GEOMETRIC AND ARITHMETIC ASPECTS OF APPROXIMATION VECTORS

URI SHAPIRA AND BARAK WEISS

Abstract. Let $\theta \in \mathbb{R}^d$. We associate three objects to each approximation $(\mathbf{p}, q) \in \mathbb{Z}^d \times \mathbb{N}$ of θ: the projection of the lattice $\mathbb{Z}^{d+1}$ to $\mathbb{R}^d$, along the approximating vector $(\mathbf{p}, q)$; the displacement vector $(\mathbf{p} - q\theta)$; and the residue classes of the components of the $(d+1)$-tuple $(\mathbf{p}, q)$ modulo all primes. All of these have been studied in connection with Diophantine approximation problems. We consider the asymptotic distribution of all of these quantities, properly rescaled, as $(\mathbf{p}, q)$ ranges over the best approximants and ε-approximants of θ, and describe limiting measures on the relevant spaces, which hold for Lebesgue a.e. θ. We also consider a similar problem for vectors θ whose components, together with 1, span a totally real number field of degree $d+1$. Our technique involve recasting the problem as an equidistribution problem for a cross-section of a one-parameter flow on an adelic space, which is a fibration over the space of $(d+1)$-dimensional lattices. Our results generalize results of many previous authors, to higher dimensions and to joint equidistribution.

Contents

1. Introduction

Our results concern the asymptotic statistics of certain geometric and arithmetic quantities associated with approximation vectors. We begin by introducing the notions and notations necessary for formulating the results (more details will be given in subsequent sections of the paper).

Throughout this paper, d and n are positive integers with $n = d+1$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$, and let $\mathbf{v} = \sum v_i \mathbf{e}_i \in \mathbb{R}^n$ with $v_n \neq 0$. We will continuously use the direct sum decomposition

$$\mathbb{R}^n = \mathbb{R}^d \oplus \operatorname{span}(\mathbf{v}), \quad \text{where } \mathbb{R}^d \stackrel{\text{def}}{=} \{\mathbf{u} \in \mathbb{R}^n : u_n = 0\}, \tag{1}$$

and the first summand in this decomposition will be called the *horizontal space*. The projection $\mathbb{R}^n \to \mathbb{R}^d$ will be denoted by $\pi^{\mathbf{v}}_{\mathbb{R}^d}$, and $\pi_{\mathbb{R}^d}$ will be an abbreviation for $\pi^{\mathbf{e}_n}_{\mathbb{R}^d}$.

Choose some norm on $\mathbb{R}^d$, and for $\mathbf{y} \in \mathbb{R}^d$, denote $\langle \mathbf{y} \rangle = \min_{\mathbf{p} \in \mathbb{Z}^d} \|\mathbf{y} - \mathbf{p}\|$. For $\theta \in \mathbb{R}^d$, we say that $(\mathbf{p}, q) \in \mathbb{Z}^d \times \mathbb{N}$ is a *best approximation* of θ if for any $q' < q$, $\langle q\theta \rangle < \langle q'\theta \rangle$, and $\langle q\theta \rangle = \|q\theta - \mathbf{p}\|$. For all $\theta \in \mathbb{R}^d \smallsetminus \mathbb{Q}^d$, the set of best approximations of θ is an infinite sequence $(\mathbf{p}_k, q_k)_{k \in \mathbb{N}}$, where the order is chosen so that $q_1 \leq q_2 \leq \cdots$. In fact there will be k_0 such that $q_{k+1} > q_k$ for all $k > k_0$, and the $\mathbf{p}_k$ will be uniquely determined for $k > k_0$. The potential ambiguity of choices for $k \leq k_0$ will have no effect on the objects we will consider in this paper. For $\theta \in \mathbb{R}^d$ and $\mathbf{v} = (\mathbf{p}, q) \in \mathbb{Z}^d \times \mathbb{N}$, we will write

$$\mathrm{disp}(\theta, \mathbf{v}) \stackrel{\mathrm{def}}{=} q^{1/d}(\mathbf{p} - q\theta) \in \mathbb{R}^d, \tag{2}$$

and refer to this vector as the *displacement*.

All measure spaces in this paper will be standard Borel spaces and all measures will be Borel measures. The collection of probability measures on a measure space X is denoted by $\mathcal{P}(X)$. Two measures on X are said to be in *the same measure class* if they are mutually absolutely continuous (i.e. have the same null-sets). If X is a locally compact second countable Hausdorff (lcsc) space, μ is a regular measure on X, and $(x_k)_{k \in \mathbb{N}}$ is a sequence in X, we say that (x_k) *equidistributes with respect to* μ if the measures $\frac{1}{N}\sum_{k=1}^{N} \delta_{x_k}$ converge weak-* to μ.

A lattice in $\mathbb{R}^n$ is a discrete subgroup of full rank. Its covolume is the volume of a fundamental domain in $\mathbb{R}^n$, and the space of all lattices of covolume 1 is denoted by $\mathscr{X}_n$. It can be identified with the quotient $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ via the map $g\mathbb{Z}^n \mapsto g\,\mathrm{SL}_n(\mathbb{Z})$, and thus acquires a natural $\mathrm{SL}_n(\mathbb{R})$-invariant probability measure, which we denote by $m_{\mathscr{X}_n}$. This measure is sometimes called the *Haar-Siegel measure.* Two lattices in $\mathbb{R}^n$ are *homothetic* if one can be obtained from the other by multiplication by a nonzero scalar, and the homothety class of each Λ contains a unique representative in $\mathscr{X}_n$, which we denote by $[\Lambda]$. If Λ is a lattice in $\mathbb{R}^n$ and $\mathbf{v} \in \Lambda \smallsetminus \mathbb{R}^d$ then $\pi^{\mathbf{v}}_{\mathbb{R}^d}(\Lambda)$ is a lattice in $\mathbb{R}^d$ and $\left[\pi^{\mathbf{v}}_{\mathbb{R}^d}(\Lambda)\right] \in \mathscr{X}_d$.

We denote by $\widehat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$, that is, the inverse limit of groups $\mathbb{Z}/m\mathbb{Z}$, with respect to the natural maps $\mathbb{Z}/m_1\mathbb{Z} \to \mathbb{Z}/m_2\mathbb{Z}$ whenever $m_2 | m_1$. This is a compact topological ring which is isomorphic to $\prod_{p \text{ prime}} \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers. We denote by $\widehat{\mathbb{Z}}^n$ the additive group which is the n-fold Cartesian product of $\widehat{\mathbb{Z}}$; it is also the inverse limit of quotient groups $\mathbb{Z}^n/\Lambda$, where Λ ranges over finite index subgroups of $\mathbb{Z}^n$. We denote by $m_{\widehat{\mathbb{Z}}^n}$ the Haar probability measure on $\widehat{\mathbb{Z}}^n$. For $\Lambda \in \mathscr{X}_n$, a vector $v \in \Lambda$ is said to be *primitive* if it is not a multiple of a vector in Λ by an integer different from ± 1, and we denote the primitive elements of Λ by Λ_{prim}. The natural diagonal embedding $\mathbb{Z}^n \hookrightarrow \widehat{\mathbb{Z}}^n$ has a dense image, and we denote the closure of the image of $\mathbb{Z}^n_{\mathrm{prim}}$ by $\widehat{\mathbb{Z}}^n_{\mathrm{prim}}$. Note that $\widehat{\mathbb{Z}}^n_{\mathrm{prim}}$ is one orbit for the natural action of the group $\mathrm{SL}_n(\widehat{\mathbb{Z}})$, and

$$\widehat{\mathbb{Z}}^n_{\mathrm{prim}} = \{(v_p)_{p \text{ prime}} : \forall p,\ \|v_p\|_p = 1\}.$$

Since $\mathrm{SL}_n(\widehat{\mathbb{Z}})$ acts transitively on $\widehat{\mathbb{Z}}^n_{\mathrm{prim}}$, there is a unique invariant probability measure for this action, which we denote by $m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}$.

Theorem 1.1. *For any norm $||\cdot||$ on $\mathbb{R}^d$ there is a probability measure $\mu = \mu_{\mathrm{best},||\cdot||}$ on $\mathscr{X}_d \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$ such that for Lebesgue almost any $\theta \in \mathbb{R}^d$, the following holds. Let $\mathbf{v}_k \in \mathbb{Z}^n$ be the sequence of best approximations to θ with respect to the norm $||\cdot||$. Then the sequence*

$$\left(\left[\pi^{\mathbf{v}_k}_{\mathbb{R}^d}(\mathbb{Z}^n)\right], \mathrm{disp}(\theta, \mathbf{v}_k), \mathbf{v}_k\right)_{k\in\mathbb{N}} \in \mathscr{X}_d \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n \tag{3}$$

equidistributes with respect to μ. The measure μ has the following properties:

1. *It is a product $\mu = \mu^{(\infty)} \times \mu^{(f)}$ where $\mu^{(\infty)} \in \mathcal{P}(\mathscr{X}_d \times \mathbb{R}^d)$, $\mu_f \in \mathcal{P}(\widehat{\mathbb{Z}}^n)$.*
2. *The measure $\mu^{(f)}$ is $m_{\widehat{\mathbb{Z}^n}_{\mathrm{prim}}}$ (and in particular, does not depend on the choice of the norm).*
3. *The projection $\mu^{(\mathscr{X}_d)}$ of $\mu^{(\infty)}$ to $\mathscr{X}_d$ is equivalent to $m_{\mathscr{X}_d}$, but is equal to it only in case $d = 1$.*
4. *The projection $\mu^{(\mathbb{R}^d)}$ of $\mu^{(\infty)}$ to $\mathbb{R}^d$ is boundedly supported, absolutely continuous w.r.t. Lebesgue with a nontrivial density (i.e., is not the restriction of Lebesgue measure to a subset of $\mathbb{R}^d$). If $||\cdot||$ is the Euclidean norm, then it is $\mathrm{SO}_d(\mathbb{R})$-invariant.*
5. *For $d > 1$, $\mu^{(\infty)} \neq \mu^{(\mathscr{X}_d)} \times \mu^{(\mathbb{R}^d)}$.*

Furthermore, each of the coordinate sequences

$$\left(\left[\pi^{\mathbf{v}_k}_{\mathbb{R}^d}(\mathbb{Z}^n)\right]\right) \subset \mathscr{X}_d, \quad (\mathrm{disp}(\theta, \mathbf{v}_k)) \subset \mathbb{R}^d, \quad (\mathbf{v}_k) \subset \widehat{\mathbb{Z}}^n \tag{4}$$

equidistributes in its respective space, with respect to the pushforward of $\mu^{(\mathscr{X}_d)}, \mu^{(\mathbb{R}^d)}, \mu^{(f)}$ respectively.

Let $\varepsilon > 0$ and fix a norm $\|\cdot\|$ on $\mathbb{R}^d$. Given $\theta \in \mathbb{R}^d$, we say that $\mathbf{w} = (\mathbf{p}, q) \in \mathbb{Z}^d \times \mathbb{N}$ is an *ε-approximation of θ (with respect to $\|\cdot\|$)* if $\|\mathrm{disp}(\theta, \mathbf{w})\| \leq \varepsilon$ and $\gcd(p_1, \ldots, p_d, q) = 1$. Standard results in Diophantine approximation imply that for a.e. θ, for all ε, there are infinitely many ε-approximations. When we refer to the sequence $(\mathbf{p}_k, q_k)$ of ε-approximations, we will always assume that they are ordered so that $q_1 \leq q_2 \leq \cdots$.

Theorem 1.2. *For any norm $||\cdot||$ on $\mathbb{R}^d$ and any $\varepsilon > 0$ there is a probability measure $\nu = \nu_{\varepsilon-\mathrm{approx},||\cdot||}$ on $\mathscr{X}_d \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$ such that for Lebesgue almost any $\theta \in \mathbb{R}^d$, the following holds. Let $\mathbf{w}_k \in \mathbb{Z}^n$ be the sequence of ε-approximations of θ with respect to the norm $||\cdot||$. Then the sequence*

$$\left(\left[\pi^{\mathbf{w}_k}_{\mathbb{R}^d}(\mathbb{Z}^n)\right], \mathrm{disp}(\theta, \mathbf{w}_k), \mathbf{w}_k\right)_{k\in\mathbb{N}} \in \mathscr{X}_d \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n \tag{5}$$

equidistributes with respect to ν. The measure ν has the following properties:

1. *The measure ν is a product $\nu = \nu^{(\mathscr{X}_d)} \times \nu^{(\mathbb{R}^d)} \times \nu^{(f)}$ where $\nu^{(\mathscr{X}_d)} \in \mathcal{P}(\mathscr{X}_d)$, $\nu^{(\mathbb{R}^d)} \in \mathcal{P}(\mathbb{R}^d)$ and $\nu^{(f)} \in \mathcal{P}(\widehat{\mathbb{Z}}^n)$.*

(2) The measure $\nu^{(\mathscr{X}_d)}$ is $m_{\mathscr{X}_d}$ and the measure $\nu^{(f)}$ is $m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}$ (in particular, these measures do not depend on the choice of ε or of the norm).

(3) The measure $\nu^{(\mathbb{R}^d)}$ is the normalized restriction of the Lebesgue measure on $\mathbb{R}^d$, to the ball of radius ε around the origin with respect to the norm $\|\cdot\|$.

Furthermore, each of the coordinate sequences

$$\left(\left[\pi_{\mathbb{R}^d}^{\mathbf{w}_k}(\mathbb{Z}^n)\right]\right) \subset \mathscr{X}_d, \quad (\mathrm{disp}(\theta, \mathbf{w}_k)) \subset \mathbb{R}^d, \quad (\mathbf{w}_k) \subset \widehat{\mathbb{Z}}^n \tag{6}$$

equidistributes in its respective space, with respect to the pushforward of $\nu^{(\mathscr{X}_d)}, \nu^{(\mathbb{R}^d)}, \nu^{(f)}$ respectively.

Remark 1.3. A comparison of the two statements reveals that for the measure ν arising in Theorem 1.2 we have a somewhat simpler description than for the measure μ in Theorem 1.1. In fact, as will be seen in §11, μ is absolutely continuous with respect to ν, and the density, which will be described explicitly, is not a product.

Remark 1.4. In the case of best approximations, our proof gives more information on the set of full measure in $\mathbb{R}^d$ for which the conclusions hold. For instance, as was pointed out to us by Yiftach Dayan, using [SW19] one obtains the equidistribution of the first two components of (3) with respect to the measures described in Theorem 1.1, for a.e. θ, with respect to the natural measure on a self-similar fractal such as Cantor's middle thirds set or the Koch snowflake. See Remark 13.1.

Various special cases of these results, dealing with the individual coordinate sequences (4), and mostly for $d = 1$, were proved in prior work. Even in case $d = 1$, the joint equidistribution of these sequences is new. We will survey these results in §3. Furthermore, in the sequel we will state a refinement (see Theorem 2.1) where equidistribution will take place in a torus bundle over the product $\mathscr{X}_d \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$.

Theorems 1.1 and 1.2 give information about typical vectors $\theta \in \mathbb{R}^d$ but as is often the case, they say nothing about concrete vectors. The following result deals with the asymptotic statistical properties of approximation vectors of certain algebraic vectors θ. It shows that there are limit laws governing the approximations but that they are actually quite different from the ones appearing in Theorems 1.1 and 1.2. To the best of our knowledge there are no prior results of this type.

Theorem 1.5. *Let $\vec{\alpha} \in \mathbb{R}^d$ be a vector of the form $\vec{\alpha} = (\alpha_1, \ldots, \alpha_d)$, where $\mathbb{K} \stackrel{\mathrm{def}}{=} \mathrm{span}_{\mathbb{Q}}(1, \alpha_1, \ldots, \alpha_d)$ is a totally real number field of degree $n \geq 3$. Let*

$$\varepsilon_0 \stackrel{\mathrm{def}}{=} \inf\left\{\varepsilon > 0 : \text{there are infinitely many } \varepsilon\text{-approximations to } \vec{\alpha}\right\}. \tag{7}$$

Let $\varepsilon > \varepsilon_0$ and let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. Then there are measures $\mu^{(\vec{\alpha})} = \mu^{(\vec{\alpha})}_{\mathrm{best},\|\cdot\|}$ and $\nu^{(\vec{\alpha})} = \nu^{(\vec{\alpha})}_{\varepsilon-\mathrm{approx},\|\cdot\|}$ on $\mathscr{X}_d \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$, such that the *following hold. Let* $(\mathbf{v}_k)_{k\in\mathbb{N}}$ *and* $(\mathbf{w}_k)_{k\in\mathbb{N}}$ *denote respectively the sequence of best approximations and* ε*-approximations of* $\vec{\alpha}$*, with respect to* $\|\cdot\|$*. Then:*

- *The sequence* (3) *equidistributes with respect to* $\mu^{(\vec{\alpha})}$*, provided*

$$\text{the norm on } \mathbb{R}^d \text{ is either the Euclidean norm or the sup-norm.} \tag{8}$$

- *For any norm, the sequence* (5) *equidistributes with respect to* $\nu^{(\vec{\alpha})}$*.*

Furthermore, the supports of the projections of $\mu^{(\vec{\alpha})}$ *and* $\nu^{(\vec{\alpha})}$ *to* $\mathscr{X}_d, \mathbb{R}^d, \widehat{\mathbb{Z}}^n$ *are null sets with respect to* $m_{\mathscr{X}_d}, m_{\mathbb{R}^d}, m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}$ *respectively, and in particular, they are singular with respect to the measures appearing in Theorems 1.1 and 1.2.*

Remarks 1.6. (1) A version of Theorem 1.5 is also true in dimension $d = 1$ but requires slightly different methods. Also, with our method one can also treat vectors $\vec{\alpha}$ for which the field $\mathbb{K}$ is real but not totally real. However this necessitates some additional arguments and some additional conditions on the norm. We hope to return to these topics in future work.

(2) Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_d)$ be as in Theorem 1.5. It is well-known that $\vec{\alpha}$ is badly approximable (see e.g. [Sch71]). That is,

$$\inf\{||\mathrm{disp}(\vec{\alpha}, \mathbf{w})|| : \mathbf{w} \in \mathbb{Z}^d \times \mathbb{N}\} > 0 \tag{9}$$

(where $\mathrm{disp}(\vec{\alpha}, \mathbf{w})$ is as in (2)). This shows that if limit measures $\mu^{(\alpha)}, \nu^{(\alpha)}$ as in Theorem 1.5 exist, then their projection to $\mathbb{R}^d$ is bounded away from 0. This is already quite different from the typical behavior described in Theorems 1.1, 1.2.

(3) The measures $\mu^{(\vec{\alpha})}_{\mathrm{best},\|\cdot\|}$ and $\nu^{(\vec{\alpha})}_{\varepsilon-\mathrm{approx},\|\cdot\|}$ admit an explicit description, see §11. Although they are very different from the measures $\mu_{\mathrm{best},\|\cdot\|}$ and $\nu_{\varepsilon-\mathrm{approx},\|\cdot\|}$ appearing in Theorem 1.1 and 1.2, in the recent paper [SZ], the first author and Zheng exhibit explicit choices of sequences of vectors $\vec{\alpha}_j$, corresponding to totally real number fields, for which $\mu^{(\vec{\alpha}_j)}_{\mathrm{best},\|\cdot\|} \longrightarrow_{j\to\infty} \mu_{\mathrm{best},\|\cdot\|}$.

(4) Several authors (see [Che13, §2.3] and references therein) have shown that in some cases of vectors whose coordinates generate cubic fields which are not totally real, best approximation denominators vectors are periodic, or satisfy higher order linear recurrences. We show (see Proposition 11.7) that for any norm, and any $\vec{\alpha}$ as in Theorem 1.5, for both best approximations and ε-approximations, the sequence of denominators $(\log(q_k))_{k\in\mathbb{N}}$ is strongly asymptotic to a one-dimensional cut-and-project set, and the sequence of approximation vectors is strongly asymptotic to a generalized cut-and-project set (see §11.2 for the definitions of these terms).

1.1. **Outline of method and structure of the paper.** Our method closely follows that used by [CC], which in turn was inspired by [AN93].

We consider a space X, a flow $a_t \curvearrowright X$, and a subset $\mathcal{S} \subset X$ intersecting all orbits along a discrete infinite countable set of times. The set $\mathcal{S}$ is called a *Poincaré section* or *cross-section*, and the flow induces a *return time function*

$$\tau : \mathcal{S} \to \mathbb{R}_+, \quad \tau(x) = \min\{t > 0 : a_t x \in \mathcal{S}\},$$

and a *first return map*

$$T_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}, \quad T_{\mathcal{S}}(x) = a_{\tau(x)} x.$$

It has been known since classical work of Ambrose and Kakutani (see [Amb41, AK42] or §4.1) that there is a bijection $\mu \longleftrightarrow \mu_{\mathcal{S}}$ between $\{a_t\}$-invariant ergodic measures on X, and $T_{\mathcal{S}}$-invariant ergodic measures on $\mathcal{S}$. These notions were intensively studied both in the setup of standard Borel spaces with Borel actions, and in the setup of smooth flows on manifolds, with sections which are codimension one submanifolds. In our setup, the space X and the section $\mathcal{S}$ are chosen so that they satisfy the following properties.

- The space X is an adelic homogeneous space, that is, a quotient $X = \mathbf{G}(\mathbb{A})/\mathbf{G}(\mathbb{Q})$ where $\mathbf{G}$ is a $\mathbb{Q}$-algebraic group, and $\{a_t\}$ is a one-parameter real subgroup acting on X by left translations. In particular, X is a locally compact and second countable topological space, but is not a manifold.
- The section $\mathcal{S}$ is chosen so that for each θ in $\mathbb{R}^d$ there is $\widetilde{\Lambda}_\theta \in X$, such that visits to $\mathcal{S}$ of the trajectory $\{a_t \widetilde{\Lambda}_\theta : t \geq 0\}$ in X, correspond in an explicit way to a sequence of approximations. In particular we will choose distinct (but closely related) sections for dealing with best approximations and ε-approximations.
- The observables we are interested in, like $[\pi^{\mathbf{v}}_{\mathbb{R}^d}(\mathbb{Z}^n)]$, $\mathrm{disp}(\theta, \mathbf{v})$, and $\mathbf{v}$ (seen as an element of $\widehat{\mathbb{Z}}^n_{\mathrm{prim}}$), are explicitly given by functions on $\mathcal{S}$.
- For certain dynamically natural measures μ on X, the corresponding measures $\mu_{\mathcal{S}}$ on $\mathcal{S}$ can be described explicitly. In particular this is true for $\mu = m_X$, the unique $\mathbf{G}(\mathbb{A})$-invariant probability measure on X.

The specific cross-section we work with in this paper involves lattices which contain a primitive vector whose nth coordinate is 1. We remark that in [CC] a different cross-section is used.

Recall that for a flow $\{a_t\}$ on a locally compact second countable space X, and an invariant measure μ, the trajectory of a point $x \in X$ is *generic for* μ if the orbital averaging measures $\frac{1}{T}\int_0^T \delta_{a_t x} dt$ converge weak-* to μ as $T \to \infty$. A similar definition can be given with $\mu_{\mathcal{S}}, T_{\mathcal{S}}$ in place of $\mu, \{a_t\}$. Although the relationship between a_t-invariant measures on X and $T_{\mathcal{S}}$-invariant measures on $\mathcal{S}$ is well-understood, for our application we need a finer understanding of the relationship between $\{a_t\}$-generic trajectories on (X, μ) and $T_{\mathcal{S}}$-generic trajectories on $(\mathcal{S}, \mu_{\mathcal{S}})$. We are not aware of any treatment which is suitable for our purposes and we develop it in detail in this paper.

In §2 we introduce a torus bundle $\mathscr{E}_n \to \mathscr{X}_d$, which we call the *space of lift functionals*. We then state strengthenings of Theorems 1.1, 1.2, 1.5 in which the measures $\mu^{(\mathscr{X}_d)}, \nu^{(\mathscr{X}_d)}$ are replaced with measures $\mu^{(\mathscr{E}_n)}, \nu^{(\mathscr{E}_n)}$ on $\mathscr{E}_n$. In §3 we state more detailed results and compare our results to those of previous authors. §4 contains some measure-theoretic preliminaries, and in §5 we introduce *reasonable cross sections* for a flow on a lcsc measure space (X, μ) (here and throughout, *lcsc* is an abbreviation for locally compact second countable Hausdorff space, and all measures on lcsc spaces are Borel regular Radon measures, but not necessarily probability measures). Roughly speaking, the definition of a reasonable cross section captures the minimal topological and measure-theoretic structure needed in order to establish a link between generic points for $\{a_t\}$ and for $T_{\mathcal{S}}$.

A further helpful notion introduced in Definition 4.7 is that of a *tempered subset* of a reasonable cross-section. Such a subset is a cross-section in its own right, and for tempered subsets, the relation between generic points is clearer: a point in X is generic for μ if and only if its $\{a_t\}$-orbit intersects $\mathcal{S}$ in points which are generic for $T_{\mathcal{S}}$. We remark that the cross-section we will use for analyzing best approximations is a tempered subset, but the cross-section we will use for analyzing ε-approximations is not (see Proposition 9.10). Correspondingly, for best approximations our results will be slightly stronger (see Remark 1.4 and Remark 1.6(3)). We remark further that the section of [CC] was also shown to be tempered.

In §6 we will discuss some useful abstract properties of reasonable cross-sections; in particular, how to lift a cross-section from a factor, and continuity properties of cross-sections measures. Taken together, sections §§4-6 constitute our contribution to the abstract theory of cross-sections in lcsc spaces.

In the subsequent sections we will apply this abstract theory to the spaces which are relevant for us. In §7 and §8 we will introduce the specific real and adelic spaces we will work with, and the flows and cross-sections we will use. Namely, for Theorems 1.1 and 1.2 we will work with $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$, the natural measure on the adelic space $\mathrm{SL}_n(\mathbb{A})/\mathrm{SL}_n(\mathbb{Q})$, and for Theorem 1.5 the measure μ will be a homogeneous measure on an adelic torus-orbit. A considerable part of the paper, comprising §8-§9, will be devoted to checking that the cross-sections we work with are reasonable, and special subsets corresponding to best approximations and ε-approximations, are Jordan measurable. In §10 we will explain how certain observables associated with approximation vectors, can be read off from intersection times with the section. In §11 we will analyze the cross-section measures in detail. In §12 we will argue that for typical θ, the trajectories $\{a_t\widetilde{\Lambda}_\theta\}$ relevant for the approximation question, give rise to generic intersection with the cross-section. In §13 we will combine all of these ingredients to conclude the proofs of the main results.

1.2. **Acknowledgements.** The authors gratefully acknowledge support of the following grants: ISF 2019/19, ISF 871/17, ISF-NSFC 3739/21, BSF 2016256. This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 Research and Innovation Program, Grant agreement no. 754475.

2. A strengthening: the bundle $\mathscr{E}_n$ of lift functionals over projected lattices

Let $\rho_{\mathscr{X}_d}$ be the map appearing in the first coordinate of (3), that is

$$\rho_{\mathscr{X}_d}(\mathbf{v}) \stackrel{\text{def}}{=} \left[\pi^{\mathbf{v}}_{\mathbb{R}^d}(\mathbb{Z}^n)\right].$$

In this section we will introduce a probability space $(\mathscr{E}_n, m_{\mathscr{E}_n})$ and maps $\pi_{\mathscr{X}_d}, \rho_{\mathscr{E}_n}$ that fit in the following commuting diagram:

$$\begin{array}{ccc} & & \mathscr{E}_n \\ & \nearrow^{\rho_{\mathscr{E}_n}} & \downarrow \pi_{\mathscr{X}_d} \\ \mathbb{Z}^n_{\text{prim}} & \xrightarrow[\rho_{\mathscr{X}_d}]{} & \mathscr{X}_d \end{array} \tag{10}$$

We first define the spaces and maps group-theoretically, and then use them to state strengthenings of our results. We will then discuss the geometric information encoded in the space $\mathscr{E}_n$.

