MINIMAL SETS FOR FLOWS ON MODULI SPACE

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Abstract. Let S be a compact orientable surface, let $\mathcal{Q}$ be the moduli space of quadratic differentials on S and let $\mathcal{M}$ be a stratum in $\mathcal{Q}$. We explicitly describe the minimal sets for the (Teichmüller) horocycle flow on $\mathcal{M}$ and on $\mathcal{Q}$, and show that these correspond to horizontal cylindrical decompositions of S.

1. Introduction

Let S be a surface of genus g. The moduli space $\mathcal{Q}$ of unit-area quadratic differentials on S is a noncompact orbifold, endowed with a natural action of the group $G = \mathrm{SL}(2, \mathbb{R})$, and any of its subgroups. The action of the one-parameter subgroup of upper triangular unipotent matrices is known as the *horocycle flow*, and the G-orbits are known as *Teichmüller disks*. The space $\mathcal{Q}$ is naturally partitioned into locally closed G-invariant sub-orbifolds called *strata*. Understanding the dynamics of these actions on $\mathcal{Q}$ and on each of the strata has been a subject of extensive research, due partly to its connection with rational polygonal billiards and interval exchange transformation. We refer the reader to [MaTa] for a recent survey.

A basic problem in topological dynamics is to understand the *minimal sets* of a given action (a minimal set is by definition a nonempty closed invariant subset which is minimal with respect to inclusion). In this note we discuss this problem for the G-action and the action of its various subgroups, and obtain a complete solution for the horocycle flow. The analogous ergodic-theoretic problem of classifying the invariant ergodic measures has attracted much attention recently and is substantially harder.

A standard application of Zorn's lemma shows that any compact dynamical system contains a minimal set, but the corresponding statement is not necessarily true for non-compact systems, see e.g. [Kul] and the references therein. After introducing our notation, in §3 we discuss the problem of existence of minimal sets for various subgroups of G. In §4 we state and prove Theorem 5, which explicitly describes all minimal sets for the horocycle flow. The proof depends essentially on the nondivergence results of [MiWe]. We remark that an analogous result for the earthquake flow may be proved by a similar argument. We conclude in §5 by recording another application of [MiWe]: any closed orbit Gq admits a finite G-invariant measure, and with a list of open questions.

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2. Notation, definitions

We first briefly introduce our objects of study.

2.1. **Strata, quadratic differentials, horizontal foliation.** Let S be a compact oriented surface of genus g. Fix a finite subset $\Sigma = \{\sigma_1, \dots, \sigma_r\} \subset S$, $\vec{k} = (k_1, \dots, k_r)$ where

$$k_i \in \{1, 3, 4, 5, \dots\} \text{ satisfy } 4(g-1) = \sum (k_i - 2), \tag{1}$$

and $\varepsilon \in \{\pm 1\}$.

Following [MaSm1] we define a *flat structure of type* $(\Sigma, \vec{k}, \varepsilon)$ *on* S to be an atlas of charts $(U_\alpha, \varphi_\alpha)$, where $S \smallsetminus \Sigma = \bigcup U_\alpha$, $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$ such that:

- the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^2$ are of the form $x \mapsto \pm x + c$.
- around $\sigma_i \in \Sigma$, the U_α glue together to form a cone type singularity of cone angle $k_i \pi$.
- The linear holonomy homomorphism $\pi_1(S \smallsetminus \Sigma) \to \mathbb{Z}/2\mathbb{Z}$ is trivial if and only if $\varepsilon = 1$.

The charts endow S with an area element, and we say that the chart has unit area if the total area of S with respect to this area element is equal to one.

We denote by $\mathrm{Mod}(S, \Sigma)$ the relative mapping class group, that is, the orientation preserving homeomorphisms of S which fix Σ pointwise, considered up to isotopy. We denote by $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}(\Sigma, \vec{k}, \tau)$ the space of flat structures of type $(\Sigma, \vec{k}, \tau)$ on S of unit area, and by $\mathcal{M}$ the quotient of $\widetilde{\mathcal{M}}$ by the natural action of $\mathrm{Mod}(S, \Sigma)$. We let $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ denote the natural quotient map. It is known that $\widetilde{\mathcal{M}}$ is a manifold and that $\mathcal{M}$ is an orbifold, called a *stratum*, with a natural finite measure (see [MaSm1, Ve1]).

