathcal{M}}$ be the special flat measure on $\mathcal{M}^{(1)}$. By mixing of the geodesic flow (Proposition 4.8), there is $T > 0$ such that for any $t > T$, we have

$$\left| \frac{1}{m_{\mathcal{M}}(B)} \int_{B_t} f \, dm_{\mathcal{M}} - \int_{\mathcal{M}} f \, dm_{\mathcal{M}} \right| < \frac{\varepsilon}{4}.$$

Applying Proposition 4.5 to the special flat measure $m_{\mathcal{M}}$, and denoting by $\boldsymbol{L}_t$ the plaque of $g_t\boldsymbol{q}$ in $\boldsymbol{B}_t$, we have

$$\left| \frac{1}{m_{\mathcal{M}}(B)} \int_{B_t} f \, dm_{\mathcal{M}} - \frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_t)} \int_{\boldsymbol{L}_t} f\circ\pi \, d\nu_{\beta_{\mathrm{x}}} \right| < \frac{\varepsilon}{4},$$

and consequently

$$\left| \int_{\mathcal{M}} f \, dm_{\mathcal{M}} - \frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_t)} \int_{\boldsymbol{L}_t} f\circ\pi \, d\nu_{\beta_{\mathrm{x}}} \right| < \frac{\varepsilon}{2}.$$

This implies $\int_{\boldsymbol{L}_t} f\circ\pi \, d\nu_{\beta_{\mathrm{x}}} > 0$ and since $\pi(\boldsymbol{L}_t)$ is contained in $W^u(q)$, we obtain that there is $p \in W^u(q)$ such that $f(p) > 0$. □

7. Closures of horospherical leaves

The goal of this section is to prove Theorem 1.3. First, in order to explain the idea, we will prove the following weaker result.

Theorem 7.1. *Let $q \in \mathcal{M}^{(1)}$ be a surface without horizontal saddle connections. Then $W^{uu}(q)$ is dense in $\mathcal{M}^{(1)}$.*

Proof. Let U be any open set contained in $\mathcal{M}^{(1)}$ and let f be a nonzero non-negative function whose support is contained in U. It is enough to show that there is $p \in W^{uu}_q$ such that $f(p) > 0$. Let $\varepsilon \stackrel{\text{def}}{=} \int_{\mathcal{M}} f \, dm_{\mathcal{M}} > 0$, let $c \stackrel{\text{def}}{=} 1$, and let K be a compact subset as in Lemma 4.6. For any $n > 0$, the surface $g_{-n}q$ does not have horizontal saddle connections and thus there is $s_n > 0$ such that $p_n \stackrel{\text{def}}{=} u_{s_n} g_{-n} q$ satisfies

$$p_n \in K \cap W^{uu}(g_{-n}q). \tag{41}$$

The horocycle flow preserves the horospheres and the geodesic flow permutes them. As a consequence $g_n p_n \in W^{uu}(q)$. Since $K \cap \mathcal{M}_1^{(1)}$ can be covered by the image by π of finitely many arbitrarily small boxes, by passing to a subsequence and using Propositions 4.3 and 4.4, we can assume that there is a box $\boldsymbol{B} \subset \pi^{-1}(\mathcal{M})$ such that the translates $\boldsymbol{B}_n = g_n(\boldsymbol{B})$ satisfy $\omega_f(\tau_{\boldsymbol{B}_n}) + 2\|f\|_\infty \delta_{\boldsymbol{B}_n} < \frac{\varepsilon}{4}$ and $p_n \in \pi(\boldsymbol{B})$, for all $n \in \mathbb{N}$. Denote by $\boldsymbol{L}_n$ a plaque of $\boldsymbol{B}_n = g_n(\boldsymbol{B})$ whose image by π contains $g_n(p_n)$. By mixing of the geodesic flow, for all large enough n:

$$\left| \frac{1}{m_{\mathcal{M}}(B)} \int_{B_n} f \, dm_{\mathcal{M}} - \int_{\mathcal{M}} f \, dm_{\mathcal{M}} \right| < \frac{\varepsilon}{4}.$$

It thus follows from Proposition 4.5 applied to the special flat measure $m_{\mathcal{M}}$ that

$$\left| \frac{1}{m_{\mathcal{M}}(B)} \int_{B_n} f \, dm_{\mathcal{M}} - \frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_n)} \int_{\boldsymbol{L}_n} f \circ \pi \nu_{\beta_{\mathrm{x}}} \right| < \frac{\varepsilon}{4}.$$

Consequently, for large enough n,

$$\left| \int_{\mathcal{M}} f \, dm_{\mathcal{M}} - \frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_n)} \int_{\boldsymbol{L}_n} f \circ \pi \nu_{\beta_{\mathrm{x}}} \right| < \frac{\varepsilon}{2},$$

and thus

$$\frac{1}{\nu_{\beta_{\mathrm{x}}}(\boldsymbol{L}_n)} \int_{\boldsymbol{L}_n} f \circ \pi \nu_{\beta_{\mathrm{x}}} > \int_{\mathcal{M}} f \, dm_{\mathcal{M}} - \frac{\varepsilon}{2} > 0.$$