Let

$$\mathscr{E}_n \stackrel{\text{def}}{=} \{\Lambda \in \mathscr{X}_n : \mathbf{e}_n \in \Lambda_{\text{prim}}\} \tag{11}$$

be the set of lattices containing $\mathbf{e}_n$ as a primitive vector, and let

$$H \stackrel{\text{def}}{=} \left\{ \left(\begin{smallmatrix} A & 0 \\ \mathbf{x} & 1 \end{smallmatrix}\right) \in \mathrm{SL}_n(\mathbb{R}) : A \in \mathrm{SL}_d(\mathbb{R}),\ \mathbf{x}^{\mathrm{t}} \in \mathbb{R}^d \right\}. \tag{12}$$

Then the lattice $\mathbb{Z}^n$ is contained in $\mathscr{E}_n$, the group H acts transitively on $\mathscr{E}_n$, and $H(\mathbb{Z})$ is the stabilizer of $\mathbb{Z}^n$ for this action. Thus we may identify $\mathscr{E}_n \simeq H/H(\mathbb{Z})$. Since $H(\mathbb{Z})$ is a lattice in H, there is a unique H-invariant probability measure on $\mathscr{E}_n$ which we denote $m_{\mathscr{E}_n}$. We define

$$\pi_{\mathscr{X}_d} : \mathscr{E}_n \to \mathscr{X}_d, \qquad \pi_{\mathscr{X}_d}(\Lambda) \stackrel{\text{def}}{=} \pi_{\mathbb{R}^d}(\Lambda); \tag{13}$$

that is, the projection of Λ along $\mathbf{e}_n$. The condition $\mathbf{e}_n \in \Lambda_{\text{prim}}$ ensures that the image of this map is indeed in $\mathscr{X}_d$. It is clear that $\pi_{\mathscr{X}_d}$ intertwines the $\mathrm{SL}_d(\mathbb{R})$-actions on $\mathscr{E}_n$ and $\mathscr{X}_d$ (where we view $\mathrm{SL}_d(\mathbb{R})$ as embedded in H by taking $\mathbf{x} = 0$ in (12)). From the uniqueness of invariant probability measures for transitive actions we obtain

$$(\pi_{\mathscr{X}_d})_* m_{\mathscr{E}_n} = m_{\mathscr{X}_d}. \tag{14}$$

We now define the map $\rho_{\mathscr{E}_n} : \mathbb{Z}^n_{\text{prim}} \to \mathscr{E}_n$. For $v \in \mathbb{R}^d$ and $t \in \mathbb{R}$, let

$$u(v) \stackrel{\text{def}}{=} \left(\begin{smallmatrix} I_d & v \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_n(\mathbb{R}) \tag{15}$$

and

$$a_t \overset{\text{def}}{=} \operatorname{diag}\left(e^t, \ldots, e^t, e^{-dt}\right) \in \mathrm{SL}_n(\mathbb{R}). \tag{16}$$

Given $\mathbf{v} = (v_1, \ldots, v_n)^{\mathrm{t}} \in \mathbb{Z}^n_{\mathrm{prim}}$, we set

$$v \overset{\text{def}}{=} -\frac{\pi_{\mathbb{R}^d}(\mathbf{v})}{v_n} = -\frac{1}{v_n}(v_1, \ldots, v_d)^{\mathrm{t}} \quad \text{and } t \overset{\text{def}}{=} -\log |v_n|,$$

and define

$$\rho_{\mathscr{E}_n}(\mathbf{v}) \overset{\text{def}}{=} a_t u(v) \mathbb{Z}^n. \tag{17}$$

That is, we use $u(v)$ and a_t to deform $\mathbb{Z}^n$ to a lattice in $\mathscr{E}_n$ by moving $\mathbf{v}$ to $\mathbf{e}_n$, using maps which do not change the homothety class of the projection to the horizontal space. Recall that $[\cdot]$ is our notation for the covolume one representative of the homothety equivalence class, and note that a_t acts by dilations on $\mathbb{R}^d$ and $u(v)$ acts trivially on $\mathbb{R}^d$. Thus, this choice ensures that

$$\left[\pi^{\mathbf{v}}_{\mathbb{R}^d}(\mathbb{Z}^n)\right] = \left[a_t u(v) \pi^{\mathbf{v}}_{\mathbb{R}^d}(\mathbb{Z}^n)\right] = \left[\pi_{\mathbb{R}^d}(a_t u(v) \mathbb{Z}^n)\right] = \left[\pi_{\mathscr{X}_d}(\rho_{\mathscr{E}_n}(\mathbf{v}))\right].$$

This is the commutativity of the diagram (10).

With this notation in hand we now state a strengthening of Theorems 1.1, 1.2 and 1.5:

Theorem 2.1. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$, let $\varepsilon > 0$, and let $\vec{\alpha}$ be as in Theorem 1.5. Then there are measures*

$$\mu^{(\mathbf{e}_n)}, \ \mu^{(\mathbf{e}_n, \vec{\alpha})}, \ \nu^{(\mathbf{e}_n)} \text{ and } \nu^{(\mathbf{e}_n, \vec{\alpha})}$$

on $\mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$ such that, denoting by $(\mathbf{v}_k)_{k\in\mathbb{N}}$, $(\mathbf{w}_k)_{k\in\mathbb{N}}$ the sequence of best approximations and ε-approximations of $\theta \in \mathbb{R}^d$, the sequences

$$(\rho_{\mathscr{E}_n}(\mathbf{v}_k), \operatorname{disp}(\theta, \mathbf{v}_k), \mathbf{v}_k)_{k\in\mathbb{N}} \in \mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n \tag{18}$$

and

$$(\rho_{\mathscr{E}_n}(\mathbf{w}_k), \operatorname{disp}(\theta, \mathbf{w}_k), \mathbf{w}_k)_{k\in\mathbb{N}} \in \mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n \tag{19}$$

equidistribute with respect to $\mu^{(\mathbf{e}_n)}$ and $\nu^{(\mathbf{e}_n)}$ respectively for Lebesgue a.e. θ, and with respect to $\mu^{(\mathbf{e}_n,\vec{\alpha})}$ (provided (8) holds) and $\nu^{(\mathbf{e}_n,\vec{\alpha})}$ (provided $\varepsilon > \varepsilon_0$, for ε_0 as in (7)) for $\theta = \vec{\alpha}$. Furthermore, the properties of these measures, listed in Theorems 1.1, 1.2 and 1.5, remain valid, provided we replace everywhere $\mathscr{X}_d$ with $\mathscr{E}_n$.

We now discuss the additional information encoded by $\rho_{\mathscr{E}_n}(\mathbf{v})$, besides the information encoded by $\rho_{\mathscr{X}_d}(\mathbf{v})$. Let $\mathbf{v} \in \mathbb{R}^n \smallsetminus \mathbb{R}^d$ and suppose $\Lambda \in \mathscr{X}_n$ contains $\mathbf{v}$ as a primitive vector (for example $\Lambda \in \mathscr{E}_n$ and $\mathbf{v} = \mathbf{e}_n$). We will introduce an additional geometric invariant associated with $\mathbf{v}$ and Λ, which we will call the *lift functional*. It records the information required to reconstruct Λ from its projection onto the hyperplane $\mathbb{R}^d$ along $\mathbf{v}$. Let $\Lambda'' \overset{\text{def}}{=} \pi^{\mathbf{v}}_{\mathbb{R}^d}(\Lambda)$ and let Λ' be the rescaling of Λ'' which has covolume one. That is, if v_n is the nth coefficient of $\mathbf{v}$, then

$$\Lambda' = |v_n|^{1/d} \pi^{\mathbf{v}}_{\mathbb{R}^d}(\Lambda) \in \mathscr{X}_d. \tag{20}$$

Note that $\Lambda' = \pi_{\mathscr{X}_d}(\Lambda)$ if $\Lambda \in \mathscr{E}_n$ and $\mathbf{v} = \mathbf{e}_n$.

We claim that there is a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\forall \mathbf{w} \in \Lambda \ \exists k \in \mathbb{Z} \text{ such that } \mathbf{w} = \pi^{\mathbf{v}}_{\mathbb{R}^d}(\mathbf{w}) + \big(f(\pi^{\mathbf{v}}_{\mathbb{R}^d}(\mathbf{w})) + k\big)\,\mathbf{v}. \tag{21}$$

To see this, complete $\mathbf{v}$ to a $\mathbb{Z}$-basis $\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_d$ of Λ, so that $\mathbf{w}_i \stackrel{\mathrm{def}}{=} \pi^{\mathbf{v}}_{\mathbb{R}^d}(\mathbf{v}_i)$ $(i = 1, \ldots, d)$ is a $\mathbb{Z}$-basis of Λ''. For each i there is $c_i \in \mathbb{R}$ such that $\mathbf{v}_i = \mathbf{w}_i + c_i\mathbf{v}$, and we can define $f \in (\mathbb{R}^d)^*$ via the requirement

$$f(\mathbf{w}_i) = c_i \quad (i = 1, \ldots, d).$$

This construction depends on the choice of the vectors $\mathbf{v}_i$, but any two functionals f_1, f_2 satisfying (21) will satisfy that $f_1(\mathbf{u}) - f_2(\mathbf{u}) \in \mathbb{Z}$ for any $\mathbf{u} \in \Lambda''$, that is they differ by an element of the dual lattice $(\Lambda'')^*$. It will be more convenient to work with lattices of covolume 1, so we will replace Λ'' with Λ' and f with

$$f'(x) \stackrel{\mathrm{def}}{=} f\left(|v_n|^{-1/d}x\right), \tag{22}$$

so that f' is well-defined as a class in the torus $\mathbb{T}_{\Lambda'}$, where for $\Delta \in \mathscr{X}_d$ we denote

$$\mathbb{T}_\Delta \stackrel{\mathrm{def}}{=} \left(\mathbb{R}^d\right)^* / \Delta^*. \tag{23}$$

This class $[f']$ is the lift functional.

Consider the set

$$\mathscr{E} \stackrel{\mathrm{def}}{=} \left\{(\Lambda', f) : \Lambda' \in \mathscr{X}_d,\ f \in \mathbb{T}_{\Lambda'}\right\},$$

which is a torus bundle over $\mathscr{X}_d$. Then a pair (Λ', f) constructed as above by projecting Λ along $\mathbf{v} \in \Lambda_{\text{prim}} \smallsetminus \mathbb{R}^d$ is an element of $\mathscr{E}$. Moreover, using (21), we can recover Λ from Λ', f and $\mathbf{v}$, as follows:

$$\Lambda = \Lambda(\Lambda', f, \mathbf{v}) = \{\mathbf{x} + (f(\mathbf{x}) + k)\mathbf{v} : \mathbf{x} \in \Lambda',\ k \in \mathbb{Z}\}. \tag{24}$$

Thus for each fixed $\mathbf{v} \in \mathbb{R}^n \smallsetminus \mathbb{R}^d$ we have an identification of $\mathscr{E}$ with the set of lattices in $\mathscr{X}_n$ which contain $\mathbf{v}$ as a primitive vector. In particular, choosing $\mathbf{v} = \mathbf{e}_n$ we obtain $\mathscr{E}_n$ as the image of the map $\Lambda(\cdot, \cdot, \mathbf{e}_n)$ as in (24).

In order to make the connection between $\mathscr{E}_n = H\mathbb{Z}^n$ and lift functionals more concrete, we write any element of $h \in H$ in the form $h = \left(\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{x} & 1 \end{smallmatrix}\right)$, as in (12). Writing $h = (A, \mathbf{x})$ we see that H is a semi-direct product $\mathrm{SL}_d(\mathbb{R}) \ltimes \mathbb{R}^d$ where $\mathrm{SL}_d(\mathbb{R})$ acts on $\mathbb{R}^d$ by right-multiplication on row vectors. Furthermore, for $h_i = (g_i, \mathbf{x}_i)$ we have $h_1\mathbb{Z}^n = h_2\mathbb{Z}^n$ if and only if $g_1 \in g_2\,\mathrm{SL}_d(\mathbb{Z})$ and $\mathbf{x}_1^{\mathrm{t}} g_1^{-1} \in \mathbf{x}_2^{\mathrm{t}} g_2^{-1} + \mathbb{Z}^d g_1^{-1}$. In particular, the lattice $\Lambda' = g_i\mathbb{Z}^d$ and the equivalence class of the functional $f(\mathbf{w}) = \mathbf{x}_i^{\mathrm{t}} g_i^{-1}\mathbf{w}$, seen as an element of $\mathbb{T}_{\Lambda'}$, are well-defined independently of $i = 1, 2$. They represent respectively the projected lattice $\pi^{\mathbf{v}}_{\mathbb{R}^d}(h_i\mathbb{Z}^n)$, and the lift functional.

3. Consequences, and prior work

Fix a norm $\|\cdot\|$ on $\mathbb{R}^d$ and fix $\varepsilon > 0$. Let μ, ν be as in Theorem 1.1 and 1.2 respectively, and let $\mu^{(\mathscr{X}_d)}, \mu^{(\mathbb{R}^d)}, \mu^{(f)}, \nu^{(\mathscr{X}_d)}, \nu^{(\mathbb{R}^d)}, \nu^{(f)}$ be the projections of these measures on the respective factors. We discuss prior work about equidistribution on these spaces. Throughout this section we fix $\theta \in \mathbb{R}^d$ for which the conclusions of Theorems 1.1 and 1.2 are valid, and let $(\mathbf{v}_k)$, $(\mathbf{w}_k)$ denote the sequence of best approximations and ε-approximations of θ.

3.1. **Equidistribution of (shapes of) projections in $\mathscr{X}_d$.** Our theorems assert that the sequence of projected lattices $\left(\left[\pi_{\mathbb{R}^d}^{\mathbf{w}_k}(\mathbb{Z}^n)\right]\right)_{k\in\mathbb{N}}$ equidistributes with respect to the natural measure $\nu^{(\mathscr{X}_d)} = m_{\mathscr{X}_d}$ on $\mathscr{X}_d$, and the sequence $\left(\left[\pi_{\mathbb{R}^d}^{\mathbf{v}_k}(\mathbb{Z}^n)\right]\right)_{k\in\mathbb{N}}$ also equidistributes with respect to a measure $\mu^{(\mathscr{X}_d)}$ on $\mathscr{X}_d$, and $\mu^{(\mathscr{X}_d)} \ll m_{\mathscr{X}_d}$. We stress that $\mu^{(\mathscr{X}_d)} \neq m_{\mathscr{X}_d}$. We will give an explicit description of $\mu^{(\mathscr{X}_d)}$ in §11. In particular we will show that $\mu^{(\mathscr{X}_d)}$ gives less weight to lattices with short vectors than $m_{\mathscr{X}_d}$.

The first equidistribution result for projected lattices appears in beautiful papers of Roelcke [Roe56] and Schmidt [Sch98]. Let

$$\mathcal{S}\mathscr{X}_d \overset{\text{def}}{=} \mathrm{SO}_d(\mathbb{R})\backslash \mathrm{SL}_d(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z})$$

denote the *space of shapes of lattices*, that is, the space of similarity classes of lattices in $\mathbb{R}^d$ (two lattices are *similar* if they differ from each other by a conformal linear map, that is a composition of a dilation and an orthogonal transformation). This space is equipped with a natural measure $m_{\mathcal{S}\mathscr{X}_d}$ constructed from the Haar measure on $\mathrm{SL}_d(\mathbb{R})$. There is a natural projection map $\mathscr{X}_d \to \mathcal{S}\mathscr{X}_d$ with compact fiber, and we can define $m_{\mathcal{S}\mathscr{X}_d}$ concretely as the image of $m_{\mathscr{X}_d}$ under this projection.

For any $\mathbf{v} \in \mathbb{Z}^n \smallsetminus \{0\}$, the projected lattice $\pi_{\mathbf{v}^\perp}^{\mathbf{v}}(\mathbb{Z}^n)$ is a lattice in the d-dimensional subspace $\mathbf{v}^\perp \subset \mathbb{R}^n$, and represents an equivalence class in $\mathcal{S}\mathscr{X}_d$ (which we continue to denote by $\pi_{\mathbf{v}^\perp}^{\mathbf{v}}(\mathbb{Z}^n)$). Schmidt showed that the uniform measures on the finite sets $\left\{\pi_{\mathbf{v}^\perp}^{\mathbf{v}}(\mathbb{Z}^n) : \mathbf{v} \in \mathbb{Z}^n_{\mathrm{prim}}, \|\mathbf{v}\| \leq T\right\}$ are equidistributed with respect to $m_{\mathcal{S}\mathscr{X}_d}$, in the limit as $T \to \infty$[1]. There has been a recent surge of interest in equidistribution results for shapes of projected lattices, see [Mar10, EMSS16, AES16] and references therein. In these finer results, one typically considers projected lattices along a sparser sequence of vectors, than the one considered by Schmidt.

Our results deal with the projections along vectors close to a line and to the best of our knowledge these have never been considered. Note that we do not work in $\mathcal{S}\mathscr{X}_d$ but in the larger space $\mathscr{X}_d$. Indeed, in our case the directions of spaces in which the projected lattices lie converges, and there is no reason to reduce lattices modulo similarity.

[1]In fact Schmidt considered the lattices obtained by interecting $\mathbb{Z}^n$ with $\mathbf{v}^\perp$, but the results are equivalent, as can be easily seen by passing to dual lattices.

3.2. **Equidistribution in the bundle of lift functionals.** In case $d = 1$ the space $\mathscr{X}_d$ reduces to a point and the bundle $\mathscr{E}_n$ is isomorphic to the fiber over this point, namely the one-dimensional torus $\mathbb{T}_1 = \mathbb{R}/\mathbb{Z}$. It thus makes sense to consider an analogous question to Schmidt's result mentioned in §3.1, namely ask whether in the limit as $T \to \infty$, the uniform measures on the finite collection of lift functionals associated with vectors of $\mathbb{Z}^2_{\mathrm{prim}}$ of length at most T, converges to the uniform measure on $\mathbb{T}_1$. Indeed, such results were obtained by Dinaburg-Sinai [DS90] and Risager-Rudnick [RR09]. The analogous question for lift functionals arising from a sequence of approximations to a line in $\mathbb{R}^2$ has not been previously considered, as far as we know. In this very special case, our result for the equidistribution of the sequence $(\rho_{\mathscr{E}_n}(\mathbf{v}_k))_{k\in\mathbb{N}}$ takes the following form:

Corollary 3.1. *Let $\varepsilon > 0$. For a.e. $\theta \in \mathbb{R}$, let (p_k, q_k) be the sequence of best approximations, let (p'_k, q'_k) be the sequence of ε-approximations, and let (u_k, v_k) and (u'_k, v'_k) denote respectively the shortest vector in $\mathbb{Z}^2$ for which*

$$\det \begin{pmatrix} p_k & u_k \\ q_k & v_k \end{pmatrix} = 1,$$

and respectively, for which

$$\det \begin{pmatrix} p'_k & u'_k \\ q'_k & v'_k \end{pmatrix} = 1.$$

Write

$$f_k \stackrel{\text{def}}{=} \frac{p_k u_k + q_k v_k}{p_k^2 + q_k^2} \quad \text{and} \quad f'_k \stackrel{\text{def}}{=} \frac{p'_k u'_k + q'_k v'_k}{(p'_k)^2 + (q'_k)^2}.$$

Then, as $N \to \infty$, the sequence $(f'_k)_{k=1}^N$ becomes equidistributed in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, with respect to Lebesgue measure; and the sequence $(f_k)_{k\in\mathbb{N}}$ becomes equidistributed in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, with respect to an absolutely continuous measure whose density is given by

$$F : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}, \quad F(t) = \frac{1}{2\log 2}\left(\frac{1}{2 - |t|} + \frac{1}{1 + |t|}\right).$$

See [DS90, RR09, NH16] for related work and for an interpretation of (u_k, v_k) as the shortest solutions to the gcd equation.

3.3. **Equidistribution of displacement vectors in $\mathbb{R}^d$.** In case $d = 1$, for best approximations, a very closely related result is the so-called *Doeblin-Lenstra conjecture*, proved by Bosman, Jager and Wiedijk (see [Doe40, BJW83] for the original papers, and [DK02, Chap. 5.3.2] and [IK02, Chap. 4] for more detailed treatments and related results). They showed that for a.e. $\theta \in \mathbb{R}$, with best approximations (q_k, p_k), the uniform measures on the sets

$$\{|q_k(q_k x - p_k)| : n = 1, \dots, N\}$$

converge weak-* to the measure ν on $[0,1]$ whose density function is given by

$$F : [0,1] \to \mathbb{R}, \quad F(t) = \begin{cases} \frac{1}{\log 2} & 0 \le t \le \frac{1}{2} \\ \frac{1/t-1}{\log 2} & \frac{1}{2} < t \le 1. \end{cases} \tag{25}$$

In contrast, our result, is about the uniform measures on

$$\{q_k(q_k x - p_k) : n = 1, \ldots, N\}$$

(i.e., we have removed the absolute values), and we show weak-* convergence to the measure on $[-1,1]$ whose density function is

$$\bar{F} : [-1,1] \to \mathbb{R}, \quad \bar{F}(t) = \frac{1}{2} F(|t|).$$

That is, the measure μ' we consider is invariant under $t \mapsto -t$ and projects to the measure μ given by (25) under the map $t \mapsto |t|$.

As far as we are aware, in dimension $d > 1$, the distribution of the displacement sequence $\mathrm{disp}(\theta, \mathbf{v}_k)$ for best approximations has not been studied. Similarly, the distribution of displacements $\mathrm{disp}(\theta, \mathbf{w}_k)$ for ε-approximations has not been studied, even in the one dimensional case. However, there has been some study of the *direction* of the displacement of ε-approximations. Namely, let $\mathbb{S}^{d-1} = \{\vec{x} \in \mathbb{R}^d : \|\vec{x}\| = 1\}$ be the unit sphere with respect to the given norm, let

$$\mathrm{Proj} : \mathbb{R}^d \to \mathbb{S}^{d-1}, \tag{26}$$

and consider the sequence

$$(\mathrm{Proj}(\mathrm{disp}(\theta, \mathbf{w}_k)))_{k \in \mathbb{N}} \subset \mathbb{S}^{d-1}. \tag{27}$$

In [AGT15], Athreya, Ghosh and Tseng showed that if $\|\cdot\|$ is the Euclidean norm, and ε-approximations are defined without requiring $\gcd(p_1, \ldots, p_d, q) = 1$, then the sequence (27) is equidistributed with respect to the unique $\mathrm{SO}_d(\mathbb{R})$-invariant measure on $\mathbb{S}^{d-1}$. From Theorem 1.2 we immediately obtain the following statements, valid for any norm:

Corollary 3.2. *For any norm $\|\cdot\|$ on $\mathbb{R}^d$, let $m|_B$ be the normalized restriction of Lebesgue measure to the unit norm-ball $\{\mathbf{u} : \|\mathbf{u}\| \le 1\}$, and let $\nu^{(\mathbb{S}^{d-1})} = (\mathrm{Proj})_*(m|_B)$. Also let $\mu^{(\mathbb{S}^{d-1})} = (\mathrm{Proj})_* \mu^{(\mathbb{R}^d)}$, where $\mu^{(\mathbb{R}^d)}$ is as in Theorem 1.1. Then for any $\varepsilon > 0$, the sequence (27) of directions of ε-approximations is equidistributed with respect to $\nu^{(\mathbb{S}^{d-1})}$, and the sequence*

$$(\mathrm{Proj}(\mathrm{disp}(\theta, \mathbf{v}_k)))_{k \in \mathbb{N}} \tag{28}$$

of directions of best approximations, is equidistributed with respect to $\mu^{(\mathbb{S}^{d-1})}$.

Note that by Theorem 1.1(4), when $\|\cdot\|$ is the Euclidean norm, the measures $\mu^{(\mathbb{S}^{d-1})}$ and $\nu^{(\mathbb{S}^{d-1})}$ coincide, but this is not true for general norms.

In another direction, Moschevitin [Mos00] has characterized the possible sets that can be obtained as the closures of the sequences (28), for various θ.

3.4. **Equidistribution in $\widehat{\mathbb{Z}}^n$, and congruence constraints.** According to (4) and (6), for Lebesgue a.e. θ, the sequences $(\mathbf{v}_k)_{k\in\mathbb{N}}, (\mathbf{w}_k)_{k\in\mathbb{N}}$ of best approximations and ε-approximations, considered as elements of $\widehat{\mathbb{Z}}^n_{\mathrm{prim}}$, are both equidistributed with respect to $m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}$. In case $d = 1$, such results have a long history. In 1962, Szüsz [Szü62] proved the following. For any a and m, there is c' such that for any $\varepsilon > 0$, for a.e. $\theta \in \mathbb{R}$, the sequence (p_k, q_k) of ε-approximations satisfies

$$\frac{1}{N}|\{1 \le k \le N : q_k \equiv a \mod m\}| \to_{N\to\infty} c'.$$

He also proved a similar statement for sequences of rationals arising from more general approximation functions. In 1982 Moeckel [Moe82] proved a dynamical result which implies the following. For a positive integer m and integers a, b, there is c such that for a.e. $\theta \in \mathbb{R}$, the sequence (p_k, q_k) of best approximations satisfies

$$\frac{1}{N}|\{1 \le k \le N : q_k \equiv a \ \&\ p_k \equiv b \mod m\}| \to_{N\to\infty} c.$$

In these results, the constants c, c' can be easily computed. See also [JL88, DK02, FS14] and references therein.

For a positive integer m and a vector $\vec{a} \in (\mathbb{Z}/m\mathbb{Z})^n$, we say that $\vec{a}$ is *primitive (mod m)* if there are no $b, d_1, \ldots, d_n \in \mathbb{Z}/m\mathbb{Z}$ with $b \neq 1$ and $a_i = bd_i$ for all i. Let $N_{m,n}$ be the cardinality of the set of primitive vectors in $(\mathbb{Z}/m\mathbb{Z})^n$; it is not hard to verify that

$$N_{m,n} = \prod_{i=1}^{k} \left(p_i^{nr_i} - p_i^{n(r_i-1)}\right),$$

where $m = p_1^{r_1} \cdots p_k^{r_k}$ is the prime factorization of m. We generalize the above results to any dimension $d \ge 1$ in the following Corollary, which follows immediately from Theorems 1.1, 1.2.

Corollary 3.3. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. Let m be a positive integer and let $\vec{a} \in (\mathbb{Z}/m\mathbb{Z})^n$. Then for any $\varepsilon > 0$, a.e. $\theta \in \mathbb{R}^d$, the sequences $(\mathbf{v}_k)_{k\in\mathbb{N}}, (\mathbf{w}_k)_{k\in\mathbb{N}}$ of best approximations and ε-approximations to θ both satisfy*

$$\frac{1}{N}|\{1 \le k \le N : \mathbf{u}_k \equiv \vec{a} \mod m\}| \to_{N\to\infty} c$$

(in both cases $\mathbf{u}_k = \mathbf{v}_k$ and $\mathbf{u}_k = \mathbf{w}_k$), where

$$c \stackrel{\mathrm{def}}{=} \begin{cases} \frac{1}{N_{n,m}} & \vec{a} \text{ is primitive (mod } m) \\ 0 & \text{otherwise.} \end{cases}$$

As far as we know, prior to this work, there were no such results for dimension $d > 1$; this despite a remark of Cassels (Math Review of [Szü62]) that 'it would be interesting to extend these results to simultaneous approximation'. A related work of Berthé, Nakada and Natsui [BNN06] establishes similar properties for the convergents arising from the d-dimensional Jacobi-Perron algorithm.