For each g there are finitely many solutions to (1), and hence, for fixed S, only finitely many possible strata. The disjoint union $\widetilde{\mathcal{Q}}$ (resp.

$\mathcal{Q}$) of all the $\widetilde{\mathcal{M}}$ (resp. the $\mathcal{M}$) is called the *space of (marked) quadratic differentials on S* (resp. the *moduli space of quadratic differentials on S*). It has a natural manifold (resp. orbifold) structure, and is stratified by the various $\widetilde{\mathcal{M}}$ (resp. $\mathcal{M}$). The strata are locally closed in $\mathcal{Q}$. In [MaSm2] all nonempty strata were listed. Each stratum has only finitely many connected components, and these components were completely listed in [KoZo], [La].

In what follows, we will use boldface letters for elements of $\widetilde{\mathcal{Q}}$ and the corresponding lowercase letters the corresponding element of $\mathcal{Q}$, that is, $q = \pi(\mathbf{q})$.

For each $\mathbf{q} \in \widetilde{\mathcal{Q}}$, the pre-image of the foliation of $\mathbb{R}^2$ by horizontal (resp. vertical) lines under the charts defining $\mathbf{q}$ is a well-defined singular foliation on S known as the *horizontal* (resp. *vertical*) foliation. The Euclidean metric on $\mathbb{R}^2$ can be used to equip both of these foliations with a transverse measure, so we obtain two transverse measured foliations on S.

We say that the horizontal foliation of $\mathbf{q}$ is *completely periodic* if for any $x \in S \smallsetminus \Sigma$, both of the horizontal rays emanating from x are either periodic (i.e. return to x) or encounter Σ. For any segment I on S which is transverse to the horizontal foliation, we may define the first return map to I along the leaves. This map is an interval exchange transformation, and, if I meets all leaves, the complete periodicity of the foliation is equivalent to the complete periodicity of the corresponding interval exchange.

2.2. **Linear action, saddle connections, holonomy vector.** G acts on each chart by postcomposition, and this action induces a well-defined action on each $\widetilde{\mathcal{M}}$ which descends equivariantly via π to an action on each $\mathcal{M}$.

For $s, t, \theta \in \mathbb{R}$ we let:

$$h_s \stackrel{\mathrm{def}}{=} \left(\begin{array}{cc} 1 & s \\ 0 & 1 \end{array} \right), \quad g_t = \left(\begin{array}{cc} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{array} \right), \quad r_\theta \stackrel{\mathrm{def}}{=} \left(\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right),$$

$$H = \{h_s : s \in \mathbb{R}\}, \quad F = \{g_t : t \in \mathbb{R}\}.$$

Let B denote the group of upper triangular matrices in G, and let B^+ and B^- be the following subsemigroups of B:

$$B^+ = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \geq 1, b \in \mathbb{R} \right\},$$

$$B^- = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : 0 < a \leq 1, b \in \mathbb{R} \right\}.$$

A *saddle connection (for* $\mathbf{q} \in \widetilde{\mathcal{Q}}$*)* is a map $\delta : [0,1] \to S$ with $\delta^{-1}(\Sigma) = \{0,1\}$, such that the image of δ is a straight segment in each chart. The set of all saddle connections for $\mathbf{q}$ is denoted by $\mathcal{L}_{\mathbf{q}}$. There is a natural identification of $\delta \in \mathcal{L}_{\mathbf{q}}$ with $\delta \in \mathcal{L}_{g\mathbf{q}}$ for any $g \in G$. For any path α in S, the local projections of $d\alpha$ to the x and y axes in charts are well-defined up to sign, and integrating them we obtain a *holonomy vector*, denoted $(x(\alpha, \mathbf{q}), y(\alpha, \mathbf{q}))$, well-defined up to a multiple of ± 1. Since the G-action transforms the charts linearly we have for $\delta \in \mathcal{L}_{\mathbf{q}}$ and $g \in G$:

$$\begin{pmatrix} x(\alpha, g\mathbf{q}) \\ y(\alpha, g\mathbf{q}) \end{pmatrix} = \pm\, g \cdot \begin{pmatrix} x(\alpha, \mathbf{q}) \\ y(\alpha, \mathbf{q}) \end{pmatrix} \tag{2}$$

(here $g \cdot v$ is the natural action of G on $\mathbb{R}^2$ by matrix multiplication). The length $l(\delta, \mathbf{q})$ is defined as $\sqrt{x(\delta, \mathbf{q})^2 + y(\delta, \mathbf{q})^2}$.

