REMARKS ON MINIMAL SETS AND CONJECTURES OF CASSELS, SWINNERTON-DYER, AND MARGULIS

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ABSTRACT. We prove that a hypothesis of Cassels, Swinnerton-Dyer, recast by Margulis as statement on the action of the diagonal group A on the space of unimodular lattices, is equivalent to several assertions about minimal sets for this action. More generally, for a maximal $\mathbb{R}$-diagonalizable subgroup A of a reductive group G and a lattice Γ in G, we give a sufficient condition for a compact A-minimal subset Y of G/Γ to be of a simple form, which is also necessary if G is $\mathbb{R}$-split. We also show that the stabilizer of Y has no nontrivial connected unipotent subgroups.

1. INTRODUCTION

Given a group A acting on a locally compact space X, we say that the action is *minimal* if every A-orbit is dense, or equivalently, if there are no proper A-invariant closed subsets. We say that a subset $Y \subset X$ is a *minimal set* if it is A-invariant and closed, and the restriction of the A-action to Y is minimal. If X is compact then any closed invariant subset contains a minimal set, and the study of minimal sets is often an important first step in the study of all A-orbits in X. The study of minimal sets may be viewed as the topological counterpart of the study of invariant ergodic measures. In many dynamical systems, there are many minimal sets defying a simple classification; however, as we will see, in some dynamical systems of algebraic origin, the classification of minimal sets might be possible and is of great interest.

This paper is devoted to the study of minimal sets for the action of A on G/Γ, where G is a connected reductive real algebraic group, Γ is a lattice (i.e. a discrete subgroup such that the quotient G/Γ supports a finite G-invariant measure), and A is a maximal connected $\mathbb{R}$-diagonalizable subgroup of G. Recall that a *root* on A is a nontrivial homomorphism $\alpha : A \to \mathbb{R}^*$ such that there is a nonzero $\mathbf{v} \in \mathfrak{g} = \mathrm{Lie}(G)$ such that for any $a \in A$, $\mathrm{Ad}(a)\mathbf{v} = \alpha(a)\mathbf{v}$, where $\mathbb{R}^*$ denotes the multiplicative group of positive real numbers. We will denote the kernel of α by A_α; this is a subgroup of A which is of codimension one.

We will pay particular attention to the action of the groups A_α on a minimal set for A.

We begin with the important special case in which $G = \mathrm{SL}_n(\mathbb{R})$, $n \geq 3$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, and A is the subgroup of all diagonal matrices with positive diagonal entries. In this case the space G/Γ is naturally identified with the space of unimodular lattices in $\mathbb{R}^n$, where the action is simply the linear action of a matrix as a linear isomorphism of $\mathbb{R}^n$. The dynamics of the A-action on G/Γ has been intensively studied, both for its intrinsic interest as a prototypical action for which one may hope to classify all minimal sets and invariant measures, and for its connections to number-theoretic questions. See [EL] for a survey of the history as well as many recent developments.

To illustrate this connection, let f be the product of three real linear forms in three variables. In [CaSD, Hypothesis A], Cassels and Swinnerton-Dyer asked whether $\inf_{\vec{x} \in \mathbb{Z}^3 \smallsetminus \{0\}} |f(\vec{x})| > 0$ implies that f is a multiple of a form with integer coefficients[1]. They showed that an affirmative answer to this question would yield an affirmative solution to a famous conjecture of Littlewood, and formulated several additional questions about products of three linear real forms, which remain open to this day. Cassels and Swinnerton-Dyer did not make an explicit conjecture regarding the correct answer to their questions, but did express the opinion that 'we tend to believe [that Hypothesis A is true]'. In 2000 Margulis [Ma2] showed the equivalence of Hypothesis A with a dynamical statement, which he stated explicitly as a conjecture:

Conjecture 1 (Margulis). *Let $n \geq 3$, $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, and let A be the group of positive diagonal matrices. Any A-orbit on G/Γ which is bounded (i.e. has compact closure) is closed (and hence compact).*

In §2 we will show that Conjecture 1 is equivalent to statements regarding minimal sets for the A-action. Namely we will show:

Theorem 2. *Let G, Γ, A be as in Conjecture 1. The following assertions are equivalent to Conjecture 1:*
  (a) *Any compact minimal set for the A-action on G/Γ is a compact orbit.*
  (b) *For any root α and any compact A-minimal set $Y \subset G/\Gamma$, Y is also minimal for the action of A_α.*
  (c) *For any root α and any compact minimal set $Y \subset G/\Gamma$ for the action of A, any A_α-minimal subset $Y_\alpha \subset Y$ satisfies $AY_\alpha = Y$.*

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[1]Actually the statement above is the negation of what Cassels and Swinnerton-Dyer called 'Hypothesis A'.

(d) *For any root α and any compact minimal set $Y \subset G/\Gamma$ for the action of A, there exists an A_α-minimal subset $Y_\alpha \subset Y$ such that $AY_\alpha = Y$.*

Now we turn to the general case. The assertion of Conjecture 1 is not true for general G/Γ. In fact, in unpublished work, Mary Rees showed that it does not hold even for $G = \mathrm{SL}_3(\mathbb{R})$ and Γ a certain cocompact lattice. An interesting feature of Rees' examples is that the groups A_α do not act minimally on A-minimal sets, see the discussion in [LW, §5]. That is, Rees' example also shows that condition (b) of Theorem 2 need not hold in general. In this paper we clarify the implications among the conditions (a–d), in the context of a general G/Γ; among other things, it will develop that it is no coincidence, that for certain G/Γ, the same constructions show the failure of Conjecture 1 and condition (b). More precisely, we will prove the following result.

Theorem 3. *Let G be a connected reductive real algebraic group, Γ a lattice in G, A a maximal connected $\mathbb{R}$-diagonalizable subgroup of G, and $Y \subset G/\Gamma$ a compact A-minimal set. Suppose that for any root α, there exists an A_α-minimal set $Y_\alpha \subset Y$ such that $AY_\alpha = Y$. Then there exists a compact subtorus T of G centralized by A such that Y is an AT-orbit.*

Note that if G is $\mathbb{R}$-split, then A is of finite index in $Z_G(A)$, and there is no nontrivial compact torus centralized by A. So we have the following corollary, which implies the equivalence of (a), (c) and (d) in Theorem 2.

Corollary 4. *Under the conditions of Theorem 3, suppose in addition that G is $\mathbb{R}$-split, then the following are equivalent:*

(a) *Y is a single (compact) A-orbit.*
(b) *For any root α, any A_α-minimal set $Y_\alpha \subset Y$ satisfies $AY_\alpha = Y$.*
(c) *For any root α, there exists an A_α-minimal set $Y_\alpha \subset Y$ such that $AY_\alpha = Y$.*

For a Lie group H, we denote its connected component of the identity by H^0. For any compact A-minimal set $Y \subset G/\Gamma$, denote

$$\mathrm{Stab}_G(Y) \overset{\mathrm{def}}{=} \{g \in G : gY = Y\}.$$

The proof of Theorem 3 is based on the following result, which is of interest in itself.

