MODIFIED SCHMIDT GAMES AND A CONJECTURE OF MARGULIS

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Abstract. We prove a conjecture of G.A. Margulis on the abundance of certain exceptional orbits of partially hyperbolic flows on homogeneous spaces by utilizing a theory of modified Schmidt games, which are modifications of (α, β)-games introduced by W. Schmidt in mid-1960s.

1. Introduction

Let X be a separable metric space and F a group or semigroup acting on X continuously. Given $Z \subset X$, denote by $E(F, Z)$ the set of points of X with F-orbits staying away from Z, that is,

$$E(F, Z) \stackrel{\text{def}}{=} \{x \in X : \overline{Fx} \cap Z = \varnothing\} .$$

If Z consists of a single point z, we will write $E(F, z)$ instead of $E(F, \{z\})$. Such sets are clearly null with respect to any F-ergodic measure of full support. Similarly, denote by $E(F, \infty)$ the set of points of X with bounded F-orbits; it is also a null set with respect to a measure as above if X is not compact. However in many 'chaotic' situations those sets can be quite big, see [Do] and [K1] for history and references.

In this paper we take $X = G/\Gamma$, where G is a Lie group and Γ a lattice in G, and consider

$$F = \{g_t : t \in \mathbb{R}\} \subset G \tag{1.1}$$

acting on X by left translations. One of the major developments in the theory of homogeneous flows is a series of celebrated results of Ratner (see the surveys [Rt, KSS] and a book [Mo]) verifying conjectures of Raghunatan, Dani and Margulis on orbit closures and invariant measures of unipotent flows. Ratner's theorems imply that for quasiunipotent subgroups (F is *quasiunipotent* if all the eigenvalues of $\mathrm{Ad}(g_1)$ are of modulus 1, and *non-quasiunipotent* otherwise), the sets $E(F, Z)$ are countable unions of submanifolds admitting an explicit algebraic description, see [St].

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In his 1990 ICM plenary lecture, just before Ratner's announcement of her results, Margulis formulated a list of conjectures on rigidity properties of unipotent flows. In the same talk Margulis conjectured that such rigid behavior is absent if F is non-quasiunipotent, providing a list of 'non-rigidity conjectures for non-unipotent flows'. For example, he conjectured [Ma, Conjecture (A)] that $E(F,\infty)$ is a *thick* set; that is, its intersection with any open subset of X has full Hausdorff dimension. In fact, to avoid obvious counterexamples, one should assume that the flow $(G/\Gamma, F)$ is *absolutely non-quasiunipotent*, that is, it has no factors $(G'/\Gamma', F')$ (i.e., homomorphisms $p: G \to G'$ with $F' = p(F)$ and $\Gamma' = p(\Gamma)$ a lattice in G') such that F' is nontrivial and quasiunipotent. In 1996 Margulis and the second-named author [KM] proved Conjecture (A) of Margulis under this hypothesis.

Note that even earlier, in the mid-1980s, Dani established the thickness of $E(F,\infty)$ in the following two special cases [Da2, Da3]:

$$\begin{gathered} G = \mathrm{SL}_k(\mathbb{R}),\ \Gamma = \mathrm{SL}_k(\mathbb{Z}),\ \text{and } F = \{g_t\}\,, \\ \text{where } g_t = \mathrm{diag}(e^{ct}, \dots, e^{ct}, e^{-dt}, \dots, e^{-dt}) \end{gathered} \tag{1.2}$$

(here $c, d > 0$ are chosen so that the determinant of g_t is 1), and

$$G \text{ is a connected semisimple Lie group with } \mathrm{rank}_{\mathbb{R}}(G) = 1. \tag{1.3}$$

In both cases Dani's approach consisted in studying the set of points $x \in G/\Gamma$ with bounded one-sided trajectories $\{g_t x : t \ge 0\}$, and showing that its intersection with any orbit of a certain subgroup of G is a *winning set* for a game invented by Schmidt [Sc1]. This property is stronger than thickness in the sense that a countable intersection of winning sets is also winning; see [Da4] for a nice survey of Dani's results.

Understanding one-sided trajectories also plays a crucial role in the approach of [KM], as well as in the present paper. In what follows we will reserve the notation F for one-parameter subgroups as in (1.1), and let $F^+ = \{g_t : t \ge 0\}$. Also throughout the paper we will denote by H^+ the *expanding horospherical subgroup* corresponding to F^+, defined by

$$H^+ \stackrel{\mathrm{def}}{=} \{g \in G : g_{-t} g g_t \to_{t\to+\infty} e\}\,. \tag{1.4}$$

Note that H^+ is nontrivial if and only if F is non-quasiunipotent. It is a connected simply connected nilpotent Lie group normalized by g_t. Moreover, it admits a one-parameter semigroup of *contracting automorphisms*

$$\mathcal{F} = \{\Phi_t : t > 0\}, \quad \text{where } \Phi_t(g) = g_{-t} g g_t\,. \tag{1.5}$$

The main step of the proof in [KM] was to show that, in the special case when G is a connected semisimple Lie group without compact factors and Γ is an irreducible lattice in G, for any $x \in G/\Gamma$ the set

$$\{h \in H^+ : hx \in E(F^+, \infty)\} \tag{1.6}$$

is thick. In the present paper we strengthen the above conclusion, replacing thickness with the winning property for a certain game. More precisely, in [KW3] the notion of *modified Schmidt games* (to be abbreviated by MSGs) induced by semigroups of contracting automorphisms was introduced. These generalize the game devised by Schmidt and share many of its appealing properties. In particular, winning sets of these games are thick and the countable intersection property also holds. All the relevant definitions and facts are discussed in §2. The following is one of our main results:

Theorem 1.1. *Let G be a connected semisimple centerfree Lie group without compact factors, Γ an irreducible lattice in G, F a one-parameter non-quasiunipotent subgroup of G. Then there exists $a' > 0$ such that for any $x \in G/\Gamma$ and $a > a'$, the set* (1.6)*is an a-winning set for the MSG induced by $\mathcal{F}$ as in* (1.5)*.*

We also show

Theorem 1.2. *Let G, F and Γ be as in Theorem 1.1. Then there exists $a'' > 0$ such that for any $x, z \in G/\Gamma$ and $a > a''$, the set*

$$\big\{h \in H^+ : hx \in E(F^+, z)\big\}$$

is an a-winning set for the MSG induced by $\mathcal{F}$ as in (1.5)*.*

This strengthens the results of [K1], where the above sets were shown to be thick. Because of the countable intersection property, Theorems 1.1 and 1.2 readily imply the following

Corollary 1.3. *Let G, Γ and F be as in Theorem 1.1, and let Z be a countable subset of G/Γ. Then for any $x \in G/\Gamma$, the set*

$$\big\{h \in H^+ : hx \in E(F^+, Z) \cap E(F^+, \infty)\big\} \tag{1.7}$$

is a winning set for the MSG induced by $\mathcal{F}$.

This extends a result of Dolgopyat [Do, Corollary 2] who established the thickness of sets

$$\big\{h \in H^+ : hx \in E(F^+, Z) \cap E(F^+, \infty)\big\}$$

for countable Z in the case when the F^+-action corresponds to a geodesic flow on the unit tangent bundle to a manifold of constant negative curvature and finite volume.

From Corollary 1.3 we deduce

Theorem 1.4. *Let G be a Lie group, Γ a lattice in G, F a one-parameter absolutely non-quasiunipotent subgroup of G, and Z a countable subset of G/Γ. Then the set*

$$\big\{x \in E(F,Z) \cap E(F,\infty) : \dim(\overline{Fx}) < \dim(G)\big\} \tag{1.8}$$

is thick.

Here and hereafter $\dim(X)$ stands for the Hausdorff dimension of a metric space X. The above theorem settles Conjecture (B) made by Margulis in the aforementioned ICM address [Ma]; in fact, that conjecture was about *finite* $Z \subset G/\Gamma$. To reduce Theorem 1.4 to Corollary 1.3, we employ a standard (see e.g. [Da2]) reduction from G semisimple to general G, pass from one-sided to two-sided orbits roughly following the lines of [KM, §1], and also use an entropy argument, which we learned from Einsiedler and Lindenstrauss, to show that the condition $\dim(\overline{Fx}) < \dim(G)$ is automatic as long as $x \in E(F,Z) \cap E(F,\infty)$.

Both Theorem 1.1 and Theorem 1.2 will be proved in a stronger form. Namely, we establish the conclusion of Theorem 1.1 with H^+ replaced with its certain proper subgroups, so-called (F^+,Γ)-expanding subgroups, introduced and studied in §5. We will show that H^+ is always (F^+,Γ)-expanding; thus Theorem 1.1 follows from its stronger version, Theorem 5.2. Such a generalization is useful for other applications as well. Namely, in §9 we apply Theorem 5.2 to Diophantine approximation with weights, strengthening results obtained earlier in [KW2, KW3].

As for Theorem 1.2, its conclusion in fact holds for any homogeneous space G/Γ, where G is a Lie group and $\Gamma \subset G$ is discrete. Further, we are able to replace H^+ by any nontrivial connected subgroup $H \subset H^+$ normalized by F, see Theorem 7.1 for a more general statement. One application of that is a possibility of intersecting sets $E(F^+,z)$ for *different* choices of groups F, see Corollary 9.2.

The structure of the paper is as follows. In §2 we describe modified Schmidt games, the main tool designed to prove our results. Then in §3 we review reduction theory and geometry of rank-one homogeneous spaces to state criteria for compactness of subsets of G/Γ. We develop further properties of the compactness criteria in the following section. Section 5 contains the definition, properties and examples of (F^+,Γ)-expanding subgroups. There we also state Theorem 5.2 which is proved in §6. Then in §7 we deal with Theorem 1.2 via its stronger version, Theorem 7.1, and derive Corollary 1.3. The reduction of Theorem 1.4

to Corollary 1.3 occupies §8. Several concluding remarks are made in the last section of the paper.

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2. Games

In what follows, all distances (diameters of sets) in various metric spaces will be denoted by 'dist' ('diam'). Let H be a connected Lie group with a right-invariant Riemannian metric, and assume that it admits a one-parameter group of automorphisms $\{\Phi_t : t \in \mathbb{R}\}$ such that Φ_t is contracting for positive t (recall that $\Phi : H \to H$ is *contracting* if for every $g \in H$, $\Phi^k(g) \to e$ as $k \to \infty$). In other words, $\Phi_t = \exp(tY)$, where $Y \in \mathrm{End}\big(\mathrm{Lie}(H)\big)$ and the real parts of all eigenvalues of Y are negative. Note that Y is not assumed to be diagonalizable. We denote by $\mathcal{F}$ the semigroup $\mathcal{F} \stackrel{\mathrm{def}}{=} \{\Phi_t : t > 0\}$ corresponding to positive values of t. In fact, in all the examples of the present paper H will be a subgroup of H^+ as in (1.4) normalized by F as in (1.1), and $\mathcal{F}$ will be of the form (1.5).

Say that a subset D_0 of H is *admissible* if it is compact and has non-empty interior. Fix an admissible D_0 and denote by $\mathcal{D}_t$ the set of all right-translates of $\Phi_t(D_0)$,

$$\mathcal{D}_t \stackrel{\mathrm{def}}{=} \{\Phi_t(D_0)h : h \in H\}\,. \tag{2.1}$$

Because the maps Φ_t, $t > 0$, are contracting and D_0 is admissible, there exists a_* such that for any $t > a_*$, the set of $D \in \mathcal{D}_t$ contained in D_0 is nonempty. Also, since D_0 is bounded, it follows that for some $c_0 > 0$ one has $\mathrm{dist}\big(\Phi_t(g), \Phi_t(h)\big) \le c_0 e^{-\sigma t}$ for all $g, h \in D_0$, where $\sigma > 0$ is such that the real parts of all the eigenvalues of Y as above are smaller than $-\sigma$. Therefore

$$\mathrm{diam}(D) \le c_0 e^{-\sigma t} \quad \forall\, D \in \mathcal{D}_t \tag{2.2}$$

(recall that the metric is chosen to be right-invariant, so all the elements of $\mathcal{D}_t$ are isometric to $\Phi_t(D_0)$).

Now pick two real numbers $a, b > a_*$ and, following [KW3], define a game, played by two players Alice and Bob. First Bob picks $t_1 \in \mathbb{R}$ and $h_1 \in H$, thus defining $B_1 \stackrel{\text{def}}{=} \Phi_{t_1}(D_0)h_1 \in \mathcal{D}_{t_1}$. Then Alice picks a translate A_1 of $\Phi_a(B_1)$ which is contained in B_1, Bob picks a translate B_2 of $\Phi_b(A_1)$ which is contained in A_1, after that Alice picks a translate A_2 of $\Phi_a(B_2)$ which is contained in B_2, and so on. In other words, for $k \in \mathbb{N}$ we set

$$t_k = t_1 + (k-1)(a+b) \text{ and } t'_k = t_1 + (k-1)(a+b) + a\,. \tag{2.3}$$

Then at the kth step of the game, Alice picks $A_k \in \mathcal{D}_{t'_k}$ inside B_k, and then Bob picks $B_{k+1} \in \mathcal{D}_{t_{k+1}}$ inside A_k etc. Note that Alice and Bob can choose such sets because $a, b > a_*$, and that the intersection

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=0}^{\infty} B_i \tag{2.4}$$

is nonempty and consists of a single point in view of (2.2). We will refer to this procedure as to the $\{\mathcal{D}_t\}$-(a, b)-*modified Schmidt game*, abbreviated as $\{\mathcal{D}_t\}$-(a, b)-MSG. Let us say that $S \subset H$ is (a, b)-*winning* for the $\{\mathcal{D}_t\}$-MSG if Alice can proceed in such a way that the point (2.4) is contained in S no matter how Bob plays. Similarly, say that S is an a-*winning* set of the game if S is (a, b)-winning for any choice of $b > a_*$, and that S is *winning* if it is a-winning for some $a > a_*$.

The game described above falls into the framework of $(\mathfrak{F}, \mathfrak{S})$-games introduced by Schmidt in [Sc1]. Taking $H = \mathbb{R}^n$, D_0 to be the closed unit ball in $\mathbb{R}^n$ and $\Phi_t = e^{-t}\mathrm{Id}$ corresponds to the set-up where $\mathcal{D}_t$ consists of balls of radius e^{-t}, the standard special case of the construction often referred to as *Schmidt's* (α, β)-*game*[1]. It was shown in [KW3] that many features of Schmidt's original game, established in [Sc1], are present in this more general situation. The next theorem summarizes the two basic properties important for our purposes:

Theorem 2.1. *Let H, $\mathcal{F}$ and admissible D_0 be as above.*

(a) [KW3, Theorem 2.4] *Let $a > a_*$, and let $S_i \subset H$, $i \in \mathbb{N}$, be a sequence of a-winning sets of the $\{\mathcal{D}_t\}$-MSG. Then $\cap_{i=1}^{\infty} S_i$ is also a-winning.*

(b) [KW3, Corollary 3.4] *Any winning set for the $\{\mathcal{D}_t\}$-MSG is thick.*

Let us record another useful observation made in [KW3]:

[1]Traditionally the game is described in multiplicative notation, where radii of balls are multiplied by fixed constants $\alpha = e^{-a}$ and $\beta = e^{-b}$.