3.5. **Growth of q_k, the Khinchin-Lévy constant, and a 'Khinchin-Lévy distribution'.** The asymptotic rate of growth of the denominators q_k (i.e., the last coordinate of the vector $\mathbf{v}_k$ in case of best approximations, and of the vector $\mathbf{w}_k$ in case of ε-approximations) has been a topic of detailed study. For both best approximations and ε-approximations, Theorem 2.1 makes it possible to obtain new information about this growth. Namely, as we will show, for a given θ, the quotient q_{k+1}/q_k can be computed explicitly from the data contained in the 3-tuple (18) (in the case of best approximations) and (19) (in the case of ε-approximations). Using this we prove:

Corollary 3.4. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and let $\varepsilon > 0$. Then there are measures $\lambda^{(KL)}_{\|\cdot\|,\mathrm{best}}$, $\lambda^{(KL)}_{\|\cdot\|,\varepsilon}$ on $\mathbb{R}_+$ such that for a.e. $\theta \in \mathbb{R}$, the following holds. Let $(\mathbf{v}_k)_{k\in\mathbb{N}}$, $(\mathbf{w}_k)_{k\in\mathbb{N}}$ denote respectively the best approximation and ε-approximation sequence, and let $q_k^{(\mathrm{best})}, q_k^{(\varepsilon)}$ denote the corresponding denominators. For any $k \geq 2$, let*

$$t_k^{(\mathrm{best})} \stackrel{\mathrm{def}}{=} \log\left(\frac{q_{k+1}^{(\mathrm{best})}}{q_k^{(\mathrm{best})}}\right) \quad \text{and} \quad t_k^{(\varepsilon)} \stackrel{\mathrm{def}}{=} \log\left(\frac{q_{k+1}^{(\varepsilon)}}{q_k^{(\varepsilon)}}\right). \tag{29}$$

Then the uniform probability measures on the sets

$$\left\{t_k^{(\mathrm{best})} : 1 \leq k \leq N\right\}, \qquad \left\{t_k^{(\varepsilon)} : 1 \leq k \leq N\right\}$$

converge respectively to $\lambda^{(KL)}_{\|\cdot\|,\mathrm{best}}$, $\lambda^{(KL)}_{\|\cdot\|,\varepsilon}$, in the weak- topology as $N \to \infty$. The expectation of $\lambda^{(KL)}_{\|\cdot\|,\varepsilon}$ is $\frac{\zeta(n)}{V_{d,\|\cdot\|}\,\varepsilon^d}$, where $V_{d,\|\cdot\|}$ is the Lebesgue measure of the unit ball of the norm, in $\mathbb{R}^d$.*

For $d = 1$, in the case of best approximations, Khinchin [Khi36] showed in 1936 that the limit $\lim_{k\to\infty} q_k^{1/k}$ exists, and in the same year, Lévy [Lév36] gave the value of the limit. In our terminology, this limit is the expectation of the measure $\lambda = \lambda^{(KL)}_{\mathrm{best}}$, and thus Corollary 3.4 provides an analogue of the Khinchin-Lévy result for ε-approximations, valid for any d and any norm. For $d = 1$ and for best approximations, in 1986 Jager proved a convergence to a measure λ as above, and gave an explicit formula for λ. He also established a joint equidistribution result for the sequence of pairs $\left(t_k^{(\mathrm{best})}, \mathrm{disp}(\theta, \mathbf{v}_k)\right)_{k\in N}$ (see [DK02, Lemma 5.3.11]). For best approximations, and for the Euclidean norm, the convergence to a limit measure $\lambda^{(KL)}_{\|\cdot\|,\mathrm{best}}$ was recently proved in arbitrary dimension by Cheung and Chevallier[CC]. Our work extends this to arbitrary norms, answering [CC, Question 1]. As a corollary, Cheung and Chevallier obtained that for a.e. θ, the denominators q_k in the sequence of best approximations in dimension $d > 1$, satisfy that the limit $\lim_{k\to\infty} q_k^{1/k}$ exists. In a subsequent work with Xieu [Xie15], the limit was evaluated in the case $d = 2$.

Lagarias [Lag82] constructed examples showing irregular growth of the sequence $q_k^{1/k}$; in particular showing that the a.e. behavior can fail dramatically for some special choices of θ. A different way to measure the growth of approximations is due to Borel and Bernstein in dimension $d = 1$, and was extended by Chevallier to the case $d > 1$. See [Che13] for more details on this, and for a survey of more results on best approximations.

As noted in [CC, Question 1], it would be interesting to compute the expectation of $\lambda^{(KL)}_{\|\cdot\|,\mathrm{best}}$ for different dimensions and different norms. It would also be interesting to obtain additional statistical informations about these distributions, such as higher moments and tail bounds.

4. Preliminaries

In this section we recall some standard results. Some of them will be valid in a general Borel measurable setup, and for some we will require some topological assumptions. To distinguish these cases we will use the following terminology. We will have a space X on which a one-parameter group $\{a_t\}$ acts, a σ-algebra $\mathcal{B}_X$ of subsets of X, and a measure μ which is a_t-invariant.

- *The Borel setup* is the one in which $(X, \mathcal{B}_X)$ is a standard Borel space, i.e., there is a Polish structure (the topology induced by a complete and separable metric) on X for which $\mathcal{B}_X$ is the Borel σ-algebra. In this setup the action map $\mathbb{R} \times X \to X, \quad (t, x) \mapsto a_t x$ is assumed to be Borel, and the measure μ is assumed to be σ-finite.
- *The lcsc setup* is the one in which in addition, X is a Hausdorff locally compact second countable (lcsc) space, $\mathcal{B}_X$ is the Borel σ-algebra for the underlying topology, and the action map is continuous. Moreover the measure μ is assumed to be Radon (regular and finite on compact sets).

The above conditions will be assumed throughout the paper without further mention. Additionally, we will usually assume that μ is finite, but will make explicit mention of this when we do.

4.1. Measurable cross-sections.

4.1. Measurable cross-sections. In this section we recall classical definitions and results in the Borel setup. We start with the definition of a Borel cross-section which do not involve a measure and continue with the definition of a μ-cross-section which is relevant when a measure is present.

Definition 4.1. Let $(X, \mathcal{B}_X, \{a_t\})$ be as in the Borel setup. A *Borel cross-section* is a Borel subset $\mathcal{S}$ with the following properties:

- For any $x \in X$, the sets of *visit times*

$$\mathcal{Y}_x \overset{\mathrm{def}}{=} \{t \in \mathbb{R} : a_t x \in \mathcal{S}\} \tag{30}$$

  are all discrete and totally unbounded; that is, for any $T > 0$,

$$\mathcal{Y}_x \cap (T,\infty) \neq \varnothing,\ \mathcal{Y}_x \cap (-\infty,-T) \neq \varnothing, \text{ and } \#\left(\mathcal{Y}_x \cap (-T,T)\right) < \infty. \tag{31}$$

- The *return time function*

$$\tau_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}_+, \qquad \tau_{\mathcal{S}}(x) \stackrel{\text{def}}{=} \min\left(\mathcal{Y}_x \cap \mathbb{R}_+\right) \tag{32}$$

  is Borel.

If $\mathcal{S}$ is a Borel cross-section we will denote by $T_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ the *first return map* defined by

$$T_{\mathcal{S}}(x) \stackrel{\text{def}}{=} a_{\tau_{\mathcal{S}}(x)} x.$$

Definition 4.2. Let $(X, \mathcal{B}_X, \{a_t\})$ be as in the Borel setup and let μ be an $\{a_t\}$-invariant measure. We will say that $\mathcal{S} \in \mathcal{B}_X$ *is a μ-cross-section* if there is an $\{a_t\}$-invariant set $X_0 \in \mathcal{B}_X$ such that

- $\mu(X \smallsetminus X_0) = 0$.
- $\mathcal{S} \cap X_0$ is a Borel cross-section for $(X_0, \mathcal{B}_{X_0})$ (where $\mathcal{B}_{X_0}$ is the restricted σ-algebra defined by $\mathcal{B}_{X_0} \stackrel{\text{def}}{=} \{A \cap X_0 : A \in \mathcal{B}_X\}$).

If $\mathcal{S}$ is a μ-cross-section and X_0 is as in Definition 4.2, then we denote $\tau_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}_+ \cup \{\infty\}$ the function

$$\tau_{\mathcal{S}}(x) \stackrel{\text{def}}{=} \begin{cases} \tau_{\mathcal{S}\cap X_0}(x) & \text{if } x \in \mathcal{S} \cap X_0, \\ \infty & \text{if } x \in \mathcal{S} \smallsetminus X_0. \end{cases}$$

We denote $T_{\mathcal{S}\cap X_0}$ by $T_{\mathcal{S}}$. It is a well defined Borel map $\mathcal{S} \cap X_0 \to \mathcal{S} \cap X_0$, and we say that a measure on $\mathcal{S}$ is $T_{\mathcal{S}}$-invariant if $\mathcal{S} \smallsetminus X_0$ is a null-set and its restriction to $\mathcal{S} \cap X_0$ is $T_{\mathcal{S}}$-invariant.

Remark 4.3. The restricted Borel space $(X_0, \mathcal{B}_{X_0})$ appearing in Definition 4.2 is a standard Borel space, see [Kec95, Chap. 13].

For $\varepsilon > 0$ we let

$$\mathcal{S}_{\geq\varepsilon} \stackrel{\text{def}}{=} \{x \in \mathcal{S} : \tau_{\mathcal{S}}(x) \geq \varepsilon\} \quad \text{and} \quad \mathcal{S}_{<\varepsilon} \stackrel{\text{def}}{=} \mathcal{S} \smallsetminus \mathcal{S}_{\geq\varepsilon}. \tag{33}$$

The sets $\mathcal{S}_{\geq\varepsilon}$ are an increasing collection of Borel sets whose union is $\mathcal{S}$. Given $E \in \mathcal{B}_X$ and $I \subset \mathbb{R}$ we let

$$E^I \stackrel{\text{def}}{=} \{a_t x : x \in E, t \in I\}, \tag{34}$$

and let $m = m_{\mathbb{R}}$ denote the Lebesgue measure on $\mathbb{R}$.

The following structure theorem is due to Ambrose and Kakutani [Amb41, AK42], see also [Nad13, Wag88]:

Theorem 4.4. *Let $(X, \mathcal{B}_X, \{a_t\})$ be as in the Borel setup and let $\mathcal{S} \in \mathcal{B}_X$. Then, for any finite $\{a_t\}$-invariant measures μ on $(X, \mathcal{B}_X)$ for which $\mathcal{S}$ is a μ-cross-section, there exists a $T_{\mathcal{S}}$-invariant measures $\mu_{\mathcal{S}}$ on $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$ such that the following holds for any Borel set $E \subset \mathcal{S}$:*

(i) *If* $E \subset \mathcal{S}_{\geq \varepsilon}$*, and* I *is an interval of length* $< \varepsilon$ *then*

$$\mu_{\mathcal{S}}(E) = \frac{\mu(E^I)}{m(I)}.$$

In particular, for any $\varepsilon > 0$ *we have* $\mu_{\mathcal{S}}(\mathcal{S}_{\geq \varepsilon}) < \infty$*.*

(ii) *For any interval* I*,* $\mu_{\mathcal{S}}(E) \geq \frac{\mu(E^I)}{m(I)}$*.*

(iii) *In general*

$$\mu_{\mathcal{S}}(E) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \, \mu\left(E^{(0,\varepsilon)}\right). \tag{35}$$

(iv) *We have* $\mu_{\mathcal{S}}(E) = 0$ *if and only if* $\mu(E^{\mathbb{R}}) = 0$*.*

(v) *If* $\mu = \int \eta \, d\Theta(\eta)$ *for a measure* Θ *on* $\mathcal{P}(X)$*, where* Θ*-a.e.* η *is* $\{a_t\}$*-invariant, then* $\mathcal{S}$ *is an* η*-cross-section for* Θ*-a.e.* η*, and* $\mu_{\mathcal{S}} = \int \eta_{\mathcal{S}} \, d\Theta(\eta)$*. Moreover,* μ *is* $\{a_t\}$*-ergodic if and only if* $\mu_{\mathcal{S}}$ *is* $T_{\mathcal{S}}$*-ergodic.*

(vi) *We have*

$$\mu(X) = \int_{\mathcal{S}} \tau_{\mathcal{S}} \, d\mu_{\mathcal{S}}.$$

(vii) *If* $K \curvearrowright (X, \mu)$ *is a group-action commuting with the* $\{a_t\}$*-action and preserving* $\mathcal{S}$*, then the* K*-action also preserves* $\mu_{\mathcal{S}}$*.*

We will refer to item (vi) as the *Kac formula.* Note that (vii) is not mentioned in the above references but follows immediately from (35).

Definition 4.5. Let $(X, \mathcal{B}_X, \{a_t\})$ be as in the Borel setup and let μ be a finite $\{a_t\}$-invariant measure. Let $\mathcal{S}$ be a μ-cross-section. Then the measure $\mu_{\mathcal{S}}$ from Theorem 4.4 is called the *cross-section measure of* μ*.*

Note that in Theorem 4.4, μ is finite but $\mu_{\mathcal{S}}$ need not be. However, from (vi) one easily sees that if $\tau_{\mathcal{S}}$ is bounded below, or more generally, if there is $c > 0$ such that $\tau_{\mathcal{S}}(x) \geq c$ for $\mu_{\mathcal{S}}$-a.e. x, then $\mu_{\mathcal{S}}$ is finite.

Remark 4.6. In Theorem 4.4 we defined a map $\mu \mapsto \mu_{\mathcal{S}}$. Under suitable conditions this map is injective and its image has an explicit description. See [Nad13].

Definition 4.7. Let $(X, \mathcal{B}_X, \{a_t\})$ be as in the Borel setup and let μ be a finite $\{a_t\}$-invariant measure. Let $\mathcal{S}$ be a μ-cross-section. Let $M \in \mathbb{N}$ and let $E \subset \mathcal{S}$ be Borel. We say that E *is* M*-tempered* if for $\mu_{\mathcal{S}}$-a.e. x,

$$\#\{t \in [0,1] : a_t x \in E\} \leq M.$$

We say that E is *tempered* if it is M-tempered for some M, and that $\mathcal{S}$ is a *tempered* μ*-cross-section* if this condition holds for $E = \mathcal{S}$.

Clearly $\mathcal{S}$ is tempered if $\mathcal{S} = \mathcal{S}_{\geq \varepsilon}$ for some $\varepsilon > 0$. It is not hard to show (see [CC, Proof of Prop. 19]) that $\mu_{\mathcal{S}}(E) < \infty$ for any tempered subset.

4.2. **Tight convergence of measures on lcsc spaces.** Let $X, \mathcal{B}_X$ be as in the lcsc setup, let $C_b(X)$ denote the collection of bounded continuous real-valued functions on X, and let $\mathcal{M}(X)$ denote the collection of finite regular Borel measures on X. Whenever we discuss convergence of measures in this paper, it will be assumed that the measures belong to $\mathcal{M}(X)$. In particular, although infinite measures may appear in the discussion, convergence of measures will only be discussed for finite measures. For $\mu \in \mathcal{M}(X)$ and $f \in C_b(X)$, we denote the integral $\int_X f\, d\mu$ by $\mu(f)$. We will use the so-called *tight topology* (sometimes referred to as *strict topology*) on $\mathcal{M}(X)$ for which convergence $\mu_k \to \mu$ is defined by either of the following equivalent requirements (see [Bou04b, §5, Prop. 9] for the equivalence):

(i) For all $f \in C_b(X)$, $\mu_k(f) \to \mu(f)$.
(ii) For any compactly supported continuous function $f : X \to \mathbb{R}$, $\mu_k(f) \to \mu(f)$ and $\mu_k(X) \to \mu(X)$.

Note that this is not equivalent to weak-* convergence, in which f is taken to be compactly supported. Due to the characterization (ii), the weak-* topology is coarser than the tight topology when X is not compact. Nevertheless, when the total masses $\mu(X), \mu_k(X)$ are the same (e.g., when they are probability measures), these notions of convergence coincide.

For a topological space X and $E \subset X$, $\mathrm{int}(E)$, $\mathrm{cl}(E)$ and $\partial(E) = \mathrm{cl}(E) \cap \mathrm{cl}(X \smallsetminus E)$ denote respectively its topological interior, closure, and boundary. Since we will work with several topological spaces, if we want to stress the dependence on X we will write $\mathrm{int}_X(E)$, $\mathrm{cl}_X(E)$, $\partial_X(E)$.

Definition 4.8. Let X be an lcsc space and $\mu \in \mathcal{M}(X)$. We say that $E \in \mathcal{B}_X$ is *Jordan measurable with respect to* μ (abbreviated μ*-JM*) if $\mu(\partial_X(E)) = 0$.

Suppose $X' \subset X$ is locally compact (in its relative topology as a subset of X) and $\mu(X \smallsetminus X') = 0$. It may happen that for $E \subset X'$ we have $\partial_{X'}(E) \subsetneq \partial_X(E)$. Nevertheless, since $\mu(X \smallsetminus X') = 0$, we still have $\mu(\partial_{X'}(E)) = \mu(\partial_X(E))$, and so the notion of Jordan measurability with respect to μ is not affected by adding or removing sets of measure zero (as long as we work with locally compact sets).

The collection of μ-JM sets forms a sub-algebra of $\mathcal{B}_X$, and this algebra is rich enough to capture the tight convergence to μ. More precisely we have the following.

Lemma 4.9 (See [Bil68], Thms. 2.1 & 2.7, or [Bou04a], Chap. 4)**.** *If $\mu_k, \mu \in \mathcal{M}(X)$ then $\mu_k \to \mu$ tightly if and only if for any μ-JM set E one has $\mu_k(E) \to \mu(E)$.*

Moreover, if Y is also an lcsc space and $\psi : X \to Y$ is a measurable function, then the push-forward map $\psi_* : \mathcal{M}(X) \to \mathcal{M}(Y)$ is continuous at a measure $\mu \in \mathcal{M}(X)$ (with respect to the tight topologies) provided that ψ is continuous μ-almost everywhere.

5. Equidistribution of visits to a cross-section

Throughout this section we let $(X, \mathcal{B}_X, \{a_t\}, \mu)$ be as in the lcsc setup and $\mathcal{S}$ be a μ-cross-section as in Definition 4.2. We further assume that

(A) μ is a probability measure.
(B) $\mu_{\mathcal{S}}$ is finite.
(C) $\mathcal{S}$ is lcsc (with respect to its subset topology induced by the topology on X).

Let χ_E be the indicator of $E \subset X$. For $x \in X$ and $E \subset \mathcal{S}$ we let

$$N(x, T, E) = \# \{t \in [0, T] : a_t x \in E\}.$$

Definition 5.1.

1. We say that a point $x \in X$ is (a_t, μ)-*generic* if $\frac{1}{T}\int_0^T \delta_{a_t x} dt \to_{T\to\infty} \mu$.
2. We say that $x \in X$ is $(a_t, \mu_{\mathcal{S}})$-*generic* if the sequence of visits of the orbit $\{a_t x : t > 0\}$ to $\mathcal{S}$ equidistributes with respect to $\frac{1}{\mu_{\mathcal{S}}(\mathcal{S})}\mu_{\mathcal{S}}$.
3. For a Borel subset $\mathcal{S}' \subset \mathcal{S}$ which is $\mu_{\mathcal{S}}$-JM and of positive $\mu_{\mathcal{S}}$-measure, we say that $x \in X$ is $(a_t, \mu_{\mathcal{S}}|_{\mathcal{S}'})$-*generic* if the sequence of visits of the orbit $\{a_t x : t > 0\}$ to $\mathcal{S}'$ equidistributes with respect to $\frac{1}{\mu_{\mathcal{S}}(\mathcal{S}')}\mu_{\mathcal{S}}|_{\mathcal{S}'}$.

Since, by (A), both cases above concern probability measures, we can understand the equidistribution equivalently as either weak-* convergence or as tight convergence. Also, by using Lemma 4.9, x is (a_t, μ)-generic if and only if

$$\text{for any } \mu\text{-JM set } E \subset X, \quad \frac{1}{T}\int_0^T \chi_E(a_t x) dt \to_{T\to\infty} \mu(E). \tag{36}$$

Similarly, by Lemma 4.9, and using (B) and (C), x is $(a_t, \mu_{\mathcal{S}})$-generic if and only if

$$\text{for any } \mu_{\mathcal{S}}\text{-JM set } E \subset \mathcal{S}, \quad \frac{N(x,T,E)}{N(x,T,\mathcal{S})} \to_{T\to\infty} \frac{\mu_{\mathcal{S}}(E)}{\mu_{\mathcal{S}}(\mathcal{S})}. \tag{37}$$

Remark 5.2. Note that we do not define genericity with respect to the first return map $T_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$, but the reader will note the relationship between $(a_t, \mu_{\mathcal{S}})$-genericity and $(T_{\mathcal{S}}, \mu_{\mathcal{S}})$-genericity: $x \in X$ is $(a_t, \mu_{\mathcal{S}})$-generic if and only if $x' = a_{\tau_{\mathcal{S}}(x)} x$ is $\left(T_{\mathcal{S}}, \frac{1}{\mu_{\mathcal{S}}(\mathcal{S})}\,\mu_{\mathcal{S}}\right)$-generic (in the natural sense).

In this section we will study the relationship between (a_t, μ)-genericity and $(a_t, \mu_{\mathcal{S}})$-genericity. As a motivating example, consider the following simplified situation. Assume that (i) for any $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}$, the thickened set $E^{(0,1)}$ (defined via (34)) is μ-JM; and (ii) the return time function $\tau_{\mathcal{S}}$ is bounded below by 1. Then it follows that if $x \in X$ is (a_t, μ) generic, then it is $(a_t, \mu_{\mathcal{S}})$-generic. Indeed, we have

$$\frac{1}{T} N(x, T, E) = \frac{1}{T}\left(\int_0^T \chi_{E^{(0,1)}}(a_t x) dt + O(1)\right)$$

and the claim can be deduced using (36), (37), and Theorem 4.4(i).

Since the notions of cross-sections in Definitions 4.1, 4.2 refer only to the Borel structure, while equidistribution is a topological notion, it is natural to expect a topological assumption like (i), and indeed, we will require some similar additional assumptions. Regarding (ii), when trying to remove it, one encounters a complication involving the relation between the continuous time parameter T and the number of visits $N(x,T,\mathcal{S})$. Suppose x is (a_t,μ)-generic but the number of visits up to time T is large compared to T (e.g., on the order of T^2). This implies frequent visits to $\mathcal{S}_{<\varepsilon}$. In Theorem 5.11 below, which is the main result of this section, we use this observation to define a concrete μ-null set, and show that for trajectories avoiding this set, (a_t,μ)-genericity implies $(a_t,\mu_{\mathcal{S}})$-genericity.

Definition 5.3. Let $\mathcal{S} \subset X$ be a μ-cross-section satisfying (A), (B) and (C). We say that $\mathcal{S}$ is a *μ-reasonable* if in addition, the following hold:

(1) For all sufficiently small ε, the sets $\mathcal{S}_{\geq\varepsilon}$ are $\mu_{\mathcal{S}}$-JM.
(2) There is a relatively open subset $\mathcal{U} \subset \mathcal{S}$ such that the following two conditions hold:
  (a) The map $(0,1)\times\mathcal{U} \to X$, $(t,x) \mapsto a_t x$ is open;
  (b) $\mu\left((\mathrm{cl}_X(\mathcal{S}) \smallsetminus \mathcal{U})^{(0,1)}\right) = 0$.

Remark 5.4. We note that the interval $(0,1)$ in Definition 5.3 can be replaced by any fixed small interval. We also note that it is possible to obtain our results while replacing Definition 5.3(1) with the weaker requirement that there exists an increasing collection of $\mu_{\mathcal{S}}$-JM sets $\mathcal{F}_k$ such that $\mathcal{S} = \bigcup_k \mathcal{F}_k$ and such that $\mathcal{F}_k \subset \mathcal{S}_{\geq\varepsilon_k}$ for some $\varepsilon_k > 0$. This more flexible framework requires slightly more involved arguments, but we will not need it and leave the details to the dedicated reader.

The following elementary lemma shows that Definition 5.3 implies the property (i) used in the preceding discussion.

Lemma 5.5. *If $\mathcal{S}$ is μ-reasonable then for any $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}$ and any interval $I \subset [0,1]$, E^I is μ-JM.*

Proof. Assume for concreteness that I is a closed interval, the other cases being similar, and write $I = [\tau_1,\tau_2]$. Let $\mathcal{U} \subset \mathcal{S}$ be the relatively open set appearing in Definition 5.3(2). We need to show that $\mu\left(\partial_X\left(E^I\right)\right) = 0$. This will follow once we show

$$\partial_X\left(E^I\right) \subset (\mathrm{cl}_X(\mathcal{S}) \smallsetminus \mathcal{U})^I \cup a_{\tau_1}\mathcal{S} \cup a_{\tau_2}\mathcal{S} \cup (\partial_{\mathcal{S}}(E))^I \tag{38}$$

Indeed, all sets on the RHS of (38) are μ-null: the first by the choice of $\mathcal{U}$, the fourth because of Theorem 4.4(iv) and the assumption that $\partial_{\mathcal{S}}(E)$ is $\mu_{\mathcal{S}}$-null, and the second and third sets are μ-null by standing assumption (B) and Theorem 4.4(ii).

We prove (38). Let $x \in \partial_X\left(E^I\right)$, so that there is a sequence $t_k \in [\tau_1,\tau_2]$ and $y_k \in E$ such that $a_{t_k}y_k \to x$. By passing to subsequences we may assume that $t_k \to t_0 \in I$ and $y_k \to y_0 \in \mathrm{cl}_X(E) \subset \mathrm{cl}_X(\mathcal{S})$. We distinguish several cases. If $y_0 \notin \mathcal{U}$, then x clearly belongs to the RHS of (38). Thus we assume that $y_0 \in \mathcal{U}$. If $t_0 \in \{\tau_1, \tau_2\}$ then again x clearly belongs to the RHS of (38). Assume that $\tau_1 < t_0 < \tau_2$ and $y_0 \in \mathcal{U}$, hence in particular $y_0 \in \mathrm{cl}_{\mathcal{S}}(E)$. If $y_0 \notin \mathrm{int}_{\mathcal{S}}(E)$ then by definition $y_0 \in \partial_{\mathcal{S}}(E)$ and again x belongs to the RHS of (38). The only remaining possibility is that $y_0 \in \mathrm{int}_{\mathcal{S}}(E)$ but this is impossible since $x \notin \mathrm{int}_X \left(E^I\right)$ but the map $(t, y) \mapsto a_t y$ is open from $(0,1) \times \mathcal{U}$ to X. □

For a set $E \subset \mathcal{S}_{\geq \varepsilon}$ the relation between $N(x, T, E)$ and $\int_0^T \chi_{E^{(0,\varepsilon)}}(a_t x) dt$ is simple:

Proposition 5.6. *Let $\mathcal{S}$ be a μ-reasonable cross-section. Then for any $x \in X$ which is (a_t, μ)-generic, for any $\varepsilon > 0$ and for any $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}_{\geq \varepsilon}$,*

$$\lim_{T \to \infty} \frac{1}{T} N(x, T, E) = \mu_{\mathcal{S}}(E). \tag{39}$$

Proof. By Lemma 5.5 we have that $E^{(0,\varepsilon)}$ is μ-JM, and thus by Lemma 4.9, $\frac{1}{T}\int_0^T \chi_{E^{(0,\varepsilon)}}(a_t x) dt \to_{T\to\infty} \mu\left(E^{(0,\varepsilon)}\right)$. Since $E \subset \mathcal{S}_{\geq\varepsilon}$, we obtain (using Theorem 4.4) that

$$\mu(E^{(0,\varepsilon)}) = \varepsilon \mu_{\mathcal{S}}(E) \quad \text{and} \quad \int_0^T \chi_{E^{(0,\varepsilon)}}(a_t x) dt = \varepsilon N(x, T, E) + O(1),$$

and (39) follows. □

The following sets will be useful for analyzing trajectories $\{a_t x : t > 0\}$ visiting $\mathcal{S}_{<\varepsilon}$ with abnormally large frequency. Let

$$\Delta_{\mathcal{S},\delta} \stackrel{\text{def}}{=} \left\{ x \in \mathcal{S} : \forall \varepsilon > 0, \ \limsup_{T\to\infty} \frac{1}{T} N(x, T, \mathcal{S}_{<\varepsilon}) > \delta \right\}, \tag{40}$$
$$\Delta_{\mathcal{S}} \stackrel{\text{def}}{=} \bigcup_{\delta > 0} \Delta_{\mathcal{S},\delta}.$$

We have the following variant of Proposition 5.6, in which we do not require that $E \subset \mathcal{S}_{\geq\varepsilon}$.