Theorem 5. *Let G, Γ, A be as in Theorem 3, and let $Y \subset G/\Gamma$ be a compact A-minimal set. Then $\mathrm{Stab}_G(Y)$ has no nontrivial connected unipotent subgroups. Equivalently, we have $\mathrm{Stab}_G(Y)^0 \subset Z_G(A)$.*

If G is $\mathbb{R}$-split, Theorem 5 asserts that $\mathrm{Stab}_G(Y)^0 = A$, i.e., A is of finite index in $\mathrm{Stab}_G(Y)$. However, in the general case, it may happen that $\mathrm{Stab}_G(Y)$ contains a full Cartan subgroup of G which is strictly larger than A. We will illustrate this with an example where G is the underlying real algebraic group of $\mathrm{SL}_2(\mathbb{C})$ at the end of Section 3.

The proof of Theorem 5 relies on work of Dani and Margulis [DM] and Prasad and Raghunathan [PR]. It also relies on a statement (Proposition 8 below) about orbit-closures of elements of an A-minimal set, under a unipotent group normalized by A. The proof of Proposition 8 employs some ideas of Mozes [M]. Throughout this paper, by a real algebraic group, we mean an open subgroup of the Lie group of real points of an algebraic group defined over $\mathbb{R}$.

2. Conjectures for the A-action

We begin with a simple proof of Theorem 2, based on the results of [LW]. We will need the following:

Proposition 6. *Let $G/\Gamma = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, let A be the group of diagonal positive matrices, and let U be a one-parameter unipotent subgroup of G normalized by A. Then any AU-orbit in G/Γ is unbounded.*

Proof. This was proved in [CaSD] and again in [LW]; we repeat the proof for completeness.

Recall Mahler's criterion, which asserts that $X \subset G/\Gamma$ is bounded if and only if there is $\varepsilon > 0$ such that any nonzero vector in any lattice in X has length at least ε. Since A normalizes U, there are distinct $i, j \in \{1, \ldots, n\}$ such that, if we define $u(s) = \exp(sE_{ij})$, then $U = \{u(s) : s \in \mathbb{R}\}$, where E_{ij} is the matrix with 1 in the (i,j)th entry and zero elsewhere. To simplify notation assume $j = 1$. Let $x \in G/\Gamma$, and let $v = (v_1, \ldots, v_n)$ be a nonzero vector in the lattice corresponding to x. If $v_1 = 0$ then we can apply the elements $a(t) \stackrel{\mathrm{def}}{=} \mathrm{diag}(e^{(n-1)t}, e^{-t}, \ldots, e^{-t})$,

and we will have $a(t)v \to_{t \to \infty} 0$, implying via Mahler's compactness criterion that Ax is unbounded. If $v_1 \neq 0$ then we can apply $u(-v_i/v_1)$ to x to obtain, in the lattice corresponding to $u(-v_i/v_1)x$, a vector with vanishing i-th entry. Now we repeat the previous step to this vector. $\square$

Proof of Theorem 2. Conjecture 1 $\iff$ (a): Suppose Conjecture 1 holds. Any orbit inside any compact minimal set Y is bounded, hence a compact orbit, which by minimality coincides with Y. Conversely, assume (a), and suppose Ax is a bounded orbit. Then $X_0 \stackrel{\mathrm{def}}{=} \overline{Ax}$ contains a minimal set and hence a compact orbit. Now an argument of [CaSD] (see also [Ma1, LW]) shows that if X_0 is not itself a compact A-orbit, then it contains an AU-orbit and hence cannot be compact, in view of Proposition 6.

(a) $\implies$ (b): This is proved in [LW, Step 6.1].

(b) $\implies$ (c) $\implies$ (d): Immediate.

(d) $\implies$ (c): Suppose Y_α is minimal with $AY_\alpha = Y$. Then the sets $\{aY_\alpha : a \in A\}$ are all A_α-minimal, and they cover Y. Thus if $Y' \subset Y$ is another A_α-minimal set, then there is $a \in A$ such that $Y' = aY_\alpha$, and hence $AY' = Y$.

(c) $\implies$ (a): If (a) is false then there is an A-minimal set Y which does not contain a compact A-orbit. Let $a_0 \in A$ be a regular element, i.e. $a_0 \in A \smallsetminus \bigcup_\alpha A_\alpha$. For any root α, let $\mathbf{u}_\alpha$ be a generator of the root space

$$\mathfrak{u}_\alpha \stackrel{\mathrm{def}}{=} \{\mathbf{v} \in \mathfrak{g} : \forall a \in A, \, \mathrm{Ad}(a)\mathbf{v} = \alpha(a)\mathbf{v}\},$$

and let $u_\alpha(s) = \exp(s\mathbf{u}_\alpha)$, $U_\alpha = \{u_\alpha(s) : s \in \mathbb{R}\}$. Let U^+ be the group generated by $\{U_\alpha : \alpha(a_0) > 0\}$. Arguing as in [LW, Steps 4.3, 4.4] (up to replacing a_0 by a_0^{-1}), we see that there are distinct $x_1, x_2 \in Y$ and $u^+ \in U^+ \smallsetminus \{e\}$ such that $x_2 = u^+ x_1$. Arguing as in [LW, Step 4.5], there is a root α, distinct $y_1, y_2 \in Y$ and $u = u_\alpha(s_0) \in U_\alpha$, $s_0 \neq 0$ such that $y_2 = uy_1$. In light of Proposition 6, it suffices to show that Y contains a U_α-orbit.

Since $AY_\alpha = Y$, and each aY_α is A_α-minimal, we can assume that $y_1 \in Y_\alpha$, and we claim that $y_2 \in Y_\alpha$. If not then $y_2 \in Y' \stackrel{\mathrm{def}}{=} a'Y_\alpha$ for some $a' \in A \smallsetminus A_\alpha$, where Y' is also A_α-minimal, and there is a positive distance between Y' and Y_α. According to Lemma 11, which we prove below in a more general setting, there is $a \in A \smallsetminus A_\alpha$ such that $aY_\alpha = Y_\alpha$. Replacing a with a^{-1} if necessary, we may assume that $\alpha(a) < 1$. Then for each $k \in \mathbb{N}$,

$$Y' \ni a^k y_2 = a^k uy_1 = \exp(\alpha(a)^k s_0 \mathbf{u}_\alpha)a^k y_1 \in \exp(\alpha(a^k)s_0\mathbf{u}_\alpha)Y_\alpha.$$

Since $\alpha(a)^k \to_{k\to\infty} 0$, the distance from Y' to Y_α is bounded above by a quantity tending to zero as $k \to \infty$, a contradiction proving the claim.

Since $y_2 = uy_1 \in Y_\alpha$, and the A_α-action commutes with that of u we find that $H \overset{\text{def}}{=} \mathrm{Stab}_G(Y_\alpha)$ contains u. Since H is a closed group, it contains all conjugates of u by a and hence contains U_α. This completes the proof. $\square$

3. Proof of Theorem 5

We will need the following statement, which is proved in [SW] in a more restricted setting:

Proposition 7. *Let G be a real algebraic group, Γ a lattice in G, L a connected reductive algebraic subgroup of G, and $y \in G/\Gamma$. Suppose that the orbit Ly is closed. Then L can be decomposed as a direct product $L = Z \times L'$ of its connected closed normal subgroups such that*

(i) *L' contains $L_y \overset{\text{def}}{=} \{\ell \in L : \ell y = y\}$ as a lattice, and contains the maximal connected normal subgroup of G with compact center.*
(ii) *Z is an $\mathbb{R}$-diagonalizable algebraic subgroup of the center of L.*
(iii) *The natural map*
$$Z \times (L'/L_y) \to Ly, \qquad (z, \ell L_y) \mapsto z\ell y$$
is a homeomorphism.