Proposition 2.2. [KW3, Proposition 3.1] *Let D_0, D_0' be admissible, and define $\{\mathcal{D}_t\}$ and $\{\mathcal{D}_t'\}$ as in* (2.1) *using D_0 and D_0' respectively. Let $s > 0$ be such that for some $h, h' \in H$,*

$$\Phi_s(D_0)h \subset D_0' \text{ and } \Phi_s(D_0')h' \subset D_0 \tag{2.5}$$

(such an s exists in light of (2.2) *and the admissibility of D_0, D_0'). Suppose that S is a-winning for the $\{\mathcal{D}_t\}$-MSG, then it is $(a+2s)$-winning for the $\{\mathcal{D}_t'\}$-MSG.*

Based on the above proposition, in what follows we are going to choose some admissible initial domain $D_0 \subset H$, not worrying about specifying it explicitly, and use it to define the game. Moreover, we will suppress D_0 from the notation and refer to the game as to the modified Schmidt game *induced by* $\mathcal{F}$. Even though the game itself, and the value of a_*, will depend on the choice of an admissible D_0, the class of winning sets will not. Hopefully this terminology will not lead to confusion.

3. Compactness criteria

For a Lie group G and its discrete subgroup Γ, we denote by $\mathfrak{g}$ the Lie algebra of G and by π the quotient map $G \to G/\Gamma$, $g \mapsto g\Gamma$. In the next three sections we will take G and Γ as in Theorem 1.1, that is, G is a connected semisimple centerfree Lie group without compact factors. It is well-known (see e.g. [Z, Prop. 3.1.6]) that under these assumptions, G is isomorphic to the connected component of the identity in the real points of an algebraic group defined over $\mathbb{Q}$; in the sequel we will use this algebraic structure without further comment. We also assume $\Gamma \subset G$ to be non-uniform and irreducible. By the Margulis Arithmeticity Theorem [Z, Cor. 6.1.10], either $\mathrm{rank}_{\mathbb{R}}(G) = 1$, or Γ is *arithmetic*, i.e. $G(\mathbb{Z})$ is commensurable with Γ. In this section we review results which give necessary and sufficient conditions for a subset of G/Γ to be precompact. We will discuss each of the above cases separately.

3.1. **Rank one spaces.** Let $\mathrm{rank}_{\mathbb{R}}(G) = 1$. Let P be a minimal $\mathbb{R}$-parabolic subgroup of G, $U = \mathrm{Rad}_u(P)$ the unipotent radical of P, $\mathfrak{u} = \mathrm{Lie}(U)$, $d = \dim \mathfrak{u}$, $V = \bigwedge^d \mathfrak{g}$. Denote

$$\rho \stackrel{\text{def}}{=} \textstyle\bigwedge^d \mathrm{Ad} : G \to \mathrm{GL}(V)\,, \tag{3.1}$$

and choose a nonzero element $\mathbf{p}_{\mathfrak{u}}$ in the one-dimensional subspace $\bigwedge^d \mathrm{Lie}(U)$ of V. Also fix a norm $\|\cdot\|$ on V. Then we have

Proposition 3.1 (Compactness criterion, rank one case)**.** *There is a finite $C \subset G$ such that the following hold:*

(1) *For any* $c \in C$, $\rho(\Gamma c^{-1})\mathbf{p}_{\mathfrak{u}}$ *is discrete.*
(2) *For any* $L \subset G$, $\pi(L) \subset G/\Gamma$ *is precompact if and only if there is* $\varepsilon > 0$ *such that for all* $\gamma \in \Gamma$, $c \in C$, $g \in L$ *one has* $\|\rho(g\gamma c^{-1})\mathbf{p}_{\mathfrak{u}}\| \geq \varepsilon$.
(3) *There exists* ε_0 *such that for any* $g \in G$, *there is at most one* $\mathbf{v} \in \rho(\Gamma C^{-1})\mathbf{p}_{\mathfrak{u}}$ *such that* $\|\rho(g)\mathbf{v}\| < \varepsilon_0$.

Proof. This result, essentially due to H. Garland and M.S. Raghunathan, is not usually stated in this representation-theoretic language. For the reader's convenience we indicate here how to deduce the Proposition from [GR].

Let K be a maximal compact subgroup of G, let A be a maximal $\mathbb{R}$-split torus contained in P (which is one-dimensional in this case), let α be the nontrivial character on A such that $\rho(a)\mathbf{p}_{\mathfrak{u}} = e^{-\alpha(a)}\mathbf{p}_{\mathfrak{u}}$, and let

$$A_\eta = \{a \in A : \alpha(a) \geq \eta\}.$$

According to [GR, Thm. 0.6], there is a finite $C \subset G$, a compact subset $\Sigma \subset U$ and $\eta \in \mathbb{R}$ such that

(i) For any $c \in C$, $c\Gamma c^{-1} \cap U$ is a lattice in U.
(ii) $G = KA_\eta \Sigma C\Gamma$.
(iii) Given a compact $\Sigma' \supset \Sigma$, there is $s > 0$ such that, if $c_1, c_2 \in C$ and $\gamma_1, \gamma_2 \in \Gamma$ are such that $KA_s\Sigma' c_1\gamma_1 \cap KA_\eta \Sigma' c_2\gamma_2 \neq \varnothing$ then $c_1 = c_2$ and $\rho(c_1\gamma_1\gamma_2^{-1}c_2^{-1})$ fixes $\mathbf{p}_{\mathfrak{u}}$.

Assertion (1) follows from (i) and a result of Dani and Margulis [KSS, Thm. 3.4.11] applied to the lattice $c\Gamma c^{-1}$. The direction $\Longrightarrow$ in (2) is immediate from (1). To deduce $\Longleftarrow$, note that if $g_n \in L$ is written as $g_n = k_n a_n \sigma_n c_n \gamma_n$ as in (ii) , and $\pi(g_n) \to \infty$ in G/Γ, then necessarily $\alpha(a_n) \to \infty$. Take an infinite subsequence for which $c_n = c$ is constant and $k_n \to k_0$. Applying $\rho(g_n)$ to $\mathbf{v}_n \stackrel{\text{def}}{=} \rho(\gamma_n^{-1}c^{-1})\mathbf{p}_{\mathfrak{u}}$ and using the fact that $\rho(\sigma_n)\mathbf{p}_{\mathfrak{u}} = \mathbf{p}_{\mathfrak{u}}$ and K is compact, we see that

$$\rho(g_n)\mathbf{v}_n = \rho(k_n a_n)\mathbf{p}_{\mathfrak{u}} = e^{-\alpha(a_n)}\rho(k_n)\mathbf{p}_{\mathfrak{u}} \to 0.$$

This proves (2).

Now let $\Sigma' \supset \Sigma$ be a compact subset of U which contains a fundamental domain for the lattices $U \cap c\Gamma c^{-1}$ for each $c \in C$. Given $g \in G, c \in C, \gamma \in \Gamma$ we can use Iwasawa decomposition to write

$$g\gamma^{-1}c^{-1} = ka_t n'\bar{n},$$

where $n' \in \Sigma', \bar{n} \in U \cap c\Gamma c^{-1}, k \in K$, and $a_t \in A$. With no loss of generality assume the norm $\|\cdot\|$ on V is $\rho(K)$-invariant. Since

$$\|\rho(g\gamma^{-1}c^{-1})\mathbf{p}_{\mathfrak{u}}\| = \|\rho(ka_t n'\bar{n})\mathbf{p}_{\mathfrak{u}}\| = \|\rho(ka_t)\mathbf{p}_{\mathfrak{u}}\| = e^{-\alpha(t)}\|\mathbf{p}_{\mathfrak{u}}\|,$$

there exists ε_0 such that if

$$\|\rho(g\gamma^{-1}c^{-1})\mathbf{p}_{\mathfrak{u}}\| < \varepsilon_0 \tag{3.2}$$

then $t \geq s$ where s is as in (iii). Writing $\bar{n} = c\bar{\gamma}c^{-1}$ we see that $g \in KA_s\Sigma'\bar{n}c\gamma = KA_s\Sigma'c\bar{\gamma}\gamma$. From (iii) it follows that for a given g, (3.2) determines $c \in C$ uniquely. Using (iii) again, and the fact that $\rho(\bar{n})$ fixes $\mathbf{p}_{\mathfrak{u}}$, we obtain (3). □

3.2. **Arithmetic homogeneous spaces.** Here we assume that Γ is commensurable with the group of integer points of G, and equip $\mathfrak{g}$ with a $\mathbb{Q}$-structure which is $\mathrm{Ad}(\Gamma)$-invariant. Fix some rational basis for $\mathfrak{g}$ and denote its $\mathbb{Z}$-span by $\mathfrak{g}_{\mathbb{Z}}$. Let $P_1, \dots, P_r$ be the $\mathbb{Q}$-parabolic subgroups containing a fixed minimal $\mathbb{Q}$-parabolic subgroup, and let $\mathfrak{u}_1, \dots, \mathfrak{u}_r$ denote the Lie algebras of their unipotent radicals. Then it is known [Bo1] that $r = 2^{\mathrm{rank}_{\mathbb{Q}}(G)} - 1$, where $\mathrm{rank}_{\mathbb{Q}}(G)$ is the dimension of a maximal $\mathbb{Q}$-split torus in G. Since G/Γ is compact when $\mathrm{rank}_{\mathbb{Q}}(G) = 0$, we will assume that $\mathrm{rank}_{\mathbb{Q}}(G) \geq 1$, and hence that $r \geq 1$. For $j = 1, \dots, r$ let $\mathcal{P}_j$ denote the set of all conjugates of P_j, let $\mathcal{R}_j$ denote all the Lie algebras of unipotent radicals of conjugates of P_j defined over $\mathbb{Q}$, let $\mathcal{P} = \bigcup_j \mathcal{P}_j$ and $\mathcal{R} = \bigcup_j \mathcal{R}_j$.

Now for a neighborhood W of 0 in $\mathfrak{g}$, $g \in G$, and $\mathfrak{u} \in \mathcal{R}$, we say that $\mathfrak{u}$ is *W-active for g* if $\mathrm{Ad}(g)\mathfrak{u} \subset \mathrm{span}\,(W \cap \mathrm{Ad}(g)\mathfrak{g}_{\mathbb{Z}})$. We will use W-active Lie algebras for formulating a compactness criterion. It says essentially that a sequence $\pi(g_n)$ leaves larger and larger compact sets in G/Γ, if and only if the conjugated lattices $\mathrm{Ad}(g_n)\mathfrak{g}_{\mathbb{Z}}$ contain smaller and smaller spanning sets for unipotent radicals of parabolics.

Proposition 3.2 (Compactness criterion, arithmetic case)**.** *For any $L \subset G$, $\pi(L) \subset G/\Gamma$ is unbounded if and only if for any neighborhood W of 0 in $\mathfrak{g}$ there is $g \in L$, and $\mathfrak{u} \in \mathcal{R}$ which is W-active for g.*

Proposition 3.2 was essentially proved in [TW, Prop. 3.5], and will be reproved in this paper. The 'if' direction follows immediately from the discreteness of $\mathfrak{g}_{\mathbb{Z}}$, and the converse is proved using reduction theory. We postpone the proof of the converse to §4, where the stronger Proposition 4.1 is proved.

For $j = 1, \dots, r$ and for $\mathfrak{u} \in \mathcal{R}_j$, let $\mathbf{p}_{\mathfrak{u}} = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{d_j} \in \hat{V}_j \stackrel{\mathrm{def}}{=} \bigwedge^{d_j} \mathfrak{g}$, where $\mathbf{u}_1, \dots, \mathbf{u}_{d_j} \in \mathfrak{g}_{\mathbb{Z}}$ form a basis for the $\mathbb{Z}$-module $\mathfrak{u} \cap \mathfrak{g}_{\mathbb{Z}}$ ($\mathbf{p}_{\mathfrak{u}}$ is uniquely determined up to a sign). Define

$$\rho_j \stackrel{\mathrm{def}}{=} \bigwedge\nolimits^{d_j} \mathrm{Ad} : G \to \mathrm{GL}(\hat{V}_j) \quad \text{and} \quad V_j = \mathrm{span}\left(\rho_j(G)\mathbf{p}_{\mathfrak{u}_j}\right). \tag{3.3}$$

Each $\mathbf{p}_{\mathfrak{u}}$ corresponding to $\mathfrak{u} \in \mathcal{R}_j$ is contained in V_j, since all elements of $\mathcal{R}_j$ are in the same $\rho_j(G)$-orbit. Also each V_j is irreducible since $\mathbf{p}_{\mathfrak{u}_j}$

is fixed by a parabolic subgroup of G. Fix a norm on each V_j. Since $\mathfrak{g}_{\mathbb{Z}}$ is discrete, $\inf_{\mathfrak{u}\in\mathcal{R}} \|\mathbf{p}_{\mathfrak{u}}\| > 0$. Hence, for any $L \subset G$ precompact, we also have

$$\min_j \inf_{\mathfrak{u}\in\mathcal{R}_j,\, g\in L} \|\rho_j(g)\mathbf{p}_{\mathfrak{u}}\| > 0.$$

For any $\varepsilon > 0$ and any j there is a neighborhood W of 0 in $\mathfrak{g}$ such that if $\mathfrak{u} \in \mathcal{R}_j$ is W-active for g, then $\|\rho_j(g)\mathbf{p}_{\mathfrak{u}}\| < \varepsilon$. For each $j = 1, \dots, r$, let

$$\ell_j(g) \stackrel{\text{def}}{=} \inf\{\|\rho(g)\mathbf{p}_{\mathfrak{u}}\|^{1/d_j} : \mathfrak{u} \in \mathcal{R}_j\}, \quad \ell(g) = \min_j \ell_j(g).$$

With this notation we can formulate another compactness criterion which follows immediately from the preceding discussion.

Corollary 3.3. *For any $L \subset G$, $\pi(L) \subset G/\Gamma$ is precompact if and only if* $\inf_{g\in L} \ell(g) > 0$.

We will say that a linear subspace of $\mathfrak{g}$ is *unipotent* if it is contained in the Lie algebra of a unipotent subgroup. Clearly any subspace of a unipotent subspace is also unipotent.