This implies that there is a $p \in \pi(\boldsymbol{L}_n) \subset W^{uu}(q)$ such that $f(p) > 0$. $\square$

In order to upgrade Theorem 7.1 to Theorem 1.3, we will need the following result:

Theorem 7.2 (Paul Apisa and Alex Wright)**.** *Let $\mathcal{M}$ be an invariant subvariety and suppose that $q \in \mathcal{M}^{(1)}$ has horizontal saddle connections, but is not horizontally periodic. Then there is $q' \in W^{uu}(q)$ such that all horizontal saddle connections on $M_{q'}$ are longer than the shortest horizontal saddle connection on M_q. If in addition M_q has no horizontal cylinders, then for any $T > 0$ there is $q' \in W^{uu}(q)$ such that the shortest horizontal saddle connection on $M_{q'}$ is longer than T.*

Proof of Theorem 1.3. We repeat the arguments given in the proof of Theorem 7.1. In that proof, the only place where we used the assumption that q has no horizontal saddle connections, is to ensure the existence of p_n satisfying condition (41). For this, using Proposition 3.18 and Lemma 4.6, it is enough to show that there is $q'_n \in W^{uu}(g_{-n}q)$ such that the shortest horizontal saddle connection in $M_{q'_n}$ has length at least one. By our assumption, M_q has no horizontal cylinders and therefore neither does $M_{g_{-n}q}$, and thus we can conclude using the second assertion of Theorem 7.2. $\square$

Remark 7.3. *Let $q \in \mathcal{M}_1^{(1)}$ be a surface with horizontal cylinders $C_1, \ldots, C_s$. Its leaf-closure $\overline{W^{uu}(q)}$ can be described as a properly embedded bundle $\mathcal{B}$ in $\mathcal{M}^{(1)}$, very similar to the one discussed in §5.2. We sketch the argument here. Any surface in*

$q' \in W^{uu}(q)$ also has corresponding cylinders $C'_1, \dots, C'_s$ (see the proof of Proposition 3.23). Let c'_j, a'_j, τ'_j denote respectively their circumferences, heights, and twists. We have $a'_j = a_j$, that is the heights of the cylinders are the same on the surfaces M_q and $M_{q'}$. Arguing as in the proof of Proposition 5.3, the numbers c'_j, τ'_j belong to a bounded subset $\Delta \subset \mathbb{R}^{2s}$. Let Δ_q be the subset of Δ describing cylinders that can arise for $q' \in W^{uu}(q)$. This set is the base of $\mathcal{B}$. The fiber over $\vec{c}, \vec{\tau} \in \Delta_q$ is described as follows. Let $q' \in W^{uu}(q)$ have cylinders $C'_1, \dots, C'_s$ whose geometry is prescribed by $\vec{c}, \vec{\tau}$, and let q'' be the surface obtained by removing the cylinders $C'_1, \dots, C'_s$ and regluing the boundary components to each other by a translation. The translation in each cylinder is chosen so that singularities on opposite sides of a cylinder are not glued to each other. This surface can be alternatively described as the limit $t \to 0$ of the cylinder stretch map $\bar{g}$, for $g = \begin{pmatrix} t & s \\ 0 & 1 \end{pmatrix}$, for some s (see Proposition A.1). The surface q'' belongs to some invariant subvariety $\mathcal{M}''$, independent of $\vec{c}, \vec{\tau}$, in a lower dimensional stratum. It has no horizontal cylinders, so by Theorem 1.3, its horosphere is dense in $\mathcal{M}''$. Note that the area of q'' is not one and so we apply Theorem 1.3 after rescaling. Thus the fibers of $\mathcal{B}$ are all isomorphic to $\mathcal{M}''$. As $\vec{c}, \vec{\tau}$ leaves compact subsets of Δ_q, the corresponding surfaces have shorter and shorter horizontal saddle connections, and thus $\mathcal{B}$ is properly embedded in $\mathcal{M}$.

Appendix A. Extending saddle connections and cylinders (Apisa and Wright)

In this section we give the proof of Theorem 7.2. We need some auxiliary statements. A *horizontal cylinder* on a translation surface is a cylinder whose core curve is horizontal. We say that a cylinder and a saddle connection are *disjoint* if they do not intersect, except perhaps at singular points. Our convention is that cylinders are closed, and thus a cylinder and a saddle connection on one of its boundary components are not considered to be disjoint.

We recall the notion of $\mathcal{M}$-equivalence of cylinders, introduced in [Wri15a]. Let $\mathcal{M}$ be an invariant subvariety, let $q \in \mathcal{M}$ and let C_1, C_2 be two parallel cylinders in M_q. The cylinders are called *$\mathcal{M}$-parallel* if there is a neighborhood $\mathcal{U}$ of q in $\mathcal{M}$, such that C_1, C_2 remain parallel for all $q' \in \mathcal{U}$. More precisely:

- there is a lift L of $\mathcal{M}$ and open $\mathcal{V} \subset L$ and $\mathcal{U} \subset \mathcal{M}$ such that $q \in \mathcal{U}$, $\pi|_{\mathcal{V}} : \mathcal{V} \to \mathcal{U}$ is a homeomorphism and dev is injective on $\mathcal{V}$;
- for $\boldsymbol{q} \in \mathcal{V}$ with $q = \pi(\boldsymbol{q})$, represented by a marking map $f : S \to M_q$, and for any $\boldsymbol{q'} \in \mathcal{V}$, represented by $f' : S \to M_{q'}$, the sets $f' \circ f^{-1}(C_i),\ i = 1, 2$ are parallel cylinders on $q' = \pi(\boldsymbol{q'})$.