Proof. Without loss of generality, we may assume that y is the projection of e in G/Γ, and hence $L_y = L \cap \Gamma$. Let S be the maximal connected normal semisimple subgroup of L without compact factors. By Ratner's theorem, there is a connected closed subgroup S' of G containing S such that $\overline{Sy} = S'y$ and $S' \cap \Gamma$ is a lattice in S'. Since Ly is closed, we have $S' \subset L$. Let K be the maximal connected normal compact subgroup of L, and let $N = KS'$. Then N is a connected closed normal subgroup of L and L/N is a simply connected abelian Lie group. Since $S'y$ is closed in Ly, $S'L_y$, and hence NL_y, is closed in L. Thus $q(L_y)$ is a discrete subgroup of L/N, where $q : L \to L/N$ is the quotient homomorphism. Let C be the unique connected subgroup of L/N which contains $q(L_y)$ as a lattice, and let $L' = q^{-1}(C)$. We prove that L' satisfies (i). It suffices to prove that L_y is a lattice in L'. Firstly, since $q(L_y)$ is a lattice in C, L'/NL_y carries an L'-invariant finite measure. On the other hand, since $S' \cap \Gamma$ is a lattice in S', it follows that $N \cap \Gamma$ is a lattice in N, and hence L_y is a lattice in NL_y (see [Rag, Lem. 1.7]). Thus L_y is a lattice in L' (see [Rag, Lem. 1.6]).

Since L' satisfies (i), we can choose Z satisfying (ii) such that $L = Z \times L'$. Moreover, the natural map in (iii) is the composition of the

natural homeomorphisms $Z \times (L'/L_y) \cong L/L_y$ and $L/L_y \cong Ly$, hence is a homeomorphism. This completes the proof. $\square$

In the rest of this paper, we assume that G, Γ, A are as in Theorem 3, i.e., G is a connected reductive real algebraic group, $\Gamma \subset G$ is a lattice, and $A \subset G$ is a maximal connected $\mathbb{R}$-diagonalizable subgroup. Let $\Phi = \Phi(A, G)$ be the root system. For $\alpha \in \Phi$, let $\mathfrak{u}_\alpha \subset \mathfrak{g}$ be the root space of α, and let $U_\alpha = \exp(\mathfrak{u}_\alpha)$ be the root group. The following statement will be important for the sequel, and is of independent interest. Its proof is inspired by ideas of Shahar Mozes [M].

Proposition 8. *Let $Y \subset G/\Gamma$ be a compact A-minimal set, U be a connected unipotent subgroup of G normalized by A. Then there exist an A-invariant residual subset $Y_0 \subset Y$ and a connected reductive closed subgroup $\widetilde{H}$ of G which contains U and is normalized by A such that*

(i) *For every $y \in Y_0$, we have $\overline{Uy} = \widetilde{H}y$.*
(ii) *The algebraic subgroup $L \stackrel{\text{def}}{=} N_G(\widetilde{H})^0$ of G is reductive, and Y is contained in a closed L-orbit of finite volume.*

Proof. We divide the proof into steps.

Step 1. Let $\pi : G \to G/\Gamma$ be the natural projection, and let $\mathcal{H}$ be the collection of connected closed subgroups H of G for which $H\pi(e) \cong H/\Gamma \cap H$ is closed and of finite volume, and such that the subgroup of H generated by its unipotent elements acts ergodically on $H\pi(e)$. It was shown by Ratner (see [R]) that $\mathcal{H}$ is countable, and for any $g \in G$, there is $H \in \mathcal{H}$ such that $\overline{U\pi(g)} = (gHg^{-1})\pi(g)$. For any $H \in \mathcal{H}$, let

$$X(H, U) \stackrel{\text{def}}{=} \{g \in G : Ug \subset gH\}.$$

Each $X(H, U)$ is an algebraic subvariety of G, and for any $g \in X(H, U)$ we have

$$\overline{U\pi(g)} \subset gH\pi(e) = (gHg^{-1})\pi(g).$$

Note that

$$N_G(U)X(H, U)N_G(H) = X(H, U).$$

In particular, $X(H, U)$ is left A-invariant. Note also that $G \in \mathcal{H}$, and thus $G/\Gamma = \bigcup_{H \in \mathcal{H}} \pi(X(H, U))$.

Let $H_0 \in \mathcal{H}$ be a group of minimal dimension for which $\pi(X(H_0, U))$ contains an open subset of Y. Since Y is A-minimal and $\pi(X(H_0, U))$ is A-invariant, we have

$$Y \subset \pi(X(H_0, U)).$$

By the choice of H_0, if $H \in \mathcal{H}$ is a proper subgroup of H_0, then $Y \smallsetminus \pi(X(H,U))$ is a residual subset of Y, and hence so is

$$Y_0 \overset{\text{def}}{=} Y \smallsetminus \bigcup_{H \in \mathcal{H}, H \subsetneqq H_0} \pi(X(H,U)).$$

Note that Y_0 is A-invariant. It is easy to see that for $y \in Y$ and $g \in X(H_0, U)$ with $y = \pi(g)$, $y \in Y_0$ if and only if $\overline{Uy} = (gH_0g^{-1})y$. In the sequel we prove that Y_0 has the desired properties.

Step 2. We prove that for any $y \in Y_0$ and $g \in X(H_0, U)$ with $y = \pi(g)$, there is a neighborhood Ω of g in G such that

$$\pi^{-1}(Y) \cap \Omega \subset X(H_0, U).$$

If not, then there is a sequence $\{g_n\} \subset \pi^{-1}(Y) \smallsetminus X(H_0, U)$ with $g_n \to g$. By the Baire category theorem, there is a compact set $K \subset X(H_0, U)$ such that $\pi(K)$ contains a nonempty open subset of Y. In view of the minimality of the A-action on Y and the left A-invariance of $X(H_0, U)$, we may assume that $\pi(K)$ contains a neighborhood of y in Y. Since $Y \ni \pi(g_n) \to y$, we may assume that $\pi(g_n) \in \pi(K)$. So there are sequences $\{k_n\} \subset K$ and $\{\gamma_n\} \subset \Gamma$ such that $g_n = k_n\gamma_n$. Since K is compact, we may also assume that $k_n \to k$ for some $k \in K$. Hence $\gamma_n \to \gamma$, where $\gamma = k^{-1}g \in \Gamma$. It follows that $\gamma_n = \gamma$ for all large n. Since $\pi(g) = \pi(k) = y \in Y_0$, we have

$$(gH_0g^{-1})y = \overline{Uy} = (kH_0k^{-1})y.$$

This implies that $\gamma \in N_G(H_0)$. Thus for all large n we have

$$g_n = k_n\gamma \in X(H_0, U)N_G(H_0) = X(H_0, U),$$

a contradiction.