2.3. **Compactness criterion.** For $\varepsilon > 0$ we let

$$K_\varepsilon \stackrel{\text{def}}{=} \pi\left(\left\{\mathbf{q} \in \widetilde{\mathcal{Q}} : \forall \delta \in \mathcal{L}_{\mathbf{q}}, l(\delta, \mathbf{q}) \geq \varepsilon\right\}\right).$$

It is known that the $\{K_\varepsilon\}_{\varepsilon > 0}$ are a nested family of compact subsets of $\mathcal{Q}$ with $\mathcal{Q} = \bigcup_{\varepsilon > 0} K_\varepsilon$. Note that there are compact subsets of $\mathcal{Q}$ which are not contained in a single K_ε, for example the union of a sequence from one stratum and its limit point in another stratum. However the following holds:

If $\mathcal{M}$ is a stratum and $X \subset \mathcal{M}$, then $\overline{X} \cap \mathcal{M}$ is compact if and only if there is $\varepsilon > 0$ such that $X \subset K_\varepsilon$.

2.4. **Cylinders.** A *cylinder (with respect to* $\mathbf{q}$*)* is an annulus in S which is isometric, with respect to the metric defined by $\mathbf{q}$, to $\mathbb{R}/w\mathbb{Z} \times [0, h]$. Here w and h are respectively the *circumference* and *height* of the cylinder, and w/h is the *modulus* of the cylinder. A *waist curve* is the image under the above isometry of the curve $\mathbb{R}/w\mathbb{Z} \times \{h_0\}$ for some $h_0 \in [0, h]$. A cylinder is *maximal* if it is not properly contained in another cylinder. Note that a maximal cylinder is closed and has singularities on each of its waist curves. A *cylindrical decomposition (with respect to* $\mathbf{q}$*)* is a finite union of maximal cylinders (with respect to $\mathbf{q}$) $C_1, \ldots, C_r$ such that $S = \bigcup C_i$ and the interiors of the C_i are disjoint. It is called *completely periodic* if all the moduli are commensurable (i.e., their ratio is a rational number).

2.5. **Tori.** A *torus* is a quotient $\mathbb{R}^k/\Lambda$, where Λ is a discrete subgroup of rank k. Any connected compact abelian Lie group is a torus, and in particular any closed connected subgroup of a torus is a torus. Suppose $\mathbb{T}$ is a torus and $\{r(t) : t \in \mathbb{R}\}$ is a one-parameter subgroup. The corresponding *one-parameter translational flow* is defined by $t \cdot \mathbf{x} = \mathbf{x} + r(t)$, where $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{T}$. The flow is minimal if and only if $\{r(t) : t \in \mathbb{R}\}$ is dense in $\mathbb{T}$.

Suppose $\mathbb{T} = \mathbb{R}^k/\mathbb{Z}^k$ and $\{r(t) : t \in \mathbb{R}\}$ is a one-parameter subgroup. The dimension of $\mathbb{T}' = \overline{\{r(t)\}}$ is equal to the dimension over $\mathbb{Q}$ of the linear span of the coefficients of $r(1)$ (see e.g. [CoFoSi]). It is easily seen that for any one-parameter subgroup $\{r(t) : t \in \mathbb{R}\} \subset \mathbb{T}$, the restriction to $\mathbb{T}' = \overline{\{r(t)\}}$ of the one-parameter translation flow is minimal. In particular the action is minimal on $\mathbb{T}$ if and only if the dimension over $\mathbb{Q}$ of the linear span of the coefficients of $r(1)$ is equal to k.