Being $\mathcal{M}$-parallel is clearly an equivalence relation.

For a cylinder C on a translation surface M, we denote by G_C the subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ fixing the holonomy of the core curve of C. Clearly $G_{C_1} = G_{C_2}$ if C_1, C_2 are parallel. If $C_1, \dots, C_r$ are parallel on M and $g \in G_{C_1}$ then the *cylinder surgery corresponding to $g, C_1, \dots, C_r$* consists of applying g to the C_i and leaving the complement $M \smallsetminus \bigcup_{i=1}^r C_i$ untouched. For example if C is horizontal then the elements of G_C are of the form $\begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix}$, with $t > 0$. The cylinder surgery of such a matrix with $t = 1$ consists of *cylinder shears* (with shear parameter s), and with $s = 0$, consists of *cylinder stretches* (with stretch parameter t). By an appropriate conjugation, the definition of cylinder shears and stretches is extended to non-horizontal cylinders.

We have:

Proposition A.1 (Wright, [Wri15a])**.** *For $\mathcal{M}$, any $q \in \mathcal{M}$, and an $\mathcal{M}$-parallel equivalence class of cylinders $C_1, \ldots, C_r$ on M_q, if $g \in G_{C_i}$ then the surface obtained from M_q by cylinder surgery corresponding to $g, C_1, \ldots, C_r$ is also in $\mathcal{M}$.*

Suppose $q \in \mathcal{M}$ and $C_1, \ldots, C_r$ are $\mathcal{M}$-parallel cylinders on q, which are not necessarily a full equivalence class of $\mathcal{M}$-parallel cylinders. Let L be a lift of $\mathcal{M}$, let $\boldsymbol{q} \in L \cap \pi^{-1}(q)$ and let $V \subset H^1(S, \Sigma; \mathbb{C})$ such that $\mathrm{dev}(L) = V$. Varying $g \in G_{C_1}$ gives rise to a two dimensional collection (in the previous example, corresponding to possible choices of the parameters s, t) of surfaces, obtained from M_q by cylinder surgery corresponding to $g, C_1, \ldots, C_r$. This collection is affine in period coordinates and its tangent direction is a one dimensional complex subspace of $H^1(S, \Sigma; \mathbb{C})$. A generator for this subspace is

$$\sigma_{\{C_i, h_i\}} \stackrel{\text{def}}{=} \sum_{i=1}^{r} h_i \gamma_i^*, \tag{42}$$

where h_i is the height of C_i, γ_i is the core curve of C_i (considered as an element of $H_1(S, \Sigma)$) and γ_i^* is the dual class in $H^1(S, \Sigma)$. Moving along the line tangent to $\sigma_{\{C_i, h_i\}}$ in $\mathcal{M}$ amounts to performing cylinder shears in each of the C_i, and moving along the line tangent to $\mathbf{i} \cdot \sigma_{\{C_i, h_i\}}$ in $\mathcal{M}$ amounts to performing cylinder stretches.

Below we will be interested in such one-parameter families of deformations, tangent to $\sigma_{\{C_i, h_i\}}$ as in equation (42), in which the C_i might not be a full equivalence class of $\mathcal{M}$-parallel cylinders, and the h_i might not be their heights. Note that for cylinder shears, such surgeries are well-defined for any value of the shear parameter s, and for cylinder stretches, they are well-defined as long as $t > 0$.

When $\sigma_{\{C_i, h_i\}} \in V$ for $\boldsymbol{q}, L, V$ as above we simply say that $\sigma_{\{C_i, h_i\}}$ *is contained in the tangent space to $\mathcal{M}$ at q*.

Proposition A.2. *If $\mathcal{M}$ is an invariant subvariety and $q \in \mathcal{M}$ is not horizontally periodic, then there is a nonempty collection of $\mathcal{M}$-parallel cylinders $C_1, \ldots, C_r$ on M_q which consists of cylinders disjoint from all horizontal saddle connections on M_q, and positive $h_1, \ldots, h_r$ such that the class $\sigma_{\{C_i, h_i\}}$ as in equation* (42) *is contained in the tangent space to $\mathcal{M}$ at q. Furthermore, there is $A_0 > 0$, depending only on $\mathcal{M}$, so that if $q \in \mathcal{M}$ has no horizontal cylinders, one can choose a collection of cylinders with these properties, so that in addition, the sum of the areas of the cylinders is at least A_0.*

We note that the h_i in Proposition A.2 might not be the heights of the C_i. The vector $\sigma_{\{C_i, h_i\}}$ is tangent to the line in $\mathcal{M}$ obtained by applying different cylinders shears to each C_i. In order to formalize this, for $g_i \in C_i$ for each i, we define the *cylinder surgery corresponding to* $\{C_i, g_i\}$ to be the map obtained by applying g_i to each C_i, leaving the complement $M \smallsetminus \bigcup_{i=1}^r C_i$ untouched. With this terminology, the line tangent to $\sigma_{\{C_i, h_i\}}$ is the collection of surfaces obtained by cylinder surgery corresponding to $\{C_i, g_i\}$, where $s \in \mathbb{R}$ and g_i performs a cylinder shear with parameter sh_i in C_i. Similarly, the line tangent to $\mathbf{i} \cdot \sigma_{\{C_i, h_i\}}$ is the collection of surfaces obtained by cylinder surgery corresponding to $\{C_i, g_i\}$, where $s \in \mathbb{R}$ and g_i performs a cylinder stretch with parameter sh_i in C_i.