Step 3. Let $V = \bigoplus_{k=1}^{\dim G} \bigwedge^k \mathfrak{g}$, and $\bar{V}$ be the projective space of V. Consider the representation $\rho = \bigoplus_{k=1}^{\dim G} \bigwedge^k \mathrm{Ad} : G \to \mathrm{GL}(V)$ and the corresponding projective representation $\bar{\rho} : G \to \mathrm{PGL}(\bar{V})$. If S is a Lie subgroup of G, we denote $\bar{\mathbf{p}}_S = \bigwedge^{\dim S} \mathrm{Lie}(S)$, which is a line in $\bigwedge^{\dim S} \mathfrak{g}$, hence an element in $\bar{V}$. By Ratner's theorem, for every $x \in G/\Gamma$, there is a connected closed subgroup S of G containing U such that $\overline{Ux} = Sx$. This defines a map

$$\varphi : G/\Gamma \to \bar{V}, \qquad \varphi(x) = \bar{\mathbf{p}}_S.$$

Note that if $\overline{Ux} = Sx$, then $\overline{U(ax)} = a\overline{Ux} = (aSa^{-1})(ax)$. This means that φ is A-equivariant, i.e.,

$$\varphi(ax) = \bar{\rho}(a)\varphi(x), \qquad \forall a \in A, x \in G/\Gamma.$$

We prove below that the map φ is continuous on Y_0.

Let $y_n, y \in Y_0$ with $y_n \to y$. We need to show that $\varphi(y_n) \to \varphi(y)$. Choose $g \in X(H_0, U)$ with $y = \pi(g)$, and choose $g_n \in G$ with $y_n = \pi(g_n)$ and $g_n \to g$. By Step 2, we may assume that $g_n \in X(H_0, U)$ for every n. Since $y_n, y \in Y_0$, we have

$$\overline{U y_n} = (g_n H_0 g_n^{-1}) y_n \quad \text{and} \quad \overline{U y} = (g H_0 g^{-1}) y.$$

This means that

$$\varphi(y_n) = \bar{\rho}(g_n) \bar{\mathbf{p}}_{H_0} \quad \text{and} \quad \varphi(y) = \bar{\rho}(g) \bar{\mathbf{p}}_{H_0}.$$

Hence $\varphi(y_n) \to \varphi(y)$.

Step 4. We prove that there is a connected closed subgroup $\widetilde{H}$ of G which contains U and is normalized by A such that for every $y \in Y_0$, we have $\overline{U y} = \widetilde{H} y$. Note that a connected subgroup S of G is normalized by A if and only if $\bar{\mathbf{p}}_S$ is fixed by $\bar{\rho}(A)$. Thus it suffices to prove that the map φ is constant on Y_0 and $\varphi(Y_0) \subset \bar{V}^A$, where $\bar{V}^A$ is the set of $\bar{\rho}(A)$-fixed points in $\bar{V}$.

Let $y \in Y_0$. If $\varphi(y) \notin \bar{V}^A$, then there is a neighborhood N of $\varphi(y)$ in $\bar{V}$ such that $\overline{N} \cap \bar{V}^A = \emptyset$. Let

$$A_0 \overset{\text{def}}{=} \{a \in A : ay \in \varphi^{-1}(N)\}.$$

Since φ is A-equivariant, we have $\bar{\rho}(A_0) \varphi(y) \subset N$. Hence

$$\overline{\bar{\rho}(A_0) \varphi(y)} \cap \bar{V}^A = \emptyset.$$

On the other hand, since the $\rho(A)$-action on V is $\mathbb{R}$-diagonalizable, there is a finite set Ψ of homomorphisms $A \to \mathbb{R}^*$ and a direct sum decomposition

$$V = \bigoplus_{\chi \in \Psi} V_\chi, \quad \text{where} \quad V_\chi = \{v \in V : \rho(a)v = \chi(a)v, \forall a \in A\}.$$

Note that $\bar{V}^A$ is equal to the union of the projective spaces of V_χ. Choose a nonzero vector $v \in \varphi(y)$, and write $v = \sum_{\chi \in \Psi'} v_\chi$, where $\emptyset \neq \Psi' \subset \Psi$ and $0 \neq v_\chi \in V_\chi$ for every $\chi \in \Psi'$. Since φ is continuous on Y_0, $\varphi^{-1}(N)$ contains a neighborhood of y in Y_0. By the minimality of the A-action on Y, A_0 is syndetic in A, i.e., there is a compact subset $C \subset A$ with $A_0 C = A$. From this it is easy to see that there exist $\chi_0 \in \Psi'$ and a sequence $\{a_n\} \subset A_0$ such that $(\chi_0^{-1}\chi)(a_n) \to 0$ for any $\chi \in \Psi' \smallsetminus \{\chi_0\}$. Thus

$$\chi_0(a_n)^{-1} \rho(a_n) v = v_{\chi_0} + \sum_{\chi \in \Psi' \smallsetminus \{\chi_0\}} (\chi_0^{-1}\chi)(a_n) v_\chi \to v_{\chi_0}.$$

This means that $\bar{\rho}(a_n)\varphi(y)$ converges to a line in V_{χ_0}, which is a point in $\bar{V}^A$. Thus $\overline{\bar{\rho}(A_0)\varphi(y)} \cap \bar{V}^A \neq \emptyset$. This is a contradiction, hence proves that $\varphi(Y_0) \subset \bar{V}^A$.

Since φ is continuous on Y_0 and is A-equivariant, and every A-orbit in Y_0 is dense in Y_0, it follows that every $\bar{\rho}(A)$-orbit in $\varphi(Y_0)$, which is a single point, is dense in $\varphi(Y_0)$. Thus $\varphi(Y_0)$ consists of only one point. Hence φ is constant on Y_0.

Step 5. We prove that $\widetilde{H}$ is reductive. For a Lie subgroup S of G, denote

$$N_G^1(S) \stackrel{\text{def}}{=} \{g \in N_G(S) : \det \mathrm{Ad}(g)|_{\mathrm{Lie}(S)} = 1\}.$$

We first prove that $A \subset N_G^1(\widetilde{H})$. Consider the character $\chi : A \to \mathbb{R}^*$ defined by $\chi(a) = \det \mathrm{Ad}(a)|_{\mathrm{Lie}(\widetilde{H})}$. Let $\mathbf{p}_{\widetilde{H}}$ be a nonzero vector in $\bar{\mathbf{p}}_{\widetilde{H}}$. Then $\rho(a)\mathbf{p}_{\widetilde{H}} = \chi(a)\mathbf{p}_{\widetilde{H}}$ for every $a \in A$. Let $g \in X(H_0, U)$ with $\pi(g) \in Y_0$. Then $\widetilde{H} = gH_0g^{-1}$, and hence $\rho(g^{-1})\mathbf{p}_{\widetilde{H}} \in \bar{\mathbf{p}}_{H_0}$. Since Y is compact, there is a compact subset $M \subset G$ such that $A\pi(g) \subset Y \subset M\pi(e)$. This implies $A \subset M\Gamma g^{-1}$. Thus

$$\chi(A)\mathbf{p}_{\widetilde{H}} = \rho(A)\mathbf{p}_{\widetilde{H}} \subset \rho(M\Gamma g^{-1})\mathbf{p}_{\widetilde{H}}.$$

By [DM, Thm. 3.4], $\rho(\Gamma g^{-1})\mathbf{p}_{\widetilde{H}}$ is discrete. So $\rho(M\Gamma g^{-1})\mathbf{p}_{\widetilde{H}}$ is a closed subset of V and does not contain 0. Hence $0 \notin \overline{\chi(A)\mathbf{p}_{\widetilde{H}}}$. This implies that $\chi(A) \neq \mathbb{R}^*$. So χ is trivial. Hence $A \subset N_G^1(\widetilde{H})$.