2.6. **Quantitative nondivergence.** The horocycle flow was shown by Veech [Ve3] not to have any divergent trajectories. A quantitative version of this result, proved in [MiWe], will be very useful:

Proposition 1 ([MiWe], Thm. 6.3)**.** *There are positive constants C, α, ρ_0, depending only on S, such that if $\mathbf{q} \in \widetilde{\mathcal{Q}}$, an interval $I \subset \mathbb{R}$, and $0 < \rho \le \rho_0$ satisfy:*

$$\text{for any } \delta \in \mathcal{L}_{\mathbf{q}}, \text{ there is } s \in I \text{ such that } l(\delta, h_s\mathbf{q}) \ge \rho, \tag{3}$$

then for any $\varepsilon > 0$ we have:

$$|\{s \in I : h_s q \notin K_\varepsilon\}| \le C \left(\frac{\varepsilon}{\rho}\right)^\alpha |I|. \tag{4}$$

3. Existence of minimal sets

In this section we collect information about the existence of minimal sets for various subgroups of G. The case of the horocycle flow is dealt with in:

Proposition 2. *Let $\mathcal{M} \subset \mathcal{Q}$ be a stratum, and let X be equal to either $\mathcal{M}$ or $\mathcal{Q}$. Then any closed invariant set for the horocycle flow on X contains a minimal set, and a minimal set is compact.*

Proof. In case $X = \mathcal{Q}$, this is precisely [MiWe, Cor. 2.7]. The case $X = \mathcal{M}$ was not considered explicitly in [MiWe], but follows using an identical proof. ☐

For G and B we have the following:

Proposition 3. *Let $\mathcal{M}$ be a stratum of $\mathcal{Q}$, and let L be equal to either G or B. Then:*

(i) *There is a compact $K \subset \mathcal{M}$ such that for any $q \in \mathcal{M}$,*

$$Lq \cap K \ne \varnothing.$$

(ii) *Let X be equal to either to $\mathcal{M}$ or to $\mathcal{Q}$. Every closed L-invariant subset of X contains a minimal set for the L-action.*

Proof. In proving (i) we may assume that $L = B$. Let ρ_0, C, α be as in Proposition 1, let ε_0 be small enough so that $C\left(\frac{\varepsilon_0}{\rho_0}\right)^{\alpha} < 1$, and set $K \stackrel{\mathrm{def}}{=} K_{\varepsilon_0}$. For $q \in \mathcal{M}$, let $\mathcal{L}_0$ be the set of all saddle connections which are contained in a horizontal leaf of $\mathbf{q}$ and let

$$\rho_1 = \min_{\delta \in \mathcal{L}_0} l(\delta, \mathbf{q}).$$

For large enough t we have $e^{t/2}\rho_1 > \rho_0$, which means that $l(\delta, g_t\mathbf{q}) \geq \rho_0$ for all $\delta \in \mathcal{L}_0$. Now let

$$\mathcal{L}_1 = \{\delta \in \mathcal{L}_{\mathbf{q}} : l(\delta, g_t\mathbf{q}) < \rho_0\},$$

a finite set. Since $\mathcal{L}_0 \cap \mathcal{L}_1 = \varnothing$, we have $y(\delta, \mathbf{q}) \neq 0$ for all $\delta \in \mathcal{L}_1$. Therefore there is $s > 0$ such that for all $\delta \in \mathcal{L}_1$, $l(\delta, h_s g_t q) \geq \rho_0$. Thus (3) holds for $\rho = \rho_0$, $I = [0, s]$ and $g_t\mathbf{q}$ in place of $\mathbf{q}$. It follows that there is $s_0 \in [0, s]$ such that $h_{s_0} g_t q \in K$. In particular $Bq \cap K \neq \varnothing$.

Now (ii) follows from (i) via Zorn's lemma and the finite intersection property for compact sets. □

Remark. Conclusion (i) is not valid if one takes either F or H for L. We do not know whether the geodesic flow satisfies conclusion (ii) – see question (I) below.

4. Description of minimal sets

In this section we first describe some minimal sets for the horocycle flow. Then we show that these are the only minimal sets for this flow.