Proposition 3.4 ([TW], Prop. 3.3)**.** *There is a neighborhood W of* 0 *in $\mathfrak{g}$ such that for any $g \in G$, the span of* $\mathrm{Ad}(g)\mathfrak{g}_{\mathbb{Z}} \cap W$ *is unipotent.*

We will need the following consequence of [TW, Prop. 5.3]:

Proposition 3.5. *Suppose for some $j \in \{1, \dots, r\}$ that $\mathfrak{u}, \mathfrak{u}' \in \mathcal{R}_j$, and that* $\mathrm{span}(\mathfrak{u}, \mathfrak{u}')$ *is unipotent. Then $\mathfrak{u} = \mathfrak{u}'$. In particular, for any unipotent $\mathfrak{v} \subset \mathfrak{g}$,*

$$\#\{\mathfrak{u} \in \mathcal{R} : \mathfrak{u} \subset \mathfrak{v}\} \le r.$$

Proof. Since $\mathrm{span}(\mathfrak{u}, \mathfrak{u}')$ is unipotent and defined over $\mathbb{Q}$, it is contained in a maximal unipotent subgroup defined over $\mathbb{Q}$, which is the unipotent radical of a minimal $\mathbb{Q}$-parabolic B. Since $\mathfrak{u}, \mathfrak{u}'$ belong to the same $\mathcal{R}_j$, there is $g \in G$ such that $\mathfrak{u}' = \mathrm{Ad}(g)(\mathfrak{u})$. Applying statement (i) of [TW, Prop. 5.3], with P the parabolic group whose unipotent radical has Lie algebra $\mathfrak{u}$, we see that $B \subset P$. Applying (i) again with gPg^{-1}, we see that $B \subset gPg^{-1}$, so by statement (ii), $g \in P$, hence $\mathfrak{u}' = \mathrm{Ad}(g)(\mathfrak{u}) = \mathfrak{u}$. This proves the first statement, and the second is an immediate consequence. $\square$

4. Active elements are essentially smallest

We will need to obtain more precise information in Proposition 3.2 and Corollary 3.3, relating the values of ℓ with the corresponding values obtained using W-active elements only. This information is contained in the following.

Proposition 4.1. *For any neighborhood W of 0 in $\mathfrak{g}$ and any $M > 0$ there is a compact $K \subset G/\Gamma$ such that if $\pi(g) \notin K$ then*

(i) *There is j and there is $\mathfrak{u} \in \mathcal{R}_j$ such that $\ell(g) = \|\rho_j(g)\mathbf{p}_{\mathfrak{u}}\|^{1/d_j}$ and $\mathfrak{u}$ is W-active for g.*

(ii) *For any $j = 1, \ldots, r$ and any $\mathfrak{u}' \in \mathcal{R}_j$ which is not W-active for g,*

$$\|\rho_j(g)\mathbf{p}_{\mathfrak{u}'}\|^{1/d_j} \geq M\,\ell(g).$$

In particular there is $c > 0$ such that

$$\ell(g) \geq c \min\left\{\|\rho_j(g)\mathbf{p}_{\mathfrak{u}}\|^{1/d_j} : \mathfrak{u} \text{ is } W\text{-active for } g\right\}$$

(where we adopt a convention $\min \varnothing = 1$*).*

Note that the 'only if' implication in Proposition 3.2 is an immediate consequence of assertion (i). For the proof of Proposition 4.1, we will require some well-known facts regarding the structure of parabolic $\mathbb{Q}$-subgroups and reduction theory, see [Bo1, Bo2] for more details. Fix a maximal $\mathbb{Q}$-split torus S in G, which we identify with its lie algebra in a way which should cause no confusion. Let $\Phi = {}_{\mathbb{Q}}\Phi \subset S^*$ denote the $\mathbb{Q}$-roots, i.e. $\lambda \in \Phi$ if and only if

$$\mathfrak{g}_\lambda \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathfrak{g} : \forall s \in S,\ \mathrm{Ad}(s)\mathbf{x} = e^{\lambda(s)}\mathbf{x}\}$$

is nontrivial. Then Φ is an indecomposable (possibly non-reduced) root system. For each j there is $\Psi_j \subset \Phi$ such that

$$\mathfrak{u}_j = \bigoplus_{\lambda \in \Psi_j} \mathfrak{g}_\lambda.$$

An order on Φ is chosen so that $\Psi_j \subset \Phi^+$ for all j. We will need to identify the sets Ψ_j explicitly. Let $\Delta \subset \Phi^+$ be a basis of positive simple roots, and given a root $\lambda \in \Phi^+$, write $\lambda = \sum_{\beta \in \Delta} n_\beta(\lambda)\beta$, and say that β *contributes to* λ if $n_\beta(\lambda) \neq 0$. Then for each j there is a nonempty $\Delta_j \subset \Delta$ such that

$$\Psi_j = \{\lambda \in \Phi^+ : \exists \beta \in \Delta_j \text{ such that } \beta \text{ contributes to } \lambda\}. \tag{4.1}$$

For each j, let

$$\chi_j \stackrel{\text{def}}{=} \sum_{\lambda \in \Psi_j} \lambda.$$

Then χ_j is the weight corresponding to $\mathfrak{u}_j$, i.e. for all $s \in S$ we have $\rho_j(s)\mathbf{p}_{\mathfrak{u}_j} = e^{\chi_j(s)}\mathbf{p}_{\mathfrak{u}_j}$.

We can write

$$g = ksnf\gamma, \tag{4.2}$$

where $\gamma \in \Gamma$, f is in a finite subset of $G(\mathbb{Q})$, n is contained in a compact subset of the minimal $\mathbb{Q}$-parabolic $\cap_j P_j$, and k is contained in a compact subset of the centralizer of S. In the above representation we have

$$s \in S_\eta \stackrel{\mathrm{def}}{=} \{s \in S : \forall \alpha \in \Delta, \alpha(s) \leq \eta\}$$

for some η. Note that the presentation (4.2) is not unique in general, but for each g there are at most finitely many such presentations. We will fix a norm on S and note that $g \to \infty$ in G/Γ, if and only if $\|s\| \to \infty$, where s is given by the decomposition (4.2), which in turn happens if and only if $\min_{\alpha \in \Delta} \alpha(s) \to -\infty$.

In the remainder of this section we will use Vinogradov notation, i.e. $A \ll B$ means that A and B may depend on g and there is a constant C, independent of g, such that $A \leq CB$. Also $A \asymp B$ means $A \ll B$ and $A \gg B$.

Proof of Proposition 4.1. Step 1. We first claim that up to constants, in the definition of $\ell(g)$, instead of letting the inf range over all $\mathfrak{u} \in \mathcal{R}$ we may use the special elements $\mathrm{Ad}(\gamma^{-1} f^{-1})\mathfrak{u}_j$ for some j, where f and γ are as in (4.2). Indeed, in each representation ρ_j, χ_j is a dominant weight which means that for any $\alpha \in S^*$ appearing in the weight decomposition

$$V_j = \bigoplus_{\alpha \in S^*} V_{j,\alpha}, \quad V_{j,\alpha} \stackrel{\mathrm{def}}{=} \{\mathbf{x} \in V_j : \forall s \in S,\ \rho_j(s)\mathbf{x} = e^{\alpha(s)}\mathbf{x}\},$$

the difference $\chi_j - \alpha$ is positive, i.e. a linear combination of elements of Δ with non-negative coefficients, implying that $\chi_j(s) - \alpha(s)$ is bounded below for $s \in S_\eta$. Moreover, we can complete $\mathbf{p}_{\mathfrak{u}_j}$ to a basis of V_j consisting of eigenvectors for $\rho_j(S)$, such that for each $\mathfrak{u} \in \mathcal{R}_j, \mathbf{p}_{\mathfrak{u}}$ is a linear combination of these basis elements with integer coefficients. Therefore, comparing with the sup-norm with respect to this basis, we see that for each $s \in S_\eta$,

$$\inf_{\mathfrak{u} \in \mathcal{R}_j} \|\rho_j(s)\mathbf{p}_{\mathfrak{u}}\| \gg e^{\chi_j(s)} = \|\rho_j(s)\mathbf{p}_{\mathfrak{u}_j}\| = \|\rho_j(sn\gamma)\mathbf{p}_{\mathrm{Ad}(\gamma^{-1}f^{-1})\mathfrak{u}_j}\|.$$

These considerations also show

$$\ell_j(g) \asymp e^{\chi_j(s)/d_j}. \tag{4.3}$$

Step 2. We will now show that there is T such that if $\alpha(s) < T$ for all $\alpha \in \Delta_j$, then $\mathrm{Ad}(\gamma^{-1} f^{-1})\mathfrak{u}_j$ is W-active for g. Indeed, for every $\lambda \in \Psi_j$ there is $\alpha \in \Delta_j$ which contributes to λ, and since $s \in S_\eta$, $\lambda(s) \ll \alpha(s)$. Since n in (4.2) preserves the subspace $\mathfrak{u}_j$, and n and k come from a fixed compact set, we find

$$\|\rho(g)X\| \ll e^T \|X\|$$

for any $X \in \mathfrak{u}_j$. The claim follows on letting $T \to -\infty$.

Step 3. Now let

$$S_0 \stackrel{\mathrm{def}}{=} \{s \in S : \|s\| = 1 \text{ and } \forall \beta \in \Delta,\ \beta(s) \leq 0\}.$$

For each fixed $s_0 \in S_0$ let

$$\Delta' \stackrel{\mathrm{def}}{=} \{\beta \in \Delta : \beta(s_0) = 0\}, \qquad \Delta'' \stackrel{\mathrm{def}}{=} \Delta \smallsetminus \Delta'$$

(so that $\Delta'' \neq \varnothing$). We claim that for all ℓ for which

$$\frac{\chi_\ell(s_0)}{d_\ell} = \min_j \frac{\chi_j(s_0)}{d_j}, \tag{4.4}$$

the set Δ_ℓ defined by (4.1) does not intersect Δ'. Indeed, let k be the index for which $\Delta_k = \Delta''$, and assume by contradiction that $\Delta' \cap \Delta_\ell \neq \varnothing$. Write Ψ_k, Ψ_ℓ for the set of roots corresponding via (4.1) to $\mathfrak{u}_k, \mathfrak{u}_\ell$ respectively and let $A = \Psi_k \cap \Psi_\ell, B = \Psi_k \smallsetminus \Psi_\ell, C = \Psi_\ell \smallsetminus \Psi_k$. Then C contains the roots in $\Delta' \cap \Delta_\ell$ so is nonempty, and for every $\lambda \in C, \lambda(s_0) = 0$. Also for every $\lambda \in B, \lambda(s_0) < 0$, since there is some $\beta \in \Delta_k$ contributing to λ. This implies that for any $\lambda_1 \in B, \lambda_2 \in C$, we have $\lambda_1(s_0) < \lambda_2(s_0)$. The numbers $\frac{\chi_k(s_0)}{d_k}, \frac{\chi_\ell(s_0)}{d_\ell}$ are the averages of the numbers $\lambda(s_0)$, where λ ranges over $A \cup B$ and $A \cup C$ respectively. From this it follows that $\frac{\chi_k(s_0)}{d_k} < \frac{\chi_\ell(s_0)}{d_\ell}$, a contradiction to (4.4).

Step 4. We now show that for all ℓ for which (4.4) holds, the eigenspace in V_ℓ for $\rho_\ell(s_0)$ corresponding to $\chi_\ell(s_0)$ is one-dimensional, i.e. is not larger than $\mathrm{span}(\mathbf{p}_{\mathfrak{u}_\ell})$. It suffices to show that if χ is any other character appearing in the weight-decomposition of V_ℓ, then $\chi(s_0) > \chi_\ell(s_0)$. To see this, note that by Step 3, for any $\lambda \in \Psi_\ell$ there is $\beta \in \Delta''$ contributing to λ. So if $\alpha \in \Delta'$, the linear combination $\lambda - \alpha$ contains β with a positive coefficient. Since roots are written as combinations of elements of Δ with all coefficients of the same sign, this implies that $\lambda - \alpha$ either is not a root, or still belongs to Ψ_ℓ. Therefore $\mathfrak{u}_\ell$ is $\mathrm{ad}(\mathfrak{g}_{-\alpha})$-invariant for any $\alpha \in \Delta''$. Since the representation V_j is equal to $\mathrm{span}(\rho_\ell(G)\mathbf{p}_{\mathfrak{u}_\ell})$, for any other weight χ of V_ℓ besides χ_ℓ, there is $\lambda \in \Delta''$ such that $-\lambda$ contributes to $\chi - \chi_\ell$. This implies that $\chi(s_0) > \chi_\ell(s_0)$ as claimed.

Step 5. With these preparations we are ready to complete the proof of Proposition 4.1. By Steps 3 and 4 and compactness of S_0, there is $c > 0$ such that for any $s_0 \in S_0$, any ℓ satisfying (4.4), the eigenspace V_{ℓ,χ_ℓ} is one dimensional and spanned by $\mathbf{p}_{\mathfrak{u}_\ell}$, and any weight χ appearing in V_ℓ and distinct from χ_ℓ we have $\chi(s_0) - \chi_\ell(s_0) > c$. By (4.3), this implies that when $s \in S_\eta$ and $\|s\|$ is large enough, if $\ell(g) = \ell_j(g)$ then the minimum in the definition of $\ell_j(g)$ is attained by the unique eigenvector corresponding to the character $\chi_j(s)$. By Steps 2 and 3, if $\|s\|$ is large enough, this unique eigenvector corresponds to a W-active $\mathfrak{u} \in \mathcal{R}$. This implies (i). Moreover, if $\mathfrak{u} \in \mathcal{R}_\ell$ is not W-active then either ℓ does not satisfy (4.4), or $\mathbf{p}_\mathfrak{u}$ has a nonzero coefficient corresponding to some weight χ for V_ℓ distinct from χ_ℓ. In both cases, by making $\|s\|$ sufficiently large, we can ensure (ii). $\square$

5. Expanding subgroups

In this section we define (F^+, Γ)-expanding subgroups, and give some examples. Unifying the notation from the two cases considered in §3, we define a representation $\rho : G \to \mathrm{GL}(V)$ as in (3.1) in the rank one case, and in the arithmetic case we let

$$V \stackrel{\mathrm{def}}{=} \bigoplus V_j \quad \text{and} \quad \rho \stackrel{\mathrm{def}}{=} \bigoplus \rho_j \, ,$$

where V_j, ρ_j are as in (3.3).

We will start by citing the following

Proposition 5.1. [Bo1, §4, §8] *For any one-parameter subgroup $F = \{g_t\}$ of G one can find one-parameter subgroups $\{k_t\}, \{a_t\}$ and $\{u_t\}$ of G, such that the following hold:*

- *$g_t = k_t u_t a_t$ for all t.*
- *All elements in all of the above one parameter subgroups commute.*
- *$\{\mathrm{Ad}(k_t) : t \in \mathbb{R}\}$ is bounded.*
- *$\mathrm{Ad}(a_t)$ is semisimple and $\mathrm{Ad}(u_t)$ is unipotent for all t.*

Note that $\mathrm{Ad}(a_t)$ is nontrivial if and only if g_t is not quasiunipotent, which will be our standing assumption for this section.