Let W be the unipotent radical of the Zariski closure of $\widetilde{H}$ in G. Since $A \subset N_G^1(\widetilde{H})$, we have $W \subset \widetilde{H}$ and $A \subset N_G^1(W)$. Since the real algebraic group AW is solvable and $\mathbb{R}$-split, by Borel's fixed point theorem [B, Prop. 15.2], AW is contained in a minimal parabolic subgroup P of G. The set $\Phi^+ = \{\alpha \in \Phi : U_\alpha \subset P\}$ is a system of positive roots. So there exists $a_0 \in A$ such that $\alpha(a_0) > 1$ for every $\alpha \in \Phi^+$. If W is nontrivial, then all eigenvalues of $\mathrm{Ad}(a_0)|_{\mathrm{Lie}(W)}$ are of the form $\alpha(a_0)$ ($\alpha \in \Phi^+$), and hence $\det \mathrm{Ad}(a_0)|_{\mathrm{Lie}(W)} > 1$. This contradicts $A \subset N_G^1(W)$. So W is trivial. Hence $\widetilde{H}$ is reductive. The proof of (i) is completed.

Step 6. We now prove (ii). Since $\widetilde{H}$ is reductive, so is L (see e.g. [LR, Lem 1.1]), and we have $L = N_G^1(\widetilde{H})^0$. Let $y \in Y_0$, $g \in X(H_0, U)$ with $\pi(g) = y$. Then

$$Ly = N_G^1(\widetilde{H})^0 y = gN_G^1(H_0)^0\pi(e).$$

By [DM, Thm. 3.4], $N_G^1(H_0)\pi(e)$ is closed. So Ly is also closed. Since $A \subset L$, we have $Y = \overline{Ay} \subset Ly$. It remains to prove that Ly is of finite volume. If not, then by Proposition 7, there is a nontrivial $\mathbb{R}$-diagonalizable connected algebraic central subgroup Z of L such that Zy is a divergent orbit in G/Γ. Since L contains A and is reductive, we

must have $Z \subset A$. Thus $Zy \subset Y$. This contradicts the compactness of Y. □

Proof of Theorem 5. Firstly, we remark that if S is a Lie subgroup of G containing A, then S has no nontrivial connected unipotent subgroups if and only $S^0 \subset Z_G(A)$. In fact, since S contains A, we have

$$\operatorname{Lie}(S) = (\operatorname{Lie}(S) \cap \operatorname{Lie}(Z_G(A))) \oplus \bigoplus_{\alpha \in \Phi} (\operatorname{Lie}(S) \cap \mathfrak{u}_\alpha).$$

If S has no nontrivial connected unipotent subgroups, then for every $\alpha \in \Phi$ we have $\operatorname{Lie}(S) \cap \mathfrak{u}_\alpha = 0$. Thus $\operatorname{Lie}(S) \subset \operatorname{Lie}(Z_G(A))$, and hence $S^0 \subset Z_G(A)$. Conversely, since $Z_G(A)$ consists of semisimple elements, if $S^0 \subset Z_G(A)$ then S has no nontrivial connected unipotent subgroups.

We now prove that $\operatorname{Stab}_G(Y)$ has no nontrivial connected unipotent subgroups by induction on $\dim G$. If $\dim G = 0$, there is nothing to prove. Assume $\dim G > 0$ and the assertion holds for groups of smaller dimension. We first prove the following:

Claim. If there exists a nontrivial connected normal algebraic subgroup N of G such that $\pi(\Gamma)$ is a lattice in G/N, where $\pi : G \to G/N$ is the projection, then any connected unipotent subgroup U of $\operatorname{Stab}_G(Y)$ is contained in N.

Let $G' = G/N$. Then G' is a reductive real algebraic group, and $\pi(A)$ is a maximal connected $\mathbb{R}$-diagonalizable subgroup of G'. Let $\bar{\pi} : G/\Gamma \to G'/\pi(\Gamma)$ be the induced projection. Then $\bar{\pi}(Y)$ is a compact $\pi(A)$-minimal subset of $G'/\pi(\Gamma)$, and $\pi(U)$ is a connected unipotent subgroup of $\operatorname{Stab}_{G'}(\bar{\pi}(Y))$. By the induction hypothesis, $\pi(U)$ is trivial. Hence $U \subset N$. This proves the claim.

We conclude the proof by considering three cases.

Case 1. Suppose G is not semisimple. By [Rag, Cor. 8.27], the group $N = C(G)^0$ satisfies the assumption of the claim. So any connected unipotent subgroup U of $\operatorname{Stab}_G(Y)$ is contained in $C(G)$. But $C(G)$ consists of semisimple elements. So U must be trivial.

Case 2. Suppose G is semisimple and Γ is reducible. Then there are nontrivial connected normal subgroups G_1, G_2 of G such that $G = G_1 G_2$, $G_1 \cap G_2$ is discrete, and the assumption of the claim is satisfied for $N = G_1, G_2$. Thus any connected unipotent subgroup of $\operatorname{Stab}_G(Y)$ is contained in $(G_1 \cap G_2)^0$, which is trivial.

Case 3. Suppose G is semisimple and Γ is irreducible. Suppose, to the contrary, that $\operatorname{Stab}_G(Y)$ has a nontrivial connected unipotent subgroup U. Since $\operatorname{Stab}_G(Y)$ contains A, we may assume that U is normalized by A. By Proposition 8, there exist $y \in Y$ and a connected

reductive subgroup $\widetilde{H}$ of G containing U such that $\overline{Uy} = \widetilde{H}y$, and such that Ly is closed of finite volume and contains Y, where $L = N_G(\widetilde{H})^0$ is a reductive real algebraic group containing $A\widetilde{H}$. There are three subcases:

(i) $L \neq G$. The fact $U \subset \mathrm{Stab}_L(Y)$ contradicts the induction hypothesis.

(ii) $L = G$ but $\widetilde{H} \neq G$. Then $\widetilde{H}$ is a proper normal subgroup of G, and has a closed orbit in G/Γ of finite volume. This contradicts the irreducibility of Γ.

(iii) $\widetilde{H} = G$. Then $Y \supset \overline{Uy} = \widetilde{H}y = G/\Gamma$. Hence A acts minimally on G/Γ. But by [PR, Thm. 2.8], if C is Cartan subgroup of G containing A, then there are compact C-orbits in G/Γ, a contradiction.