Proposition 5.7. *Let $\mathcal{S}$ be a μ-reasonable cross-section. Assume that $x \in X \smallsetminus \Delta_{\mathcal{S}}^{\mathbb{R}}$ is (a_t, μ)-generic (where $\Delta_{\mathcal{S}}^{\mathbb{R}}$ is the thickening of $\Delta_{\mathcal{S}}$ as in (34)). Then (39) holds for any $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}$.*

Proof. By Definition 5.3, for any small enough $\varepsilon > 0$ we have that $E \cap \mathcal{S}_{\geq\varepsilon}$ is $\mu_{\mathcal{S}}$-JM, as an intersection of two such sets and so by Proposition 5.6,

$$\liminf_{T\to\infty} \frac{1}{T} N(x, T, E) \geq \lim_{T\to\infty} \frac{1}{T} N(x, T, E \cap \mathcal{S}_{\geq\varepsilon}) = \mu_{\mathcal{S}}(E \cap \mathcal{S}_{\geq\varepsilon}).$$

Since the sets $\mathcal{S}_{\geq\varepsilon}$ exhaust $\mathcal{S}$, and ε can be chosen arbitrarily small, we find

$$\liminf_{T\to\infty} \frac{1}{T} N(x, T, E) \geq \mu_{\mathcal{S}}(E). \tag{41}$$

Fix $\delta > 0$. Since $x \notin \Delta_{\mathcal{S}}^{\mathbb{R}}$, we have $x \notin \Delta_{\mathcal{S},\delta}^{\mathbb{R}}$. By (40) there exists $\varepsilon > 0$ so that

$$\limsup_{T\to\infty} \frac{1}{T} N(x, T, \mathcal{S}_{<\varepsilon}) \le \delta, \tag{42}$$

and clearly we may take ε arbitrarily small to ensure that $\mathcal{S}_{\ge\varepsilon}$ is $\mu_{\mathcal{S}}$-JM. Taking (42) into account and applying again Proposition 5.6 we get

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T} N(x,T,E) &= \limsup_{T\to\infty} \frac{1}{T}\left(N(x,T,E\cap\mathcal{S}_{\ge\varepsilon}) + N(x,T,E\cap\mathcal{S}_{<\varepsilon})\right)\\ &\le \lim_{T\to\infty} \frac{1}{T} N(x,T,E\cap\mathcal{S}_{\ge\varepsilon}) + \limsup_{T\to\infty}\frac{1}{T} N(x,T,\mathcal{S}_{<\varepsilon}) \\ &\le \mu_{\mathcal{S}}(E\cap\mathcal{S}_{\ge\varepsilon}) + \delta \le \mu_{\mathcal{S}}(E) + \delta.\end{aligned} \tag{43}$$

Since δ was arbitrary, we get (39). □

The following lemma shows that the extra assumption in Proposition 5.7, namely that $x \notin \Delta_{\mathcal{S}}^{\mathbb{R}}$, is almost harmless.

Lemma 5.8. *Let $\mathcal{S}$ be a μ-reasonable cross-section. Then*

$$\mu_{\mathcal{S}}(\Delta_{\mathcal{S}}) = 0 \quad \textit{and} \quad \mu\left(\Delta_{\mathcal{S}}^{\mathbb{R}}\right) = 0.$$

Proof. By Theorem 4.4(iv) it is enough to show that $\mu_{\mathcal{S}}(\Delta_{\mathcal{S}}) = 0$. For this it is enough to show that for any fixed $\delta > 0$, $\mu_{\mathcal{S}}(\Delta_{\mathcal{S},\delta}) = 0$. Take $0 < \varepsilon_1 < \varepsilon_0$ small enough so that $\mu_{\mathcal{S}}(\mathcal{S}_{\ge\varepsilon_0}) > 0$ and $\mu_{\mathcal{S}}(\mathcal{S}_{<\varepsilon_1}) < \delta$. This is possible by (B) and because $\mathcal{S} = \bigcup_{\varepsilon>0} \mathcal{S}_{\ge\varepsilon}$. Let

$$\mu = \int \nu \, d\Theta(\nu)$$

be the ergodic decomposition of μ; that is, Θ is a probability measure on $\mathcal{P}(X)$, and for Θ-a.e. ν, ν is an $\{a_t\}$-invariant ergodic measure on X. Moreover, Θ-a.e. ν satisfies that $\mathcal{S}_{\ge\varepsilon_0}, \mathcal{S}_{<\varepsilon_1}$ are both ν-JM, and $\mathcal{S}$ is a cross-section for $(X, \nu, \{a_t\})$. We will show that for such ν, for ν-a.e. x, the cross-section measure $\nu_{\mathcal{S}}$ satisfies

$$\lim_{T\to\infty} \frac{1}{T} N(x,T,\mathcal{S}_{<\varepsilon_1}) = \nu_{\mathcal{S}}(\mathcal{S}_{<\varepsilon_1}). \tag{44}$$

This will imply $\nu_{\mathcal{S}}(\Delta_{\mathcal{S},\delta}) = 0$ for such ν, and hence that $\mu_{\mathcal{S}}(\Delta_{\mathcal{S},\delta}) = 0$.

Note that for $E \subset \mathcal{S}$, the ratio $\frac{N(x,T,E)}{N(x,T,\mathcal{S})}$ is an ergodic average for χ_E in the ergodic dynamical system $(\mathcal{S}, \frac{1}{\nu_{\mathcal{S}}(\mathcal{S})}\nu_{\mathcal{S}}, T_{\mathcal{S}})$. Applying the pointwise ergodic theorem to the characteristic functions of the sets $\mathcal{S}_{\ge\varepsilon_0}, \mathcal{S}_{<\varepsilon_1}$ we deduce that there is a set $F \subset \mathcal{S}$ of full ν-measure such that for any $y \in F$,

$$\frac{N(y,T,\mathcal{S}_{\ge\varepsilon_0})}{N(y,T,\mathcal{S})} \to_{T\to\infty} \frac{\nu_{\mathcal{S}}(\mathcal{S}_{\ge\varepsilon_0})}{\nu_{\mathcal{S}}(\mathcal{S})}$$
$$\frac{N(y,T,\mathcal{S}_{<\varepsilon_1})}{N(y,T,\mathcal{S})} \to_{T\to\infty} \frac{\nu_{\mathcal{S}}(\mathcal{S}_{<\varepsilon_1})}{\nu_{\mathcal{S}}(\mathcal{S})},$$

and thus

$$\lim_{T\to\infty} \frac{\frac{1}{T}N(y,T,\mathcal{S}_{<\varepsilon_1})}{\frac{1}{T}N(y,T,\mathcal{S}_{\geq\varepsilon_0})} = \lim_{T\to\infty} \frac{N(y,T,\mathcal{S}_{<\varepsilon_1})}{N(y,T,\mathcal{S}_{\geq\varepsilon_0})} = \frac{\nu_{\mathcal{S}}(\mathcal{S}_{<\varepsilon_1})}{\nu_{\mathcal{S}}(\mathcal{S}_{\geq\varepsilon_0})}. \tag{45}$$

Moreover, replacing F with a smaller set of full ν-measure, we can assume that each $y \in F$ is also (a_t,ν)-generic. Applying Proposition 5.6 for $E = \mathcal{S}_{\geq\varepsilon_0}$ we get that the denominator on the LHS of (45) converges to the denominator on the RHS. This implies (44) for $y \in F$. □

For tempered subsets of $\mathcal{S}$ (see Definition 4.7) we can prove a version of Proposition 5.6 without reference to the problematic set $\Delta_{\mathcal{S}}$.

Proposition 5.9. *Let $\mathcal{S}$ be a μ-reasonable cross-section and let $E \subset \mathcal{S}$ be a tempered subset which is $\mu_{\mathcal{S}}$-JM. If $x \in X$ is (a_t,μ)-generic then (39) holds.*

For the proof we will need the following lemma which provides the substitute for the assumption $x \notin \Delta_{\mathcal{S}}^{\mathbb{R}}$.

Lemma 5.10. *Let $\mathcal{S}$ be a μ-reasonable cross-section for (X,a_t,μ) and let $F \subset \mathcal{S}$ be a $\mu_{\mathcal{S}}$-JM set which is M-tempered. Then for any x which is (a_t,μ)-generic we have*

$$\limsup_{T\to\infty} \frac{1}{T}N(x,T,F) \leq M\mu\left(F^{(0,1)}\right). \tag{46}$$

Proof. Let $I = (0,1)$. By Lemma 5.5, the set F^I is μ-JM and hence $\frac{1}{T}\int_0^T \chi_{F^I}(a_t x)dt \to_{T\to\infty} \mu\left(F^I\right)$. Thus (46) will follow from

$$N(x,T,F) \leq M \cdot m\left(\left\{t \in [0,T] : a_t x \in F^I\right\}\right) + M, \tag{47}$$

for each $T > 0$ (where m is the Lebesgue measure on $\mathbb{R}$). Let $k = N(x,T,F)$ and let $t_1 < \cdots < t_k$ be an ordering of $\{t \in [0,T] : a_t x \in F\}$. Then, the M-temperedness implies that there is a subset $J \subset \{1,\dots,k\}$ of cardinality at least $\frac{k}{M}$, such that for any $j_1 < j_2$ in J one has $t_{j_2} - t_{j_1} \geq 1$. For each $t_j \in J$ except perhaps the largest, and for any $t \in (t_j, t_j+1)$, we have $t \leq T$ and $a_t x \in F^I$. This implies that $\left\{t \in [0,T] : a_t x \in F^I\right\}$ contains at least $\left\lfloor \frac{k}{M} \right\rfloor - 1$ disjoint intervals of length 1, which implies (47). □

Proof of Proposition 5.9. The proof is very similar to that of Proposition 5.7. The inequality $\liminf_{T\to\infty} \frac{1}{T}N(x,T,E) \geq \mu_{\mathcal{S}}(E)$ follows as in (41). On the other hand, similarly to (43), using Proposition 5.6, for any sufficiently small $\varepsilon > 0$ we have

$$\limsup_{T\to\infty} \frac{1}{T}N(x,T,E) = \mu_{\mathcal{S}}(E \cap \mathcal{S}_{\geq\varepsilon}) + \limsup_{T\to\infty} \frac{1}{T}N(x,T,E_\varepsilon),$$

where $E_\varepsilon \stackrel{\text{def}}{=} E \cap \mathcal{S}_{<\varepsilon}$. The sets E_ε are M-tempered and so by Lemma 5.10, $\limsup_{T\to\infty} \frac{1}{T}N(x,T,E) \leq \mu_{\mathcal{S}}(E) + M\,\mu\left(E_\varepsilon^{(0,1)}\right)$. But since $\bigcap_{\varepsilon>0} E_\varepsilon^{(0,1)} = \varnothing$, we have

$$\mu\left(E_\varepsilon^{(0,1)}\right) \to_{\varepsilon\to 0} 0,$$

and hence $\limsup_{T\to\infty} \frac{1}{T} N(x,T,E) \leq \mu_{\mathcal{S}}(E)$. Putting these inequalities together gives (39). □

We summarize the results of this section in the following theorem.

Theorem 5.11. *Let $\mathcal{S}$ be a μ-reasonable cross-section, and let $\mathcal{S}' \subset \mathcal{S}$ be a subset which is lcsc, $\mu_{\mathcal{S}}$-JM and satisfies $\mu_{\mathcal{S}}(\mathcal{S}') > 0$. Suppose $x \in X$ is (a_t, μ)-generic. Assume in addition that one of the following also hold:*

(i) $x \notin \Delta^{\mathbb{R}}_{\mathcal{S}}$;
(ii) $\mathcal{S}'$ is tempered.

Then x is $(a_t, \mu_{\mathcal{S}}|_{\mathcal{S}'})$-generic.

Proof. Let x be (a_t, μ)-generic and assume (i). Then Proposition 5.7 and Lemma 4.9 imply that x is $(a_t, \mu_{\mathcal{S}})$-generic. Showing that x is $(a_t, \mu_{\mathcal{S}}|_{\mathcal{S}'})$-generic is equivalent to showing that for any $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}'$,

$$\lim_{T\to\infty} \frac{N(x,T,E)}{N(x,T,\mathcal{S}')} = \frac{\mu_{\mathcal{S}}(E)}{\mu_{\mathcal{S}}(\mathcal{S}')},$$

which readily follows using (39) in both numerator and denominator.

Now assume (ii). Since $\mathcal{S}'$ is tempered, so is $E \subset \mathcal{S}'$, and by Proposition 5.9, we once again have (39) for both the numerator and the denominator. □

6. Lifting reasonable cross-sections

The goal of this section is to show that our conditions on cross-sections behave well with respect to factor maps.

Let $\{a_t\} \curvearrowright (X, \mathcal{B}_X, \mu)$ and $\{a_t\} \curvearrowright (\widetilde{X}, \mathcal{B}_{\widetilde{X}}, \tilde{\mu})$ be two actions as in the lcsc setup, and assume that $\mu, \tilde{\mu}$ are both probability measures. We say that $\pi : \widetilde{X} \to X$ is a *continuous factor map* if it is continuous and satisfies $\mu = \pi_* \tilde{\mu}$ and $a_t \circ \pi = \pi \circ a_t$ for any $t \in \mathbb{R}$.

Proposition 6.1. *In the above setup, let $\pi : \widetilde{X} \to X$ be a continuous factor map. Assume furthermore that π is open. Let $\mathcal{S} \subset X$ be a μ-reasonable cross-section and let $\widetilde{\mathcal{S}} \stackrel{\mathrm{def}}{=} \pi^{-1}(\mathcal{S})$. Then:*

(a) $\widetilde{\mathcal{S}}$ is a $\tilde{\mu}$-cross-section.
(b) $\pi_* \tilde{\mu}_{\widetilde{\mathcal{S}}} = \mu_{\mathcal{S}}$.
(c) $\tilde{\mu}_{\widetilde{\mathcal{S}}}$ is finite.
(d) $\widetilde{\mathcal{S}}$ is $\tilde{\mu}$-reasonable.
(e) For any $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}$, $\pi^{-1}(E) \subset \widetilde{\mathcal{S}}$ is $\mu_{\widetilde{\mathcal{S}}}$-JM.
(f) If $E \subset \mathcal{S}$ is M-tempered then $\pi^{-1}(E)$ is M-tempered.

Proof. Since $a_t \circ \pi = \pi \circ a_t$, we have

$$\left(\pi^{-1}(E)\right)^I = \pi^{-1}\left(E^I\right) \tag{48}$$

for the thickened sets as in (34). In addition, for $x = \pi(\tilde{x})$ we have

$$\left\{t : a_t \tilde{x} \in \widetilde{\mathcal{S}}\right\} = \{t : a_t x \in \mathcal{S}\}, \tag{49}$$

and hence

$$\mathcal{Y}_{\tilde{x}} = \mathcal{Y}_x, \quad \tau_{\mathcal{S}}(x) = \tau_{\widetilde{\mathcal{S}}}(\tilde{x}), \quad \text{and} \quad \widetilde{\mathcal{S}}_{\geq\varepsilon} = \pi^{-1}(\mathcal{S}_{\geq\varepsilon}). \tag{50}$$

We prove (a): since $\mathcal{S}$ is a μ-cross-section, there exists an $\{a_t\}$-invariant set $X_0 \in \mathcal{B}_X$ so that $(X_0, \mathcal{B}_{X_0})$ is a standard Borel space such that $\mu(X_0) = 1$ and $\mathcal{S} \cap X_0$ is a Borel cross-section (see Definitions 4.1, 4.2). It follows that if $\widetilde{X}_0 \overset{\text{def}}{=} \pi^{-1}(X_0)$, then $\widetilde{X}_0 \in \mathcal{B}_{\widetilde{X}}$ is an $\{a_t\}$-invariant set with $\tilde{\mu}(\widetilde{X}_0) = 1$. Finally, it follows from (50) that $\widetilde{\mathcal{S}} \cap \widetilde{X}_0$ is a Borel cross-section according to Definition 4.1. This finishes the verification of Definition 4.2 and proves (a).

Item (b) follows from (35), (48), and the assumption that $\pi_*\tilde{\mu} = \mu$.

Item (c) follows from (b) since the μ-reasonability of $\mathcal{S}$ implies that $\mu_{\mathcal{S}}$ is finite.

Since $\pi : \widetilde{\mathcal{S}} \to \mathcal{S}$ is continuous, for any $E \subset \mathcal{S}$,

$$\partial_{\widetilde{\mathcal{S}}}(\pi^{-1}(E)) \subset \pi^{-1}(\partial_{\mathcal{S}} E).$$

It follows that if $E \subset \mathcal{S}$ is $\mu_{\mathcal{S}}$-JM then

$$\tilde{\mu}_{\widetilde{\mathcal{S}}}(\partial_{\widetilde{\mathcal{S}}}(\pi^{-1}(E))) \leq \tilde{\mu}_{\widetilde{\mathcal{S}}}(\pi^{-1}(\partial_{\mathcal{S}} E)) = \mu_{\mathcal{S}}(\partial_{\mathcal{S}} E) = 0,$$

and (e) follows.

Item (f) follows from the equivariance of π.

It remains to prove (d). We verify Definition 5.3. We note that $\widetilde{\mathcal{S}}$ satisfies (A), (B) and (C). Indeed, (A) is part of our assumptions, (B) follows from (b) and (C) follows easily from the fact that $\widetilde{X}$ and $\mathcal{S}$ are lcsc and π is continuous. We now check the further conditions of Definition 5.3. Condition 5.3(1) follows from (e) and (50). For the technical condition 5.3(2), let $\mathcal{U} \subset \mathcal{S}$ be the subset appearing in the definition for $\mathcal{S}$, and let $\widetilde{\mathcal{U}} = \pi^{-1}(\mathcal{U})$. First,

$$\mathrm{cl}_{\widetilde{X}}(\widetilde{\mathcal{S}}) \smallsetminus \widetilde{\mathcal{U}} \subset \pi^{-1}\left(\mathrm{cl}_X(\mathcal{S}) \smallsetminus \mathcal{U}\right)$$

and since $\pi_*\tilde{\mu} = \mu$,

$$\tilde{\mu}\left(\mathrm{cl}_{\widetilde{X}}(\widetilde{\mathcal{S}}) \smallsetminus \widetilde{\mathcal{U}}\right) \leq \mu\left(\mathrm{cl}_X(\mathcal{S}) \smallsetminus \mathcal{U}\right) = 0.$$

In order to verify that the map $(0,1) \times \widetilde{\mathcal{U}} \to \widetilde{X}$, $(t, y) \mapsto a_t y$ is open, note that $\widetilde{\mathcal{U}}^{(0,1)} = \pi^{-1}(\mathcal{U}^{(0,1)})$ is open in $\widetilde{X}$, and thus it is enough to show that the map $(0,1) \times \widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}}^{(0,1)}$ is open. Consider the commutative diagram

$$\begin{array}{ccc} (0,1) \times \widetilde{\mathcal{U}} & \xrightarrow{(t,\tilde{x}) \mapsto a_t \tilde{x}} & \widetilde{\mathcal{U}}^{(0,1)} \\ \Big\downarrow{\scriptstyle (\mathrm{id},\pi)} & & \Big\downarrow{\scriptstyle \pi} \\ (0,1) \times \mathcal{U} & \xrightarrow[(t,x) \mapsto a_t x]{} & \mathcal{U}^{(0,1)} \end{array}$$

in which the vertical maps are open by assumption on π and the bottom map is open by the reasonability of $\mathcal{S}$. It follows that the top horizontal map is open as well. $\square$

6.1. **Continuity of the cross-section measure construction.** In this subsection we prove the following continuity property of the map $\mu \mapsto \mu_{\mathcal{S}}$:

Proposition 6.2. *Let $X, \mathcal{B}, \{a_t\}$ be as in the lcsc setup, and let $\mu_k, \mu \in \mathcal{P}(X)$. Suppose μ and each μ_k are $\{a_t\}$-invariant, and $\mu_k \to_{k\to\infty} \mu$ in the strict topology (or equivalently, since they are in $\mathcal{P}(X)$, in the weak-* topology). Also assume that $\mathcal{S}', \mathcal{S}$ satisfy the following, for $\nu = \mu$ and for $\nu = \mu_k$, for any $k \in \mathbb{N}$:*

(1) $\mathcal{S}$ is a ν-reasonable cross-section.
(2) $\mathcal{S}' \subset \mathcal{S}$ is $\nu_{\mathcal{S}}$-JM.
(3) $\mathcal{S}'$ is tempered.

Then $(\mu_k)_{\mathcal{S}}|_{\mathcal{S}'} \to_{k\to\infty} \mu_{\mathcal{S}}|_{\mathcal{S}'}$, with respect to the strict topology.

Proof. By Lemma 4.9 we need to show that for any $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}'$ we have

$$(\mu_k)_{\mathcal{S}}(E) \to_{k\to\infty} \mu_{\mathcal{S}}(E). \tag{51}$$

Take $E \subset \mathcal{S}'$ a $\mu_{\mathcal{S}}$-JM set. For any $\varepsilon > 0$ we can decompose $E = (E \cap \mathcal{S}_{\geq\varepsilon}) \cup (E \cap \mathcal{S}_{<\varepsilon})$ and hence for $\nu \in \{\mu, \mu_k\}$ we have

$$\nu_{\mathcal{S}}(E) = \nu_{\mathcal{S}}(E \cap \mathcal{S}_{\geq\varepsilon}) + \nu_{\mathcal{S}}(E \cap \mathcal{S}_{<\varepsilon}).$$

By Lemma 5.5 we have that $(E \cap \mathcal{S}_{\geq\varepsilon})^{(0,\varepsilon)}$ is μ-JM, and so by Theorem 4.4(i),

$$\begin{aligned}\lim_{k\to\infty} (\mu_k)_{\mathcal{S}}(E \cap \mathcal{S}_{\geq\varepsilon}) &= \lim_{k\to\infty} \frac{1}{\varepsilon}\, \mu_k \left((E \cap \mathcal{S}_{\geq\varepsilon})^{(0,\varepsilon)} \right) \\ =& \frac{1}{\varepsilon}\, \mu \left((E \cap \mathcal{S}_{\geq\varepsilon})^{(0,\varepsilon)} \right) = \mu_{\mathcal{S}} \left(E \cap \mathcal{S}_{\geq\varepsilon} \right).\end{aligned}$$

Moreover $\mu_{\mathcal{S}}(E \cap \mathcal{S}_{<\varepsilon}) \to_{\varepsilon\to 0} 0$, because $\mu_{\mathcal{S}}$ is a finite measure, and hence

$$\liminf_{k\to\infty} (\mu_k)_{\mathcal{S}}(E) \geq \mu_{\mathcal{S}}(E),$$

and (51) will follow once we establish that

$$\limsup_{k\to\infty} (\mu_k)_{\mathcal{S}}(E \cap \mathcal{S}_{<\varepsilon}) \to_{\varepsilon\to 0} 0. \tag{52}$$

Clearly, it is enough to show (52) for $E = \mathcal{S}'$. Assume first that each μ_k is ergodic. Since $\mathcal{S}'$ is μ_k-JM, there is $x_k \in X$ which is (a_t, μ_k)-generic. Choose M so that $\mathcal{S}'$ is M-tempered, then by (46) we have

$$(\mu_k)_{\mathcal{S}}(\mathcal{S}' \cap \mathcal{S}_{<\varepsilon}) = \lim_{T\to\infty} \frac{1}{T} N(x_k, T, \mathcal{S}' \cap \mathcal{S}_{<\varepsilon}) \leq M \mu_k \left((\mathcal{S}' \cap \mathcal{S}_{<\varepsilon})^{(0,1)} \right). \tag{53}$$

Using ergodic decomposition and Theorem 4.4(v), we see that (53) also holds without assuming that μ_k is ergodic. This gives

$$\limsup_{k\to\infty} (\mu_k)_{\mathcal{S}}(\mathcal{S}' \cap \mathcal{S}_{<\varepsilon}) \leq M \mu \left((\mathcal{S}' \cap \mathcal{S}_{<\varepsilon})^{(0,1)} \right),$$

and taking the limit as $\varepsilon \to 0$ we obtain (52). ∎

7. Homogeneous spaces and homogeneous measures

7.1. **Guide to the rest of the paper.** In order to obtain our results, we will apply the theory developed in §§4-6 in order to get equidistributed sequences on cross-sections in various spaces, and then interpret them as being related to best approximations and ε-approximations. For our results we will have three distinctions which give rise to $8 = 2^3$ cases. The first distinction is between best approximations and ε-approximations (compare Theorems 1.1 and 1.2), the second distinction is between Lebesgue a.e. vector and vectors with entries in a totally real field (compare both of the above, with Theorem 1.5) and the third is between results on equidistribution in the real spaces $\mathscr{E}_n \times \mathbb{R}^d$ and in the larger locally compact space $\mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$ (compare the first two coordinates of (18) with all three of them). In all cases we will specify the dynamical system, define our cross-section, give an expression for the cross-section measures, and check that the axioms used in §§4-6 are satisfied. Some of these verifications will be routine but others will require detailed argumentation.

Since the reader may not be equally interested in all eight cases, we preface this discussion with a short guide. We will first consider the *real homogeneous space* $\mathscr{X}_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, where the action is given by left multiplication by the one-parameter group $\{a_t\}$ in (16). Understanding the space $\mathscr{X}_n$ will only give information about the real components $\mathscr{E}_n \times \mathbb{R}^d$ in (18). On $\mathscr{X}_n$ we will define a cross-section $\mathcal{S}_{r_0}$ (defined below in (78)). We will consider two kinds of measures μ on $\mathscr{X}_n$. The first is the Haar-Siegel measure $m_{\mathscr{X}_n}$. This measure will give information about the properties of Lebesgue a.e. $\theta \in \mathbb{R}^d$. When discussing this measure we will say that we are in *Case I*. In the second case, related to approximation in totally real fields, which we will refer to as *Case II*, we will have a homogeneous measure $m_{\vec{\alpha}}$ (see Proposition 7.5), where by *homogeneous* we mean that there is a closed subgroup $L \subset \mathrm{SL}_n(\mathbb{R})$, such that $m_{\vec{\alpha}}$ is L-invariant and $\mathrm{supp}\,(m_{\vec{\alpha}})$ is a closed L-orbit. In our case L is a conjugate of the diagonal group A, where the conjugating matrix will depend on the algebraic vector $\vec{\alpha}$, and $\mathrm{supp}\,(m_{\vec{\alpha}})$ is compact. We will prove that $\mathcal{S}_{r_0}$ is μ-reasonable in both Case I and Case II (see §8.3 and §8.4 respectively).

In order to derive information about best approximations, we will work with a subset $\mathcal{B} \subset \mathcal{S}_{r_0}$, and for ε-approximations, we will work with a subset $\mathcal{S}_\varepsilon \subset \mathcal{S}_{r_0}$. We will show (see §9.1, §9.2) that these sets are $\mu_{\mathcal{S}_{r_0}}$-JM in both cases. As remarked in the introduction, $\mathcal{B}$ will be a tempered subset (see Proposition 9.8), but $\mathcal{S}_\varepsilon$ will not be.