Proof of Proposition A.2. Since the U-action is linear in charts, preserves horizontal saddle connections, and maps $\mathcal{M}$-equivalent cylinders to $\mathcal{M}$-equivalent cylinders, preserving their area, the validity of both statements is unchanged if we replace q with some surface q' in its horocycle orbit Uq. According to [SW04], there is q_∞ in the closure of Uq which is horizontally periodic, and we will see that the required properties hold for all q' sufficiently close to q_∞. Let M_∞ be the underlying surface of q_∞, let $C_1, \ldots, C_r$ be the horizontal cylinders on M_∞, and for each i, A_i, c_i, h_i, γ_i denote respectively the area, circumference, height, and core curve of C_i. Here we consider γ_i as an element of $H_1(S, \Sigma)$ by using a marking $f : S \to M_\infty$ corresponding to $\boldsymbol{q}_\infty \in \pi^{-1}(q_\infty)$. By Proposition A.1, $\sigma_{\{C_i,h_i\}}$ belongs to the tangent space of $\mathcal{M}$ at q.

For any $\theta_0 > 0$ there is a neighborhood $\mathcal{U} = \mathcal{U}(\theta_0)$ of q_∞ in $\mathcal{M}$ such that if $q' \in \mathcal{U}$ then the underlying surface has r cylinders $C_1', \ldots, C_r'$ of circumferences and areas satisfying

$$c_i' < \bar{c} \stackrel{\text{def}}{=} 2 \max_{i=1,\ldots,r} c_i, \qquad A_i' > \underline{A} \stackrel{\text{def}}{=} \frac{1}{2} \min_{i=1,\ldots,r} A_i, \tag{43}$$

and with directions of core curves in $(-\theta_0, \theta_0)$.

If C and σ are a cylinder and a straight segment on a translation surface, we say that σ *crosses* C if it intersects both of the boundary components of C (possibly at singular points). Since a cylinder contains no singularities in its interior, if a saddle connection intersects the interior of a cylinder, then it must cross it. Let s be the maximal length of a horizontal saddle connection on M_q and let θ_0 be small enough so that a horizontal segment of length s cannot cross a cylinder of direction θ satisfying $0 < |\theta| < \theta_0$, with circumference at most $\bar{c}$ and area at least $\underline{A}$.

By making $\mathcal{U}$ smaller, so that it is an evenly covered neighborhood of q_∞, we can ensure that $\sigma_{\{C_i',h_i\}}$ belongs to the tangent space of $\mathcal{M}$ at q'. Indeed, if $\mathcal{V}$ is a connected component of $\pi^{-1}(\mathcal{U})$ and $\boldsymbol{q}', \boldsymbol{q}_\infty \in \mathcal{V}$ are preimages of q', q_∞ respectively, then the core curves of the cylinders C_i', C_i map to the same elements $\gamma_i \in H_1(S, \Sigma)$ under the corresponding marking maps, and thus $\sigma_{\{C_i',h_i\}} = \sigma_{\{C_i,h_i\}}$.

Now suppose that $q' \in Uq \cap \mathcal{U}$, and let M' be the underlying surface. Since q is not horizontally periodic, neither is q'. Therefore there is an equivalence class $C_1, \ldots, C_r$ of $\mathcal{M}$-parallel cylinders on M_∞, so that the corresponding cylinders $C_1', \ldots, C_r'$ are not horizontal cylinders on M', and satisfy the bounds in equation (43). Furthermore the maximal length of a horizontal saddle connection on M' is s, since the horocycle flow maps horizontal saddle connections to horizontal saddle connections of the same length. By choice of θ_0, the cylinders in this equivalence class are all disjoint from horizontal saddle connections on M'. This proves the first assertion.