Proposition 4. *Suppose $\mathbf{q}_0 \in \widetilde{\mathcal{Q}}$ is such that the corresponding horizontal foliation is completely periodic. Let $\mathcal{O} = \overline{Hq_0}$. Then*

1. *S admits a cylinder decomposition $S = C_1 \cup \cdots \cup C_r$, where each C_i is a cylinder whose interior is a union of horizontal waist curves.*
2. *There is an isomorphism between $\mathcal{O}$ and a d-dimensional torus, where d is the dimension of the $\mathbb{Q}$-linear subspace of $\mathbb{R}$ spanned by the moduli of $C_1, \ldots, C_r$. This isomorphism conjugates the H-action on $\mathcal{O}$ with a one-parameter translational flow.*
3. *The restriction of the H-action to $\mathcal{O}$ is minimal.*

Proof. Let Ξ denote the union of horizontal saddle connections and rays emanating from points in Σ. By complete periodicity, Ξ is a union of saddle connections. Consider any connected component C of $S \smallsetminus \Xi$. The boundary of C consists of straight lines, making an interior angle of π at each singularity, and C does not contain singular points in its interior. This implies that C is a metric cylinder, with horizontal boundary. This proves (1).

Now let $\mathbf{q} \in \widetilde{\mathcal{Q}}$ correspond to a flat structure on S which admits a decomposition $S = C_1 \cup \cdots \cup C_r$ for some $C_i = C_i(\mathbf{q})$ which are cylinders as in (1). For $i = 1, \ldots, r$, let α_i, β_i be two singular points which belong to the boundary of C_i, one on each boundary component, and choose a segment γ_i connecting α_i to β_i and contained in C_i. Let $w_i = w_i(\mathbf{q})$ be the circumference of C_i, and let $(x_i(\mathbf{q}), y_i(\mathbf{q}))$ by the holonomy vector of γ_i. We think of (C_i, α_i, β_i) as a surface with two marked points. Then γ_i determines a marking on (C_i, α_i, β_i) and the data (w_i, x_i, y_i) determines $\mathbf{q}$ uniquely as a flat structure on (C_i, α_i, β_i). Thus $\mathbf{q}$ may be reconstructed from $\{w_i(\mathbf{q}), x_i(\mathbf{q}), y_i(\mathbf{q}) : i = 1, \ldots, r\}$. The modulus of P_i is defined to be $m_i = \frac{y_i}{w_i}$.

By (2), the action of h_s preserves w_i and y_i, and maps x_i to $x_i + sy_i$ for all i. A (positive) Dehn twist d_j about a horizontal edge of any of the C_j leaves the data $\{w_i, y_i\}_{i=1,\ldots,r}, \{x_i : i \neq j\}$ invariant and maps x_j to $x_j + w_j$.

Let $X' \subset \widetilde{\mathcal{Q}}$ denote the set of all $\mathbf{q} \in \widetilde{\mathcal{Q}}$ which admit a cylindrical decomposition $C_1, \ldots, C_r$ as above. The function

$$X' \to \mathbb{R}^{3r}, \quad \mathbf{q} \mapsto (w_i(\mathbf{q}), x_i(\mathbf{q}), y_i(\mathbf{q}))_{i=1,\ldots,r}$$

can be chosen to depend continuously on $\mathbf{q}$. Now define

$$X \stackrel{\text{def}}{=} \{\mathbf{q} \in X' : w_i(\mathbf{q}) = w_i(\mathbf{q}_0),\ y_i(\mathbf{q}) = y_i(\mathbf{q}_0)\}$$

and

$$\Psi : X \to \mathbb{R}^r, \quad \Psi(\mathbf{q}) \stackrel{\text{def}}{=} \left(\frac{x_i(\mathbf{q})}{w_i(\mathbf{q}_0)}\right)_{i=1,\ldots,r}.$$

Since each $\mathbf{q} \in X$ can be reconstructed from the values of $\{x_i(\mathbf{q})\}$, Ψ is a bijection, and by the above discussion

$$\Psi(h_s\mathbf{q}) = \Psi(\mathbf{q}) + s\mathbf{m}(\mathbf{q}_0), \quad \text{where } \mathbf{m}(\mathbf{q}) \stackrel{\text{def}}{=} (m_i(\mathbf{q}))_{i=1,\ldots,r}$$

and

$$\Psi(d_j\mathbf{q}) = \Psi(\mathbf{q}) + \mathbf{e}_j$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_r$ are the standard basis vectors in $\mathbb{R}^r$. In particular, there is a finite index subgroup $\Lambda \subset \mathbb{Z}^r$ such that if $\Psi(\mathbf{q}_1) \in \Psi(\mathbf{q}_2) + \Lambda$ then $\pi(\mathbf{q}_1) = \pi(\mathbf{q}_2)$.