In order to understand all three components in (18), we will consider the *adelic homogeneous space* $\mathscr{X}_n^{\mathbb{A}} \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{A})/\mathrm{SL}_n(\mathbb{Q})$, and the factor map $\pi : \mathscr{X}_n^{\mathbb{A}} \to \mathscr{X}_n$. These will be defined in §7.4. The group $\{a_t\}$ defined in (16) is contained in $\mathrm{SL}_n(\mathbb{R})$ and thus in $\mathrm{SL}_n(\mathbb{A})$, and hence acts on $\mathscr{X}_n^{\mathbb{A}}$, and the map π is $\{a_t\}$-equivariant. We will lift $\mathcal{S}_{r_0}$ to a cross-section $\widetilde{\mathcal{S}_{r_0}} =$

$\pi^{-1}(\mathcal{S}_{r_0})$ in $\mathscr{X}_n^{\mathbb{A}}$, and define relevant a_t-invariant measures μ on $\mathscr{X}_n^{\mathbb{A}}$ which descend under π to the correct measure on $\mathscr{X}_n$ according to the case at hand. Namely, in Case I, the measure $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$ is the unique $\mathrm{SL}_n(\mathbb{A})$-invariant probability measure on $\mathscr{X}_n^{\mathbb{A}}$, and in Case II we have will take measures $\mu = \widetilde{m}_{\vec{\alpha}} = m_{\widetilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}}}$ corresponding to $\vec{\alpha}$, which are homogeneous measures supported on a compact adelic torus-orbit. These measures satisfy $\pi_* m_{\mathscr{X}_n^{\mathbb{A}}} = m_{\mathscr{X}_n}$ and $\pi_*\widetilde{m}_{\vec{\alpha}} = m_{\vec{\alpha}}$. Using the results of §6, we will obtain that the lifted cross-section $\widetilde{\mathcal{S}}_{r_0} \overset{\mathrm{def}}{=} \pi^{-1}(\mathcal{S}_{r_0})$ is $\widetilde{\mu}$-reasonable in both cases. Similarly we will obtain that the lifted subsets $\widetilde{\mathcal{B}} \overset{\mathrm{def}}{=} \pi^{-1}(\mathcal{B})$, $\widetilde{\mathcal{S}}_{\varepsilon} \overset{\mathrm{def}}{=} \pi^{-1}(\mathcal{S}_{\varepsilon})$ are $\widetilde{\mu}$-JM. Throughout the discussion we will take care to obtain explicit description of the maps and measures that arise.

In the subsection below we will introduce the spaces and measures (X, μ), and in the subsequent sections we will introduce the cross-sections and their JM-subsets.

7.2. **The real homogeneous space, Case I.** We will work with the space of lattices $\mathscr{X}_n \overset{\mathrm{def}}{=} \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, and the Haar-Siegel measure $m_{\mathscr{X}_n}$. Let $\mathrm{SL}_n^{(\pm)}(\mathbb{R})$ denote the $n \times n$ real matrices of determinant ± 1. The following simple observation will be useful.

Proposition 7.1. *$\mathscr{X}_n$ is isomorphic to the quotient $\mathrm{SL}_n^{(\pm)}(\mathbb{R})/\mathrm{SL}_n^{(\pm)}(\mathbb{Z})$, via a map which is $\mathrm{SL}_n(\mathbb{R})$-equivariant.*

Proof. Let $\tau : \mathrm{SL}_n(\mathbb{R}) \hookrightarrow \mathrm{SL}_n^{(\pm)}(\mathbb{R})$ be the embedding. Then $\tau(\mathrm{SL}_n(\mathbb{Z})) \subset \mathrm{SL}_n^{(\pm)}(\mathbb{Z})$ and hence τ induces an $\mathrm{SL}_n(\mathbb{R})$-equivariant map $\bar{\tau} : \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n^{(\pm)}(\mathbb{R})/\mathrm{SL}_n^{(\pm)}(\mathbb{Z})$. We leave it to the reader to verify that $\bar{\tau}$ is a bijection. □

Note that elements of $\mathrm{SL}_n^{(\pm)}(\mathbb{R})$ act on $\mathscr{X}_n$ via their action on $\mathbb{R}^n$; in terms of the isomorphism given in Proposition 7.1, the left-action of $\mathrm{SL}_n^{(\pm)}(\mathbb{R})$ by matrix multiplication on lattices, is given by left multiplication on cosets.

7.2.1. *Contracting horospherical group.* Let G be an lcsc group, $\{a_t\} \subset G$ a one-parameter subgroup, and $\Gamma < G$ a lattice. Let $\mathscr{X} = G/\Gamma$ and let μ be an $\{a_t\}$-invariant measure on $\mathscr{X}$. The group

$$H^- \overset{\mathrm{def}}{=} \{g \in G : a_t g a_{-t} \to_{t\to+\infty} e\}, \tag{54}$$

is known as the *contracting horospherical subgroup* of G, corresponding to $\{a_t\}$. Also we denote the centralizer of $\{a_t\}$ by H^0, that is,

$$H^0 \overset{\mathrm{def}}{=} \{g \in G : \forall t, a_t g = g a_t\}. \tag{55}$$

We will need the following well-known fact. We leave the proof to the reader.

Proposition 7.2. *The product $H^{\leq} = H^0 H^-$ is a group. If x_0 is (a_t, μ)-generic and $h^0 \in H^0$, $h^- \in H^-$, then $h^0 h^- x_0$ is $(a_t, h^0_*\mu)$-generic.*

7.3. **The real homogeneous space, Case II.** Let $d \geq 2$, and let $\mathbb{K} = \mathrm{span}_{\mathbb{Q}}\{\alpha_1, \ldots, \alpha_d, 1\}$ be a totally real field of degree n over $\mathbb{Q}$. Let $\sigma_1, \ldots, \sigma_n : \mathbb{K} \to \mathbb{R}$ denote the distinct field embeddings. Departing slightly from common conventions, we let $\sigma_n = \mathrm{Id}$. Let

$$\vec{\alpha} \stackrel{\mathrm{def}}{=} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \in \mathbb{R}^d. \tag{56}$$

We point out a slight abuse of notation: what we now denote by $\vec{\alpha}$ is a column vector whereas in the introduction the same notation was used for a row vector with the same entries. In this section it will be more convenient to work with column vectors, and this should cause no confusion.

For $\beta \in \mathbb{K}$, let $\boldsymbol{\sigma} : \mathbb{K} \to \mathbb{R}^n$ be the $\mathbb{Q}$-linear map defined by

$$\boldsymbol{\sigma}(\beta) \stackrel{\mathrm{def}}{=} \begin{pmatrix} \sigma_1(\beta) \\ \vdots \\ \sigma_{n-1}(\beta) \\ \beta \end{pmatrix} \in \mathbb{R}^n. \tag{57}$$

We will refer to $\boldsymbol{\sigma}(\beta)$ as the *geometric embedding* of β. A lattice $\Lambda \in \mathscr{X}_n$ is called *of type* $(\boldsymbol{\sigma}, \mathbb{K})$ if there exist a basis $\beta_1, \ldots, \beta_n$ of $\mathbb{K}$ over $\mathbb{Q}$ such that Λ is homothetic to the lattice

$$\{\boldsymbol{\sigma}(\beta) : \beta \in \mathrm{span}_{\mathbb{Z}}\{\beta_1, \ldots, \beta_n\}\} = \begin{pmatrix} | & \cdots & | \\ \boldsymbol{\sigma}(\beta_1) & \cdots & \boldsymbol{\sigma}(\beta_n) \\ | & \cdots & | \end{pmatrix} \mathbb{Z}^n. \tag{58}$$

Let $A \subset \mathrm{SL}_n(\mathbb{R})$ denote the group of diagonal matrices with positive diagonal entries. We say that an orbit $A\Lambda \subset \mathscr{X}_n$ is *an orbit of type* $(\boldsymbol{\sigma}, \mathbb{K})$ if there is $x^*_{\vec{\alpha}}$ of type $(\boldsymbol{\sigma}, \mathbb{K})$ such that $A\Lambda = Ax^*_{\vec{\alpha}}$.

Lemma 7.3. *Orbits of type* $(\boldsymbol{\sigma}, \mathbb{K})$ *are compact, and all compact* A*-orbits are of type* $(\boldsymbol{\sigma}, \mathbb{K})$*, for some totally real number field* $\mathbb{K}$ *and some* $\boldsymbol{\sigma}$*. If* $x^*_{\vec{\alpha}}, \Lambda \in \mathscr{X}_n$ *are lattices such that* $x^*_{\vec{\alpha}}$ *is of type* $(\boldsymbol{\sigma}, \mathbb{K})$*, and* $\mathrm{Stab}_A(\Lambda) = \mathrm{Stab}_A(x^*_{\vec{\alpha}})$*, then* $A\Lambda$ *is an orbit of type* $(\boldsymbol{\sigma}, \mathbb{K})$ *(for the same field* $\mathbb{K}$*).*

Proof. For the first two assertions see [LW01, §6]. For the third assertion, let $\Delta \stackrel{\mathrm{def}}{=} \mathrm{Stab}_A(x^*_{\vec{\alpha}})$. It is well-known that the $\mathbb{Q}$-linear span of Δ, in the linear space of $n \times n$ real matrices, satisfies

$$\widetilde{\mathbb{K}} := \mathrm{span}_{\mathbb{Q}}(\Delta) = \{\mathrm{diag}\,(\sigma_1(\alpha), \ldots, \sigma_n(\alpha)) : \alpha \in \mathbb{K}\}.$$

This follows from the fact that any finite index subgroup of the group of units in the ring of integers of $\mathbb{K}$ spans $\mathbb{K}$. For $\alpha \in \mathbb{K}$ let us denote $\tilde{\alpha} := \mathrm{diag}\,(\sigma_1(\alpha), \ldots, \sigma_n(\alpha))$. Fix a vector $\mathbf{w} \in \Lambda$ all of whose coordinates are strictly positive. Consider the map

$$\iota : \mathbb{K} \to \mathbb{R}^n, \quad \iota(\alpha) \stackrel{\mathrm{def}}{=} \tilde{\alpha}\mathbf{w}.$$

Note that this is an injective $\mathbb{Q}$-linear map. We claim that its image is the $\mathbb{Q}$-span of Λ, which we denote by $\Lambda_{\mathbb{Q}}$. Indeed, any $\tilde{\alpha} \in \widetilde{\mathbb{K}}$ is a linear combination over $\mathbb{Q}$ of elements from Δ, and this implies that $\iota(\mathbb{K}) \subset \Lambda_{\mathbb{Q}}$. Since $\mathbb{K}$ has dimension n over $\mathbb{Q}$ we conclude that ι is a linear isomorphism between $\mathbb{K}$ and $\Lambda_{\mathbb{Q}}$.

Now let $M = \iota^{-1}(\Lambda)$ and let $\beta_1, \ldots, \beta_n$ be a basis of M as a $\mathbb{Z}$-module. It is clear from the construction that Λ is obtained from $\boldsymbol{\sigma}(M)$ by applying the diagonal matrix whose diagonal entries are the coordinates of $\mathbf{w}$. This shows that $A\Lambda = A\Lambda'$, where Λ' is the lattice of type $(\boldsymbol{\sigma}, \mathbb{K})$ obtained from $\boldsymbol{\sigma}(M)$ by normalizing its covolume to be one. $\square$

Let $\vec{\alpha}$ be as in (56). We set

$$g_{\vec{\alpha}} \stackrel{\text{def}}{=} \begin{pmatrix} | & \cdots & | & | \\ \boldsymbol{\sigma}(\alpha_1) & \cdots & \boldsymbol{\sigma}(\alpha_d) & \boldsymbol{\sigma}(1) \\ | & \cdots & | & | \end{pmatrix}, \tag{59}$$

so the bottom row of $g_{\vec{\alpha}}$ is $(\vec{\alpha}^{\mathbf{t}}, 1)$, where $\vec{\alpha}^{\mathbf{t}}$ denotes the transpose of $\vec{\alpha}$. It is well-known that the matrix $g_{\vec{\alpha}}$ is invertible. Let

$$x_{\vec{\alpha}} \stackrel{\text{def}}{=} |\det g_{\vec{\alpha}}|^{-1/n} g_{\vec{\alpha}} \mathbb{Z}^n \in \mathscr{X}_n. \tag{60}$$

For $M \in \mathrm{GL}_n(\mathbb{R})$ we denote by $M^* = (M^{-1})^{\mathbf{t}}$ the inverse of the transpose of M. For a lattice $x = g\mathbb{Z}^n \in \mathscr{X}_n$, the dual lattice is defined by $x^* \stackrel{\text{def}}{=} g^*\mathbb{Z}^n$. We then have that

$$x^*_{\vec{\alpha}} = c g^*_{\vec{\alpha}} \mathbb{Z}^n, \tag{61}$$

where $c > 0$ is chosen so that $x^*_{\vec{\alpha}} \in \mathscr{X}_n$.

Note that the one-parameter group $\{a_t\}$ defined in (16) is contained in A. We will need the following well-known fact:

Proposition 7.4. *The orbit $Ax^*_{\vec{\alpha}}$ is of type $(\boldsymbol{\sigma}, \mathbb{K})$, and $\{a_t\}$ acts uniquely ergodically on $Ax^*_{\vec{\alpha}}$. For any $\Lambda \in Ax^*_{\vec{\alpha}}$, and any $v \in \Lambda \smallsetminus \{0\}$, all the coordinates of v are nonzero.*

Proof. Let $c' \stackrel{\text{def}}{=} |\det(g_{\vec{\alpha}})|^{-1/n} > 0$, so that $x_{\vec{\alpha}} \stackrel{\text{def}}{=} c' g_{\vec{\alpha}} \mathbb{Z}^n \in \mathscr{X}_n$. Then by Lemma 7.3, $Ax_{\vec{\alpha}} \subset \mathscr{X}_n$ is a compact orbit. The map $M \mapsto M^*$ is a continuous group automorphism of $\mathrm{SL}_n(\mathbb{R})$ which maps the groups A and $\mathrm{SL}_n(\mathbb{Z})$ to themselves. Therefore it induces an automorphism Ψ of $\mathscr{X}_n$, such that $x^*_{\vec{\alpha}} = \Psi(x_{\vec{\alpha}})$. Since $a^* = a^{-1}$ for $a \in A$, the stabilizers satisfy $\mathrm{Stab}_A(x^*_{\vec{\alpha}}) = \mathrm{Stab}_A(x_{\vec{\alpha}})$, and Lemma 7.3 implies that $Ax^*_{\vec{\alpha}}$ is an orbit of type $(\boldsymbol{\sigma}, \mathbb{K})$.

We now prove the second assertion. The group A is isomorphic (as a Lie group) to $\mathbb{R}^d$, and we can realize this group isomorphism explicitly using the exponential map $\mathrm{Lie}(A) \to A$. The orbit $Ax^*_{\vec{\alpha}}$ is isomorphic to $\mathbb{T} = \mathbb{R}^d/\Delta$ for some lattice Δ in $\mathbb{R}^d$, and the action $a_t \curvearrowright Ax^*_{\vec{\alpha}}$ is mapped by this isomorphism to a straightline flow on $\mathbb{T}$, that is, a flow of the form

$$tP(x) = P\left(x + t\vec{\ell}\right), \quad \text{where } \vec{\ell} \in \mathbb{R}^d \smallsetminus \{0\} \text{ and } P : \mathbb{R}^d \to \mathbb{T}$$

is the projection. Recall that such a straightline flow is uniquely ergodic unless the straightline orbit of $0 \in \mathbb{T}$ is contained in a proper subtorus of $\mathbb{T}$. Let $\mathbb{L}$ be the Galois closure of $\mathbb{K}/\mathbb{Q}$ and let $\mathscr{G} = \mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ denote the corresponding Galois group. Then $\mathscr{G}$ acts transitively on the field embeddings $\sigma_1, \dots, \sigma_n$ by post-composition. This gives an identification of $\mathscr{G}$ with a transitive subgroup of the group S_n of permutations of $\{1, \dots, n\}$. In turn, this allows $\mathscr{G}$ to act on the group of diagonal matrices by permuting the coordinates on the diagonal. As the discussion in [LW01, §6] shows (in particular, from [LW01, Step 6.1]), for $N \subset \mathbb{R}^d \cong \mathrm{Lie}(A)$, $P(N)$ is a compact subtorus of $\mathbb{T}$ if and only if $A_0 \stackrel{\mathrm{def}}{=} \exp(N) \subset A$ is $\mathscr{G}$-invariant. Thus if $\{a_t\} \subset A_0$, then A_0 contains any group obtained from $\{a_t\}$ by acting on it with $\mathscr{G}$. Because of the transitivity mentioned above, A_0 must contain all the subgroups

$$\left\{a_t^i : t \in \mathbb{R}\right\}, \quad \text{where } a_t^i = \mathrm{diag}\left(e^t, \dots, e^t, \underbrace{e^{-dt}}_{i\text{th position}}, e^t, \dots, e^t\right).$$

Since the groups $\left\{a_t^i\right\}$ generate A, we must have $A_0 = A$, and this establishes unique ergodicity.

For the third assertion, note that if $\boldsymbol{\sigma}(\beta)$ has one of its coordinates equal to zero, then $\beta = 0$. This observation, along with (58), implies the third assertion for $\Lambda = x^*_{\vec{\alpha}}$. The statement now follows for general $\Lambda = a x^*_{\vec{\alpha}}$ by the definition of the A-action. □

We will denote the A-invariant probability measure on $Ax^*_{\vec{\alpha}}$ by $m_{Ax^*_{\vec{\alpha}}}$.

Proposition 7.5. *Let*

$$\Lambda_{\vec{\alpha}} \stackrel{\mathrm{def}}{=} \begin{pmatrix} I_d & -\vec{\alpha} \\ \mathbf{0}^{\mathbf{t}} & 1 \end{pmatrix} \mathbb{Z}^n \in \mathscr{X}_n, \tag{62}$$

where $\mathbf{0} \in \mathbb{R}^d$ is the zero (column) vector, and let

$$B_{\vec{\alpha}} \stackrel{\mathrm{def}}{=} (b_{ij})_{i,j=1,\dots,d} \quad \textit{where} \quad b_{ij} \stackrel{\mathrm{def}}{=} \sigma_j(\alpha_i) - \alpha_i.$$

Then $B_{\vec{\alpha}}$ is invertible, and for $c_1 \stackrel{\mathrm{def}}{=} \left|\det\left(B_{\vec{\alpha}}\right)\right|^{-1/n}$, the matrix

$$\bar{h}_{\vec{\alpha}} \stackrel{\mathrm{def}}{=} c_1 \begin{pmatrix} B_{\vec{\alpha}} & \mathbf{0} \\ \mathbf{0}^{\mathbf{t}} & 1 \end{pmatrix} \in \mathrm{SL}_n^{(\pm)}(\mathbb{R}) \tag{63}$$

satisfies that the trajectory $\{a_t \Lambda_{\vec{\alpha}} : t > 0\}$ is generic for the measure

$$m_{\vec{\alpha}} \stackrel{\mathrm{def}}{=} (\bar{h}_{\vec{\alpha}})_* m_{Ax^*_{\vec{\alpha}}}. \tag{64}$$

Proof. Note that

$$B^{\mathbf{t}}_{\vec{\alpha}} = \left(\sigma_i(\alpha_j) - \alpha_j\right)_{i,j=1,\dots,d}.$$

Also note that the entries of the right-most column of (59) are all equal to 1, and the first d entries of the bottom row in (59) are $\vec{\alpha}^{\mathbf{t}}$. Thus, letting

$\mathbf{1} \in \mathbb{R}^d$ be the column vector all of whose entries are 1, we find

$$g_{\vec{\alpha}} = \begin{pmatrix} B_{\vec{\alpha}}^{\mathbf{t}} & \mathbf{1} \\ \mathbf{0}^{\mathbf{t}} & 1 \end{pmatrix} \begin{pmatrix} I_d & \mathbf{0} \\ \vec{\alpha}^{\mathbf{t}} & 1 \end{pmatrix}, \tag{65}$$

which implies $\det(B_{\vec{\alpha}}) \neq 0$, and thus $\det\left(\bar{h}_{\vec{\alpha}}\right) = \pm 1$.

From (65) we have that

$$g_{\vec{\alpha}}^* = \begin{pmatrix} B_{\vec{\alpha}}^{\mathbf{t}} & \mathbf{1} \\ \mathbf{0}^{\mathbf{t}} & 1 \end{pmatrix}^* \begin{pmatrix} I_d & -\vec{\alpha} \\ \mathbf{0}^{\mathbf{t}} & 1 \end{pmatrix} = \begin{pmatrix} B_{\vec{\alpha}}^{-1} & \mathbf{0} \\ \mathbf{0}^{\mathbf{t}} & 1 \end{pmatrix} q \begin{pmatrix} I_d & -\vec{\alpha} \\ \mathbf{0}^{\mathbf{t}} & 1 \end{pmatrix}, \tag{66}$$

where q is of the form

$$q = \begin{pmatrix} I_d & \mathbf{0} \\ \mathbf{x}^{\mathbf{t}} & 1 \end{pmatrix}, \text{ for some } \mathbf{x} \in \mathbb{R}^d.$$

In particular we have $\lim_{t\to\infty} a_t q a_{-t} = e$; i.e., $q \in H^-$, the contracting horospherical subgroup of $\{a_t\}$.

Let $x_{\vec{\alpha}}^* = c_1^{-1} g_{\vec{\alpha}}^* \mathbb{Z}^n$. By (66), $c_1 = \left|\det\left(g_{\vec{\alpha}}^*\right)\right|^{-1/n}$, and thus $x_{\vec{\alpha}}^*$ is the lattice as in Proposition 7.4 and satisfies

$$x_{\vec{\alpha}}^* = (\bar{h}_{\vec{\alpha}})^{-1} q \begin{pmatrix} I_d & -\vec{\alpha} \\ \mathbf{0}^{\mathbf{t}} & 1 \end{pmatrix} \mathbb{Z}^n = (\bar{h}_{\vec{\alpha}})^{-1} q \Lambda_{\vec{\alpha}}. \tag{67}$$

The lattice $x_{\vec{\alpha}}^*$ is $(a_t, m_{Ax_{\vec{\alpha}}^*})$-generic by Proposition 7.4. Since $\Lambda_{\vec{\alpha}} = q^{-1}\bar{h}_{\vec{\alpha}} x_{\vec{\alpha}}^*$, $\bar{h}$ commutes with the $\{a_t\}$ action, and q^{-1} belongs to the contracting horospherical group for $\{a_t\}$, we have by Proposition 7.2 that Λ_α is $(a_t, m_{\vec{\alpha}})$-generic. □

7.4. **The adelic homogeneous space, Case I.** We briefly recall facts and notation regarding the rational adeles. See [Wei82, PR94] for more details on adeles and arithmetic groups, and see [Gui14] for a gentle recent introduction. Let $\mathbf{P}$ be the set of (rational) primes. Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f = \mathbb{R} \times \prod'_{p\in\mathbf{P}} \mathbb{Q}_p$ be the ring of adeles. Here $\prod'$ stands for the restricted product — that is, a sequence $\underline{\beta} = (\beta_\infty, \beta_f) = (\beta_\infty, \beta_2, \beta_3, \dots, \beta_p, \dots)$ belongs to $\mathbb{A}$ if and only if $\beta_p \in \mathbb{Z}_p$ for all but finitely many p. As suggested by the notation, we denote the real coordinate of a sequence $\underline{\beta} \in \mathbb{A}$ by β_∞ and the sequence of p-adic coordinates by $\beta_f = (\beta_p)_{p\in\mathbf{P}}$. The rational numbers $\mathbb{Q}$ are embedded in $\mathbb{A}$ diagonally, that is, $q \in \mathbb{Q}$ is identified with the constant sequence $(q, q, \cdots)$. We let $\mathrm{SL}_n(\mathbb{A}) = \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{A}_f) = \mathrm{SL}_n(\mathbb{R}) \times \prod'_{p\in\mathbf{P}} \mathrm{SL}_n(\mathbb{Q}_p)$ and use similar notation $(g_\infty, g_f) = (g_\infty, (g_p)_{p\in\mathbf{P}})$ to denote elements of $\mathrm{SL}_n(\mathbb{A})$. It is well-known that the diagonal embedding of $\mathrm{SL}_n(\mathbb{Q})$ in $\mathrm{SL}_n(\mathbb{A})$ is a lattice in $\mathrm{SL}_n(\mathbb{A})$. Let

$$K_f \stackrel{\text{def}}{=} \prod_{p\in\mathbf{P}} \mathrm{SL}_n(\mathbb{Z}_p) \tag{68}$$

and

$$\pi_f : \mathrm{SL}_n(\mathbb{A}) \to \mathrm{SL}_n(\mathbb{A}_f), \quad \pi_f(g_\infty, g_f) \stackrel{\text{def}}{=} g_f.$$

Then K_f is a compact open subgroup of $\mathrm{SL}_n(\mathbb{A}_f)$. Via the embedding $\mathrm{SL}_n(\mathbb{A}_f) \cong \{e\} \times \mathrm{SL}_n(\mathbb{A}_f)$ we also think of K_f as a subgroup of $\mathrm{SL}_n(\mathbb{A})$. We shall use the following two basic facts (see [PR94, Chap. 7]):

(i) The intersection $K_f \cap \pi_f(\mathrm{SL}_n(\mathbb{Q}))$ is equal to $\pi_f(\mathrm{SL}_n(\mathbb{Z}))$.
(ii) The projection $\pi_f(\mathrm{SL}_n(\mathbb{Q}))$ is dense in $\mathrm{SL}_n(\mathbb{A}_f)$.

Let

$$\mathscr{X}_n^{\mathbb{A}} = \mathrm{SL}_n(\mathbb{A}) / \mathrm{SL}_n(\mathbb{Q}),$$

and let $m_{\mathscr{X}_n^{\mathbb{A}}}$ denote the $\mathrm{SL}_n(\mathbb{A})$-invariant probability measure on $\mathscr{X}_n^{\mathbb{A}}$. There is a natural projection $\pi : \mathscr{X}_n^{\mathbb{A}} \to \mathscr{X}_n$ which we now describe in two equivalent ways.

First definition of π: Given $\tilde{x} = (g_\infty, g_f) \mathrm{SL}_n(\mathbb{Q}) \in \mathscr{X}_n^{\mathbb{A}}$, using (ii) and the fact that K_f is open, we may replace the representative (g_∞, g_f) by another $(g_\infty\gamma, g_f\gamma)$, where $\gamma \in \mathrm{SL}_n(\mathbb{Q})$ is such that $g_f\gamma \in K_f$. We then define $\pi(\tilde{x}) = g_\infty\gamma \mathrm{SL}_n(\mathbb{Z})$. This is well-defined since, if $g_f\gamma_1, g_f\gamma_2 \in K_f$, then by (i), $\pi_f(\gamma_1^{-1}\gamma_2) \in K_f \cap \pi_f(\mathrm{SL}_n(\mathbb{Q})) = \pi_f(\mathrm{SL}_n(\mathbb{Z}))$, and so $g_\infty\gamma_1 \mathrm{SL}_n(\mathbb{Z}) = g_\infty\gamma_2 \mathrm{SL}_n(\mathbb{Z})$.

Second definition of π: View K_f as a subgroup of $\mathrm{SL}_n(\mathbb{A})$. We claim that we may identify the double coset space $K_f \backslash \mathrm{SL}_n(\mathbb{A}) / \mathrm{SL}_n(\mathbb{Q})$ with $\mathrm{SL}_n(\mathbb{R}) / \mathrm{SL}_n(\mathbb{Z})$. Indeed, by (ii) and since K_f is open (as a subgroup of $\mathrm{SL}_n(\mathbb{A}_f)$), each double coset $K_f(g_\infty, g_f) \mathrm{SL}_n(\mathbb{Q})$ contains representatives with $g_f = e_f$ (the identity element in $\mathrm{SL}_n(\mathbb{A}_f)$). The real coordinates of all such representatives form a single left coset of $K_f \cap \mathrm{SL}_n(\mathbb{Q}) = \mathrm{SL}_n(\mathbb{Z})$. With this identification π is simply the projection from the coset space $\mathrm{SL}_n(\mathbb{A}) / \mathrm{SL}_n(\mathbb{Q})$ to the double coset space $K_f \backslash \mathrm{SL}_n(\mathbb{A}) / \mathrm{SL}_n(\mathbb{Q})$.