This completes the proof of Theorem 5. $\qquad\qquad\qquad\square$

We conclude this section with an example[2]. Let G be the underlying real algebraic group of $\mathrm{SL}_2(\mathbb{C})$. Consider the order $\mathcal{O} = \mathbb{Z}[\sqrt{-d}]$ in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, where $d > 0$ is a square-free integer. Then $\Gamma = \mathrm{SL}_2(\mathcal{O})$ is a lattice in G. The subgroup $A \subset G$ consisting of positive real diagonal matrices is a maximal connected $\mathbb{R}$-diagonalizable subgroup, and the subgroup $C \subset G$ consisting of all diagonal matrices is a Cartan subgroup. Given a compact C-orbit Cy in G/Γ, the A-action on Cy admits two possibilities: either A acts minimally on Cy, or all A-orbits in Cy are closed. In what follows, we classify all compact C-orbits on which A acts minimally. Note that the stabilizer of such a C-orbit always contains C. Thus the conclusion of Theorem 5 cannot be replaced by the stronger statement '$\mathrm{Stab}_G(Y)$ has no connected subgroups not contained in A'.

We first parameterize compact C-orbits in G/Γ. Let $\gamma \in \Gamma$ be a semisimple element of infinite order. Then there exists $g \in G$ with $g\gamma g^{-1} \in C$. It follows that $C \cap g\Gamma g^{-1}$ is infinite. This implies that $C\pi(g)$ is compact, where $\pi : G \to G/\Gamma$ is the projection. Note that $C\pi(g)$ is determined by γ up to left translation by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. By abuse of language, we say that $C\pi(g)$ is *the* compact orbit defined by γ. Every compact C-orbit arises in this way. In fact, if $C\pi(g)$ is compact, then $C \cap g\Gamma g^{-1}$ is infinite. Any element $\gamma \in \Gamma$ such that $g\gamma g^{-1}$ lies in C and is of infinite order defines $C\pi(g)$.

The compact C-orbits which are A-minimal can be characterized as follows.

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[2]The idea of constructing this example is due to Uri Shapira.

Proposition 9. *Let $\gamma \in \Gamma$ be a semisimple element of infinite order. The following statements are equivalent:*

(i) *The A-action on the compact C-orbit defined by γ is not minimal.*

(ii) *There is a positive integer n such that γ^n has real eigenvalues.*

(iii) *The extension field of $\mathbb{Q}$ generated by an eigenvalue of γ is Galois over $\mathbb{Q}$.*

(iv) *There exists $k \in \{0, 1, 2, 3, 4\}$ such that the Pell equation*

$$(4 - k)a^2 - kdb^2 = k(4 - k)$$

holds, where $a, b \in \mathbb{Z}$ are such that $\operatorname{tr}(\gamma) = a + b\sqrt{-d}$.

Proof. (i) $\Longleftrightarrow$ (ii): Let $C\pi(g)$ be the compact C-orbit defined by γ. Then $g\gamma g^{-1} \in C$. Note that the cyclic group $\langle g\gamma g^{-1} \rangle$ is infinite, hence is of finite index in $C \cap g\Gamma g^{-1}$. It follows that

$C\pi(g)$ is not A-minimal

$\Longleftrightarrow A\pi(g)$ is compact

$\Longleftrightarrow A \cap g\Gamma g^{-1}$ is infinite

$\Longleftrightarrow A \cap \langle g\gamma g^{-1} \rangle$ is infinite

$\Longleftrightarrow$ there is a positive integer n such that $g\gamma^n g^{-1} \in A$

$\Longleftrightarrow$ there is a positive integer n such that γ^n has real eigenvalues.

This proves the equivalence of (i) and (ii).

In order to prove the equivalence of (ii), (iii), and (iv), we consider the polynomial

$$F(x) = (x^2 - \operatorname{tr}(\gamma)x + 1)(x^2 - \overline{\operatorname{tr}(\gamma)}x + 1)$$
$$= x^4 - 2ax^3 + (a^2 + b^2 d + 2)x^2 - 2ax + 1,$$

where a, b are as in (iv). Let λ be an eigenvalue of γ. Then the roots of F are $\{\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}\}$. It is easy to see that if $b = 0$ then (ii), (iii), and (iv) hold. In what follows, we assume that $b \neq 0$. Then λ is neither real nor a root of unity. We claim that in this case F is irreducible over $\mathbb{Q}$. In fact, otherwise it would follow from $\lambda \notin \mathbb{R}$ that $\mathbb{Q}(\lambda)$ is an imaginary quadratic field, and hence λ, as an algebraic unit in $\mathbb{Q}(\lambda)$, would be a root of unity, a contradiction. The irreducibility of F implies that $[\mathbb{Q}(\lambda) : \mathbb{Q}] = 4$. Let K be the splitting field of F over $\mathbb{Q}$. We identify $\operatorname{Gal}(K/\mathbb{Q})$ with a permutation group on the roots of F. Then $\operatorname{Gal}(K/\mathbb{Q})$ is contained in the group D of permutations that send $\{\lambda, \lambda^{-1}\}$ onto $\{\lambda, \lambda^{-1}\}$ or $\{\bar{\lambda}, \bar{\lambda}^{-1}\}$. The group D is a dihedral group of order 8. So $[K : \mathbb{Q}] = 4$ or 8. Note that $\mathbb{Q}(\lambda)$ is Galois over

$\mathbb{Q}$ if and only if $[K : \mathbb{Q}] = 4$. Note also that $\mathrm{Gal}(K/\mathbb{Q})$ contains the complex conjugation, which we denote by c.

(ii) $\Longrightarrow$ (iii): Suppose to the contrary that $\mathbb{Q}(\lambda)$ is not Galois over $\mathbb{Q}$. Then $[K : \mathbb{Q}] = 8$, and hence $\mathrm{Gal}(K/\mathbb{Q}) = D$. There is a nontrivial element $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ of order 2 that pointwise fixes $\mathbb{Q}(\lambda)$, which must be the transposition of $\bar{\lambda}$ and $\bar{\lambda}^{-1}$. The subfield $\mathbb{Q}(\lambda) \cap \mathbb{R}$ is pointwise fixed by σ and c. It is easy to see that σ and c generate D. So $\mathbb{Q}(\lambda) \cap \mathbb{R} = \mathbb{Q}$. It follows from (ii) that $\lambda^n \in \mathbb{Q}(\lambda) \cap \mathbb{R} = \mathbb{Q}$ for some $n \in \mathbb{N}$. As a rational algebraic unit, λ^n must be ± 1. So λ is a root of unity, a contradiction.

(iii) $\Longrightarrow$ (iv): It follows from (iii) that $|\mathrm{Gal}(K/\mathbb{Q})| = 4$. So $\mathrm{Gal}(K/\mathbb{Q})$ consists of even permutations in D (which form the only order 4 subgroup of D containing c), and hence the discriminant $\Delta(F)$ of F is a square. It is easy to show that

$$\Delta(F) = 16b^4 d^2 ((a+2)^2 + b^2 d)((a-2)^2 + b^2 d)$$

(see [Co, Example 13.1.3]). The case of $k \in \{0, 4\}$ in (iv) corresponds to $ab = 0$. Suppose $ab \neq 0$. Then

$$(a+2)^2 + b^2 d = km^2, \qquad (a-2)^2 + b^2 d = kn^2$$

for some positive integers k, m, n with k square-free and $m \neq n$. It follows that

$$8a = k(m+n)(m-n), \tag{1}$$

$$4(a^2 + b^2 d + 4) = k(m+n)^2 + k(m-n)^2. \tag{2}$$

By eliminating a, we obtain

$$(k(m+n)^2 - 16)(k(m-n)^2 - 16) + 64b^2 d = 0.$$

Thus

$$k(m-n)^2 < 16. \tag{3}$$

Since k is square-free, it follows from (1) that $m - n$ is even. In view of (3), we have $m - n = \pm 2$, and hence $k \in \{1, 2, 3\}$. Now (1) becomes $k(m+n) = \pm 4a$. It follows from (2) that

$$k(a^2 + b^2 d + 4) = 4a^2 + k^2.$$

This is the Pell equation in (iv).