Let t be an upper bound on the number of horizontal cylinders for a surface in $\mathcal{M}$ and let $A_0 \stackrel{\text{def}}{=} \frac{1}{2t}$. The argument above works for any collection of $\mathcal{M}$-parallel cylinders $C_1, \ldots, C_r$ which are horizontal on q_∞ and are not horizontal on q'. If q has no horizontal cylinders then neither does q', and we can apply the argument with any equivalence class of $\mathcal{M}$-parallel horizontal cylinders $C_1, \ldots, C_r$ on M_∞. One of these classes must have total area at least $\frac{1}{t}$, and thus for $\mathcal{U}$ sufficiently small, the sums of the areas of the corresponding cylinders $C_1', \ldots, C_r'$ is at least A_0. $\square$

Proof of Theorem 7.2. We first prove the first assertion. Let $C_1, \ldots, C_r$ be the $\mathcal{M}$-parallel cylinders on M_q, and $h_1, \ldots, h_r$ the positive numbers provided by Proposition A.2. We define q' as $g \circ \varphi(q)$, where $g \in \mathrm{GL}_2^+(\mathbb{R})$ and φ is a cylinder surgery corresponding to $\{C_i, g_i\}$, and g_i is the cylinder stretch with parameter sh_i for some $s < 0$. Neither of the maps g, φ preserves the area of the surface, we will choose parameters so that their composition does. Moreover both of these maps will not change the vertical component of the holonomy of any curve. The $\mathrm{GL}_2^+(\mathbb{R})$-action preserves $\mathcal{M}$, and by Proposition A.2, so does φ. The map g will increase the length of all horizontal saddle connections on q, and the cylinder surgery φ will not affect their length, since the cylinders C_i are disjoint from the horizontal saddle connections on q.

The area of $\varphi(q)$ is smaller than the area of q since all of the cylinders C_i are stretched by a negative parameter. Let A be the sum of the areas of the cylinders $C_1, \ldots, C_r$. Then by choosing the parameter s appropriately, we can arrange so that the area of $\varphi(q)$ is $1 - \frac{A}{2}$. We now set

$$t \stackrel{\text{def}}{=} \left(1 - \frac{A}{2}\right)^{-1} \quad \text{and } g \stackrel{\text{def}}{=} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}. \tag{44}$$

Then g increases the lengths of horizontal saddle connections by a factor $t > 1$, and multiplies the area of $\varphi(q)$ by t. This completes the proof of the first assertion.

For the second assertion, we use the second assertion in Proposition A.2 to choose the cylinders so the sum of their areas satisfies $A \geq A_0$. This ensures that the horizontal saddle connections on $q' \stackrel{\text{def}}{=} g \circ \varphi(q)$ are longer than the horizontal saddle connections on q by a factor of at least t, where $t > 1$ is as in equation (44) and $t - 1$ is bounded away from 0. In light of Proposition 3.23, $q' \in W^{uu}(q)$ also does not have horizontal cylinders. So we can apply the above argument iteratively, at each stage obtaining surfaces in $W^{uu}(q)$ with longer and longer horizontal saddle connections. Since the lengths of these horizontal saddle connections grows by a definite amount in each step, after finitely many steps they will all be longer than T. $\square$

Appendix B. Measures on $\mathcal{H}$ and $\mathrm{Mod}(S, \Sigma)$-invariant measures on $\mathcal{H}_{\mathrm{m}}$

The goal of this section is to prove a result on the correspondence between Radon measures on $\mathcal{H}$ and $\mathrm{Mod}(S, \Sigma)$-invariant Radon measures on $\mathcal{H}_{\mathrm{m}}$. This result is part of the folklore but we were not able to find a reference; see [Fur73, Prop. 1.3] for an analogous result in a restricted setting.

We state the result in a general setting. Let $\tilde{X}$ be a paracompact manifold and Γ a discrete group acting properly discontinuously on $\tilde{X}$. We will write the Γ-action as an action on the right. Let $X = \tilde{X}/\Gamma$ be the quotient space and $\pi : \tilde{X} \to X$ the quotient map. If Γ acts freely then X is a manifold and π is a covering map. If Γ does not act freely then we can view X as an orbifold and π as a regular orbifold covering map (although no knowledge of orbifolds is assumed in this section). We do not assume that the action on Γ is faithful but, since the action of Γ is proper, the subgroup of Γ that acts trivially on $\tilde{X}$ must be finite.

For $\tilde{q} \in \tilde{X}$ let $\Gamma(\tilde{q})$ be the stabilizer of $\tilde{q}$ in Γ. For any $q \in X$, we define a measure on $\tilde{X}$ by

$$\theta_q \stackrel{\mathrm{def}}{=} \sum_{\tilde{q} \in \pi^{-1}(q)} |\Gamma(\tilde{q})| \cdot \delta_{\tilde{q}}, \tag{45}$$

where $\delta_{\tilde{q}}$ is the Dirac mass at $\tilde{q}$. The measure θ_q is supported on $\pi^{-1}(q)$. For any $f \in C_c(\tilde{X})$ and $\tilde{q} \in \tilde{X}$ we have

$$\int_{\tilde{X}} f \, d\theta_{\pi(\tilde{q})} = \sum_{\gamma \in \Gamma} f(\tilde{q} \cdot \gamma). \tag{46}$$

It follows from the fact that Γ acts properly discontinuously on $\tilde{X}$ that the sum on the right-hand side is finite.