Now let $\mathbb{T}$ be the closure of the image of $\{s\mathbf{m}(\mathbf{q}_0) : s \in \mathbb{R}\}$ in $\mathbb{R}^r/\mathbb{Z}^r$. It is a d-dimensional rational subtorus, where d is the dimension of the $\mathbb{Q}$-linear space spanned by $m_1, \ldots, m_r$. It is clear from the construction that

$$\pi^{-1}(\mathcal{O}) = \Psi^{-1}(\mathbb{T}),$$

and (2) follows. Since the translational flow on $\mathbb{T}$ admits a dense orbit, it is minimal, and (3) follows. ☐

Remark. From the arguments of [MaSm2] it follows that any stratum of quadratic differentials contains $\mathbf{q}$ admitting a horizontal cylinder decomposition. See [Cal] for explicit examples in genus 2, $\varepsilon = 1$. Note that by perturbing the moduli of the cylinders, one may obtain examples where the dimension d of the torus is greater than one. This yields examples of H-orbits which are non-periodic and whose closures are not G-invariant.

Theorem 5. *If $\mathbf{q} \in \widetilde{\mathcal{Q}}$ is such that $\mathcal{O} \stackrel{\mathrm{def}}{=} \overline{Hq}$ is contained in a compact subset of a single stratum, then the flow along $\mathbf{q}$-horizontal leaves is completely periodic; in particular, any minimal set for the horocycle flow is as described in Proposition 4.*

Proof. To prove the first assertion, suppose there is $\varepsilon > 0$ such that

$$\mathcal{O} \subset K_\varepsilon. \tag{5}$$

Let Ξ be the union of all horizontal saddle connections and rays based at points of Σ. Arguing as in the proof of Proposition 4(1), it is enough to prove that Ξ consists of saddle connections, that is, it does not contain infinite rays.

Supposing otherwise, let $\sigma \in \Sigma$ and let ℓ be an infinite horizontal ray with one endpoint at σ. It follows from standard facts about interval exchange transformations (see e.g. [Bo]) that the accumulation points of ℓ in S contain a singularity. We denote one such singularity by σ' (we may have $\sigma = \sigma'$). Let $p \in \ell$ such that $d(p, \sigma') < \varepsilon$. Let γ_1 denote the path from σ to p along ℓ, let γ_2 denote a path from p to σ' of length less than ε, and let γ denote the concatenation of γ_1, γ_2. We assume, by moving p, that the length of γ_1 is greater than ε and that γ_2 is vertical.

Define

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{\mathrm{def}}{=} \begin{pmatrix} x(\gamma, \mathbf{q}) \\ y(\gamma, \mathbf{q}) \end{pmatrix}.$$

Note that $|x|$ is the length of γ_1 and $|y|$ is the length of γ_2. Thus

$$|y| < \varepsilon.$$

Now set

$$\theta_0 \stackrel{\mathrm{def}}{=} \arctan \frac{y}{x}, \quad M \stackrel{\mathrm{def}}{=} \sqrt{x^2 + y^2}$$

and assume to simplify notation that $\theta_0 > 0$. For $\theta \in [0, \theta_0]$, let $\mathcal{F}_\theta$ denote the rotation by θ of the horizontal foliation of $\mathbf{q}$, and let v_θ be rotation by an angle θ of the initial direction of ℓ. Let θ' be the smallest $\theta \in [0, \theta_0)$ for which there is a saddle connection in $\mathcal{F}_\theta$ with initial direction v_θ of length at most M (if there are no such θ we set $\theta' = \theta_0$). The minimal value of such θ is achieved by an easy compactness argument.

Let δ be the shortest saddle connection in $\mathcal{F}_{\theta'}$ which starts at σ and has initial direction $v_{\theta'}$. In case $\theta' = \theta_0$, it may be that δ connects σ to σ' and has length M, or δ may be shorter. Defining

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \stackrel{\mathrm{def}}{=} \begin{pmatrix} x(\delta, \mathbf{q}) \\ y(\delta, \mathbf{q}) \end{pmatrix},$$

we have by construction $|y'| \leq |y| < \varepsilon$. Also $y' \neq 0$ since by assumption the ray ℓ is infinite.