We leave it to the reader to check that these two definitions agree and that π intertwines the actions of $G_\infty \stackrel{\mathrm{def}}{=} \mathrm{SL}_n(\mathbb{R})$ on $\mathscr{X}_n^{\mathbb{A}}, \mathscr{X}_n$. Since there is a unique G_∞-invariant probability measure on $\mathscr{X}_n$, we have that $\pi_* m_{\mathscr{X}_n^{\mathbb{A}}} = m_{\mathscr{X}_n}$. In particular, the 1-parameter group $\{a_t\} \subset G_\infty$ acts on both of these spaces and π is a factor map for these actions. The following standard statement will be important for us:

Lemma 7.6. *The group* $\{a_t\}$ *acts ergodically on* $\left(\mathscr{X}_n^{\mathbb{A}}, m_{\mathscr{X}_n^{\mathbb{A}}}\right)$.

Proof. By the Mautner phenomenon (see e.g. [EW11]), it is enough to show that G_∞ acts ergodically on $\left(\mathscr{X}_n^{\mathbb{A}}, m_{\mathscr{X}_n^{\mathbb{A}}}\right)$. By duality, this is equivalent to the ergodicity of the action by right translations, of $\pi_f(\mathrm{SL}_n(\mathbb{Q}))$ on $G_\infty \backslash \mathrm{SL}_n(\mathbb{A}) = \mathrm{SL}_n(\mathbb{A}_f)$. Since the stabilizer of a measure is a closed group, by property (ii) above, any $\pi_f(\mathrm{SL}_n(\mathbb{Q}))$-invariant measure must be $\mathrm{SL}_n(\mathbb{A}_f)$-invariant, and thus is the Haar measure of $\mathrm{SL}_n(\mathbb{A}_f)$. In particular the action of $\pi_f(\mathrm{SL}_n(\mathbb{Q}))$ is uniquely ergodic, and hence ergodic. □

7.4.1. *The adelic homogeneous space, Case II.* Let A be the diagonal group, let $x^*_{\vec{\alpha}}$ be as in (61), let $\bar{h}_{\vec{\alpha}}$ be as in (63) and let

$$\bar{A}_{\vec{\alpha}} \stackrel{\mathrm{def}}{=} \bar{h}_{\vec{\alpha}} A \bar{h}_{\vec{\alpha}}^{-1}, \qquad y_{\vec{\alpha}} \stackrel{\mathrm{def}}{=} \bar{h}_{\vec{\alpha}} x^*_{\vec{\alpha}}. \tag{69}$$

The measure $m_{\vec{\alpha}}$ defined in (64) is $\bar{A}_{\vec{\alpha}}$-homogeneous, more precisely, by Proposition 7.4, we have that the orbit $\bar{A}_{\vec{\alpha}} y_{\vec{\alpha}}$ is compact and $m_{\vec{\alpha}}$ is the $\bar{A}_{\vec{\alpha}}$-invariant probability measure on $\bar{A}_{\vec{\alpha}} y_{\vec{\alpha}}$.

Let

$$\mathrm{Stab}_{\bar{A}_{\vec{\alpha}}}(y_{\vec{\alpha}}) \stackrel{\mathrm{def}}{=} \left\{ \bar{a} \in \bar{A}_{\vec{\alpha}} : \bar{a} y_{\vec{\alpha}} = y_{\vec{\alpha}} \right\} \tag{70}$$

be the stabilizer group of $y_{\vec{\alpha}}$ in $\bar{A}_{\vec{\alpha}}$. Write $y_{\vec{\alpha}} = g_\infty \mathbb{Z}^n$ for some $g_\infty \in \mathrm{SL}_n(\mathbb{R})$, so that $g_\infty^{-1} \mathrm{Stab}_{\tilde{L}_{\vec{\alpha}}}(\tilde{y}_{\vec{\alpha}}) g_\infty \subset \mathrm{SL}_n(\mathbb{Z})$ is cocompact in the conjugated group $g_\infty^{-1} \bar{A}_{\vec{\alpha}} g_\infty$. Let Δ be the diagonal embedding of $\mathrm{SL}_n(\mathbb{Z})$ in $\mathrm{SL}_n(\mathbb{A})$, let $M_{\vec{\alpha}} \subset K_f$ denote the closure of $\pi_f \circ \Delta \left(g_\infty^{-1} \mathrm{Stab}_{\tilde{L}_{\vec{\alpha}}}(\tilde{y}_{\vec{\alpha}}) g_\infty \right)$, and let

$$\tilde{L}_{\vec{\alpha}} \stackrel{\mathrm{def}}{=} \bar{A}_{\vec{\alpha}} \times M_{\vec{\alpha}} \subset \mathrm{SL}_n(\mathbb{A}).$$

For $\bar{a} \in \mathrm{Stab}_{\tilde{L}_{\vec{\alpha}}}(\tilde{y}_{\vec{\alpha}})$ we let

$$\gamma_{\bar{a}} \stackrel{\mathrm{def}}{=} \pi_f \circ \Delta(g_\infty^{-1} \bar{a} g_\infty) \in M_{\vec{\alpha}},$$

and

$$\tilde{y}_{\vec{\alpha}} \stackrel{\mathrm{def}}{=} (g_\infty, e_f) \, \mathrm{SL}_n(\mathbb{Q}) \in \mathscr{X}_n^{\mathbb{A}}. \tag{71}$$

Note that $M_{\vec{\alpha}}$ is a compact abelian group. Note also that the group $M_{\vec{\alpha}}$, the homomorphism $\bar{a} \mapsto \gamma_{\bar{a}}$, and the point $\tilde{y}_{\vec{\alpha}}$ all depend on the choice of the representative g_∞ of $y_{\vec{\alpha}}$. This dependence will not matter to us and we suppress it from the notation.

Proposition 7.7. *The orbit $\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}} \subset \mathscr{X}_n^{\mathbb{A}}$ is compact, and supports a finite $\tilde{L}_{\vec{\alpha}}$-invariant measure $m_{\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}}$. The action of $\{a_t\}$ on $\left(\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}, m_{\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}} \right)$ is uniquely ergodic, and $\pi_* \left(m_{\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}} \right) = m_{\vec{\alpha}}$. Moreover, $M_{\vec{\alpha}}$ acts transitively on the fibers of $\pi|_{\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}}$.*

Proof. In order to show that $\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}$ is compact we need to show that $\mathrm{Stab}_{\tilde{L}_{\vec{\alpha}}}(\tilde{y}_{\vec{\alpha}})$ is a lattice in $\tilde{L}_{\vec{\alpha}}$. We claim that

$$\left\{ (\bar{a}, \gamma_{\bar{a}}) : \bar{a} \in \mathrm{Stab}_{\tilde{L}_{\vec{\alpha}}}(\tilde{y}_{\vec{\alpha}}) \right\} \subset \mathrm{Stab}_{\tilde{L}_{\vec{\alpha}}}(\tilde{y}_{\vec{\alpha}}). \tag{72}$$

Indeed, for any $\bar{a} \in \mathrm{Stab}_{\bar{A}_{\vec{\alpha}}}(y_{\vec{\alpha}})$ we have

$$\begin{aligned} (\bar{a}, \gamma_{\bar{a}}) \tilde{y}_{\vec{\alpha}} =& (\bar{a}, \gamma_{\bar{a}})(g_\infty, e_f) \, \mathrm{SL}_n(\mathbb{Q}) = (g_\infty g_\infty^{-1} \bar{a} g_\infty, \gamma_{\bar{a}}) \, \mathrm{SL}_n(\mathbb{Q}) \\ =& (g_\infty, e_f)(g_\infty^{-1} \bar{a} g_\infty, \pi_f \circ \Delta(g_\infty^{-1} \bar{a} g_\infty)) \, \mathrm{SL}_n(\mathbb{Q}) \\ =& (g_\infty, e_f) \, \mathrm{SL}_n(\mathbb{Q}) = \tilde{y}_{\vec{\alpha}}. \end{aligned}$$

We remark that the other inclusion in (72) is also true, but we will not need it. Now, since $M_{\vec{\alpha}}$ is compact and $\mathrm{Stab}_{\bar{A}_{\vec{\alpha}}}(y_{\vec{\alpha}})$ is a lattice in $\bar{A}_{\vec{\alpha}}$, the graph $\left\{ (\bar{a}, \gamma_{\bar{a}}) : \bar{a} \in \mathrm{Stab}_{\bar{A}_{\vec{\alpha}}}(y_{\vec{\alpha}}) \right\}$ is a lattice in $\tilde{L}_{\vec{\alpha}}$.

Since $M_{\vec{\alpha}} \subset K_f$ we have that $\pi(\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}) = \bar{A}_{\vec{\alpha}} y_{\vec{\alpha}}$ and since π intertwines the $\bar{A}_{\vec{\alpha}}$-action, we have that $\pi_* m_{\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}}$ is an $\bar{A}_{\vec{\alpha}}$-invariant probability measure supported on $\bar{A}_{\vec{\alpha}} y_{\vec{\alpha}}$. As $m_{\vec{\alpha}}$ is the unique such measure, we have $\pi_* m_{\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}} = m_{\vec{\alpha}}$.

It remains to establish the unique ergodicity of the action of a_t on $\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}$. By standard facts on translation flows on compact abelian groups (see e.g. [EW11, Thm. 4.14]), this is equivalent to showing that any character on $\tilde{L}_{\vec{\alpha}} / \operatorname{Stab}_{\tilde{L}_{\vec{\alpha}}}(\tilde{y}_{\vec{\alpha}})$ which is trivial on the image of $\{a_t\}$ is trivial. This is in turn equivalent to the fact that any character on $\tilde{L}_{\vec{\alpha}}$ which is trivial on $\{a_t\}$ and on $\operatorname{Stab}_{\tilde{L}_{\vec{\alpha}}}(\tilde{y}_{\vec{\alpha}})$ is trivial. To this end, let $\chi : \tilde{L}_{\vec{\alpha}} \to \mathbb{S}^1$ be a character such that

$$\chi|_{\{a_t\}} \equiv \chi|_{\operatorname{Stab}_{\tilde{L}_{\vec{\alpha}}}(\tilde{y}_{\vec{\alpha}})} \equiv 1.$$

There are characters $\chi_1 : \bar{A}_{\vec{\alpha}} \to \mathbb{S}^1$, $\chi_2 : M_{\vec{\alpha}} \to \mathbb{S}^1$ such that

$$\chi(\bar{a}, h) = \chi_1(\bar{a}) \cdot \chi_2(h).$$

We claim that there exists $k \in \mathbb{N}$ such that χ_2^k is trivial. Let $\mathcal{W}$ be a neighborhood of 1 in $\mathbb{S}^1$ which does not contain any nontrivial subgroups. The group K_f has a collection of clopen subgroups that give a basis of the topology at the identity, and thus the same is true for $M_{\vec{\alpha}}$. By continuity of χ_2 there is a clopen subgroup M' of $M_{\vec{\alpha}}$ such that $\chi_2(M') \subset \mathcal{W}$, and hence $M' \subset \ker \chi_2$. This implies that χ_2 factors through the finite quotient of $M_{\vec{\alpha}}/M'$, proving the claim.

It follows that χ_1 is trivial on $\operatorname{Stab}_{\bar{A}_{\vec{\alpha}}}(y_{\vec{\alpha}})^k$ and on $\{a_t\}$ and therefore induces a character on $\bar{A}_{\vec{\alpha}} / \operatorname{Stab}_{\bar{A}_{\vec{\alpha}}}(y_{\vec{\alpha}})^k$ which is trivial on the image of $\{a_t\}$. Since $\{a_t\}$ acts ergodically on $\bar{A}_{\vec{\alpha}} / \operatorname{Stab}_{\bar{A}_{\vec{\alpha}}}(y_{\vec{\alpha}})$ and $\bar{A}_{\vec{\alpha}}$ is connected, $\{a_t\}$ also acts ergodically on $\bar{A}_{\vec{\alpha}} / \operatorname{Stab}_{\bar{A}_{\vec{\alpha}}}(y_{\vec{\alpha}})^k$. Hence χ_1 is trivial. This in turn implies that χ_2 is trivial on $\left\{\gamma_{\bar{a}} : \bar{a} \in \operatorname{Stab}_{\tilde{L}_{\vec{\alpha}}}(\tilde{y}_{\vec{\alpha}})\right\}$, which is a dense subgroup of M. Therefore χ_2 is also trivial.

We now establish the transitivity of the $M_{\vec{\alpha}}$-action on the fibers of $\pi|_{\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}}$. We need to show that if $x, y \in \tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}$ are such that $\pi(x) = \pi(y)$, then there exists $m \in M_{\vec{\alpha}}$ such that $mx = y$. Since $\tilde{L}_{\vec{\alpha}}$ commutes with π, acts transitively, and is commutative we may assume without loss of generality that $x = \tilde{y}_{\vec{\alpha}}$. Recall that $\tilde{y}_{\vec{\alpha}} = (g_\infty, e_f)\Gamma_{\mathbb{A}}$, where we set $\Gamma_{\mathbb{A}} \stackrel{\text{def}}{=} \operatorname{SL}_n(\mathbb{Q})$. Since $\bar{A}_{\vec{\alpha}} \times M_{\vec{\alpha}}$ acts transitively, we can write

$$y = (\bar{a}, m_f)\tilde{y}_{\vec{\alpha}} = (\bar{a}, m_f)(g_\infty, e_f)\Gamma_{\mathbb{A}} = (\bar{a} g_\infty, m_f)\Gamma_{\mathbb{A}}.$$

Applying π we see that

$$y_{\vec{\alpha}} = g_\infty \operatorname{SL}_n(\mathbb{Z}) = \pi(x) = \pi(y) = \bar{a} g_\infty \operatorname{SL}_n(\mathbb{Z}),$$

and therefore by definition of $M_{\vec{\alpha}}$, there exists $\gamma \in M \cap \operatorname{SL}_d(\mathbb{Z})$ such that $\bar{a} g_\infty = g_\infty \gamma$. It follows that

$$y = (g_\infty \gamma, m_f)\Gamma_{\mathbb{A}} = (g_\infty, m_f \gamma^{-1})\Gamma_{\mathbb{A}} = (e_\infty, m_f \gamma^{-1})(g_\infty, e_f)\Gamma_{\mathbb{A}} \in M \tilde{y}_{\vec{\alpha}}.$$

□

8. Some reasonable cross-sections

We will now define the cross-sections $\mathcal{S}$ we will need for our applications, as well as the cross-section measures $\mu_{\mathcal{S}}$. In this section we will work with the real space $\mathscr{X}_n$, and the adelic space $\mathscr{X}_n^{\mathbb{A}}$ will be discussed in §9.3.

8.1. **The cross-section $\mathcal{S}_{r_0}$.** We introduce some convenient notation. Recall that the set of primitive vectors in Λ is denoted by Λ_{prim}. Given a subset $W \subset \mathbb{R}^n$ and $k \geq 1$ we let

$$\mathscr{X}_n(W,k) \stackrel{\mathrm{def}}{=} \{\Lambda \in \mathscr{X}_n : \#(\Lambda_{\mathrm{prim}} \cap W) \geq k\}. \tag{73}$$

For $k = 1$ we will omit k and denote

$$\mathscr{X}_n(W) \stackrel{\mathrm{def}}{=} \mathscr{X}_n(W,1).$$

We will be interested in the case when there is a unique primitive vector in W, and thus we let

$$\mathscr{X}_n^{\sharp}(W) \stackrel{\mathrm{def}}{=} \mathscr{X}_n(W) \smallsetminus \mathscr{X}_n(W,2).$$

There is a natural map

$$v : \mathscr{X}_n^{\sharp}(W) \to W, \quad \text{defined by } \{v(\Lambda)\} = \Lambda_{\mathrm{prim}} \cap W. \tag{74}$$

With this notation we have:

Lemma 8.1. *Let $W \subset \mathbb{R}^n$ be a compact set, $V \subset W$ a relatively open subset and $k \geq 1$ an integer.*

1. *The set $\mathscr{X}_n(W,k)$ is closed in $\mathscr{X}_n$.*
2. *The set $\mathscr{X}_n^{\sharp}(W) \cap \mathscr{X}_n(V)$ is open in $\mathscr{X}_n(W)$.*
3. *The map $v : \mathscr{X}_n^{\sharp}(W) \to W$ is continuous.*

Proof. Let $\Lambda_i \in \mathscr{X}_n(W,k)$ such that $\Lambda_i \to \Lambda$. We can choose $g_i \to h$ such that $\Lambda_i = g_i\mathbb{Z}^n$ and $\Lambda = h\mathbb{Z}^n$. We need to show that Λ contains at least k primitive vectors in W. Since $g_i \to h$ and W is compact, there is a compact subset of $\mathbb{R}^n$ containing all of the sets $g_i^{-1}W$. Since $\Lambda_i \in \mathscr{X}_n(W,k)$, each $g_i^{-1}W$ contains at least k distinct elements of $\mathbb{Z}^n_{\mathrm{prim}}$. After passing to a subsequence if necessary, there are distinct $\mathbf{p}_1, \ldots, \mathbf{p}_k \in \mathbb{Z}^n_{\mathrm{prim}}$, such that $g_i^{-1}\mathbf{p}_j \in W$ for $j = 1, \ldots, k$. Since W is closed, their limits $h^{-1}\mathbf{p}_j$ belong to $\Lambda_{\mathrm{prim}} \cap W$, and are distinct. This proves (1).

The complement of $\mathscr{X}_n^{\sharp}(W) \cap \mathscr{X}_n(V)$ in $\mathscr{X}_n(W)$ consists of lattices that either contain at least two distinct primitive vectors in W or else, contain a primitive vector in $W \smallsetminus V$ (these cases are not mutually exclusive). That is,

$$\mathscr{X}_n(W) \smallsetminus (\mathscr{X}_n^{\sharp}(W) \cap \mathscr{X}_n(V)) = \mathscr{X}_n(W,2) \cup \mathscr{X}_n(W \smallsetminus V),$$

which by part (1) of the Lemma, is a union of two closed subsets of $\mathscr{X}_n$. This proves (2).

Suppose $\Lambda_i = g_i\mathbb{Z}^n \to \Lambda = h\mathbb{Z}^n$ is a converging sequence in $\mathscr{X}_n^{\sharp}(W)$, with $g_i \to h$. As before, the sets $g_i^{-1}W$ are all contained in a fixed compact set in $\mathbb{R}^n$. Since Λ_i and Λ belong to $\mathscr{X}_n^\sharp(W)$, the sets $g_i^{-1}W \cap \mathbb{Z}^n_{\mathrm{prim}}$ and $h^{-1}W \cap \mathbb{Z}^n_{\mathrm{prim}}$ are singletons. Passing to a subsequence if needed, we may assume that $g_i^{-1}W = \{\mathbf{p}\}$ for a fixed $\mathbf{p}$. Therefore $v(\Lambda_i) = g_i\mathbf{p} \to h\mathbf{p} \in \Lambda_{\mathrm{prim}} \cap W$ and $h\mathbf{p} = v(\Lambda)$, proving (3). □

The following Lemma is proved using similar arguments:

Lemma 8.2. *Let $W \subset \mathbb{R}^n$ be an open set. For any $k \geq 1$, $\mathscr{X}_n(W,k)$ is open in $\mathscr{X}_n$.*

□

Fix a norm $\|\cdot\|$ on $\mathbb{R}^d$, and denote the Lebesgue measure on $\mathbb{R}^n$ by m. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we refer to $\pi_{\mathbb{R}^d}(\mathbf{x})$ as the *horizontal component* of $\mathbf{x}$ and to x_n as the *vertical component* of $\mathbf{x}$. For positive numbers r and s, let

$$C_r(s) \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n : \|\pi_{\mathbb{R}^d}(\mathbf{x})\| \leq r, \ |x_n| \leq s\} \text{ and } C_r \stackrel{\text{def}}{=} C_r(1). \tag{75}$$

Note that this cylinder depends on the choice of the norm; we consider the norm as fixed and thus it does not appear in the notation. Choose $r_0 > 0$ large enough so that

$$m(C_{r_0}) \geq 2^n. \tag{76}$$

By Minkowski's convex body theorem, this implies that for any $\Lambda \in \mathscr{X}_n$, $\Lambda_{\mathrm{prim}} \cap C_{r_0} \neq \varnothing$. In other words, $\mathscr{X}_n(C_{r_0}) = \mathscr{X}_n$. Also set

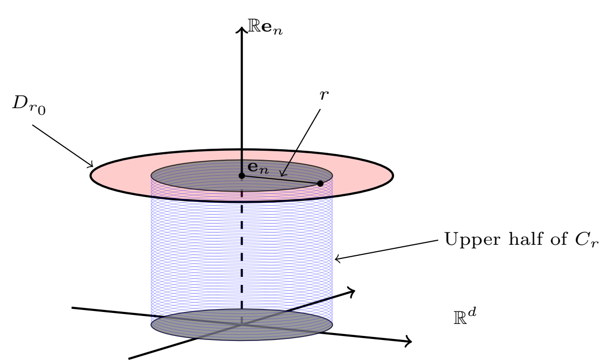
$$D_r \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n : \|\pi_{\mathbb{R}^d}(\mathbf{x})\| \leq r, \ x_n = 1\}, \tag{77}$$

and define

$$\mathcal{S}_r = \mathscr{X}_n(D_r) \quad \text{and} \quad \mathcal{S}_r^\sharp = \mathscr{X}_n^\sharp(D_r). \tag{78}$$

The set $\mathcal{S}_{r_0}$ will be our cross-section.

Remark 8.3. The reader will note that we have not specified r_0 explicitly, e.g. we did not specify an equality $m(C_{r_0}) = 2^n$. Two situations in which this additional flexibility will be useful, are when dealing with ε-approximations for large ε (in which case we will require $r_0 \geq \varepsilon$), and in the proof of Proposition 12.5.



Lemma 8.4. *Let μ be any $\{a_t\}$-invariant probability measure on $\mathscr{X}_n$. Then $\mathcal{S}_{r_0}$ is a μ-cross-section for $(\mathscr{X}_n, \mu, \{a_t\})$. Furthermore, the cross-section measure satisfies*

$$\mu_{\mathcal{S}_{r_0}}\left((\mathscr{X}_n(D_{r_0}, 2)\right) = 0. \tag{79}$$

Proof. We verify Definition 4.2. We take X_0 to be the set of lattices which intersect the horizontal space and the vertical axis span $\{\mathbf{e}_n\}$ only at $\{0\}$. It is easy to see that this is a G_δ-set and therefore $(X_0, \mathcal{B}_{X_0})$ is a standard Borel space. In order to show that $\mathcal{S}$ is a μ-cross-section we need to check three things: that $\mu(\mathscr{X}_n \smallsetminus X_0) = 0$, that for any $\Lambda \in X_0$ the set of visit times $\mathcal{Y}_\Lambda$ is discrete and unbounded from below and above, and finally that the return time function $\tau_{\mathcal{S}} : \mathcal{S} \cap X_0 \to \mathbb{R}_+$ is measurable.

Let $\Lambda \in \mathscr{X}_n$. We say that Λ is *divergent in positive (negative) time* if for any compact $K \subset \mathscr{X}_n$ there is $t_0 \in \mathbb{R}$ such that for all $t > t_0$ (respectively, for all $t < t_0$) we have $a_t\Lambda \notin K$. Recall that by Mahler's compactness criterion, a closed subset $K \subset \mathscr{X}_n$ is compact if and only if there is $\varepsilon > 0$ such that for all $\Lambda \in K$, any nonzero $v \in \Lambda$ satisfies $\|v\| \geq \varepsilon$. In particular, if $\Lambda \in \mathscr{X}_n \smallsetminus X_0$ then either it is divergent in positive time or in negative time. By Poincaré recurrence, $\mathscr{X}_n \smallsetminus X_0$ is a μ-null set.

Let $\Lambda \in X_0$. We verify (31), that is, we show that $\{t \in \mathbb{R} : a_t\Lambda \in \mathcal{S}_{r_0}\}$ is discrete and unbounded from below and from above.

Note that $a_t\Lambda \in \mathcal{S}_{r_0}$ if and only if Λ_{prim} contains a vector in $a_{-t}(D_{r_0})$. Discreteness of the set of visit times readily follows from this, and the fact that Λ_{prim} is a discrete set in $\mathbb{R}^n$. Now suppose by contradiction that there is $T > 0$ such that for all $s \geq T$, $a_s\Lambda \notin \mathcal{S}_{r_0}$. The set $F \overset{\mathrm{def}}{=} a_T\Lambda \cap C_{r_0}$ is finite and since $\Lambda \in X_0$, all vectors in F have non-zero horizontal component. It follows that for all large enough $t > T$, $a_{t-T}(F) \cap C_{r_0} = \varnothing$. By Minkowski's convex body theorem and the choice of r_0, the lattice $a_t\Lambda$ contains a primitive vector $v = (v_1, \ldots, v_n)$ in the cylinder C_{r_0}. Since $\Lambda \in X_0$, we have $v_n \neq 0$, and we can assume without loss of generality that $v_n > 0$. Let $v = a_t v_0$ for $v_0 \in \Lambda_{\mathrm{prim}}$. By (16) there is a unique s such that $a_s v_0 \in D_{r_0}$, and since the vertical component of v is at most 1, we have $s \leq t$. This means that $a_s\Lambda \in \mathcal{S}_{r_0}$ and by choice of T we have $s \leq T$, so that the vertical component of $a_T v_0$ is at most 1. On the other hand

$$\|\pi_{\mathbb{R}^d}(a_T v_0)\| = \|\pi_{\mathbb{R}^d}(a_{T-t} v)\| = e^{t-T}\|\pi_{\mathbb{R}^d}(v)\| \leq r_0,$$

so that $a_T v_0 \in C_{r_0}$. This shows that $a_T v_0 \in F$ and hence $v \in a_{t-T}F$, a contradiction. The argument showing unboundedness from below is similar.

To complete the verification of Definition 4.2 we need to show that the return time function is Borel measurable, or equivalently, that the sub-level sets $\mathcal{S}_{r_0,<\varepsilon}$ defined in (33) are Borel. The thickened set

$$W_\varepsilon \overset{\mathrm{def}}{=} D_{r_0}^{(-\varepsilon,0)} = \bigcup_{t\in(0,\varepsilon)} a_{-t}(D_{r_0}) \tag{80}$$

is a Borel subset of $\mathbb{R}^n$, and therefore

$$\mathcal{S}_{r_0,<\varepsilon} = \{\Lambda \in \mathcal{S}_{r_0} : \Lambda_{\text{prim}} \cap W_\varepsilon \neq \varnothing\} = \mathscr{X}_n(D_{r_0}) \cap \mathscr{X}_n(W_\varepsilon) \tag{81}$$

is Borel as well.