Let T be a maximal $\mathbb{R}$-split torus of G containing $\{a_t\}$ and let $\mathcal{X}(T)$ be the set of $\mathbb{R}$-algebraic homomorphisms $T \to \mathbb{R}$. Let Ψ be the set of weights for ρ. Then we have $V = \bigoplus_{\lambda \in \Psi} V^\lambda$, where $\Psi \subset \mathcal{X}(T)$ and

$$V^\lambda \stackrel{\mathrm{def}}{=} \left\{ \mathbf{v} \in V : \forall g \in T, \ \rho(g)\mathbf{v} = e^{\lambda(g)}\mathbf{v} \right\}$$

is nonzero for each $\lambda \in \Psi$. Since G is semisimple, $\Psi = -\Psi$. We write

$$V^> \stackrel{\mathrm{def}}{=} \bigoplus_{\lambda \in \Psi, \, \lambda(a_1) > 0} V^\lambda, \quad V^0 \stackrel{\mathrm{def}}{=} \bigoplus_{\lambda \in \Psi, \, \lambda(a_1) = 0} V^\lambda, \quad V^< \stackrel{\mathrm{def}}{=} \bigoplus_{\lambda \in \Psi. \, \lambda(a_1) < 0} V^\lambda \, ,$$

and also let $V^\leq \stackrel{\mathrm{def}}{=} V^0 \oplus V^<$. For any other representation $\tau : G \to \mathrm{GL}(W)$ defined over $\mathbb{R}$ we will denote by Ψ_τ the set of weights for τ, and similarly will write W^λ for $\lambda \in \Psi_\tau$, $W^>$, $W^\leq$, etc.

Let $\mathcal{W}$ denote the union of the G-orbits of the vectors representing the appropriate Lie algebras of unipotent radicals of parabolic subgroups. I.e., in the rank one case, let $\mathcal{W} = \rho(G)\mathbf{p}_{\mathfrak{u}}$, and in the arithmetic case, $\mathcal{W} = \bigcup_{j=1}^{r} \rho(G)\mathbf{p}_{\mathfrak{u}_j}$. Now let us say that a subgroup $H \subset G$ is (F^+, Γ)-*expanding* if

$$\rho(H)\mathbf{p} \not\subset V^{\leq} \quad \text{for any } \mathbf{p} \in \mathcal{W}. \tag{5.1}$$

Having introduced this definition, we can state our main theorem:

Theorem 5.2. *Suppose G is a connected centerfree semisimple Lie group with no compact factors, $\Gamma \subset G$ is an irreducible lattice, and F is a non-quasiunipotent one-parameter subgroup. Let H be an (F^+, Γ)-expanding subgroup of H^+ normalized by F, and let $\mathcal{F}$ be as in (1.5). Then there exists $a' > 0$ such that for any $x \in G/\Gamma$ and $a > a'$, the set*

$$\{h \in H : hx \in E(F^+, \infty)\} \tag{5.2}$$

is a-winning for the MSG induced by $\mathcal{F}$.

Note that since H is normalized by F, any $\Phi_t \in \mathcal{F}$ can be viewed as a (contracting) automorphism of H, that is, the MSG considered in the above theorem is in fact induced by the restriction of $\mathcal{F}$ to H.

Theorem 5.2 will be proved in §6. Meanwhile, our goal in this section is to give some examples of (F^+, Γ)-expanding subgroups. We begin with the following sufficient condition:

Proposition 5.3. *Let A be the semisimple part of F as in Proposition 5.1. Suppose H is a unipotent group normalized by A, and suppose there is a semisimple subgroup $L \subset G$ containing A and H such that:*

- (a) *for any representation $\tau : L \to \mathrm{GL}(W)$ defined over $\mathbb{R}$, if $\mathbf{w} \in W$ is invariant under $\tau(H)$, then either $\mathbf{w}$ is fixed by $\tau(L)$ or $\mathbf{w} \notin W^{\leq}$;*
- (b) *A projects nontrivially onto any simple factor of L;*
- (c) *for any $\mathbf{p} \in \mathcal{W}$, $\rho(L)$ does not fix $\mathbf{p}$.*

Then H is (F^+, Γ)-expanding.

Proof. Take $\mathbf{p} \in \mathcal{W}$, let

$$\hat{V} \stackrel{\text{def}}{=} \operatorname{span} \rho(H)\mathbf{p},$$

and suppose, in contradiction to (5.1), that $\hat{V} \subset V^{\leq}$. Put $d = \dim(\hat{V})$, consider the representation $\tau \stackrel{\text{def}}{=} \bigwedge^d \rho$ on the space $W \stackrel{\text{def}}{=} \bigwedge^d V$, and take a nonzero vector $\mathbf{w} \in \bigwedge^d \hat{V}$.

Since $\hat{V}$ is $\rho(H)$-invariant, the line spanned by $\mathbf{w}$ is fixed by $\tau(H)$. Since H is unipotent, it has no multiplicative algebraic characters, and hence $\mathbf{w}$ is fixed by $\tau(H)$. The fact that $\hat{V} \subset V^{\leq}$ implies that $\mathbf{w} \in W^{\leq}$. By property (a), $\mathbf{w}$ is $\tau(L)$-invariant, i.e. the subspace $\hat{V}$ is $\rho(L)$-invariant. Also, since $A \subset L$, $\mathbf{w}$ is fixed by $\tau(A)$, therefore $\hat{V} \subset V^0$. Now let N denote the kernel of the action of L on $\hat{V}$. Then N is a normal subgroup of L, and it contains A, since $\hat{V} \subset V^0$. By (b) $N = L$, a contradiction to (c). $\square$

We remark that condition (a) in the above proposition is equivalent to the statement that the subgroup AH generated by A and H is *epimorphic* in L, see [Wei].

We are now ready to exhibit subgroups which are (F^+, Γ)-expanding.

Proposition 5.4. *Let G and Γ be as in Theorem 1.1, and let F be non-quasiunipotent. Then the expanding horospherical subgroup H^+ defined in* (1.4) *is (F^+, Γ)-expanding.*

Note that this proposition, together with Theorem 5.2, immediately implies Theorem 1.1. For the proof of Proposition 5.4 we will need the following fact, which will be used later on as well:

Proposition 5.5. *Suppose G is a semisimple centerfree Lie group with no compact factors, Γ is an irreducible lattice, and L is a nontrivial normal subgroup of G. Then the L-action on G/Γ is minimal and uniquely ergodic.*

Proof. The fact that $L\Gamma$ is dense in G follows immediately from the irreducibility of Γ. Since L is normal, so is $gL\Gamma = Lg\Gamma$ for any $g \in G$, so the action is minimal. If μ is any L-invariant probability measure on G/Γ, then it lifts to a Radon measure $\hat{\mu}$ on G which is left-invariant by L and right-invariant by Γ. Since L is normal in G, this means that $\hat{\mu}$ is right-invariant by $L\Gamma$. Since the stabilizer of a Radon measure is closed and $L\Gamma$ is dense, this means that $\hat{\mu}$ is G-invariant, i.e. it is a Haar measure on G. In particular μ is the unique G-invariant probability measure. $\square$

Proof of Proposition 5.4. We apply Proposition 5.3 with L being the smallest normal subgroup of G which contains A. We need to show that H^+ is contained in L and normalized by A, and verify conditions (a), (b) and (c).

Since the projection of H^+ onto G/L is expanded by conjugation under the projection of A, which is trivial, H^+ is contained in L. Note that if we consider the representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G, then, using the notation above, the Lie algebra $\mathfrak{h}$ of H^+ is precisely $\mathfrak{g}^{>}$. In particular, it is $\mathrm{Ad}(A)$-invariant, so H is normalized by A.

Condition (b) is immediate from the definition of L. To uphold (c), suppose that $\rho(L)$ fixes $\mathbf{p} \in \mathcal{W}$. There is some $g \in G$ so that $\mathbf{p} = \rho(g)\mathbf{p}_{\mathfrak{u}}$, where $\mathbf{p}_{\mathfrak{u}}$ is a vector representing a unipotent radical U of a parabolic subgroup. So $\rho(g^{-1}Lg) = \rho(L)$ fixes $\mathbf{p}_{\mathfrak{u}}$. On the other hand, by Proposition 5.5 we have

$$\rho(G)\mathbf{p}_{\mathfrak{u}} \subset \overline{\rho(\Gamma L)\mathbf{p}_{\mathfrak{u}}} = \overline{\rho(\Gamma)\mathbf{p}_{\mathfrak{u}}}\,.$$

Using either Proposition 3.1(1) in the rank-one case, or rationality of ρ and $\mathbf{p}_{\mathfrak{u}}$ in the arithmetic case, we see that after replacing U with a conjugate we may assume that $\rho(\Gamma)\mathbf{p}_{\mathfrak{u}}$ is discrete. Therefore $\rho(G)$ fixes $\mathbf{p}_{\mathfrak{u}}$, thus $\mathrm{Lie}(U)$ is $\mathrm{Ad}(G)$-invariant. Hence U is a normal unipotent subgroup of G, which is impossible.

It remains to verify (a). Let $\tau : L \to \mathrm{GL}(W)$ be a representation, let $\mathfrak{l} = \mathrm{Lie}(L)$, and let Φ be the set of roots, i.e. weights of $\mathrm{Ad} : L \to \mathrm{GL}(\mathfrak{l})$. Recall that

$$d\tau(\mathfrak{l}_\alpha)W^\lambda \subset W^{\lambda+\alpha}, \tag{5.3}$$

where $\lambda \in \Psi_\tau$, $\alpha \in \Phi$, $\mathfrak{l}_\alpha$ is the root subspace corresponding to the root α of L, and $d\tau$ is the derivative map $\mathfrak{l} \to \mathrm{End}(W)$. Let Q be the parabolic subgroup of L with Lie algebra $\mathfrak{q} = \mathfrak{l}^{\leq}$, then $\mathfrak{q} \oplus \mathfrak{h} = \mathfrak{l}$ (as vector spaces), hence QH^+ is Zariski dense in L. Also by (5.3), $W^{\leq}$ is $\tau(Q)$-invariant. Now suppose $\mathbf{w} \in W$ is a vector fixed by $\tau(H^+)$ and contained in $W^{\leq}$. Then $\tau(QH^+)\mathbf{w} \subset W^{\leq}$. Therefore $\tau(L)\mathbf{w}$ is a subset of $W^{\leq}$, hence so is $\hat{V} \stackrel{\mathrm{def}}{=} \operatorname{span} \tau(L)\mathbf{w}$, a $\tau(L)$-invariant subspace. Since L is semisimple, $\Phi = -\Phi$, which implies that $\hat{V} \subset W^0$. In particular $L \mapsto \mathrm{End}(\hat{V})$ has a kernel N which contains A. Since A projects nontrivially onto every factor of L we must have $L \subset N$, i.e. L acts trivially on $\hat{V}$, in particular $\mathbf{w}$ is $\tau(L)$-invariant. $\square$

As described in the introduction, Dani [Da3] proved that $E(F, \infty)$ is thick whenever G is a rank one semisimple Lie group and F is non-quasiunipotent. He did this by considering the Schmidt game played on the group H corresponding to a certain subgroup of H^+. It turns out that Dani's result is a special case of Theorem 5.2:

Proposition 5.6. *Suppose* $\mathrm{rank}_{\mathbb{R}}(G) = 1$. *Then any nontrivial connected subgroup H of H^+ which is normalized by F is (F^+, Γ)-expanding.*

Proof. Clearly any group containing an (F^+, Γ)-expanding subgroup is also (F^+, Γ)-expanding; therefore it suffices to prove the proposition when $\dim(H) = 1$. By the Jacobson-Morozov lemma one can find a one-dimensional subgroup H^- which is opposite to H in the following sense: the group L generated by H and H^- is locally isomorphicto $\mathrm{SL}_2(\mathbb{R})$, and its diagonal subgroup F', which normalizes both H and

H^-, centralizes F. Since $\mathrm{rank}_{\mathbb{R}}(G) = 1$, we must have $F = F'$. In particular L is generated by unipotent elements, and contains FH as a Borel subgroup. The representation theory of $\mathfrak{sl}_2(\mathbb{R})$ now showsthat condition (a) of Proposition 5.3 holds, and (b) is immediate. It remains to check (c). Suppose that $\rho(L)$ fixes $\mathbf{p}$, that is L normalizes a parabolic P. Since it is simple, it fixes its volume element, hence so does F. Since the normalizer of a parabolic has elements of G which do not preserve its volume element, we find that the normalizer of P contains a torus of rank 2, contradicting the fact that $\mathrm{rank}_{\mathbb{R}}(G) = 1$. □

We conclude the section with a higher rank example related to Diophantine approximation. Let $m, n \in \mathbb{N}$ be positive integers, put $k = m + n$, and consider $G = \mathrm{SL}_k(\mathbb{R})$, $\Gamma = \mathrm{SL}_k(\mathbb{Z})$, and $F = \{g_t\}$, where

$$g_t = \mathrm{diag}(e^{t/m}, \ldots, e^{t/m}, e^{-t/n}, \ldots, e^{-t/n})$$

(see case (1.2) discussed in the introduction). Clearly the group

$$H = \{u_Y : Y \in M_{m,n}\}, \tag{5.4}$$

where

$$u_Y \stackrel{\mathrm{def}}{=} \begin{pmatrix} I_m & Y \\ 0 & I_n \end{pmatrix} \tag{5.5}$$

(here and hereafter $M_{m,n}$ is the space of $m \times n$ matrices with real entries and I_ℓ stands for the $\ell \times \ell$ identity matrix), is expanding horospherical with respect to g_1, hence (F^+, Γ)-expanding by Proposition 5.4. Now, more generally, consider $F_{\mathbf{r},\mathbf{s}} \stackrel{\mathrm{def}}{=} \{g_t^{(\mathbf{r},\mathbf{s})}\}$ where

$$g_t^{(\mathbf{r},\mathbf{s})} = \mathrm{diag}(e^{r_1 t}, \ldots, e^{r_m t}, e^{-s_1 t}, \ldots, e^{-s_n t}), \tag{5.6}$$

with

$$r_i, s_j > 0 \quad \text{and} \quad \sum_{i=1}^{m} r_i = 1 = \sum_{j=1}^{n} s_j\,. \tag{5.7}$$

Then H is contained in the expanding horospherical subgroup with respect to $g_1^{(\mathbf{r},\mathbf{s})}$, and the containment is proper when some components of either $\mathbf{r}$ or $\mathbf{s}$ are different. Nevertheless, the following holds:

Proposition 5.7. *The group* (5.4) *is* $(F^+_{\mathbf{r},\mathbf{s}}, \Gamma)$*-expanding for any* $\mathbf{r}, \mathbf{s}$ *as in* (5.7).