Now letting $s \stackrel{\mathrm{def}}{=} \frac{-x'}{y'}$, one has by (2):

$$\begin{pmatrix} x(\delta, h_s\mathbf{q}) \\ y(\delta, h_s\mathbf{q}) \end{pmatrix} = \begin{pmatrix} 1 & -x'/y' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ y' \end{pmatrix}.$$

We have shown

$$l(\delta, h_s\mathbf{q}) = |y'| < \varepsilon,$$

in contradiction to (5). This proves the first assertion.

For the second assertion, note that since all strata are locally closed, a minimal set $\mathcal{O}$ is contained in a single stratum. Also, by Proposition 2, $\mathcal{O}$ is compact. Thus the second assertion follows from the first. □

Putting together Proposition 2 and Theorem 5 yields:

Corollary 6. *Every orbit-closure for the horocycle flow (on either $\mathcal{Q}$ or its stratum $\mathcal{M}$) contains a point with a horizontal cylindrical decomposition.*

Corollary 7. *Let X be equal either to $\mathcal{Q}$ or to a stratum $\mathcal{M} \subset \mathcal{Q}$. Let L be equal to one of the semigroups B^+, B^-. Then there are no bounded trajectories under L in X.*

Proof. If $L = B^+$ the proof is easy: suppose first that X is a stratum. Given $q \in X$, let $\delta \in \mathcal{L}_\mathbf{q}$ which is not horizontal, that is $y(\delta, \mathbf{q}) \neq 0$. Since $x(\delta, h_s\mathbf{q}) = x(\delta, \mathbf{q}) + sy(\delta, \mathbf{q})$ there is s_0 such that $x(\delta, h_{s_0}\mathbf{q}) = 0$ and hence

$$l(\delta, g_t h_{s_0}\mathbf{q}) \to_{t \to +\infty} 0,$$

showing that Lq is not contained in any K_ε.

Now if $X = \mathcal{Q}$ we let $\mathcal{M}$ be the stratum of $\mathcal{Q}$ of smallest dimension which intersects $\overline{Lq}$; since lower dimensional strata do not accumulate on higher dimensional ones, $\mathcal{M} \cap \overline{Lq}$ is a compact subset of $\mathcal{M}$, so we may repeat the previous argument with $X = \mathcal{M}$.

Now suppose $L = B^-$, and let $q \in X$. Applying Corollary 6, we find $q' \in \overline{Lq}$ such that S admits a $\mathbf{q}'$-horizontal cylindrical decomposition; in particular there is $\delta' \in \mathcal{L}_{\mathbf{q}'}$ with $y(\delta', \mathbf{q}') = 0$. This implies that

$$l(\delta', g_t\mathbf{q}') \to_{t \to -\infty} 0,$$

and we may now repeat the previous argument. □

5. Closed trajectories and lattice examples

Propositions 3, 2 of this note depend on the nondivergence behavior of the horocycle flow (see Proposition 1). We take this opportunity to record a proof of the following result which also depends on this nondivergence behavior. Note that this approach is somewhat different from that sketched in [Ve2].

Proposition 8. *A closed G-orbit in $\mathcal{Q}$ necessarily carries a finite G-invariant measure. In other words, for any $q \in \mathcal{Q}$*

$$Gq \text{ is closed} \iff G_q \stackrel{\text{def}}{=} \{g \in G : gq = q\} \text{ is a lattice in } G.$$

Proof. Since G_q is discrete, it is unimodular, and hence there is a locally finite G-invariant measure on G/G_q. Since Gq is closed, the orbit map $g \mapsto gq$ descends to a homeomorphism $G/G_q \to Gq$. In particular Gq supports a locally finite G-invariant measure, which by [MiWe, Cor. 2.6] is necessarily finite. This implies that the measure of G/G_q is finite, that is, G_q is a lattice in G. □

We conclude with a list of open questions:

- (I) Does conclusion (ii) of Proposition 3 hold for $L = F$, i.e., does every closed invariant subset for the geodesic flow contain a minimal set?
- (II) Are there minimal sets for the G-action which are not closed orbits (and hence lattice examples)?
- (III) Is a minimal set for B necessarily also G-invariant?
- (IV) Can one weaken the hypothesis of Theorem 5 and assume only that $\mathcal{O}$ is compact in $\mathcal{Q}$? That is, are there horocycle orbits which are bounded in $\mathcal{Q}$ but not in their stratum?

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