Definition B.1. *Given a Radon measure ν on X we define a Radon measure $\tilde{\nu}$ on $\tilde{X}$, called the* pre-image *of ν, by the formula*

$$\int_{\tilde{X}} f \; d\tilde{\nu} = \int_X \left(\int_{\tilde{X}} f \; d\theta_q \right) \; d\nu(q) \quad \textit{for any } f \in C_c(\tilde{X}). \tag{47}$$

Equation (47) defines a unique Radon measure $\tilde{\nu}$ on $\tilde{X}$ in light of the Riesz Representation Theorem. To see that (47) converges, note that the integrand $q \mapsto F(q) \stackrel{\mathrm{def}}{=} \int_{\tilde{X}} f \; d\theta_q$ is a Borel function, which is supported on the compact set $\pi(\operatorname{supp} f)$, and is bounded by $D \, \|f\|_\infty$, where $D \stackrel{\mathrm{def}}{=} \#\{\gamma \in \Gamma : (\operatorname{supp} f) \cdot \gamma \cap \operatorname{supp} f \neq \emptyset\}$ is finite since the Γ-action is properly discontinuous.

By equation (46) the measures θ_q are all Γ-invariant, and since $\tilde{\nu}$ is an average of the measures θ_q, we have:

Lemma B.2. *The measure $\tilde{\nu}$ is Γ-invariant.*

The following converse can be understood as a disintegration theorem for Γ-invariant Radon measures on $\tilde{X}$.

Proposition B.3. *Let m be a Γ-invariant Radon measure on $\tilde{X}$. There is a unique Radon measure μ on X such that m is the pre-image of μ.*

We call μ the *image* of m.

Proof. Let m be given. We are claiming the existence of a Radon measure μ so that equation (47) holds (with $\tilde{\nu}, \nu$ replaced with m, μ). The idea of the proof is to build μ on small neighborhoods using the fact that π is an orbifold cover. This will be made rigorous using a partition of unity. Let $\tilde{q} \in \tilde{X}$ and let $\Gamma(\tilde{q})$ be the stabilizer of $\tilde{q}$ in Γ. Since Γ acts properly discontinuously on $\tilde{X}$, $\Gamma(\tilde{q})$ is finite, and there is connected $\Gamma(\tilde{q})$-invariant neighborhood $\mathcal{V}$ of $\tilde{q}$ and a neighborhood $\mathcal{U}$ of $\pi(\tilde{q})$ such that π induces a homeomorphism $\mathcal{V}/\Gamma(\tilde{q}) \to \mathcal{U}$, and

$$\pi^{-1}(\mathcal{U}) = \bigsqcup_{\gamma \in \Gamma(\tilde{q}) \backslash \Gamma} \mathcal{V} \cdot \gamma,$$

where γ ranges over a set of coset representatives, and where the sets $\mathcal{V} \cdot \gamma$ are disjoint. We say that such a $\mathcal{U} \subset X$ is *evenly covered (in the orbifold sense).*

Let $(\mathcal{U}_i)_{i \in I}$ be a locally finite cover of X by evenly covered neighborhoods. Such a cover exists by the paracompactness of $\tilde{X}$ and the considerations above. For each $i \in I$ choose a connected component $\mathcal{V}_i$ of $\pi^{-1}(\mathcal{U}_i)$. Denote by $\Gamma_i = \Gamma(\mathcal{V}_i)$ the stabilizer of $\mathcal{V}_i$ in Γ. Let $(\rho_i)_{i\in I}$ be a partition of unity subordinate to the cover $(\mathcal{U}_i)_{i\in I}$ and define a Radon measure μ on X (by using the Riesz Representation Theorem) such that for any $f \in C_c(X)$,

$$\int_X f\ d\mu = \sum_{i\in I} \frac{1}{|\Gamma_i|} \int_{\mathcal{V}_i} (\rho_i f) \circ \pi\, dm. \tag{48}$$

We want to show that the measure μ satisfies equation (47). We claim first that for any $\tilde{q} \in \tilde{X}$, there is a neighborhood $\mathcal{V}$ around $\tilde{q}$ such that equation (47) holds for any $f \in C_c(\tilde{X})$ with support contained in $\mathcal{V}$. Indeed, let $\tilde{q} \in \tilde{X}$ and let $\mathcal{V}$ be a neighborhood of $\tilde{q}$ small enough so that for any $i \in I$, $\mathcal{V}$ intersects at most one connected component of $\pi^{-1}(\mathcal{U}_i)$. This is possible since the cover by the $\mathcal{U}_i$ is locally finite. Let $J = \{i \in I : \pi(\mathcal{V}) \cap \mathcal{U}_i \neq \emptyset\}$ and for $j \in J$, let $\gamma_j \in \Gamma$ be such that $\mathcal{V}_j \cdot \gamma_j \cap \mathcal{V} \neq \emptyset$. By the assumption on $\mathcal{V}$, the coset $\Gamma_j \cdot \gamma_j$ is uniquely determined. Let $f \in C_c(\tilde{X})$ with support contained in $\mathcal{V}$. We compute:

$$\begin{aligned}
\int_X \left(\int_{\tilde{X}} f\ d\theta_q \right) d\mu(q) &= \sum_{i\in I} \frac{1}{|\Gamma_i|} \int_{\mathcal{V}_i} \sum_{\gamma\in\Gamma} \rho_i(\pi(\tilde{q})) f(\tilde{q}\cdot\gamma)\ dm(\tilde{q}) \\
&= \sum_{j\in J} \frac{1}{|\Gamma_j|} \int_{\mathcal{V}_j\cdot\gamma_j} \sum_{\gamma\in\Gamma_j} \rho_j(\pi(\tilde{q})) f(\tilde{q}\cdot\gamma)\ dm(\tilde{q}) \\
&= \sum_{j\in J} \frac{1}{|\Gamma_j|} \int_{\tilde{X}} \sum_{\gamma\in\Gamma_j} \rho_j(\pi(\tilde{q})) f(\tilde{q})\ dm(\tilde{q}) \\
&= \sum_{j\in J} \int_{\tilde{X}} \rho_j(\pi(\tilde{q})) f(\tilde{q})\ dm(\tilde{q}) = \int_{\tilde{X}} f\, dm.
\end{aligned} \tag{49}$$