To finish the proof we show (79) holds. This follows since $\mathscr{X}_n(D_{r_0}, 2)^{\mathbb{R}} \subset \mathscr{X}_n \smallsetminus X_0$ (where we use the notation in (34)) is μ-null and so by Corollary 4.4(iv), $\mathscr{X}_n(D_{r_0}, 2)$ is $\mu_{\mathcal{S}_{r_0}}$-null. □

8.2. **Parameterizing $\mathcal{S}_{r_0}$.** One of the advantages of the cross-section $\mathcal{S}_{r_0}$ is that it has a nice description in terms of orbits and groups. Let

$$H = \left\{ \left(\begin{smallmatrix} A & 0 \\ w^{\mathbf{t}} & 1 \end{smallmatrix}\right) : A \in \mathrm{SL}_d(\mathbb{R}),\, w \in \mathbb{R}^d \right\}, \quad \text{and} \tag{82}$$

$$U = \left\{ u(v) : v \in \mathbb{R}^d \right\}, \quad \text{where } u(v) \text{ is as in (15)} ; \tag{83}$$

that is, $H = H^{\leq}$ is the group defined in (12) and Proposition 7.2, the orbit $H\mathbb{Z}^n$ is the space $\mathscr{E}_n = \mathscr{X}_n(\mathbf{e}_n)$ of lattices which contain $\mathbf{e}_n$ as a primitive vector (see (11)), and U is the *expanding horospherical group of a_t in positive time.*

We let $\bar{B}_r \subset \mathbb{R}^d$ denote the closed ball centered at $\mathbf{0} \in \mathbb{R}^d$, with respect to our chosen norm (note that the norm is suppressed from the notation). Consider the map

$$\varphi : \mathscr{E}_n \times \bar{B}_{r_0} \to \mathcal{S}_{r_0}, \quad \varphi(\Lambda, v) \stackrel{\text{def}}{=} u(v)\Lambda.$$

Note that the map $v \mapsto u(v)\mathbf{e}_n$ is a bijection between $\bar{B}_{r_0}$ and D_{r_0}. It follows that φ is onto $\mathcal{S}_{r_0}$, and

$$\text{for any } r \in (0, r_0), \quad \varphi(\mathscr{E}_n \times \bar{B}_r) = \mathcal{S}_r. \tag{84}$$

Furthermore, for $\Lambda \in \mathcal{S}_{r_0}$,

$$\#\, \varphi^{-1}(\Lambda) = \#\left(\Lambda_{\text{prim}} \cap D_{r_0}\right). \tag{85}$$

Indeed, for any $v \in \Lambda_{\text{prim}} \cap D_{r_0}$,

$$\left(u(\pi_{\mathbb{R}^d}(v))^{-1}\Lambda, \pi_{\mathbb{R}^d}(v)\right) \in \varphi^{-1}(\Lambda),$$

and this assignment is easily seen to be a bijection. Let

$$\begin{aligned} &\psi : \mathcal{S}^{\sharp}_{r_0} \to \mathscr{E}_n \times \bar{B}_{r_0}, \\ &\psi(\Lambda) \stackrel{\text{def}}{=} \left(u(v_\Lambda)^{-1}\Lambda, v_\Lambda\right), \end{aligned} \tag{86}$$

where

$$v_\Lambda \stackrel{\text{def}}{=} \pi_{\mathbb{R}^d}(v(\Lambda)) \ \text{ and } \{v(\Lambda)\} = \Lambda \cap D_{r_0}. \tag{87}$$

Lemma 8.5. *The set $\mathcal{S}_{r_0}$ is closed in $\mathscr{X}_n$, ψ as in (86) is the inverse of $\varphi|_{\varphi^{-1}(\mathcal{S}^{\sharp}_{r_0})}$, and is a homeomorphism between $\mathcal{S}^{\sharp}_{r_0}$ and $\varphi^{-1}(\mathcal{S}^{\sharp}_{r_0})$.*

Proof. The assertions that $\mathcal{S}_{r_0}$ is closed, and that ψ is continuous, follow from Lemma 8.1. It is clear that φ is continuous. To see that ψ and $\varphi|_{\varphi^{-1}(\mathcal{S}^\sharp_{r_0})}$ are mutual inverses, we see easily that $\varphi\circ\psi = \mathrm{Id}_{\mathcal{S}^\sharp_{r_0}}$. By (85), φ is injective on $\varphi^{-1}(\mathcal{S}^\sharp_{r_0})$, and thus we also have $\psi\circ\varphi|_{\varphi^{-1}(\mathcal{S}^\sharp_{r_0})} = \mathrm{Id}_{\varphi^{-1}(\mathcal{S}^\sharp_{r_0})}$. □

In the following subsections we will use the map φ to describe the cross-section measure $\mu_{\mathcal{S}_{r_0}}$, corresponding to certain invariant measures μ.

8.3. **The cross-section measure, real homogeneous space, Case I.** The goal of this subsection is the following result:

Theorem 8.6. *The cross-section $\mathcal{S}_{r_0}$ is $m_{\mathscr{X}_n}$-reasonable.*

For the proof we will need some preparations. The first is an explicit description of the cross-section measure $\mu_{\mathcal{S}_{r_0}}$. Let $m_{\mathbb{R}^d}$ be the Lebesgue measure on $\mathbb{R}^d$.

Proposition 8.7. *In Case I,*

$$\mu_{\mathcal{S}_{r_0}} = \frac{1}{\zeta(n)}\,\varphi_*\left(m_{\mathscr{E}_n}\times m_{\mathbb{R}^d}|_{\bar{B}_{r_0}}\right)$$

(where $\zeta(n)=\sum_{k\in\mathbb{N}} k^{-n}$). In particular, $\mu_{\mathcal{S}_{r_0}}$ is finite and $\mathrm{supp}(\mu_{\mathcal{S}_{r_0}}) = \mathcal{S}_{r_0}$.

The proof is a straightforward but lengthy computation which is postponed to §8.5.

Here is another result used in the proof of Theorem 8.6. We state it in a general form which will be useful in the sequel.

Lemma 8.8. *Let $L\subset \mathrm{SL}_n(\mathbb{R})$ be a closed subgroup, with left Haar measure m_L. Let $\Lambda_0\in\mathscr{X}_n$ such that $L\Lambda_0$ is a closed orbit supporting a finite L-invariant measure $m_{L\Lambda_0}$. Let $W\subset\mathbb{R}^n$ such that for any $v\in W\cap L\Lambda_0$ there is $\varepsilon>0$ such that*

$$m_L\left(\left\{\ell\in B^L_\varepsilon : \ell v\in W\right\}\right)=0 \tag{88}$$

(where B^L_ε denotes the ε-ball around the identity element of L, with respect to some metric inducing the topology). Then $m_{L\Lambda_0}\left(\mathscr{X}_n(W)\right)=0$.

Proof. By covering W with countably many bounded sets we may assume that W is bounded. Since W is bounded, for any $\Lambda\in L\Lambda_0$, the cardinality of $\Lambda_{\mathrm{prim}}\cap W$ is finite, and bounded for Λ in a compact subset of $L\Lambda_0$. Using the hypothesis we deduce that for each $\Lambda\in L\Lambda_0\cap\mathscr{X}_n(W)$ there exist $\varepsilon>0$ such that (88) holds for any $v\in\Lambda_{\mathrm{prim}}\cap W$. Since $m_{L\Lambda_0}$ is the restriction of m_L to a fundamental domain for the action of the stabilizer L_{Λ_0}, it follows that $m_{L\Lambda_0}\left(\mathscr{X}_n(W)\cap B^L_\varepsilon\Lambda\right)=0$. We can cover $L\Lambda_0$ by countably many sets $\{B^L_\varepsilon\Lambda_i\}_{i\in\mathbb{N}}$, and therefore $m_{L\Lambda_0}(\mathscr{X}_n(W))=0$. □

For positive r and ε, define

$$D^\circ_r \overset{\mathrm{def}}{=} \{\mathbf{x}\in\mathbb{R}^n : \|\pi_{\mathbb{R}^d}(\mathbf{x})\|<r,\ x_n=1\}$$

(compare with the set D_r defined in (77)), and

$$F_{\varepsilon,r} \stackrel{\text{def}}{=} (D_r \smallsetminus D_r^\circ)^{[-\varepsilon,0]}.$$

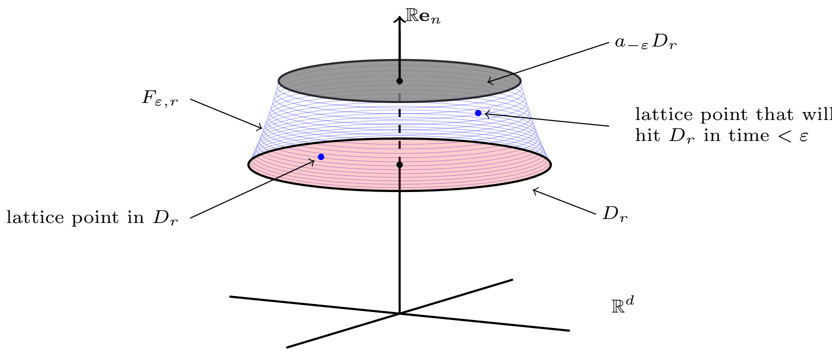


FIGURE 1. The set $D_r^{(-\varepsilon,0)}$ bounded by the surface $F_{\varepsilon,r}$ and the disks D_r and $a_\varepsilon(D_r)$. Lattice points in $D_r^{(0,\varepsilon)}$ correspond to visits to $\mathcal{S}_{r,<\varepsilon}$.

See Figure 1. We will need the following:

Lemma 8.9. *For any $r \geq r_0$, $\varepsilon > 0$ we have that*

$$\partial_{\mathcal{S}_r}(\mathcal{S}_{r,<\varepsilon}) \subset \mathscr{X}_n(D_r, 2) \cup (\mathscr{X}_n(D_r) \cap \mathscr{X}_n(a_{-\varepsilon}(D_r))) \cup \mathscr{X}_n(F_{\varepsilon,r}), \tag{89}$$

and the set $\mathcal{S}_{r,<\varepsilon}$ is $\mu_{\mathcal{S}_r}$-JM.

Proof. Let W_ε be as in (80) and let $\overline{W}_\varepsilon$ be its closure in $\mathbb{R}^n$. Note that

$$\overline{W}_\varepsilon = W_\varepsilon \cup D_r \cup a_{-\varepsilon}(D_r) \cup F_{\varepsilon,r}. \tag{90}$$

Recall from (81) that $\mathcal{S}_{r,<\varepsilon}$ consists of the lattices containing a primitive point in D_r and another primitive point in W_ε. In particular, $\mathcal{S}_{r,<\varepsilon}$ is contained in

$$E_1 \stackrel{\text{def}}{=} \mathscr{X}_n(D_r) \cap \mathscr{X}_n(\overline{W}_\varepsilon, 2),$$

which is closed by Lemma 8.1. By Lemma 8.2, $\mathscr{X}_n(W_\varepsilon)$ is open and in conjunction with Lemma 8.1 we deduce that the set

$$E_2 \stackrel{\text{def}}{=} \mathscr{X}_n^{\#}(D_r) \cap \mathscr{X}_n(D_r^\circ) \cap \mathscr{X}_n(W_\varepsilon)$$

is open in $\mathscr{X}_n(D_r)$. It follows that

$$\partial_{\mathcal{S}_r}(\mathcal{S}_{r,<\varepsilon}) \subset E_1 \smallsetminus E_2.$$

We now show that $E_1 \smallsetminus E_2$ is contained in the RHS of (89). Suppose $\Lambda \in E_1$ and hence Λ_{prim} contains two distinct vectors v_1, v_2 such that $v_1 \in D_r$ and $v_2 \in \overline{W}_\varepsilon$. If Λ does not belong to the RHS of (89), then v_1 must be the unique primitive vector in D_r, it cannot lie in $F_{\varepsilon,r}$ and so it must lie in

D_r° and v_2 cannot lie in $F_{\varepsilon,r}$ or $a_{-\varepsilon}(D_r)$ and so it must lie in W_ε. That is, $\Lambda \in E_2$. This concludes the proof of (89).

In order to show that $\mathcal{S}_{r,<\varepsilon}$ is $\mu_{\mathcal{S}_r}$-JM, it suffices to show that the sets

$$\mathscr{X}_n(D_r, 2),\ \mathscr{X}_n(F_{\varepsilon,r}),\ \mathscr{X}_n(D_r) \cap \mathscr{X}_n(a_{-\varepsilon}(D_r))$$

are all $\mu_{\mathcal{S}_r}$-null. By Theorem 4.4 it is enough to show that the sets

$$\mathscr{X}_n(D_r, 2)^{\mathbb{R}},\ \mathscr{X}_n(F_{\varepsilon,r})^{\mathbb{R}},\ (\mathscr{X}_n(D_r) \cap \mathscr{X}_n(a_{-\varepsilon}(D_r)))^{\mathbb{R}}$$

are all $m_{\mathscr{X}_n}$-null.

The set $\mathscr{X}_n(D_r, 2)^{\mathbb{R}}$ is $m_{\mathscr{X}_n}$-null because of (79). Next, if we set

$$M_r \stackrel{\text{def}}{=} \left\{ \mathbf{x} \in \mathbb{R}^n : x_n \cdot ||\pi_{\mathbb{R}^d}(\mathbf{x})||^d = r^d \right\} = \bigcup_{t \in \mathbb{R}} a_t \left(D_r \smallsetminus D_r^\circ \right), \tag{91}$$

then

$$\mathscr{X}_n(D_r \smallsetminus D_r^\circ)^{\mathbb{R}} = \mathscr{X}_n(F_{\varepsilon,r})^{\mathbb{R}} = \mathscr{X}_n(M_r). \tag{92}$$

Since $\mathrm{SL}_n(\mathbb{R})$ contains elements which expand the vertical component x_n without affecting $||\pi_{\mathbb{R}^d}(\mathbf{x})||$, condition (88) is satisfied for $W = M_r$, so applying Lemma 8.8 we have that $\mathscr{X}_n(M_r)$ is $m_{\mathscr{X}_n}$-null.

It remains to show that

$$E_\varepsilon \stackrel{\text{def}}{=} (\mathscr{X}_n(D_r) \cap \mathscr{X}_n(a_{-\varepsilon}(D_r)))^{\mathbb{R}} = \bigcup_{t \in \mathbb{R}} (\mathscr{X}_n(a_t(D_r)) \cap \mathscr{X}_n(a_{t+\varepsilon}(D_r))$$

is $m_{\mathscr{X}_n}$-null. If $\Lambda \in \mathscr{X}_n(a_t(D_r)) \cap \mathscr{X}_n(a_{t+\varepsilon}(D_r))$ for some t, and v, w are primitive vectors in Λ such that $v \in a_t(D_r)$ and $w \in a_{t+\varepsilon}(D_r)$, then v, w are two linearly independent vectors in Λ and the ratio between their vertical components satisfies $\frac{w_n}{v_n} = e^\varepsilon$.

We use a result of Siegel [Sie45], according to which for a null set in $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$, the function

$$\widehat{\chi_\Omega}(\Lambda) = \sum_{\substack{v,w \in \Lambda \\ \text{linearlty independent}}} \chi_\Omega(v, w)$$

has integral zero with respect to $m_{\mathscr{X}_n}$. We apply this with

$$\Omega \stackrel{\text{def}}{=} \left\{ (v, w) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{w_n}{v_n} = e^\varepsilon \right\}.$$

Since $\widehat{\chi_\Omega}$ bounds the characteristic function of E_ε from above, we obtain that E_ε is $m_{\mathscr{X}_n}$-null. □

Let

$$\mathcal{U}_{r_0} \stackrel{\text{def}}{=} \mathscr{X}_n^{\#}(D_{r_0}) \cap \mathscr{X}_n(D_{r_0}^\circ). \tag{93}$$

Lemma 8.10. *The set $\mathcal{U}_{r_0}$ is open in $\mathcal{S}_{r_0}$, the set $(\mathrm{cl}_{X_n}(\mathcal{S}_{r_0}) \smallsetminus \mathcal{U}_{r_0})^{(0,1)}$ is $m_{\mathscr{X}_n}$-null, and the map $(t, \Lambda) \mapsto a_t\Lambda$ from $(0,1) \times \mathcal{U}_{r_0}$ to $\mathscr{X}_n$ is open.*

Proof. Item (2) of Lemma 8.1 shows that $\mathcal{U}_{r_0}$ is open in $\mathcal{S}_{r_0}$, and item (1) shows that $\mathcal{S}_{r_0} = \mathscr{X}_n(D_{r_0})$ is closed in $\mathscr{X}_n$. Therefore

$$\begin{aligned} \mathrm{cl}_{\mathscr{X}_n}(\mathcal{S}_{r_0}) \smallsetminus \mathcal{U}_{r_0} =& \mathscr{X}_n(D_{r_0}) \smallsetminus (\mathscr{X}_n^{\#}(D_{r_0}) \cap \mathscr{X}_n(D^{\circ}_{r_0})) \\ \subset& \mathscr{X}_n(D_{r_0}, 2) \cup \mathscr{X}_n(D_{r_0} \smallsetminus D^{\circ}_{r_0}). \end{aligned} \tag{94}$$

In the proof of Lemma 8.9 we showed that $\mathscr{X}_n(D_{r_0}, 2)^{\mathbb{R}}$ as well as $\mathscr{X}_n(D_{r_0} \smallsetminus D^{\circ}_{r_0})^{\mathbb{R}}$ are $m_{\mathscr{X}_n}$-null. This proves the second assertion.

For the third assertion, let U and H be as in (82), let $Q \stackrel{\mathrm{def}}{=} \{a_t\} \ltimes U$, and let $\mathfrak{h}, \mathfrak{q}, \mathfrak{g}$ denote respectively the Lie algebras of H, Q and $\mathrm{SL}_n(\mathbb{R})$. The product map $(t, u) \mapsto a_t u$ is a homeomorphism $\mathbb{R} \times U \to Q$, and since $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$, the product map $\mathbb{R} \times U \times H \to G$ is open. This implies that the map

$$\mathbb{R} \times U \times H\mathbb{Z}^n \to \mathscr{X}_n, \quad (t, u, h\mathbb{Z}^n) \mapsto a_t u h\mathbb{Z}^n$$

is open, and the map in the statement of the Lemma is its restriction to an open set because $\mathscr{X}_n(D^{\circ}_{r_0}) = B^U_{r_0} \cdot H\mathbb{Z}^n$ (see §8.2). □

Proof of Theorem 8.6. Property (A) of §5 is immediate, property (B) follows from Proposition 8.7, and property (C) follows from Lemma 8.5. Item (1) of Definition 5.3 follows from Lemma 8.9, and item (2) follows from Lemma 8.10. □

8.4. **The cross-section measure, real homogeneous space, Case II.** In this subsection, the notation is as in §7.3, and we write $\mu = m_{\vec{\alpha}}$ (see (64)). We denote by $\mu_{\mathcal{S}_{r_0}}$ the corresponding measure on $\mathcal{S}_{r_0}$, defined via Theorem 4.4. The goal of this subsection is the following result:

Theorem 8.11. *With this choice of μ, the cross-section $\mathcal{S}_{r_0}$ is μ-reasonable.*

Remark 8.12. Recall that in Case II we always assume $d \geq 2$. The reason is that for $d = 1$, for some choices of r_0, $\mathcal{S}_{r_0}$ may fail to be μ-reasonable.

We will need a detailed description of $\mu_{\mathcal{S}_{r_0}}$. Let $x^*_{\vec{\alpha}}$ be as in (61), let $\bar{A}_{\vec{\alpha}}$, $y_{\vec{\alpha}}$ be as in (69), so that

$$\mathrm{supp}\ m_{\vec{\alpha}} = \bar{h} A x^*_{\vec{\alpha}} = \bar{A}_{\vec{\alpha}} y_{\vec{\alpha}}.$$

Also let

$$\bar{A}^{(1)}_{\vec{\alpha}} \stackrel{\mathrm{def}}{=} \{\bar{a} \in \bar{A}_{\vec{\alpha}} : \bar{a}\mathbf{e}_n = \mathbf{e}_n\}, \tag{95}$$

and let $m_{\bar{A}^{(1)}_{\vec{\alpha}}}$ denote the Haar measure on $\bar{A}^{(1)}_{\vec{\alpha}}$. Note that both $\bar{h}$ and $\bar{A}_{\vec{\alpha}}$ act on $\mathbb{R}^n$ without changing the vertical component of any vector.

Proposition 8.13. *The following hold:*

(a) The support of $\mu_{\mathcal{S}_{r_0}}$ is the compact set $\bar{A}_{\vec{\alpha}} y_{\vec{\alpha}} \cap \mathcal{S}_{r_0}$;

(b) The support of $\mu_{\mathcal{S}_{r_0}}$ is equal to a finite union $\bigcup_{i=1}^k \mathcal{O}_i \bar{\Lambda}_i$ where each $\mathcal{O}_i \subset \bar{A}^{(1)}_{\vec{\alpha}}$ is homeomorphic to a closed ball, and $\bar{\Lambda}_i \in \bar{A}_{\vec{\alpha}} y_{\vec{\alpha}}$;

(c) *The restriction of* $\mu_{\mathcal{S}_{r_0}}$ *to each of the subsets* $\mathcal{O}_i\bar{\Lambda}_i$ *in* (b) *is the push-forward of the restriction of the Haar measure* $m_{\bar{A}^{(1)}_{\vec{\alpha}}}|_{\mathcal{O}_i}$ *under the orbit map* $\bar{a} \mapsto \bar{a}y_{\vec{\alpha}}$.

(d) $\bar{A}_{\vec{\alpha}}y_{\vec{\alpha}} \cap \mathcal{S}_{r_0} \subset \mathcal{S}_{r_0,\geq\varepsilon}$ *for some* $\varepsilon > 0$;

(e) $\mu_{\mathcal{S}_{r_0}}$ *is finite.*

Proof. Item (a) follows from Corollary 4.4(iv). For item (b), we have from Proposition 7.4 that the orbit $Ax^*_{\vec{\alpha}}$ is of type $(\boldsymbol{\sigma}, \mathbb{K})$. Define $N : \mathbb{R}^n \to \mathbb{R}$ by

$$N(\mathbf{x}) \stackrel{\text{def}}{=} \prod_{i=1}^{n} s_i, \quad \text{where } (s_1, \dots, s_n)^{\mathbf{t}} = \bar{h}_{\vec{\alpha}}^{-1}\mathbf{x}. \tag{96}$$

In other words, the restriction of N to $y_{\vec{\alpha}}$ is obtained from the norm $N_{\mathbb{K}/\mathbb{Q}}$ by the geometric embedding and a change of variables. Then N is $\bar{A}_{\vec{\alpha}}$-invariant and, since the $N_{\mathbb{K}/\mathbb{Q}}$ takes a discrete set of values on $x^*_{\vec{\alpha}}$, there is a sequence $\alpha_i \to \infty$ such that the vectors comprising $y_{\vec{\alpha}}$ all lie in the countable union of hypersurfaces $\bigcup_{i=1}^{\infty} N^{-1}(\{\alpha_i\})$, and each $N^{-1}(\{\alpha_i\})$ is a finite union of $\bar{A}_{\vec{\alpha}}$-orbits. Only finitely many of these hypersurfaces intersect the bounded set D_{r_0} and therefore there is a finite set $\{v_1, \dots, v_k\} \subset y_{\vec{\alpha}}$ such that if we denote

$$\bar{\mathcal{O}}_i \stackrel{\text{def}}{=} \left\{\bar{a} \in \bar{A}_{\vec{\alpha}} : \bar{a}v_i \in D_{r_0}\right\} \quad \text{and} \quad W_i \stackrel{\text{def}}{=} \bar{\mathcal{O}}_i y_{\vec{\alpha}},$$

then

$$\bar{A}_{\vec{\alpha}}y_{\vec{\alpha}} \cap \mathcal{S}_{r_0} = \bigcup_{i=1}^{k} W_i.$$

Let $L_1 \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n : x_n = 1\}$, so that $\bar{A}^{(1)}_{\vec{\alpha}}$ is the subgroup of $\bar{A}_{\vec{\alpha}}$ leaving L_1 invariant, and $D_r \subset L_1$. Thus we may write $\bar{\mathcal{O}}_i = \mathcal{O}_i\bar{a}_i$ for some $\bar{a}_i \in \bar{A}_{\vec{\alpha}}$ and with $\mathcal{O}_i \subset \bar{A}^{(1)}_{\vec{\alpha}}$. Setting $\bar{\Lambda}_i = \bar{a}_i y_{\vec{\alpha}}$ we obtain (b).

We show (c). Note that the orbit map $\bar{a} \mapsto \bar{a}\bar{\Lambda}_i$ is injective on each $\mathcal{O}_i$ since otherwise $y_{\vec{\alpha}}$ would contain two distinct vectors with the same vertical component, contradicting the boundedness of the orbit $\{a_t y_{\vec{\alpha}} : t \in \mathbb{R}\}$. It follows that $\mathcal{O}_i$ is contained in a fundamental domain for the orbit $\bar{A}_{\vec{\alpha}}y_{\vec{\alpha}}$. Since $m_{\vec{\alpha}}$ can be identified with the restriction of the Haar measure of $\bar{A}_{\vec{\alpha}}$ to a fundamental domain via the orbit map, and since

$$\bar{A}_{\vec{\alpha}} = \bar{A}^{(1)}_{\vec{\alpha}} \times \{a_t\},$$

item (c) follows from (35).

To prove item (d), suppose by contradiction that for any $j \in \mathbb{N}$ there exists $\Lambda_j \in \bar{A}y_{\vec{\alpha}} \cap \mathcal{S}_{r_0,<1/j}$. Then by compactness we can take a convergent sequence to conclude that there is $\bar{\Lambda}' \in \bar{A}_{\vec{\alpha}}y_{\vec{\alpha}} \cap \mathscr{X}_n(D_{r_0}, 2)$. Hence $\bar{\Lambda}'$ contains a nonzero vector with zero vertical component. The map $\bar{h}_{\vec{\alpha}}$ preserves the horizontal space $\mathbb{R}^d$, and thus $\bar{h}_{\vec{\alpha}}^{-1}\bar{\Lambda}' \in Ax^*_{\vec{\alpha}}$ also contains a nonzero vector with zero vertical component, contradicting Proposition 7.4. Item (e) now follows from Theorem 4.4(i). ∎

Lemma 8.14. *For any norm $\|\cdot\|$, any $r > 0$, and any $\vec{\alpha}$ as in (56), the set M_r defined in (91) satisfies $m_{\vec{\alpha}}\left(\mathscr{X}_n(M_r)\right) = 0$.*

Proof. We apply Lemma 8.8, with $L = \bar{A}_{\vec{\alpha}}$ and $W = M_r$. We need to check (88). Let $m_{\bar{A}_{\vec{\alpha}}}$, $m_{\bar{A}^{(1)}_{\vec{\alpha}}}$ denote respectively the Haar measure on $\bar{A}_{\vec{\alpha}}$ and $\bar{A}^{(1)}_{\vec{\alpha}}$. Assume by way of contradiction that there exists $v \in \mathbb{R}^n \cap y_{\vec{\alpha}} \cap M_r$ such that

$$m_{\bar{A}_{\vec{\alpha}}}\left(\left\{\bar{a} \in \bar{A}_{\vec{\alpha}} : \bar{a}v \in M_r\right\}\right) > 0.$$

Since M_r is a_t-invariant, and the action of $\bar{A}^{(1)}_{\vec{\alpha}}$ commutes with the projection $\pi_{\mathbb{R}^d}$ this implies that for $\bar{v} \stackrel{\mathrm{def}}{=} \pi_{\mathbb{R}^d}(v)$ we have

$$m_{\bar{A}^{(1)}_{\vec{\alpha}}}(B) > 0, \qquad \text{where } B \stackrel{\mathrm{def}}{=} \left\{\bar{a} \in \bar{A}^{(1)}_{\vec{\alpha}} : \|\bar{a}\bar{v}\| = \|\bar{v}\|\right\}.$$

We can make a change of variables to replace $\bar{A}^{(1)}_{\vec{\alpha}}$ with the group of diagonal matrices of positive diagonal entries and determinant 1 in $\mathrm{SL}_d(\mathbb{R})$. We will denote this group by A_2. By Proposition 7.4, after this change of variables, the coordinates of $\bar{v}$ are nonzero. Thus, by applying another linear change of variables on $\mathbb{R}^d$ and changing the norm, we may assume that $\bar{v} = \mathbf{1}$ is the vector all of whose coordinates are equal to 1. Furthermore, we can replace B with $b_0 B$ for some b_0, in order to assume that the identity is a Lebesgue density point for B. To get a contradiction we will show that there is $b \in B$ such that $||b\mathbf{1}|| > ||\mathbf{1}||$.