Proof. One can use Proposition 5.3 with $L = G$. Conditions (b) and (c) are immediate, and (a) follows from the proof of [KW2, Lemma 2.3]. □

See Corollary 9.1 for a Diophantine application of the above proposition.

6. Bounded trajectories are winning

In this section we will prove Theorem 5.2; throughout this section the assumptions will be as in the theorem. Let $\|\cdot\|$ denote a norm on $\mathbb{R}^n$, and for a polynomial $\varphi : \mathbb{R}^\ell \to \mathbb{R}^n$, let $\|\varphi\|$ denote the maximum of the absolute values of the coefficients of φ. We will need the following lemmas.

Lemma 6.1. *Given natural numbers r, ℓ, n, d and a ball $B' \subset \mathbb{R}^\ell$, there are positive constants η, c such that the following holds. Suppose for $j = 1, \ldots, r$ that $\varphi_j : \mathbb{R}^\ell \to \mathbb{R}^{n_j}$ is a polynomial map in ℓ variables of degree at most d, with $n_j \le n$. Then there is a ball $B \subset B'$ of diameter c such that for $j = 1, \ldots, r$,*

$$\mathbf{x} \in B \implies \|\varphi_j(\mathbf{x})\| \ge \eta \|\varphi_j\|.$$

Proof. By induction we may assume that $r = 1$, i.e. that there is only one polynomial map, which we denote by φ. It is easy to see that the general case follows from the case $n = 1$.

Note that the space of all possible φ's is a finite dimensional real vector space. Since the two norms $\|\varphi\|$ and $\varphi \mapsto \max_{\mathbf{x}\in B'} |\varphi(\mathbf{x})|$ are equivalent, there is $\eta' > 0$ (independent of φ) such that B' contains a point $\mathbf{x}'$ such that $|\varphi(\mathbf{x}')| \ge \eta' \|\varphi\|$. By continuity and compactness, there is $C > 0$ such that for any φ, $\|\nabla\varphi\| \le C\|\varphi\|$. Now take $\eta = \eta'/2$ and $c = \min\big(\eta'/2C, \operatorname{diam}(B')\big)$. On any ball of diameter c the variation of φ is at most $\eta'\|\varphi\|/2$. Thus we may take B to be a sub-ball of B' of diameter c containing $\mathbf{x}'$. $\square$

In our application the polynomial maps will come from the representation ρ defined at the beginning of §5. Given $\mathbf{v} \in V$, define

$$\varphi = \varphi^{\mathbf{v}} : H \to \mathbb{R}^n, \quad \varphi(h) \stackrel{\text{def}}{=} \rho(h)\,\mathbf{v}\,.$$

We will think of φ as a polynomial map in the following sense. Since H is unipotent, the exponential map $\mathfrak{h} \stackrel{\text{def}}{=} \operatorname{Lie}(H) \to H$ is a polynomial. Fix a basis $\mathbf{x}_1, \ldots, \mathbf{x}_\ell$ for $\mathfrak{h}$, where $\ell = \dim H$, and for φ as above consider

$$\widetilde{\varphi} : \mathbb{R}^\ell \to \mathbb{R}^n, \quad \widetilde{\varphi}(s_1, \ldots, s_\ell) = \varphi\left(\exp\left(\sum^{\ell} s_i \mathbf{x}_i\right)\right).$$

When discussing φ as a polynomial (e.g. when discussing the degree or coefficients of φ) we will actually mean $\widetilde{\varphi}$. Since $\rho \circ \exp = \exp \circ\, d\rho$, the degree of φ is no more than $\dim(V) - 1$. Let $\Pi : V \to V^{>}$ be the projection corresponding to the direct sum decomposition $V =$

$V^{\leq} \oplus V^{>}$. One can fix the norm $\|\cdot\|$ on V and choose $a_0 > 0$ so that

$$\mathbf{v} \in V,\ t \geq a_0 \quad \Longrightarrow \quad \|\mathbf{v}\| \geq \|\Pi\mathbf{v}\|,\ \|\rho(g_t)\Pi\mathbf{v}\| \geq \|\Pi\mathbf{v}\|. \tag{6.1}$$

Lemma 6.2. *There is $c' > 0$ such that for any $\mathbf{v} \in \mathcal{W}$,*

$$\|\Pi \circ \varphi^{\mathbf{v}}\| \geq c'\|\mathbf{v}\|$$

(the norm on the right hand side is the norm on V and the one on the left is the norm on polynomial functions $H \to \mathbb{R}^n$).

Proof. The map $\mathbf{v} \mapsto \varphi^{\mathbf{v}}$ is continuous. For any $\mathbf{v} \in \mathcal{W}$, the line span$(\mathbf{v})$ is contained in $\mathcal{W}$ and invariant under a parabolic subgroup of G. This implies that the image of $\mathcal{W}$ in $\mathbb{P}(V)$ (the space of lines in V) is compact. Moreover in proving the statement we may replace $\mathbf{v}$ by any multiple of it by a nonzero scalar. Thus, if the statement is false, then by compactness there is $\mathbf{v} \in \mathcal{W}$ for which $\varphi^{\mathbf{v}}$ is zero. But unravelling definitions, one sees that this contradicts (5.1). □

Proof of Theorem 5.2. We are given an admissible D_0. Our goal is to find $a > 0$ such that for any $g \in G$, the set (5.2) is a-winning for the $\{\mathcal{D}_t\}$-game with $\{\mathcal{D}_t\}$ as in (2.1), where $x = \pi(g)$. Let $B' \subset D_0$ be a ball, let $n = \dim V$, let $d = n - 1$, and take $r = 1$ if $\mathrm{rank}_{\mathbb{R}}(G) = 1$, and $r = 2^{\mathrm{rank}_{\mathbb{Q}}(G)} - 1$ otherwise. Then let c, η be the constants as in Lemma 6.1 corresponding to r, ℓ, n, d and B'. Replacing g with hg for an appropriate $h \in H$ we may assume that $e \in B'$. Since the maps Φ_t are contracting, we may choose $a' > a_0$, with a_0 as in (6.1), large enough so that for any $a > a'$, any ball in B' of diameter c contains a right-translate of $\Phi_a(D_0)$.

Now given any choice of $b > 0$, an initial $t_0 > 0$, $x = \pi(g) \in G/\Gamma$ and any B_1 which is a right-translate of $\Phi_{t_0}(D_0)$, we need to describe a compact $K \subset G/\Gamma$ and a strategy for Alice, such that for the choices of sets $B_1, B_2, \ldots$ by Bob, there are choices of sets $A_i \subset B_i$ for Alice with

$$h \in A_i,\ i \in \mathbb{N} \quad \Longrightarrow \quad g_i h x \in K, \tag{6.2}$$

where

$$g_i \stackrel{\text{def}}{=} g_{t_0 + i(a+b)}\,. \tag{6.3}$$

This will ensure that the trajectory $F^+h_\infty x$ is bounded, where h_∞ is the intersection point (2.4). The definition of K and the description of the strategy will now be given separately for the rank-one and arithmetic cases.

The rank-one case. Assume the notation of §3.1. By boundedness of B_1 and Proposition 3.1(2) there is $\delta_1 > 0$ so that

$$c \in C,\ h \in B_1,\ \gamma \in \Gamma \quad \Longrightarrow \quad \|\rho(g_1 h g \gamma c^{-1})\mathbf{p}_{\mathrm{u}}\| \geq \delta_1. \tag{6.4}$$

Let ε_0 be as in Proposition 3.1(3). Since D_0 is bounded, by making δ_1 smaller we can assume that $\delta_1 < \varepsilon_0$ and

$$h \in g_a D_0 D_0^{-1} g_a^{-1},\ g' \in G,\ \mathbf{v} \in V,\ \|\rho(hg')\mathbf{v}\| < \delta_1 \implies \|\rho(g')\mathbf{v}\| < \varepsilon_0. \tag{6.5}$$

Using Lemma 6.2, choose δ_2 so that

$$\mathbf{v} \in \mathcal{W},\ \|\mathbf{v}\| \geq \delta_1 \implies \|\Pi \circ \varphi^{\bar{\mathbf{v}}}\| \geq \delta_2, \quad \text{where } \bar{\mathbf{v}} = g_b \mathbf{v}. \tag{6.6}$$

Now let

$$\delta_3 = \min\{\eta\delta_2, \delta_1\},$$

where η is as in Lemma 6.1, and

$$K = \pi\left(\left\{g' \in G : \forall \gamma \in \Gamma, c \in C, \|\rho(g'\gamma c^{-1})\mathbf{p}_{\mathfrak{u}}\| \geq \delta_3\right\}\right),$$

which is compact in light of Proposition 3.1(2).

We claim Alice can make moves so that for each $i \in \mathbb{N}$, each $h \in A_i$, each $c \in C$, and each $\gamma \in \Gamma$, at least one of the following holds:

(a) $\|\rho(g_i h g \gamma c^{-1})\mathbf{p}_{\mathfrak{u}}\| \geq \delta_1$;
(b) $\|\Pi \circ \rho\left(g_i h g \gamma c^{-1}\right)\mathbf{p}_{\mathfrak{u}}\| \geq \delta_3$.

Clearly this will imply (6.2) and conclude the proof.

The claim is proved by induction. From (6.4) it follows that (a) holds for $i = 1$ and for all $h \in B_1$, so Alice can choose A_1 at her whim. Assume the claim is true for $i-1$, where $i \geq 2$. We claim that (a) can fail for i with at most one vector. That is, there is at most one vector $\mathbf{v} = \rho(\gamma c^{-1})\mathbf{p}_{\mathfrak{u}}$ for which there is $h \in B_{i-1}$ with $\|\rho(g_i h g)\mathbf{v}\| < \delta_1$. For suppose, for $\ell = 1, 2$, that $\mathbf{v}_\ell = \rho(\gamma_\ell c_\ell^{-1})\mathbf{p}_{\mathfrak{u}}$ are two such vectors, with $h_\ell \in B_{i-1}$ the corresponding elements. Then the h_ℓ are both in B_{i-1} which is a right translate of $g_b^{-1} g_{i-1}^{-1} D_0 g_{i-1} g_b$, i.e.

$$g_b g_{i-1} h_1 h_2^{-1} g_{i-1}^{-1} g_b^{-1} \in D_0 D_0^{-1}.$$

Therefore $g_i h_1 = h' g_i h_2$ where $h' = g_i h_1 h_2^{-1} g_i^{-1} \in g_a D_0 D_0^{-1} g_a^{-1}$. In view of (6.5) and Proposition 3.1(3) we must have $\mathbf{v}_1 = \mathbf{v}_2$.

So no matter how Alice chooses A_i, (a) will hold except possibly for one vector $\mathbf{v} = \rho(\gamma c^{-1})\mathbf{p}_{\mathfrak{u}} \in \mathcal{W}$. By the induction hypothesis, either $\|\Pi \circ \rho(g_{i-1} h g)\mathbf{v}\| \geq \delta_3$ or $\|\rho(g_{i-1} h g)\mathbf{v}\| \geq \delta_1$. If the former occurs, it also occurs for i because of (6.1), implying (b). If the latter occurs, write

$$B_{i-1} = g_b^{-1} g_{i-1}^{-1} D_0 g_{i-1} g_b h_0, \quad \text{where } h_0 \in H,$$

and set

$$\mathbf{v}' = \rho(g_{i-1} h_0 g)\mathbf{v} \in \mathcal{W}, \quad \bar{\mathbf{v}} = \rho(g_b)\mathbf{v}'.$$

Then $\|\mathbf{v}'\| \geq \delta_1$ so by (6.6) we have $\|\Pi \circ \varphi^{\bar{\mathbf{v}}}\| \geq \delta_2$. By Lemma 6.1 we can choose a ball $B \subset B'$ of diameter c so that

$$h \in B \implies \|\Pi \circ \varphi^{\bar{\mathbf{v}}}(h)\| \geq \eta\delta_2 \geq \delta_3. \tag{6.7}$$

By the choice of a,

$$D_0 \supset B' \supset B \supset g_a^{-1} D_0 g_a h_1$$

for some $h_1 \in H$. Since

$$B_{i-1} \supset g_{i-1}^{-1} g_b^{-1} B' g_b g_{i-1} h_0,$$

Alice can choose

$$A_i = (g_b g_{i-1})^{-1} g_a^{-1} D_0 g_a h_1 g_b g_{i-1} h_0 = g_i^{-1} D_0 g_i h_2\,,$$

where $h_2 = g_i^{-1} g_a h_1 g_a^{-1} g_i h_0 \in H$. Now for all

$$h = (g_b g_{i-1})^{-1} h' g_b g_{i-1} h_0 \in A_i\,,$$

where $h' \in g_a^{-1} D_0 g_a h_1 \subset B$, we have

$$\begin{aligned} \|\Pi \circ \rho(g_i h g c \gamma)\mathbf{p}_{\mathfrak{u}}\| &\geq \|\Pi \circ \rho(g_b g_{i-1} h g)\mathbf{v}\| \\ &= \|\Pi \circ \rho(g_b g_{i-1} h h_0^{-1} g_b^{-1} g_{i-1}^{-1})\bar{\mathbf{v}}\| \\ &= \|\Pi \circ \rho(h')\bar{\mathbf{v}}\| \underset{(6.7)}{\geq} \delta_3\,, \end{aligned}$$

as required.

The arithmetic case. Let W be a neighborhood of 0 in $\mathfrak{g}$ as in Proposition 3.4. Since the set $D_0 D_0^{-1}$ is a bounded subset of G, there is $W^{(1)} \subset W$, a neighborhood of 0 in $\mathfrak{g}$, such that

$$x \in D_0 D_0^{-1},\ \mathbf{v} \in W^{(1)} \implies \mathrm{Ad}(x)\mathbf{v} \in W. \tag{6.8}$$

Since B is compact and in view of Corollary 3.3, there is $\varepsilon_0 > 0$ such that $\ell(hg) \geq \varepsilon_0$ for all $h \in B_1$. Let η, c' be as in Lemmas 6.1 and 6.2 respectively, and assume with no loss of generality that $c'\eta < 1$. Let $d \stackrel{\mathrm{def}}{=} \min_j d_j$ so that

$$M_1 \stackrel{\mathrm{def}}{=} (c'\eta)^{1/d} \leq (c'\eta)^{1/\dim \mathfrak{u}}$$

for any $\mathfrak{u} \in \mathcal{R}$. Let M_2 be a Lipschitz constant for $\rho(g_{a+b})^{-1}$, that is for any $\mathbf{v} \in V$,

$$\|\rho(g_{a+b})\mathbf{v}\| \geq \frac{\|\mathbf{v}\|}{M_2}.$$

Applying Proposition 4.1 with $M = \dfrac{M_2}{M_1}$, we can choose $\varepsilon_0 < 1$ small enough so that for any $\mathfrak{u} \in \mathcal{R}$, and any $g \in G$ for which $\ell(g) < M_2\varepsilon_0$, either $\mathfrak{u}$ is $W^{(1)}$-active for g, or

$$\|\rho(g)\mathbf{p}_{\mathfrak{u}}\|^{1/\dim \mathfrak{u}} \geq M\ell(g). \tag{6.9}$$

Now let

$$\varepsilon_1 \stackrel{\mathrm{def}}{=} M_1\varepsilon_0 \quad \text{and} \quad K \stackrel{\mathrm{def}}{=} \pi\left(\{g \in G : \ell(g) \geq \varepsilon_1\}\right). \tag{6.10}$$

Then K is compact by Corollary 3.3.