Now let f be an arbitrary compactly supported continuous function and let K denote its support. Using a covering argument and the computation above, we can find finitely many $(\mathcal{W}_i)_i$ that cover K and such that equation (48) holds for continuous functions with support contained in $\mathcal{W}_i$. Let $(\psi_i)_i$ be a partition of unity associated with this cover. We can write $f = \sum_i \psi_i f$. By construction, each of the $\psi_i f$ has support contained in $\mathcal{W}_i$ and the result follows by equation (49) and the linearity of the integral.

To prove uniqueness of the measure μ, we proceed as follows. Let μ_1 and μ_2 be two Radon measures on X that satisfy equation (47) (with m, μ_i instead of $\tilde{\nu}, \nu$). Let $f \in C_c(X)$ be a compactly supported continuous function whose support is contained in an evenly covered neighborhood $\mathcal{U}$ and let $\mathcal{V}$ be a connected component of $\pi^{-1}(\mathcal{U})$. We denote by $\Gamma(\mathcal{V})$ the stabilizer in Γ of $\mathcal{V}$. Let h be the function on $\tilde{X}$ that is equal to $f \circ \pi$ on $\mathcal{V}$ and vanishes outside of $\mathcal{V}$. Since the support of f is contained in $\mathcal{U}$, the function h is continuous and has compact support. Furthermore, it is easy to see that for any $q \in X$, $\int_{\tilde{X}} h\ d\theta_q = |\Gamma(\mathcal{V})| f(q)$. This implies

$$\int_X f\ d\mu_1 = \frac{1}{|\Gamma(\mathcal{V})|} \int_{\tilde{X}} h\ dm = \int_X f\ d\mu_2.$$

To deal with the case when f is an arbitrary function of compact support we appeal once more to existence of partitions of unity. □

Let G be a group acting on $\tilde{X}$ so that the action commutes with the action of Γ. The group G induces an action on X so that π is G-equivariant.

Proposition B.4. *A Radon measure μ on X is invariant under $g \in G$ if and only if its pre-image $\tilde{\mu}$ is invariant under the action of g on $\tilde{X}$.*

Proof. We start by proving two formulas showing the naturality of the pre-image construction.

Claim B.5. $g_*(\theta_q) = \theta_{g(q)}$.

$$\begin{aligned} g_*(\theta_q) &= \sum_{\pi(\tilde{q})=q} |\Gamma(\tilde{q})| \cdot \delta_{g(\tilde{q})} \\ &= \sum_{\pi(g(\tilde{q}))=g(q)} |\Gamma(\tilde{q})| \cdot \delta_{g(\tilde{q})} \qquad (50) \\ &= \sum_{\pi(g(\tilde{q}))=g(q)} |\Gamma(g(\tilde{q}))| \cdot \delta_{g(\tilde{q})} = \theta_{g(q)}. \qquad (51) \end{aligned}$$

In line (50) we used the fact that $\pi(g(\tilde{q})) = g(\pi(\tilde{q})) = g(q)$. In line (51) we used the fact that the Γ action commutes with g, which implies that $|\Gamma(g(\tilde{q}))| = |\Gamma(\tilde{q})|$.

Claim B.6. $g_*(\tilde{\mu}) = \widetilde{g_*(\mu)}$.

It suffices to show that both measures assign the same integrals to continuous functions of compact support on $\tilde{X}$. Let f be such a function.

$$\begin{aligned} \int_{\tilde{X}} f \, d\widetilde{g_*(\nu)} &= \int_X \left(\int_{\tilde{X}} f \; d\theta_q \right) dg_*(\nu)(q) \\ &= \int_X \left(\int_{\tilde{X}} f \; d\theta_{g(q)} \right) d\nu(q) = \int_X \left(\int_{\tilde{X}} f \; dg_*(\theta_q) \right) d\nu(q) \qquad (52) \\ &= \int_X \left(\int_{\tilde{X}} f \circ g \, d\theta_q \right) d\nu(q) \\ &= \int_{\tilde{X}} f \circ g \, d\tilde{\nu} = \int_{\tilde{X}} f \, dg_*(\tilde{\nu}). \qquad (53) \end{aligned}$$

In line (52) we used Claim B.5. In line (53) we used the definition of the pre-image applied to the function of compact support $f \circ g$.