(iv) $\Longrightarrow$ (ii): The case of $k \in \{0, 4\}$ is obvious. Suppose $k \in \{1, 2, 3\}$. It is straightforward to check from (iv) that

$$\lambda = \left(\frac{a}{\sqrt{k}} + \frac{\sqrt{d}b}{\sqrt{4-k}} \right) \frac{\sqrt{k} + \sqrt{k-4}}{2}$$

is an eigenvalue of γ. Note that $\frac{\sqrt{k}+\sqrt{k-4}}{2} = e^{\frac{\pi i}{3}}, e^{\frac{\pi i}{4}}$, or $e^{\frac{\pi i}{6}}$ when $k = 1, 2$, or 3. Thus (ii) follows. $\qquad\square$

4. Proof of Theorem 3

The root system Φ may be non-reduced. For $\alpha \in \Phi$, denote $[\alpha] = \{c\alpha : c > 0\} \cap \Phi$. Then $[\alpha]$ has three possibilities: $\{\alpha\}$, $\{\alpha, 2\alpha\}$, and $\{\alpha, \alpha/2\}$. Let $\mathfrak{u}_{[\alpha]} = \sum_{\beta \in [\alpha]} \mathfrak{u}_\beta$, and let $U_{[\alpha]} = \exp(\mathfrak{u}_{[\alpha]})$ be the unipotent group with Lie algebra $\mathfrak{u}_{[\alpha]}$. We first prove:

Lemma 10. *Let $R \subset G$ be a closed subset invariant under the conjugation of A such that $R \cap U_{[\alpha]} \subset \{e\}$ for every $\alpha \in \Phi$. Then there exists a neighborhood Ω of e in G such that $R \cap \Omega \subset Z_G(A)$.*

Proof. Suppose the conclusion of the lemma is not true. Then there exists a sequence $\mathbf{r}_n \in \exp^{-1}(R) \smallsetminus \mathrm{Lie}(Z_G(A))$ with $\mathbf{r}_n \to 0$. Since $\mathfrak{g} = \mathrm{Lie}(Z_G(A)) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{u}_\alpha$, we can write

$$\mathbf{r}_n = \mathbf{z}_n + \sum_{\alpha \in \Phi} \mathbf{u}_{n,\alpha},$$

where $\mathbf{z}_n \in \mathrm{Lie}(Z_G(A))$, $\mathbf{u}_{n,\alpha} \in \mathfrak{u}_\alpha$. Note that $\mathbf{z}_n$ and $\mathbf{u}_{n,\alpha}$ converge to 0 as $n \to \infty$. Identifying a character $A \to \mathbb{R}^*$ with its differential, we can think of a root α as an element in $\mathfrak{a}^*$. By passing to a subsequence, we may assume that there exists $\alpha_0 \in \Phi$ such that

$$|\mathbf{u}_{n,\alpha}|^{\frac{1}{|\alpha|}} \leq |\mathbf{u}_{n,\alpha_0}|^{\frac{1}{|\alpha_0|}}, \qquad \forall \alpha \in \Phi, n \in \mathbb{N},$$

where $|\mathbf{u}_{n,\alpha}|$ (resp. $|\alpha|$) is the norm of $\mathbf{u}_{n,\alpha}$ (resp. α) with respect to a fixed inner product on $\mathfrak{g}$ (resp. $\mathfrak{a}^*$). Since $\mathbf{r}_n \notin \mathrm{Lie}(Z_G(A))$, we have $|\mathbf{u}_{n,\alpha_0}| > 0$. Let $t_n \in \mathbb{R}$ be such that $e^{t_n} = |\mathbf{u}_{n,\alpha_0}|^{-\frac{1}{|\alpha_0|^2}}$, and let $\mathbf{a} \in \mathfrak{a}$ be such that $\alpha(\mathbf{a}) = \langle \alpha, \alpha_0 \rangle$ for every $\alpha \in \Phi$. Then

$$\mathrm{Ad}(\exp(t_n\mathbf{a}))\mathbf{r}_n = \mathbf{z}_n + \sum_{\alpha \in \Phi} e^{t_n\alpha(\mathbf{a})}\mathbf{u}_{n,\alpha}$$

$$= \mathbf{z}_n + |\mathbf{u}_{n,\alpha_0}|^{-1}\mathbf{u}_{n,\alpha_0} + \sum_{\alpha \in \Phi \smallsetminus \{\alpha_0\}} |\mathbf{u}_{n,\alpha_0}|^{-\frac{\langle \alpha, \alpha_0 \rangle}{|\alpha_0|^2}}\mathbf{u}_{n,\alpha}.$$

By passing to a subsequence, we may assume that

$$|\mathbf{u}_{n,\alpha_0}|^{-1}\mathbf{u}_{n,\alpha_0} \to \mathbf{u}_1$$

for some $\mathbf{u}_1 \in \mathfrak{u}_{\alpha_0}$ with $|\mathbf{u}_1| = 1$. For $\alpha \in \Phi \smallsetminus \{\alpha_0\}$, we have

$$|\mathbf{u}_{n,\alpha_0}|^{-\frac{\langle \alpha, \alpha_0 \rangle}{|\alpha_0|^2}}|\mathbf{u}_{n,\alpha}| \leq |\mathbf{u}_{n,\alpha_0}|^{\frac{|\alpha||\alpha_0|-\langle \alpha, \alpha_0 \rangle}{|\alpha_0|^2}}.$$

If $\alpha \in \Phi \smallsetminus [\alpha_0]$, then $\langle \alpha, \alpha_0 \rangle < |\alpha||\alpha_0|$, and hence

$$|\mathbf{u}_{n,\alpha_0}|^{-\frac{\langle \alpha, \alpha_0 \rangle}{|\alpha_0|^2}} \mathbf{u}_{n,\alpha} \to 0.$$

If $[\alpha_0] \neq \{\alpha_0\}$ and $\alpha \in [\alpha_0] \smallsetminus \{\alpha_0\}$, then $\langle \alpha, \alpha_0 \rangle = |\alpha||\alpha_0|$, and we have

$$|\mathbf{u}_{n,\alpha_0}|^{-\frac{\langle \alpha, \alpha_0 \rangle}{|\alpha_0|^2}} |\mathbf{u}_{n,\alpha}| \leq 1.$$