We will show how Alice can make moves so that for each $i \in \mathbb{N}$, and each $\mathfrak{u} \in \mathcal{R}$ for which there exists $h' \in A_i$ such that $\mathfrak{u}$ is $W^{(1)}$-active for $g_i h' g$,

$$\|\Pi \circ \rho\left(g_i h g\right) \mathbf{p}_{\mathfrak{u}}\|^{1/\dim \mathfrak{u}} \geq \varepsilon_1 \text{ for any } h \in A_i. \tag{6.11}$$

In view of (6.1), the choice of ε_0 and Proposition 4.1, this will imply (6.2) and conclude the proof.

In order to describe Alice's first move, let $\mathcal{U}$ denote the set of $\mathfrak{u} \in \mathcal{R}$ for which there exist $h = h(\mathfrak{u}) \in B_1$ such that $\mathfrak{u}$ is $W^{(1)}$-active for some $g_1 h g$. We claim that

$$\#\mathcal{U} \leq r. \tag{6.12}$$

In view of Proposition 3.5, it suffices to show that the subalgebra $\mathfrak{v}$ spanned by all $\mathfrak{u} \in \mathcal{U}$ is unipotent. Fix $h_0 \in B_1$, and suppose $\mathfrak{u} \in \mathcal{U}$ and $h = h(\mathfrak{u})$. Then there is $Q = Q(\mathfrak{u}) \subset \mathfrak{g}_{\mathbb{Z}}$ such that $\mathrm{Ad}(g_1 h g)Q \subset W^{(1)}$ and $\mathfrak{u} = \mathrm{span}(Q)$. The set B_1 is a right-translate of $g_1^{-1} D_0 g_1$, and hence

$$g_1 h_0 h^{-1} g_1^{-1} \in g_1(B_1 B_1^{-1})g_1^{-1} = D_0 D_0^{-1}.$$

Therefore, by (6.8),

$$\mathrm{Ad}(g_1 h_0 g)Q = \mathrm{Ad}(g_1 h_0 h^{-1} g_1^{-1})\mathrm{Ad}(g_1 h g)Q \subset W.$$

By the choice of W, this implies that the subalgebra spanned by

$$\bigcup_{\mathfrak{u} \in \mathcal{U}} \mathrm{Ad}\left(g_1 h_0 g\right) Q(\mathfrak{u})$$

is unipotent, hence so is $\mathfrak{v}$. This proves (6.12).

The choice of ε_0 implies that for any $\mathfrak{u} \in \mathcal{U}$, $\|\rho(hg)\mathbf{p}_{\mathfrak{u}}\|^{1/\dim \mathfrak{u}} \geq \varepsilon_0$ for any $h \in B_1$. In particular this holds for $h = e$, so writing $\mathbf{v} \stackrel{\mathrm{def}}{=} \rho(g)\mathbf{p}_{\mathfrak{u}}$ we get $\|\mathbf{v}\| \geq \varepsilon_0^{\dim \mathfrak{u}}$. Applying Lemma 6.2 we get $\|\Pi \circ \varphi^{\mathbf{v}}\| \geq c'\varepsilon_0^{\dim \mathfrak{u}}$, so by Lemma 6.1 there is $B \subset B'$ of diameter c such that for any $h \in B$, $\|\Pi \circ \varphi^{\mathbf{v}}(h)\| \geq c'\eta\varepsilon_0^{\dim \mathfrak{u}}$. Alice can choose $A_1 \subset B$ and for any $h \in A_1$ we find by (6.1) that

$$\|\Pi \circ \rho(g_1 h g)\mathbf{p}_{\mathfrak{u}}\|^{1/\dim \mathfrak{u}} \geq \|\Pi \circ \rho(hg)\mathbf{p}_{\mathfrak{u}}\|^{1/\dim \mathfrak{u}} \geq (c'\eta)^{1/\dim \mathfrak{u}}\varepsilon_0 \geq M_1\varepsilon_0 = \varepsilon_1.$$

We continue inductively, assuming for $i \geq 2$ that Alice and Bob have made their choices up to step $i-1$, and Bob has also chosen B_i. Let $\mathcal{U}'$ denote the set of $\mathfrak{u} \in \mathcal{R}$ for which there exist $h = h(\mathfrak{u}) \in B_i$ such that $\mathfrak{u}$ is $W^{(1)}$-active for $g_i hg$. Arguing as in the proof of (6.12), we find that

$$\#\mathcal{U}' \le r. \tag{6.13}$$

Now let $\mathcal{U}_1$ be the subset of those $\mathfrak{u} \in \mathcal{U}'$ for which there is some $h \in A_i$ such that $\mathfrak{u}$ is $W^{(1)}$-active for $g_{i-1}hg$, and let $\mathcal{U}_2 \stackrel{\text{def}}{=} \mathcal{U}' \smallsetminus \mathcal{U}_1$. If $\mathfrak{u} \in \mathcal{U}_1$ then by induction, $\|\Pi \circ \rho\,(g_{i-1}hg)\,\mathbf{p}_\mathfrak{u}\|^{1/\dim\mathfrak{u}} \ge \varepsilon_1$ for any $h \in A_{i-1}$. Since $A_i \subset A_{i-1}$, this implies (6.11) via (6.1).

We will now show that for any $\mathfrak{u} \in \mathcal{U}_2$ and any $h \in B_i$,

$$\|\rho(g_i hg)\mathbf{p}_\mathfrak{u}\|^{1/\dim\mathfrak{u}} \ge \varepsilon_0. \tag{6.14}$$

By the choice of M_2 it suffices to show that

$$\|\rho(g_{i-1}hg)\mathbf{p}_\mathfrak{u}\|^{1/\dim\mathfrak{u}} \ge M_2\varepsilon_0. \tag{6.15}$$

If $\ell(g_{i-1}hg) \ge M_2\varepsilon_0$ then (6.15) is immediate. Otherwise, the induction hypothesis and (6.1) ensure that $\ell(g_{i-1}hg) \ge \varepsilon_1$, and since $\mathbf{p}_\mathfrak{u}$ is not $W^{(1)}$-active for $g_{i-1}hg$, (6.9) implies $\|\rho(g_{i-1}hg)\mathbf{p}_\mathfrak{u}\|^{1/\dim\mathfrak{u}} \ge M\varepsilon_1 = M_2\varepsilon_0$. This yields (6.15). Having proved (6.14) we proceed as in the previous case, using Lemmas 6.2 and 6.1 to make Alice's choices. □

7. Escaping points: proof of Theorem 1.2

In this and the next sections we will analyze the local geometry of G/Γ. In what follows, $B(x,r)$ stands for the open ball centered at x of radius r. To avoid confusion, we will sometimes put subscripts indicating the underlying metric space. If the metric space is a group and e is its identity element, we will simply write $B(r)$ instead of $B(e,r)$. For a subset K of a metric space, $K^{(\varepsilon)}$ will denote the ε-neighborhood of K.

Let G be a real Lie group, $\mathfrak{g}$ its Lie algebra, let F as in (1.1) be non-quasiunipotent, and denote the Lie algebra of F by $\mathfrak{f}$. Using the notation introduced in §5, let $\mathfrak{g}^>$ be the expanding subalgebra corresponding to the representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. Then $\mathfrak{g}^>$ is exactly the Lie algebra of H^+ as in (1.4). The fact that F is not quasiunipotent implies that $\mathfrak{g}^>$ is nontrivial. Let us fix a Euclidean structure on $\mathfrak{g}$ so that $\mathfrak{f}$ and $\mathfrak{g}^>$ are mutually orthogonal subspaces, and transport it via right-multiplication to define a right-invariant Riemannian metric on G. This way, the exponential map $\exp : \mathfrak{g} \to G$ becomes almost an isometry locally around $0 \in \mathfrak{g}$. More precisely, for any positive ε one can choose a neighborhood U of $0 \in \mathfrak{g}$ such that $\exp|_U$ is $(1+\varepsilon)$-bilipschitz.

If Γ is a discrete subgroup of G, we will equip G/Γ with the Riemannian metric coming from G. Then for any $x \in G/\Gamma$ the orbit map

$G \to G/\Gamma$, $g \mapsto gx$ is an isometry when restricted to a sufficiently small (depending on x) neighborhood of $e \in G$.

Our main goal in this section is to prove the following generalization of Theorem 1.2:

Theorem 7.1. *Let G be a Lie group, $\Gamma \subset G$ discrete, F as in* (1.1)*, H^+ as in* (1.4)*. Let H be a nontrivial connected subgroup of H^+ normalized by F, and let $\mathcal{F}$ be as in* (1.5) *to H. Then there exists $a'' > 0$ such that for any $a > a''$ and any $x, z \in G/\Gamma$, the set*

$$\big\{h \in H : hx \in E(F^+, z)\big\} \tag{7.1}$$

is an a-winning set for the MSG induced by $\mathcal{F}$.

Proof. Recall that we are given an admissible $D_0 \subset H$, and our goal is to find a such that the set (7.1) is a-winning for the $\{\mathcal{D}_t\}$-game with $\{\mathcal{D}_t\}$ as in (2.1). From the admissibility of D_0 and contracting properties of $\{\Phi_t\}$ it follows that there exist positive a'' and ε_0 such that for any $a > a''$,

$$\begin{gathered} D_0 \text{ contains two translates } D_1, D_2 \text{ of } \Phi_a(D_0) \\ \text{of distance at least } \varepsilon_0 \text{ from each other.} \end{gathered} \tag{7.2}$$

We claim Alice has a winning strategy for $a > a''$.

Given $T > 0$, define $Z_T \stackrel{\text{def}}{=} \{g_t z : 0 \le t \le T\}$. This is a compact curve in G/Γ, with the crucial property that for any $x \in Z_T$, the tangent vector to Z_T at x (the flow direction) is orthogonal to the tangent space to Hx at x. This is exploited in the following

Lemma 7.2. *For any $z \in G/\Gamma$ and $T > 0$ there exist $\delta = \delta(z, T) > 0$ with the following property: for any $0 < \varepsilon \le \delta$ and any $x \in G/\Gamma$, the intersection of $B_H(\delta)x$ with $Z_T^{(\varepsilon/8)}$ has diameter at most ε.*

Proof. Let $\mathfrak{l}$ be the subspace of $\mathfrak{g}$ perpendicular to $\mathfrak{f} \oplus \mathfrak{h}$, so that we may write

$$\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{l} \oplus \mathfrak{h}.$$

The map

$$\psi : \mathfrak{g} \to G, \quad \psi(\mathbf{f} + \mathbf{l} + \mathbf{h}) = \exp(\mathbf{f}) \exp(\mathbf{l}) \exp(\mathbf{h})$$

(where $\mathbf{f} \in \mathfrak{f}$, $\mathbf{l} \in \mathfrak{l}$, $\mathbf{h} \in \mathfrak{h}$) has the identity map as its derivative at $0 \in \mathfrak{g}$. Hence in a sufficiently small neighborhood of 0, it is a homeomorphism onto its image and is 2-bilipschitz. Since the map $G \to G/\Gamma$, $g \mapsto gx$ is a local isometry for each $x \in G/\Gamma$, there is a neighborhood $W = W_x$ of 0 in $\mathfrak{g}$ such that the map

$$p_x : W \to G/\Gamma, \ \mathbf{v} \mapsto \psi(\mathbf{v})x$$

is injective, has an open image, and is 2-bi-Lipschitz. By compactness of Z_T we can choose W uniformly for all $x \in Z_T^{(1)}$. Choose $\delta < 1$ small enough so that $B_G(2\delta) \subset \psi(W)$.

The order of factors in the definition of ψ was chosen so that if $y_1, y_2 \in \psi(W)$ are in the same F-orbit (i.e. $y_1 = g_t y_2$ for some t) then $\psi^{-1}(y_1) - \psi^{-1}(y_2) \in \mathfrak{f}$. Using this, we can now make δ and W small enough so that if $B_H(\delta)x \cap Z_T^{(\delta)} \neq \varnothing$, then also $Z_T \cap p_x(W) \neq \varnothing$, and $y_1, y_2 \in p_x(W) \cap Z_T$ implies $p_x^{-1}(y_1) - p_x^{-1}(y_2) \in \mathfrak{f}$.

If the claim is false then there are $\varepsilon \leq \delta$, $x \in G/\Gamma$, $h_1, h_2 \in B_H(\delta)$, $y_1, y_2 \in p_x(W) \cap Z_T$ such that

$$\operatorname{dist}(h_i x, y_i) < \varepsilon/8, \quad i = 1, 2$$

and

$$\operatorname{dist}(h_1 x, h_2 x) > \varepsilon.$$

Let $\mathbf{v}_i = p_x^{-1}(y_i), \mathbf{h}_i = p_x^{-1}(h_i x) = \exp^{-1}(h_i)$. Since the map p_x is 2-bi-Lipschitz,

$$\|\mathbf{h}_i - \mathbf{v}_i\| < \varepsilon/4 \ \text{ and } \|\mathbf{h}_1 - \mathbf{h}_2\| > \varepsilon/2\,. \tag{7.3}$$

By the previous observation, $\mathbf{v}_1, \mathbf{v}_2$ belong to the same affine subspace $\mathbf{v}_0 + \mathfrak{f}$. Without affecting (7.3), we can replace each $\mathbf{v}_i$ with the point in $\mathbf{v}_0 + \mathfrak{f}$ closest to $\mathbf{h}_i$. But since $\mathbf{h}_i \in \mathfrak{h}$ and $\mathfrak{h}$ is orthogonal to $\mathfrak{f}$, these two points are the same, i.e. $\mathbf{v}_1 = \mathbf{v}_2$. But now (7.3) yields a contradiction to the triangle inequality. $\square$

Now given any choice of $b > 0$, an initial $t_1 > 0$ and any B_1 which is a right-translate of $\Phi_{t_1}(D_0)$, we are going to find $\eta > 0$ and $N \in \mathbb{N}$ such that for any choices made by Bob, there are choices $A_i \subset B_i$ for Alice such that

$$h \in A_{N+k},\ k \in \mathbb{N} \quad \Longrightarrow \quad \operatorname{dist}(g_{kT} h x, Z_T) > \eta\,, \tag{7.4}$$

where $T = a + b$. It follows from (7.4) that the closure of the trajectory $F^+ h_\infty x$, where h_∞ is the intersection point (2.4), does not contain z. Indeed, if $g_{t_i} h_\infty x$ converges to z, then we would have $\operatorname{dist}(g_{\ell_i T} h_\infty x, g_{t_i'} z) \to 0$ where $t_i = \ell_i T - t_i'$ with $0 \leq t_i' < T$ and $\ell_i \in \mathbb{N}$, contradicting (7.4).