Using these formulas we now prove the Proposition. If $g_*(\mu) = \mu$ then $g_*(\tilde{\mu}) = \widetilde{g_*(\mu)} = \tilde{\mu}$ by Claim B.6. So $\tilde{\mu}$ is invariant under the action of g on $\tilde{X}$. If $g_*(\tilde{\mu}) = \tilde{\mu}$ then $\widetilde{g_*(\mu)} = g_*(\tilde{\mu}) = \tilde{\mu}$ so the pre-images of $g_*(\mu)$ and μ are equal. It follows from the uniqueness assertion in Proposition B.3 that $g_*(\mu) = \mu$. $\square$

References

[AG10] A. Avila and S. Gouëzel. Small eigenvalues of the Laplacian for algebraic measures in moduli space, and mixing properties of the Teichmüller flow. *Ann. of Math. (2)*, 178:385–442, 2010.

[AGY06] Artur Avila, Sébastien Gouëzel, and Jean-Christophe Yoccoz. Exponential mixing for the Teichmüller flow. *Publ. Math. Inst. Hautes Études Sci.*, (104):143–211, 2006.

[BSW22] Matt Bainbridge, John Smillie, and Barak Weiss. Horocycle dynamics: new invariants and eigenform loci in the stratum $\mathcal{H}(1,1)$. *Mem. Amer. Math. Soc.*, 280(1384):v+100, 2022.

[Cal04] Kariane Calta. Veech surfaces and complete periodicity in genus two. *J. Amer. Math. Soc.*, 17(4):871–908, 2004.

[CC03] Alberto Candel and Lawrence Conlon. *Foliations. II*, volume 60 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[Dan78] S. G. Dani. Invariant measures of horospherical flows on noncompact homogeneous spaces. *Invent. Math.*, 47(2):101–138, 1978.

[EM18] Alex Eskin and Maryam Mirzakhani. Invariant and stationary measures for the $\mathrm{SL}(2,\mathbb{R})$ action on moduli space. *Publications mathématiques de l'Institut des hautes études scientifiques*, 127(1):95–324, 2018.

[EMM15] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi. Isolation, equidistribution, and orbit closures for the $\mathrm{SL}(2,\mathbb{R})$ action on moduli space. *Ann. of Math. (2)*, 182(2):673–721, 2015.

[EMM22] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi. Effective counting of simple closed geodesics on hyperbolic surfaces. *J. Eur. Math. Soc. (JEMS)*, 24(9):3059–3108, 2022.

[FM14] Giovanni Forni and Carlos Matheus. Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards. *J. Mod. Dyn.*, 8(3-4):271–436, 2014.

[Fur73] Harry Furstenberg. The unique ergodicity of the horocycle flow. Recent Advances topol. Dynamics, Proc. Conf. topol. Dynamics Yale Univ. 1972, Lect. Notes Math. 318, 95-115 (1973)., 1973.

[Gol] William Goldman. Geometric structures on manifolds. AMS Graduate Studies in Mathematics, accepted for publication.

[Ham09] Ursula Hamenstädt. Invariant Radon measures on measured lamination space. *Inventiones mathematicae*, 176(2):223–273, 2009.

[Kre85] Ulrich Krengel. *Ergodic theorems*, volume 6 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. With a supplement by Antoine Brunel.

[LL12] John M Lee and John M Lee. *Smooth manifolds*. Springer, 2012.

[LM08] Elon Lindenstrauss and Maryam Mirzakhani. Ergodic theory of the space of measured laminations. *International mathematics research notices*, 2008(1), 2008.

[Mar04] Grigoriy A. Margulis. *On some aspects of the theory of Anosov systems*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows, Translated from the Russian by Valentina Vladimirovna Szulikowska.

[McM07] Curtis T. McMullen. Dynamics of $\mathrm{SL}_2(\mathbb{R})$ over moduli space in genus two. *Ann. of Math. (2)*, 165(2):397–456, 2007.

[MT02] Howard Masur and Serge Tabachnikov. Rational billiards and flat structures. In *Handbook of dynamical systems, Vol. 1A*, pages 1015–1089. North-Holland, Amsterdam, 2002.

[MW02] Yair Minsky and Barak Weiss. Nondivergence of horocyclic flows on moduli space. *Journal für die reine und angewandte Mathematik*, 2002(552):131–177, 2002.

[MW14] Yair Minsky and Barak Weiss. Cohomology classes represented by measured foliations, and Mahler's question for interval exchanges. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(2):245–284, 2014.

[SW04] John Smillie and Barak Weiss. Minimal sets for flows on moduli space. *Israel J. Math.*, 142:249–260, 2004.

[SY] John Smillie and Florent Ygouf. Geometric structures and orbit closures in moduli spaces of translation surfaces. In preparation.

[Vee86] William A Veech. The Teichmüller geodesic flow. *Ann. of Math. (2)*, 124(3):441–530, 1986.

[Wri15a] Alex Wright. Cylinder deformations in orbit closures of translation surfaces. *Geom. Topol.*, 19(1):413–438, 2015.

[Wri15b] Alex Wright. Translation surfaces and their orbit closures: an introduction for a broad audience. *EMS Surv. Math. Sci.*, 2(1):63–108, 2015.

[Zor06] Anton Zorich. Flat surfaces. In *Frontiers in number theory, physics, and geometry. I*, pages 437–583. Springer, Berlin, 2006.

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