Hence by passing to a further subsequence, we may assume that

$$|\mathbf{u}_{n,\alpha_0}|^{-\frac{\langle \alpha, \alpha_0 \rangle}{|\alpha_0|^2}} \mathbf{u}_{n,\alpha} \to \mathbf{u}_2$$

for some $\mathbf{u}_2 \in \mathfrak{u}_\alpha$. In summary, a subsequence of $\mathrm{Ad}(\exp(t_n \mathbf{a}))\mathbf{r}_n$ converges to a nonzero element $\mathbf{u} \in \mathfrak{u}_{[\alpha_0]}$, where $\mathbf{u} = \mathbf{u}_1$ if $[\alpha_0] = \{\alpha_0\}$, and $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ if $[\alpha_0] \neq \{\alpha_0\}$. Since $\exp^{-1}(R)$ is closed and $\mathrm{Ad}(A)$-invariant, we have $\mathbf{u} \in \exp^{-1}(R)$. Thus $e \neq \exp(\mathbf{u}) \in R \cap U_{[\alpha_0]}$, a contradiction. $\qquad\square$

Lemma 11. *Let $Y \subset G/\Gamma$ be a compact A-minimal set, $\alpha \in \Phi$, and $Y_\alpha \subset Y$ be an A_α-minimal set. Suppose $AY_\alpha = Y$. Then $\mathrm{Stab}_A(Y_\alpha)$ is cocompact in A.*

Proof. We first prove that for any open subset B of A, BY_α is open in Y. It suffices to prove that for every $b \in B$, bY_α is contained in the interior $\mathrm{int}(BY_\alpha)$ of BY_α. Let C be a compact neighborhood of e in A such that $bC^{-1}C \subset B$, and let $\{a_n\}$ be a sequence in A such that $\bigcup_{n=1}^\infty a_n C = A$. Then $\bigcup_{n=1}^\infty a_n CY_\alpha = Y$. By the Baire category theorem, some $a_n CY_\alpha$, and hence CY_α, has an interior point. Let $c \in C$ be such that $\mathrm{int}(CY_\alpha) \cap cY_\alpha \neq \emptyset$. Then

$$\mathrm{int}(BY_\alpha) \cap bY_\alpha \supset \mathrm{int}(bc^{-1}CY_\alpha) \cap bY_\alpha = bc^{-1}(\mathrm{int}(CY_\alpha) \cap cY_\alpha) \neq \emptyset.$$

Since $\mathrm{int}(BY_\alpha)$ is open A_α-invariant and bY_α is A_α-minimal, we have $bY_\alpha \subset \mathrm{int}(BY_\alpha)$. Thus BY_α is open in Y.

Now we prove that $\mathrm{Stab}_A(Y_\alpha)$ is cocompact in A. Since $\mathrm{Stab}_A(Y_\alpha) \supset A_\alpha$, it suffices to prove that $\mathrm{Stab}_A(Y_\alpha) \neq A_\alpha$. Let $B_1 \subset B_2 \subset \cdots$ be a nested sequence of bounded open subsets of A such that $\bigcup_{n=1}^\infty B_n = A$. Then $\{B_n Y_\alpha\}$ is an open cover of Y. Since Y is compact, there exists $n_0 \in \mathbb{N}$ such that $Y = B_{n_0} Y_\alpha$. Let $a \in A$ be such that $a^{-1}B_{n_0} \cap A_\alpha = \emptyset$. Since aY_α is A_α-minimal and $Y = B_{n_0} Y_\alpha$, there exists $b \in B_{n_0}$ with $aY_\alpha = bY_\alpha$. It follows that $a^{-1}b \in \mathrm{Stab}_A(Y_\alpha)$. But $a^{-1}b \notin A_\alpha$. So $\mathrm{Stab}_A(Y_\alpha) \neq A_\alpha$. This proves the lemma. $\qquad\square$

We are now prepared to prove Theorem 3.

Proof of Theorem 3. The assumption implies that every A_α-minimal subset of Y is of the form aY_α $(a \in A)$, hence by Lemma 11, has a cocompact stabilizer in A. Let

$$R = \{g \in G : gY \cap Y \neq \emptyset\}.$$

Then R is closed and invariant under the conjugation of A. We first prove that $R \cap U_{[\alpha]} = \{e\}$ for every $\alpha \in \Phi$. Suppose the contradiction. Then there exists $e \neq u \in U_{[\alpha]}$ such that $uY \cap Y \neq \emptyset$. Since u commutes with A_α, the compact set $uY \cap Y$ is A_α-invariant, hence contains an A_α-minimal set Y'. Since $\mathrm{Stab}_A(Y')$ is cocompact in A, there exists a sequence $\{a_n\} \subset \mathrm{Stab}_A(Y')$ such that $a_n u a_n^{-1} \to e$. Note that $u^{-1}Y' \subset Y$ is also A_α-minimal. So $u^{-1}Y' = aY'$ for some $a \in A$, i.e., $ua \in \mathrm{Stab}_G(Y')$. It follows that

$$\mathrm{Stab}_G(Y') \ni (ua)a_n(ua)^{-1}a_n^{-1} = u(a_n u a_n^{-1})^{-1} \to u.$$

By the closedness of $\mathrm{Stab}_G(Y')$, we have $u \in \mathrm{Stab}_G(Y')$. So $a_n u a_n^{-1} \in \mathrm{Stab}_G(Y')$. This implies that the group $U_{[\alpha]} \cap \mathrm{Stab}_G(Y')$ is non-discrete. So $U \stackrel{\mathrm{def}}{=} (U_{[\alpha]} \cap \mathrm{Stab}_G(Y'))^0$ is nontrivial. On the other hand, since $U_{[\alpha]}$ and $\mathrm{Stab}_G(Y')$ are normalized by $\mathrm{Stab}_A(Y')$, so is U. But $N_G(U)$ is Zariski closed in G and $\mathrm{Stab}_A(Y')$ is Zariski dense in A. So A normalizes U. Hence

$$UY = UAY' = AUY' = AY' = Y.$$

It follows that $U \subset \mathrm{Stab}_G(Y)$. This conflicts Theorem 5. Hence $R \cap U_{[\alpha]} = \{e\}$ for every $\alpha \in \Phi$.

Let $Z = Z_G(A)^0$. Then Z is the direct product of A and a connected compact subgroup M of G centralized by A. Let $y \in Y$. We claim that Zy is compact and contains Y. By Lemma 10 and the preceding paragraph, there exists an open neighborhood Ω of e in G such that $R \cap \Omega \subset Z$. It follows that $Y \cap \Omega y \subset (R \cap \Omega)y \subset Zy$. Since Y is A-minimal and $Y \cap \Omega y$ is a neighborhood of y in Y, we have $Y = A(Y \cap \Omega y) \subset Zy$. This in turn implies that $Zy = ZY = MY$ is compact, proving the claim.

Let $\Lambda = \{g \in Z : gy = y\}$. Then Λ is a lattice in Z and the natural map $Z/\Lambda \to Zy$ is a homeomorphism. Let $p : Z = A \times M \to M$ be the projection, and let $T = (\overline{p(\Lambda)})^0$. By [Rag, Thm. 8.24], T is solvable, hence is a compact torus. Note that $AT = (\overline{Ap(\Lambda)})^0 = (\overline{A\Lambda})^0$. So ATy is closed and contains Ay as a dense subset. It follows that $Y = ATy$. This completes the proof. $\square$

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