To this end, let N be an integer such that the diameter of $\Phi_{NT}(D_0)$ is less than $\delta = \delta(z, T)$ as in Lemma 7.2, let

$$\lambda \stackrel{\text{def}}{=} \inf_{h_1, h_2 \in D_0, h_1 \neq h_2} \frac{\operatorname{dist}\big(\Phi_{NT}(h_1), \Phi_{NT}(h_2)\big)}{\operatorname{dist}(h_1, h_2)}$$

be the Lipschitz constant for $\Phi_{NT}|_{D_0}$, and choose

$$\varepsilon = \min\left(\tfrac{1}{2}\lambda\varepsilon_0, \delta\right), \quad \eta = \varepsilon/8,$$

where ε_0 is as in (7.2).

We now describe Alice's strategy. For her first N moves she makes arbitrary choices. For $k \in \mathbb{N}$, suppose that Bob's choice is $B_{N+k} = \Phi_{(N+k)T}(D_0)h$. Let D_1, D_2 be as in (7.2). Then $E_i = \Phi_{(N+k)T}(D_i)h$, $i = 1, 2$, are both valid options for Alice's next move. Also,

$$g_{kT}E_i x = g_{kT}\Phi_{(N+k)T}(D_i)hx = \Phi_{NT}(D_i)g_{kT}hx\,, \tag{7.5}$$

and $\Phi_{NT}(D_i)$ are two elements in $\mathcal{D}_{NT+a}$ inside $\Phi_{NT}(D_0)$ of distance at least $\lambda\varepsilon_0$ from each other. Furthermore, the diameter of $\Phi_{NT}(D_0)$ is less than δ, so by Lemma 7.2, the intersection of

$$\Phi_{NT}(D_0)g_{kT}hx = g_{kT}B_{N+k}x$$

with $Z_T^{(\varepsilon/8)}$ has diameter less than $\frac{1}{2}\lambda\varepsilon_0$. Therefore at least one of the sets (7.5) is disjoint from $Z_T^{(\varepsilon/8)}$, i.e. is at distance at least η from Z_T. Alice chooses the corresponding E_i, and (7.4) holds. □

We conclude the section with the

Proof of Corollary 1.3. In view of the countable intersection property of winning sets, see Theorem 2.1(a), the choice $a > a_0 \stackrel{\text{def}}{=} \max(a', a'')$, where a', a'' are the constants furnished by Theorems 1.1 and 1.2 respectively, shows that the set (1.7) is an a-winning set for the MSG induced by $\mathcal{F}$. □

8. Proof of Theorem 1.4

The goal of this section is to reduce Theorem 1.4 to Corollary 1.3. We will do it in several steps. We denote by

$$\underline{\dim}_B(X) \stackrel{\text{def}}{=} \liminf_{\varepsilon\to 0} \frac{\log N_X(\varepsilon)}{-\log\varepsilon}, \quad \overline{\dim}_B(X) \stackrel{\text{def}}{=} \limsup_{\varepsilon\to 0} \frac{\log N_X(\varepsilon)}{-\log\varepsilon} \tag{8.1}$$

the lower and upper box dimension of a metric space X, where $N_X(\varepsilon)$ is the smallest number of sets of diameter ε needed to cover X. Note that one always has

$$\dim(X) \le \underline{\dim}_B(X) \le \overline{\dim}_B(X)\,. \tag{8.2}$$

We will need two classical facts about Hausdorff dimension:

Lemma 8.1. (Marstrand Slicing Theorem, see [F, Theorem 5.8] or [KM, Lemma 1.4]) *Let X be a Borel subset of a manifold such that $\dim(X) \ge \alpha$, let Y be a metric space, and let Z be a subset of the direct product $X \times Y$ such that*

$$\dim\big(Z \cap (\{x\} \times Y)\big) \ge \beta$$

for all $x \in X$. Then $\dim(Z) \ge \alpha + \beta$.

Lemma 8.2. (Wegmann Product Theorem, see [Weg] or [F, Formulae 7.2 and 7.3]) *For any two metric spaces X, Y,*

$$\dim(X) + \dim(Y) \le \dim(X \times Y) \le \dim(X) + \overline{\dim}_B(Y)\,.$$

8.1. **Two-sided orbits.** First let us show that Corollary 1.3 implies that there is a thick set of points with F-orbits bounded and staying away from a countable set.

Proposition 8.3. *Let G, Γ and F be as in Theorem 1.1, and let Z be a countable subset of G/Γ. Then $E(F,Z) \cap E(F,\infty)$ is thick.*

Proof. The argument follows the lines of [KM, §1]. Recall that we denoted by H^+ the expanding horospherical subgroup with respect to g_1. Similarly one can consider the group H^-, which is expanding horospherical with respect to g_{-1}, and another group H^0 such that the multiplication map

$$(h^-, h^+, h^0) \mapsto h^- h^+ h^0, \quad H^- \times H^+ \times H^0 \to G\,, \tag{8.3}$$

is locally (in the neighborhood of identity) very close to an isometry. More precisely, given a nonempty open $W \subset G/\Gamma$, choose a point $x \in W$ and take $U \subset G$ of the form $U^-U^+U^0$, where U^-, U^+ and U^0 are neighborhoods of identity in H^-, H^+ and H^0 respectively, such that the map (8.3) is bi-Lipschitz on U, the orbit map $\pi_x : U \to G/\Gamma$, $g \mapsto gx$, is injective and its image is contained in W. Then it is enough to show that

$$\dim\big(\{g \in U : gx \in E(F,Z) \cap E(F,\infty)\}\big) = \dim(G)\,. \tag{8.4}$$

Denote

$$C^{\pm} \stackrel{\mathrm{def}}{=} \{g \in U : gx \in E(F^{\pm}, Z) \cap E(F^{\pm}, \infty)\}\,.$$

First we fix $h^0 \in U^0$. From Corollary 1.3 and Theorem 2.1(b) it follows that

$$\dim(C^+ \cap U^+h^0) = \dim(H^+)\,. \tag{8.5}$$

We now claim that for any $g \in C^+$, the set $U^-g \smallsetminus C^+$ is at most countable. Indeed, because orbits of H^- are stable manifolds with respect to the action of F^+, for any $h^- \in U^-$ the trajectories F^+gx and F^+h^-gx have the same sets of accumulation points. Since $g \in C^+$, we conclude that either $h^-gx \in E(F^+, Z) \cap E(F^+, \infty)$, or $g_th^-gx \in Z$ for some $t \ge 0$, which can only happen for countably many h^-.

Applying Corollary 1.3 and Theorem 2.1(b) (with F^- in place of F^+ and H^- in place of H^+), one concludes that for all $h \in C^+ \cap U^+h^0$,

$$\dim(\{h^- \in U^- : h^-h \in C^-\}) = \dim(H^-)\,.$$

Using the claim, and since a countable set of points does not affect Hausdorff dimension, we find that for all $h \in C^+ \cap U^+ h^0$,

$$\begin{aligned} &\dim\left(\left\{h^- \in U^- : h^- h \in C^+ \cap C^-\right\}\right) \\ =&\dim\left(\left\{h^- \in U^- : h^- hx \in E(F,Z) \cap E(F,\infty)\right\}\right) = \dim(H^-)\,. \end{aligned} \tag{8.6}$$

Using Lemma 8.1 and (8.5),(8.6) we find that

$$\dim\left((U^- \times U^+)h^0 \cap C^+ \cap C^-\right) = \dim(H^+) + \dim(H^-)\,.$$

Now using Lemma 8.1 again (with $X = U^0$) we obtain (8.4). □

8.2. **Entropy argument.** In this subsection we will show that the requirement $\dim(\overline{Fx}) < \dim(G)$ is fulfilled automatically whenever Fx is a bounded and nondense orbit on G/Γ, where G, Γ and F are as in Theorem 1.1. Combined with the result of the previous section, this will establish

Proposition 8.4. *Let G, Γ and F be as in Theorem 1.1, and let Z be a countable subset of G/Γ. Then the set* (1.8) *is thick.*

The argument contained in this subsection was explained to us by Manfred Einsiedler and Elon Lindenstrauss. It relates the Hausdorff dimension to the topological and metrical entropy of the action of F^+. For a detailed recent exposition we refer the reader to the survey [EL].

Proposition 8.5. *Suppose G is a semisimple Lie group with no compact factors, Γ is an irreducible lattice, and $x \in G/\Gamma$ such that F^+x is bounded but not dense. Then necessarily* $\dim(\overline{F^+x}) < \dim(G)$.

We will need the following statement, whose proof is postponed to the end of this section:

Lemma 8.6. *Suppose g_1 is a non-quasiunipotent element of G and $K \subset G/\Gamma$ is compact and g_1-invariant, with* $\dim(K) = \dim(G)$. *Then the topological entropy $h_{\mathrm{top}}(g_1)$of the g_1-action on K is equal to the measure theoretic entropy of the action of g_1 on G/Γ.*

Proof of Proposition 8.5 assuming Lemma 8.6. Suppose by contradiction that $\dim(K) = \dim(G)$, where $K = \overline{F^+x}$ is compact. By Lemma 8.6 we find that the topological entropy $h_{\mathrm{top}}(g_1)$ is equal to $h_{\max}$, the measure theoretic entropy of the action of g_1 on G/Γ. By the variational principle [EL, Prop. 3.21] there are g_1-invariant measures μ_i on K with entropy $h(\mu_i)$ tending to $h_{\max}$. Since K is compact, we can take a weak-$*$ limit ν of these measures, then by [EL, Prop. 3.15] we have $h(\nu) = h_{\max}$, so by [EL, Thm. 7.9], ν is invariant under H^+. Since the stabilizer of a measure is a group, ν is also invariant under g_1^{-1}, and since the entropy of a map is the same as that of its inverse, we see that ν is also invariant under the horospherical subgroup H^- of g_1^{-1}. Let G_0 be the subgroup of G generated by H^+ and H^-; it is sometimes called the *Auslander subgroup* corresponding to g_1, see [KSS, §2.1.a]. It is normal in G, so by Proposition 5.5 it acts uniquely ergodically on G/Γ. Therefore ν is the Haar measure which is of full support. But ν is supported on K, so $K = G/\Gamma$, contrary to assumption. □

Proof of Lemma 8.6. We will use the following formula for $h_{\max}$ (see e.g. [EL]):

$$h_{\max} = \log \left| \det \mathrm{Ad}(g_1)|_{\mathfrak{h}^+} \right| = \sum_{\alpha \in \Phi^+} \dim(\mathfrak{g}_\alpha)\, \alpha(a_1), \tag{8.7}$$

where $g_1 = k_1 u_1 a_1$ is the decomposition of g_1 into its compact, semisimple and unipotent parts as in Proposition 5.1, $\mathfrak{h}^+$ is the Lie algebra of the expanding horospherical subgroup H^+, Φ is the set of roots, and $\Phi^+ = \{\alpha \in \Phi : \alpha(a_1) > 0\}$. Recall that x_1, x_2 are said to be (n, ε)-*separated* if there is $j \in \{0, \dots, n-1\}$ such that $d(g_1^j x_1, g_1^j x_2) \geq \varepsilon$. Then one has

$$h_{\mathrm{top}}(g_1) = \sup_{\varepsilon > 0} \lim_{n \to \infty} \frac{\log N_K(n, \varepsilon)}{n},$$

where $N_K(n, \varepsilon)$ is the maximal cardinality of a set of mutually (n, ε)-separated points of K. Suppose by contradiction that $h_{\mathrm{top}}(g_1) = h_{\max} - \delta_1$ for some $\delta_1 > 0$. Then there is $\delta_2 > 0$, and for any ε, there exists a sequence $n_k \to \infty$ so that for all k, there are at most

$$M_k \stackrel{\mathrm{def}}{=} e^{n_k (h_{\max} - \delta_2)}$$

mutually (n_k, ε)-separated points of K. This means that K is covered by M_k sets of the form $B(x, n_k, \varepsilon)$, where

$$B(x, n, \varepsilon) \stackrel{\mathrm{def}}{=} \bigcap_{j=0}^{n-1} g_1^{-j} \left(B(g_1^j x, \varepsilon) \right).$$

Applying an argument similar to that of [EKL, Prop. 8.3], using the fact that K is compact and the exponential map is bi-Lipschitz in a sufficiently small neighborhood of the identity, we find: for any small enough ε there is $\varepsilon' > 0$ so that if we represent $y = \exp(\mathbf{v})x$, with $\mathbf{v} \in \mathfrak{g}$, $\|\mathbf{v}\| \leq \varepsilon'$ and $x, y \in K$,then for the first index j for which $\|\mathrm{Ad}^j(g_1)\mathbf{v}\| \geq \varepsilon'$, we will have $d(g_1^j x, g_1^j y) \geq \varepsilon$. This implies that any set $B(x, n_k, \varepsilon)$ as above is covered by a set $\exp\big(C(n_k, \varepsilon')\big)x$, where

$$C(n, \eta) \stackrel{\mathrm{def}}{=} \bigcap_{j=0}^{n-1} \{\mathbf{v} \in \mathfrak{g} : \|\mathrm{Ad}^j(g_1)\mathbf{v}\| \leq \eta\}.$$

We may simplify calculations by assuming that the norm in the above formula is the sup norm with respect to a basis

$$\{\mathbf{v}_i^{(\alpha)} : \alpha \in \Phi, i = 1, \ldots, \dim(\mathfrak{g}_\alpha)\}$$

of simultaneous eigenvectors for the action of $\mathrm{Ad}(a_1)$ on $\mathfrak{g}$. Since $\mathrm{Ad}^j(g_1) = \mathrm{Ad}^j(k_1 n_1) \circ \mathrm{Ad}^j(a_1)$ and $\mathrm{Ad}^j(k_1 n_1)$ is a quasi-unipotent map, whose norm grows at most polynomially in j, for any sufficiently small $\delta_3 > 0$ and for any sufficiently large j, $\|\mathrm{Ad}^j(g_1)\mathbf{v}\|$ is bounded above by

$$\max\left(\left\{e^{j(\alpha(a_1)-\delta_3)}|c_i^{(\alpha)}| : \alpha \in \Phi^+\right\} \cup \left\{e^{j\delta_3}|c_i^{(\alpha)}| : \alpha \in \Phi \smallsetminus \Phi^+\right\}\right),$$

where

$$\mathbf{v} = \sum_{\alpha\in\Phi} \sum_{i=1}^{\dim(\mathfrak{g}_\alpha)} c_i^{(\alpha)} \mathbf{v}_i^{(\alpha)}.$$

Therefore $C(n, \eta) \subset D(n, \eta)$, where the latter is defined as

$$\left\{\mathbf{v} : \forall\alpha \in \Phi^+,\ |c_i^{(\alpha)}| \le e^{n(\delta_3-\alpha(a_1))}\eta \text{ and } \forall\alpha \in \Phi \smallsetminus \Phi^+,\ |c_i^{(\alpha)}| \le e^{n\delta_3}\eta\right\}.$$

We cover $D(n, \eta)$ by cubes in $\mathfrak{g}$ whose sidelength is the smallest of these dimensions, namely $e^{n(\delta_3-\alpha_{\max})}\eta$, where

$$\alpha_{\max} = \max\{\alpha(a_1) : \alpha \in \Phi^+\}.$$

Then the number of these cubes is at most

$$M'_n = \exp\left[n\left(\alpha_{\max}\dim(G) + 2\delta_3\dim(\mathfrak{g}^{\le}) - \sum_{\alpha\in\Phi^+}\dim(\mathfrak{g}_\alpha)\,\alpha(a_1)\right)\right],$$

where $\mathfrak{g}^{\le} = \bigoplus_{\alpha(a)\le 0}\mathfrak{g}_\alpha$. Using the images of these cubes under the exponential map, we find that for some constant c, we have a covering of K by sets whose diameter is at most $ce^{n_k(\delta_3-\alpha_{\max})}$, and whose number is bounded by $M_{n_k}M'_{n_k}$, i.e. is not greater than

$$\exp\left[n_k\left(h_{\max} - \delta_2 + \alpha_{\max}\dim(G) + 2\delta_3\dim(\mathfrak{g}^{\le}) - \sum_{\alpha\in\Phi^+}\dim(\mathfrak{g}_\alpha)\alpha(a_1)\right)\right]$$
$$\underset{(8.7)}{=} c\exp\left[n_k\left(-\delta_2 + \alpha_{\max}\dim(G) + 2\delta_3\dim(\mathfrak{g}^{\le})\right)\right].$$

Thus we have proved that

$$N_K(ce^{n_k(\delta_3-\alpha_{\max})}) \le c\exp\left(n_k\left(\alpha_{\max}\dim(G) - \delta_2 + 2\delta_3\dim(\mathfrak{g}^{\le})\right)\right).$$

This, in view of (8.1), implies

$$\begin{aligned}\dim(K) &\le \frac{\log c + n_k\big(\alpha_{\max}\dim(G) - \delta_2 + 2\delta_3\dim(\mathfrak{g}^{\le})\big)}{-\log c + n_k(\alpha_{\max}-\delta_3)}\\ &\to_{k\to\infty} \frac{\alpha_{\max}\dim(G) + 2\delta_3\dim(\mathfrak{g}^{\le}) - \delta_2}{\alpha_{\max}-\delta_3}\\ &\to_{\delta_3\to 0} \dim(G) - \delta_2/\alpha_{\max} < \dim(G)\,,\end{aligned}$$

contradicting the assumption of the lemma. □

8.3. **Completion of the proof.** Here we reduce Theorem 1.4 to the set-up of Theorem 1.1 and Proposition 8.4. There are several additional steps left, as we will see below.

Proof of Theorem 1.4. Recall that we have taken G to be an arbitrary Lie group and Γ a lattice in G. It is easy to see that G can be assumed to be connected, since F happens to be a subgroup of the connected component G^0 of identity, and connected components of G/Γ are copies of $G^0/(G^0\cap\Gamma)$.

Now let $R(G)$ be the radical of G. Then $G/R(G) = G_0\times\widehat{G}$, where G_0 is compact and $\widehat{G}$ is connected semisimple without compact factors. Let $\pi : G\to\widehat{G}$ be the canonical projection, then (see [Ra, Chapter 9] or [Da1, Lemma 5.1]) $\hat\Gamma \stackrel{\text{def}}{=} \pi(\Gamma)$ is a lattice in $\hat G$. Also, recall that F is assumed to be absolutely non-quasiunipotent; clearly the same can be said about $\pi(F)$. If we denote by $\bar\pi$ the induced map of homogeneous spaces $G/\Gamma\to\widehat{G}/\widehat{\Gamma}$, then it is known (see the second part of [Da1, Lemma 5.1]) that $\bar\pi$ has compact fibers and $\operatorname{Ker}\pi\,\cap\,\Gamma\subset\operatorname{Ker}\pi$ is a uniform lattice. Suppose we knew that the set

$$\{x\in E\big(\pi(F),\bar\pi(Z)\big)\cap E\big(\pi(F),\infty\big) : \dim(\overline{\pi(F)x}) < \dim(\widehat{G})\} \tag{8.8}$$

is thick. We claim that the $\bar\pi$-preimage of the above set is contained in the set (1.8); thus the latter is also thick. Indeed, if $\bar\pi(x)$ belongs to $E\big(\pi(F),\infty\big)$, then Fx is bounded by the compactness of the fibers of $\bar\pi$. Also, by the continuity of $\bar\pi$, if z is in the closure of Fx, then $\bar\pi(z)$ must be in the closure of $\pi(F)\bar\pi(x) = \bar\pi(Fx)$. Finally, the orbit closure $\overline{Fx}$ is contained in the preimage of $\overline{\pi(F)\bar\pi(x)}$ and therefore has codimension not less than the codimension of $\overline{\pi(F)\bar\pi(x)}$, in view of Lemma 8.2.

Thus we have reduced Theorem 1.1 to the case when G is connected semisimple without compact factors. Next note that without loss of generality we can assume that the center $C(G)$ of G is trivial. Indeed, let us denote the quotient group $G/C(G)$ by G', the homomorphism $G\to G'$ by p, and the induced map $G/\Gamma\to G'/p(\Gamma)$ by $\bar p$. Since

$\Gamma C(G)$ is discrete [Ra, Corollary 5.17], $p(\Gamma)$ is also discrete, hence the quotient $C(G)/\big(\Gamma\cap C(G)\big)$ is finite. This means that $(G/\Gamma,\bar p)$ is a finite covering of $G'/p(\Gamma)$, and an argument similar to that of the previous step completes the reduction.

Finally we are ready to reduce to the set-up of Proposition 8.4. Let $G_1,\dots,G_\ell$ be connected normal subgroups of G such that $G=\prod_{i=1}^{l} G_i$, $G_i\cap G_j=\{e\}$ if $i\neq j$, $\Gamma_i=G_i\cap\Gamma$ is an irreducible lattice in G_i for each i, and $\prod_{i=1}^{\ell}\Gamma_i$ has finite index in Γ. Denote by p_i the projection $G\to G_i$; we know that for any $i\in\{1,\dots,\ell\}$, the group $p_i(F)$ is either trivial or not quasiunipotent. Also define maps

$$G/\Gamma \xrightarrow{\tilde p} \prod_{i=1}^{\ell}(G_i/\Gamma_i) = G/\prod_{i=1}^{\ell} p_i(\Gamma) \xrightarrow{\bar p_i} G_i/\Gamma_i\,,$$

and let $\tilde p_i \stackrel{\mathrm{def}}{=} \tilde p\circ\bar p_i$. Applying Proposition 8.4 to each of the spaces G_i/Γ_i, we conclude that for all i, the sets

$$A_i \stackrel{\mathrm{def}}{=} \{x\in E\big(p_i(F),\tilde p_i(Z)\big)\cap E\big(p_i(F),\infty\big) : \dim(\overline{p_i(F)x}) < \dim(G_i)\}$$

are thick. We now claim that the set (1.8) contains $\tilde p^{-1}(A_1\times\cdots\times A_\ell)$. Indeed, if Fx is unbounded, then so is $F\tilde p(x)$, and hence at least one of its projections onto G_i/Γ_i. Likewise, if z is in the closure of Fx, then clearly its projection is in the closure of the projection of Fx. Finally, the orbit closure $\overline{Fx}$ is contained in $\tilde p^{-1}\big(\prod_{i=1}^{\ell}\overline{p(F)\tilde p_i(x)}\big)$; and as long as the codimension of at least one of the sets $\overline{p(F)\tilde p_i(x)}$ is positive, the same can be said about $\overline{Fx}$, again by Lemma 8.2. We conclude that the set (1.8) is thick, as claimed. $\square$

9. Concluding remarks

Let m,n be positive integers, and let $\mathbf r\in\mathbb R^m$ and $\mathbf s\in\mathbb R^n$ be as in (5.7). One says that $Y\in M_{m,n}$ (interpreted as a system of m linear forms in n variables) is $(\mathbf r,\mathbf s)$-*badly approximable*, denoted by $Y\in\mathbf{Bad}(\mathbf r,\mathbf s)$, if

$$\inf_{\mathbf p\in\mathbb Z^m,\,\mathbf q\in\mathbb Z^n\smallsetminus\{0\}} \max_i |Y_i\mathbf q-p_i|^{1/r_i}\cdot\max_j|q_j|^{1/s_j}>0\,, \tag{9.1}$$

where Y_i, $i=1,\dots,m$ are rows of Y (linear forms $\mathbf q\mapsto Y_i\mathbf q$). The choice of constant weights $r_i\equiv 1/m$ and $s_j\equiv 1/n$ corresponds to the classical notion of *badly approximable* systems. It has been observed by Dani [Da2] that $Y\in M_{m,n}$ is badly approximable if and only if $u_Y\Gamma\in E(F^+,\infty)$ where G, Γ and F are as in (1.2), $k=m+n$, and u_Y is as in (5.5). This way he could use Schmidt's result on thickness of the set of badly approximable matrices to construct bounded trajectories in this particular case.

Now, more generally, consider G, Γ as in (1.2), and let $F = F_{\mathbf{r},\mathbf{s}}$ be as in (5.6). A rather straightforward generalization of Dani's result [K2] shows that $Y \in \mathbf{Bad}(\mathbf{r},\mathbf{s})$ if and only if $u_Y\Gamma \in E(F^+_{\mathbf{r},\mathbf{s}}, \infty)$. Thus Theorem 5.2, in view of Proposition 5.7, implies

Corollary 9.1. $\mathbf{Bad}(\mathbf{r},\mathbf{s})$ *is winning for the MSG induced by the semigroup of contractions* $\Phi_t : (y_{ij}) \mapsto (e^{-(r_i+s_j)t}y_{ij})$ *of* $M_{m,n}$.

The case $n = 1$ of the above corollary is the main result of [KW3], which was proved via a variation of Schmidt's methods, not using homogeneous dynamics. The fact that the set $\mathbf{Bad}(\mathbf{r},\mathbf{s})$ is thick was known before, see [PV] for the case $n = 1$ and [KW2, Corollary 4.5] for the general case. Note that a famous problem dating back to W. Schmidt [Sc3] is to determine whether or not for *different* pairs $(\mathbf{r},\mathbf{s})$ and $(\mathbf{r}',\mathbf{s}')$, the intersection of $\mathbf{Bad}(\mathbf{r},\mathbf{s})$ and $\mathbf{Bad}(\mathbf{r}',\mathbf{s}')$ could be empty. Schmidt conjectured that the intersection is non-empty in the special case $n = 2$, $m = 1$, $\mathbf{r} = (\frac{1}{3}, \frac{2}{3})$ and $\mathbf{r}' = (\frac{2}{3}, \frac{1}{3})$. Recently this conjecture was proved in a much stronger form by Badziahin, Pollington and Velani [BPV]: they established that the intersection of countably many of those sets has Hausdorff dimension 2, as long as the weights (r_1, r_2) are bounded away from the endpoints of the interval $\{r_1 + r_2 = 1\}$. This was done without using Schmidt games. Unfortunately, the results of the present paper do not give rise to any further progress related to Schmidt's Conjecture, since each pair $(\mathbf{r},\mathbf{s})$ defines a different MSG, and we are unable to show that winning sets of different games must have nonempty intersection.

However, our technique can be applied to a similar problem of constructing points whose orbits under different one-parameter semigroups F_j^+ stay away from a countable subset of G/Γ. Namely, the following can be established:

Corollary 9.2. *Let* G *be a Lie group,* Γ *a discrete subgroup of* G*,* Z *a countable subset of* G/Γ*, and let* $\{F_j\}$ *be a countable collection of one-parameter subgroups of* G *such that there exists a one-dimensional subgroup* H *of* G *which is contained in the expanding horospherical subgroups corresponding to* F_j^+ *and is normalized by* F_j *for each* j*. Then the set* $\bigcap_j E(F_j^+, Z)$ *is thick.*

Proof. By Theorem 7.1, for any $x \in G/\Gamma$, each of the sets

$$\{h \in H : hx \in E(F_j^+, Z)\} \tag{9.2}$$

is a winning set for the modified Schmidt game induced by the semigroup of contracting automorphisms of H given by the conjugation by elements of F_j^+. However, since H is one-dimensional, one can reparameterize each of the groups F_j so that they all give rise to the same semigroup $\mathcal{F} = \{\Phi_t|_H\}$ and induce the same modified game (isomorphic to the original Schmidt's game). Then one can choose an interval D_0 and $a'' > 0$ such that (7.2) is satisfied, and, as in the proof of Theorem 7.1, conclude that each of the sets (9.2) is a-winning for any $a > a''$; thus the claim follows from Theorem 2.1 and Lemma 8.1. $\square$

An example of a situation described in the above corollary is furnished by the action of groups $F = F_{\mathbf{r},\mathbf{s}}$ as in (5.6) on homogeneous spaces G/Γ of $G = \mathrm{SL}_{m+n}(\mathbb{R})$, where Γ is any discrete subgroup of G and $(\mathbf{r}, \mathbf{s})$ is as in (5.7). It is easy to see that the group

$$\{u_y \stackrel{\mathrm{def}}{=} I_{m+n} + yE_{1,m+n} : y \in \mathbb{R}\}\,,$$

where $E_{1,m+n}$ stands for the matrix with 1 in the upper-right corner and 0 elsewhere, satisfies the assumptions of the corollary. Thus for any countable set of pairs $(\mathbf{r}_j, \mathbf{s}_j)$, any countable subset Z of G/Γ and any $x \in G/\Gamma$, the set $\bigcap_j \{y \in \mathbb{R} : u_y x \in E(F^+_{\mathbf{r}_j,\mathbf{s}_j}, Z)$ is winning.

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