Define

$$\Xi : A_2 \to \mathbb{R}^d, \qquad \Xi\left(\mathrm{diag}\left(x_1, \ldots, x_d\right)\right) \stackrel{\mathrm{def}}{=} (x_1, \ldots, x_d)^{\mathrm{t}},$$

and $\nu \stackrel{\mathrm{def}}{=} \Xi_* m_{A_2}$. Then Ξ is a diffeomorphism between A_2 and the hypersurface $\left\{x \in \mathbb{R}^d : x_i > 0,\ \prod_1^d x_i = 1\right\}$. Note that $\mathbf{1} = \Xi(e)$ and the ray $\mathbb{R}_+\mathbf{1}$ is transverse to $\Xi(A)$. Let B_0 be the image under Ξ of the intersection of B with a bounded neighborhood of e. Then $\mathbf{1}$ is a density point for ν, i.e.

$$\lim_{r \to 0+} \frac{\nu(B_0 \cap B(\mathbf{1}, r))}{\nu(B(\mathbf{1}, r))} = 1. \tag{97}$$

This implies that the interior of the convex hull of B_0 intersects the ray $\mathbb{R}_+\mathbf{1}$, since if this did not hold there would be a linear functional vanishing on $\mathbf{1}$ and non-negative on B_0, and the left hand side of (97) would be at most $\frac{1}{2}$.

It follows that there are distinct $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1} \in \Xi(B)$ and positive scalars $\beta_1, \ldots, \beta_{d+1}, c$, such that

$$\sum_{i=1}^{d+1} \beta_i = 1 \quad \text{and} \quad \sum_{i=1}^{d+1} \beta_i \mathbf{x}_i = c\mathbf{1}.$$

Write $\mathbf{x}_i = (x_{i1}, \ldots, x_{id})$. By the inequality of means in the jth coordinate, for each $1 \le j \le d$ we have

$$\prod_{i=1}^{d+1} x_{ij}^{\beta_i} \le \sum_{i=1}^{d+1} \beta_i x_{ij} = c,$$

with strict inequality for at least one j since the $\mathbf{x}_i$ are distinct. Multiplying these inequalities for $1 \le j \le d$ and taking into account that for each i, $\prod_{j=1}^{d} x_{ij} = 1$, we get

$$1 = \prod_{j=1}^{d} \prod_{i=1}^{k} x_{ij}^{\beta_i} < c^n,$$

and therefore $c > 1$.

Since $\mathbf{x}_i \in \Xi(B)$ we have $||\mathbf{x}_i|| = ||\mathbf{1}||$ for each i. By the triangle inequality

$$c||\mathbf{1}|| = \left\| \sum_{1}^{k} \beta_i \mathbf{x}_i \right\| \le \sum \beta_i ||\mathbf{x}_i|| = \sum \beta_i ||\mathbf{1}|| = ||\mathbf{1}||,$$

a contradiction.

□

Proof of Theorem 8.11. Property (A) of §5 is immediate, property (B) follows from Proposition 8.13(e), and property (C) follows from Lemma 8.5. Item (1) of Definition 5.3 follows from Proposition 8.13(d). It remains to prove (2). Define $\mathcal{U}_{r_0}$ using (93). In light of Lemma 8.10, we only need to show that

$$m_{\vec{\alpha}} \left((\mathrm{cl}_{\mathscr{X}_n}(\mathcal{S}_{r_0}) \smallsetminus \mathcal{U}_{r_0})^{(0,1)} \right) = 0. \tag{98}$$

For (98) it is enough to show that the two sets on the RHS of (94) are $\mu_{\mathcal{S}_{r_0}}$-null, and hence, by Theorem 4.4(iv), that

$$m_{\vec{\alpha}} \left(\mathscr{X}_n(D_{r_0}, 2)^{\mathbb{R}} \right) = 0 \quad \text{and} \quad m_{\vec{\alpha}} \left(\mathscr{X}_n(D_{r_0} \smallsetminus D_{r_0}^{\circ})^{\mathbb{R}} \right) = 0.$$

Since any lattice in $\mathscr{X}_n(D_{r_0}, 2)^{\mathbb{R}}$ contains a nonzero horizontal vector, and $\bar{h}_{\vec{\alpha}}$ preserves the horizontal subspace, Proposition 7.4 implies $\mathrm{supp}\, m_{\vec{\alpha}} \cap \mathscr{X}_n(D_{r_0}, 2)^{\mathbb{R}} = \varnothing$. The second equality follows from (92) and Lemma 8.14.

□

8.5. **Proof of Proposition 8.7.** We will need the following:

Lemma 8.15. *Let L be Lie group and let L_1, L_2 be closed subgroups such that $L_1 \cap L_2 = \{e\}$ and $\dim L_1 + \dim L_2 = \dim L$. Then:*

(1) the map $\alpha : L_1 \times L_2 \to L$ given by $\alpha(\ell_1, \ell_2) = \ell_1 \cdot \ell_2$ is a diffeomorphism onto an open subset $\mathcal{U} \subset L$.

(2) If furthermore L is unimodular, and $m_{L_1}^{\mathrm{left}}, m_{L_2}^{\mathrm{right}}$ denote left and right Haar measures on L_1, L_2 respectively, then $\alpha_* \left(m_{L_1}^{\mathrm{left}} \times m_{L_2}^{\mathrm{right}} \right)$ is proportional to the restriction to $\mathcal{U}$ of a Haar measure on L.

Lemma 8.15 is standard, see e.g. [EW11, Lemma 11.31].

Proof of Proposition 8.7. Let $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma_G \overset{\mathrm{def}}{=} \mathrm{SL}_n(\mathbb{Z})$, $\Gamma_H \overset{\mathrm{def}}{=} H(\mathbb{Z})$, $\pi_G : G \to G/\Gamma_G$ and $\pi_H : H \to H/\Gamma_H$ the projections, and let m_G and m_H denote respectively the Haar measures on G and on H. Recall that a fundamental domain for G/Γ in G is a Borel subset $\Omega \subset G$ for which $\pi_G|_\Omega$ is a bijection, and that every Borel set on which π_G is injective is contained in fundamental domain. One can describe the measure $m_{G/\Gamma}$ by $m_{G/\Gamma}(A) = m_G(\Omega \cap \pi_G^{-1}(A))$, and in particular, this formula does not depend on the choice of Ω. The same facts hold for H in place of G.

We will first show that there is $c > 0$ such that

$$\mu_{\mathcal{S}_{r_0}} = c\,\varphi_*\left(m_{\mathscr{E}_n} \times m_{\mathbb{R}^d}|_{\bar{B}_{r_0}}\right) \tag{99}$$

and then we will determine the constant. It suffices to show (99) locally, that is, to prove that for a.e. $x_0 = \varphi(\Lambda_0, v_0) \in \mathcal{S}_{r_0}$ there is a neighborhood $\mathcal{U} = \varphi(\mathcal{V})$ of x, where $\mathcal{V}$ is open in $\mathscr{E}_n \times \bar{B}_{r_0}$ and contains (Λ_0, v_0), and $c > 0$, such that $\mu_{\mathcal{S}_{r_0}}|_{\mathcal{U}} = c\,\varphi_*\left((m_{\mathscr{E}_n} \times m_{\mathbb{R}^d})\,|_{\mathcal{V}}\right)$. Using Lemmas 8.1 and 8.5, and the discreteness of Γ_H, Γ_G, we see that for any $x_0 \in \mathcal{S}^{\sharp}_{r_0}$ there are open sets $\mathcal{W}_H \subset H$, $\mathcal{W}_U \subset U$, $\mathcal{W}_G \subset G$, and $\varepsilon > 0$, such that π_H is injective on $\mathcal{W}_H$, π_G is injective on $\mathcal{W}_G$, and

$$\bigcup_{t \in [0,\varepsilon]} a_t\left(\mathcal{W}_U \mathcal{W}_H\right) \subset \mathcal{W}_G.$$

We can assume that $\mathcal{W}_G \subset \Omega_G$ and $\mathcal{W}_H \subset \Omega_H$. By Theorem 4.4(iii), it is enough to show that the product map

$$\Psi : \mathbb{R} \times \mathbb{R}^d \times H \to G, \quad \Psi(t, v, h) \overset{\mathrm{def}}{=} a_t u(v) h$$

pushes the measure $m_{\mathbb{R}} \times m_{\mathbb{R}^d} \times m_H$ to a multiple of $m_G|_{\mathbf{P}}$, where $\mathbf{P}$ is the image of Ψ. (The attentive reader will have noted that we have switched the order of the factors, that is we work with $\mathbb{R} \times \mathbb{R}^d \times H$ rather than with $\mathbb{R} \times H \times \mathbb{R}^d$.) To see this, let m_U be the Haar measure on U, i.e., m_U is the image of $m_{\mathbb{R}^d}$ under the map $v \mapsto u(v)$. Note also that $m_Q \overset{\mathrm{def}}{=} m_{\mathbb{R}} \times m_U$ is a left Haar measure on the group $Q \overset{\mathrm{def}}{=} \{a_t\} \ltimes U$, and, since H is unimodular, m_H is a right Haar measure on H. By Lemma 8.15, $\nu \overset{\mathrm{def}}{=} \Psi_*(m_Q \times m_H)$ is a multiple of $m_G|_{\mathbf{P}}$. This proves (99) and moreover shows that $\mathrm{supp}(\mu_{\mathcal{S}_{r_0}}) = \mathcal{S}_{r_0}$.

We now claim that

$$c = \frac{1}{\zeta(n)}. \tag{100}$$

We normalize the Haar measures used above by requiring $m_G(\Omega_G) = m_H(\Omega_H) = 1$, and similarly normalize m_U by requiring that a fundamental domain for $U/U(\mathbb{Z})$ has measure one. The preceding discussion shows that c is the scalar for which $m_G = c\nu$. Write

$$q(t, x_1, \ldots, x_d) \stackrel{\text{def}}{=} \begin{pmatrix} e^t & 0 & \cdots & e^t x_1 \\ 0 & e^t & \cdots & e^t x_2 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & e^{-dt} \end{pmatrix} \in Q,$$

and note, using the fact that m_U induces a probability measure on $U/U(\mathbb{Z})$, that the map

$$Q \to \mathbb{R}^n, \quad q = q(t, x_1, \ldots, x_d) \mapsto q\mathbf{e}_n = \left(e^t x_1, \ldots, e^t x_d, e^{-dt}\right)^{\mathbf{t}}$$

sends m_Q to the restriction of $m_{\mathbb{R}^n}$ to the upper half space

$$\mathbb{R}^n_+ \stackrel{\text{def}}{=} \{(y_1, \ldots, y_n)^{\mathbf{t}} \in \mathbb{R}^n : y_n > 0\}.$$

Also, the map

$$Q \times H \to G, \quad (q, h) \mapsto qh$$

is injective, since for $g = qh$ we can reconstruct q uniquely from the vector $q\mathbf{e}_n$. It is easy to verify that the image $QH \subset G$ is open.

Let $f = \mathbf{1}_E : \mathbb{R}^n \to \mathbb{R}$ be the indicator of the set E defined by

$$E \stackrel{\text{def}}{=} \left\{ \left(e^t x_1, \ldots, e^t x_d, e^{-dt}\right)^{\mathbf{t}} : t \in [0, 1], \forall i, \ |x_i| \leq \frac{1}{2} \right\} \subset \mathbb{R}^n_+.$$

For $\Lambda \in \mathscr{X}_n$, write

$$\widehat{f}(\Lambda) \stackrel{\text{def}}{=} \sum_{v \in \Lambda \smallsetminus \{0\}} f(v) \quad \text{and} \quad \widehat{f}^p(\Lambda) \stackrel{\text{def}}{=} \sum_{v \in \Lambda_{\text{prim}}} f(v).$$

By the Siegel summation formula [Sie45],

$$1 = m_{\mathbb{R}^n}(E) = \int_{\mathbb{R}^n} f \, dm_{\mathbb{R}^n} = \int_{\mathscr{X}_n} \widehat{f} \, dm_{\mathscr{X}_n} = \zeta(n) \int_{\mathscr{X}_n} \widehat{f}^p \, dm_{\mathscr{X}_n}. \tag{101}$$

We define a lift of f to $QH \subset G$ by

$$F : QH \to \mathbb{R}, \quad F(qh) = f(q\mathbf{e}_n) \cdot \mathbf{1}_{\Omega_H}(h).$$

It is easily checked that this definition implies

$$\sum_{\gamma \in \Gamma} F(g\gamma) = \hat{f}^p(g\mathbb{Z}^n). \tag{102}$$

Then

$$c \int_G F \, dm_G = \int_G F \, d\nu = \int_{\mathbb{R}^n_+} f \, dm_{\mathbb{R}^n} = 1. \tag{103}$$

Using Fubini and 'folding', we have

$$\int_G F \, dm_G = \sum_{\gamma \in \Gamma} \int_{\Omega_G \gamma} F \, dm_G = \int_{\mathscr{X}_n} \sum_{\gamma \in \Gamma} F(g\gamma) \, dm_{\mathscr{X}_n} \stackrel{(102)}{=} \int_{\mathscr{X}_n} \hat{f}^p \, dm_{\mathscr{X}_n}. \tag{104}$$

Comparing (101), (103) and (104) gives (100). □

9. Special subsets of the section

The cross-section $\mathcal{S}_{r_0}$ contains two subsets of Diophantine significance, related respectively to best approximations and to ε-approximations. In this section we will introduce these subsets, establish their Jordan measurability under suitable hypotheses, and discuss their temperedness.

9.1. **The set $\mathcal{B}$ for best approximations.**

Recall from (87) that for $\Lambda \in \mathcal{S}^{\sharp}_{r_0}$ we denote by $v_\Lambda \in \mathbb{R}^d$ the horizontal component of the unique vector $v(\Lambda) \in \Lambda_{\mathrm{prim}} \cap D_{r_0}$. Let

$$r(\Lambda) \stackrel{\mathrm{def}}{=} ||v_\Lambda|| \tag{105}$$

be its distance from the vertical axis, and let

$$\mathcal{B} = \left\{ \Lambda \in \mathcal{S}^{\sharp}_{r_0} : C_{r(\Lambda)} \cap \Lambda_{\mathrm{prim}} = \{\pm v(\Lambda)\} \right\}, \tag{106}$$

where C_r is defined in (75).

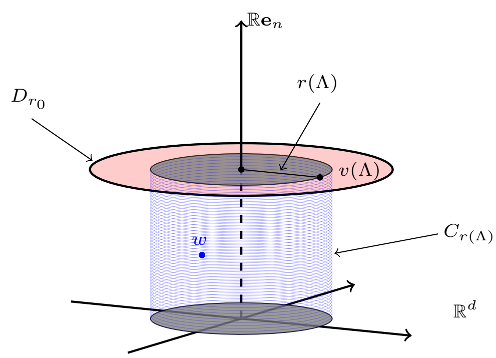


Figure 2. If the cylinder $C_{r(\Lambda)}$ defined by the unique vector $v(\Lambda) \in \Lambda \cap D_{r_0}$ contains another lattice point w, then $\Lambda \notin \mathcal{B}$.

The set $\mathcal{B}$ will detect best approximations. It consists of lattices in the cross-section $\mathcal{S}_{r_0}$ with a unique vector in D_{r_0}, such that the cylinder $C_{r(\Lambda)}$ they define contains no lattices points besides $\{0, \pm v(\Lambda)\}$ (see Figure 2). Note that $\mathcal{B}$ depends on the norm, although this is not reflected in the notation.

Lemma 9.1. *The set $\mathcal{B}$ is open in $\mathcal{S}_{r_0}$. The boundary $\partial_{\mathcal{S}_{r_0}}\mathcal{B}$ is contained in the union of $\mathscr{X}_n(D_{r_0}, 2)$ and*

$$\mathscr{Z} \stackrel{\mathrm{def}}{=} \{\Lambda \in \mathscr{X}_n : \exists v, w \in \Lambda_{\mathrm{prim}} \ s.t.\ v \neq \pm w \ and\ ||\pi_{\mathbb{R}^d}(v)|| = ||\pi_{\mathbb{R}^d}(w)||\}. \tag{107}$$

Proof. We show $\mathcal{B}$ is open by showing its complement is closed. Let

$$K \stackrel{\mathrm{def}}{=} \left\{ \Lambda \in \mathcal{S}^{\sharp}_{r_0} : \#(C_{r(\Lambda)} \cap \Lambda_{\mathrm{prim}}) \geq 3 \right\};$$

that is, K consists of the lattices $\Lambda \in \mathcal{S}_{r_0}^\sharp$ such that the cylinder $C_{r(\Lambda)}$ contains at least one extra primitive vector besides $\pm v(\Lambda)$. By definition

$$\mathcal{S}_{r_0} \smallsetminus \mathcal{B} = \mathscr{X}_n(D_{r_0}, 2) \cup K.$$

By Lemma 8.1 $\mathscr{X}_n(D_{r_0}, 2)$ is closed, so we let $\Lambda_i \in K$ be a sequence converging to some Λ, and show that $\Lambda \in \mathscr{X}_n(D_{r_0}, 2) \cup K$.

By Lemma 8.1, $\Lambda \in \mathscr{X}_n(D_{r_0})$, and suppose that $\Lambda \notin \mathscr{X}_n(D_{r_0}, 2)$, so that $\Lambda \in \mathcal{S}_{r_0}^\sharp$. Since $\Lambda_i \in K$, the cylinders $C_{r(\Lambda_i)}$ contain at least three primitive vectors. By Lemma 8.1, $v(\Lambda_i) \to v(\Lambda)$ and so $r(\Lambda_i) \to r(\Lambda)$. Let $r' > r(\Lambda)$, then for all large enough i, $r(\Lambda_i) < r'$ and so for all large enough i, $\Lambda_i \in \mathscr{X}_n(C_{r'}, 3)$. Using Lemma 8.1 again we see that $\Lambda \in \mathscr{X}_n(C_{r'}(1), 3)$. Since this is true for all $r' > r(\Lambda)$ we deduce that $\Lambda \in \mathscr{X}_n(C_{r(\Lambda)}, 3)$, and so $\Lambda \in K$.

We now prove the second assertion. Let

$$U_0 \stackrel{\text{def}}{=} \left\{\Lambda \in \mathcal{S}_{r_0}^\sharp : \Lambda_{\text{prim}} \cap C^\circ_{r(\Lambda)} \neq \varnothing\right\}$$

(where C°_r denotes the interior of C_r). We claim that U_0 is open in $\mathcal{S}_{r_0}$. From this, and since $\mathcal{S}_{r_0} \smallsetminus \mathcal{B}$ is closed, it will follow that

$$\partial_{\mathcal{S}_{r_0}} \mathcal{B} = \partial_{\mathcal{S}_{r_0}} (\mathcal{S}_{r_0} \smallsetminus \mathcal{B}) \subset (\mathscr{X}_n(D_{r_0}, 2) \cup K) \smallsetminus U_0.$$

Since $K \smallsetminus U_0 \subset \mathscr{Z}$, this will imply the second assertion of the Lemma.

To prove that U_0 is open, let Λ_i be a sequence in $\mathscr{X}_n(D_{r_0})$ that converges to a lattice $\Lambda \in U_0$. We need to show that for all large enough i, $\Lambda_i \in U_0$. Since Λ does not belong to the closed set $\mathscr{X}_n(D_{r_0}, 2)$, $\Lambda_i \notin \mathscr{X}_n(D_{r_0}, 2)$ for all large i. By Lemma 8.1, $v(\Lambda_i) \to v(\Lambda)$ and thus $r(\Lambda_i) \to r(\Lambda)$. Since there exists a primitive vector $w \in \Lambda_{\text{prim}}$ in the open set $C^\circ_{r(\Lambda)}$, there is some $r' < r(\Lambda)$ such that $w \in C^\circ_{r'}$. In other words, $\Lambda \in \mathscr{X}_n(C^\circ_{r'})$. The latter set is open in $\mathscr{X}_n$ by Lemma 8.2, and hence $\Lambda_i \in \mathscr{X}_n(C^\circ_{r'})$ for all large i. But for all large i we also have $r(\Lambda_i) > r'$, so Λ_i must contain a primitive vector in the open cylinder $C^\circ_{r(\Lambda_i)}$, i.e., $\Lambda_i \in U_0$. $\square$

Lemma 9.2. *In Case I, for any norm on $\mathbb{R}^d$, the set $\mathcal{B}$ is $\mu_{\mathcal{S}_{r_0}}$-JM and $\mu_{\mathcal{S}_{r_0}}(\mathcal{B}) > 0$.*

Proof. The fact that $\mu_{\mathcal{S}_{r_0}}(\mathcal{B}) > 0$ follows from the openness of $\mathcal{B}$ in $\mathcal{S}_{r_0}$ (see Lemma 9.1) and the fact that $\mu_{\mathcal{S}_{r_0}}$ has full support by Proposition 8.7.

By Lemmas 8.4 and 9.1, it is enough to show that $\mu_{\mathcal{S}_{r_0}}(\mathscr{Z}) = 0$, for $\mathscr{Z}$ as in (107). Since $\mathscr{Z}$ is $\{a_t\}$-invariant, this is equivalent to showing that $m_{\mathscr{X}_n}(\mathscr{Z}) = 0$. Let $G = \mathrm{SL}_n(\mathbb{R})$ and let m_G denote the Haar measure on G, so that $m_{\mathscr{X}_n}$ is the restriction of m_G to a fundamental domain. For $g \in G$ denote the column vectors of g by $g_1, \ldots, g_n$, and for $v \in \mathbb{R}^n$, denote $\bar{v} = \pi_{\mathbb{R}^d}(v)$. Let

$$\widetilde{\mathscr{Z}} \stackrel{\text{def}}{=} \{g \in G : ||\bar{g_1}|| = ||\bar{g_n}||\}.$$

Then the image of $\widetilde{\mathscr{Z}}$ in $\mathscr{X}_n$ under the projection $g \mapsto g\mathbb{Z}^n$ contains $\mathscr{Z}$. Also $m_G(\widetilde{\mathscr{Z}}) = 0$, as can be seen using the invariance of m_G under right-multiplication by elements of $\{a_t\}$. This implies $m_{\mathscr{X}_n}(\mathscr{Z}) = 0$. □

Lemma 9.3. *In case II with $\mu = m_{\vec{\alpha}}$, $\mu_{\mathcal{S}_{r_0}}(\mathcal{B}) > 0$.*

Proof. Assume by way of contradiction that $\mu_{\mathcal{S}_{r_0}}(\mathcal{B}) = 0$. Since $\mathcal{B}$ is open in $\mathcal{S}_{r_0}$ by Lemma 9.1, it follows from Proposition 8.13(b), that $\bar{A}_{\vec{\alpha}} y_{\vec{\alpha}} \cap \mathcal{B} = \varnothing$. The contradiction follows since it is easy to see that for any lattice Λ without non-zero vectors on the vertical axis, the trajectory $\{a_t\Lambda : t > 0\}$ always visits $\mathcal{B}$.

□

Let $\bar{A}_{\vec{\alpha}}$ be the group defined in (69).

Definition 9.4. We say that the norm $\|\cdot\|$ on $\mathbb{R}^d$ is $\bar{A}_{\vec{\alpha}}$-*analytic*, if for any $v, w \in \mathbb{R}^n$, the set

$$\bar{A}_{v,w} \stackrel{\text{def}}{=} \left\{\bar{a} \in \bar{A}_{\vec{\alpha}} : \|\pi_{\mathbb{R}^d}(\bar{a}v)\| = \|\pi_{\mathbb{R}^d}(\bar{a}w)\|\right\}$$

is an analytic subset of $\bar{A}_{\vec{\alpha}} \cong \mathbb{R}^d$; that is, the zero-set of an analytic function.

For example, the Euclidean norm is analytic.

Lemma 9.5. *In Case II, if the norm is $\bar{A}_{\vec{\alpha}}$-analytic then the set $\mathcal{B}$ is $\mu_{\mathcal{S}_{r_0}}$-JM.*

Proof. As in the proof of Lemma 9.2, we need to show that $m_{\vec{\alpha}}(\mathscr{Z}) = 0$. Let $\bar{h}_{\vec{\alpha}}$, $y_{\vec{\alpha}}$ and $\bar{A}^{(1)}_{\vec{\alpha}}$ be as in (63), (69) and (95). Let $m_{\bar{A}_{\vec{\alpha}}}$ denote the Haar measure on $\bar{A}_{\vec{\alpha}}$, so that $m_{\vec{\alpha}}$ is the pushforward under the map $\bar{a} \mapsto \bar{a}y_{\vec{\alpha}}$, of the restriction of $m_{\bar{A}_{\vec{\alpha}}}$ to a measurable set.

Suppose by way of contradiction that $m_{\vec{\alpha}}(\mathscr{Z}) > 0$. Then there are fixed $v, w \in (y_{\vec{\alpha}})_{\text{prim}}$ such that $v \neq \pm w$, and

$$m_{\bar{A}_{\vec{\alpha}}}\left(\bar{A}_{v,w}\right) > 0.$$

Since the norm $\|\cdot\|$ is $\bar{A}_{\vec{\alpha}}$-analytic, we must have $\bar{A}_{\vec{\alpha}} = \bar{A}_{v,w}$. Let

$$\mathbf{t}_i = \bar{h}_{\vec{\alpha}}(\mathbf{e}_i), \quad i = 1, \dots, n$$

be a basis of $\mathbb{R}^n$ consisting of simultaneous eigenvectors for $\bar{A}_{\vec{\alpha}}$. Write

$$v = \sum_{i=1}^{n} r_i \mathbf{t}_i, \quad w = \sum_{i=1}^{n} s_i \mathbf{t}_i.$$

By acting with

$$\bar{a}_t \stackrel{\text{def}}{=} \bar{h}_{\vec{\alpha}} \operatorname{diag}\left(e^{dt}, e^{-t}, \dots, e^{-t}\right) \bar{h}_{\vec{\alpha}}^{-1} \subset \bar{A}_{\vec{\alpha}} = \bar{A}_{v,w}$$

we see that that $|r_1| = |s_1|$; indeed,

$$\frac{|r_1|}{|s_1|} = \frac{\|\bar{a}_t(r_1\mathbf{t}_1)\|}{\|\bar{a}_t(s_1\mathbf{t}_1)\|} = \frac{\|\pi_{\mathbb{R}^d}(\bar{a}_t v)\| + o(t)}{\|\pi_{\mathbb{R}^d}(\bar{a}_t w)\| + o(t)} = 1 + o(t).$$

In particular, the first coordinate (with respect to the eigenbasis $\mathbf{t}_i$) of $v \pm w \in y_{\vec{\alpha}}$ is zero. Therefore the nonzero vector $\bar{h}_{\vec{\alpha}}^{-1}(v \pm w) \in (x^*_{\vec{\alpha}})_{\mathrm{prim}}$ has one of its coordinates equal to zero, which contradicts Proposition 7.4. $\square$

Lemma 9.6. *In Case II, if the norm on $\mathbb{R}^d$ is the sup-norm, then the set $\mathcal{B}$ is $\mu_{\mathcal{S}_{r_0}}$-JM.*

Proof. Once again we need to show that $m_{\vec{\alpha}}(\mathscr{Z}) = 0$. Supposing by contradiction that $m_{\vec{\alpha}}(\mathscr{Z}) > 0$, and using the explicit description of $m_{\vec{\alpha}}$ given in Propositions 7.4 and 7.5, we see that there is a lattice Λ_1 of type $\boldsymbol{\sigma}$, and two linearly independent primitive vectors $v, w \in \Lambda_1$, such that for a set of positive measure of $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$,

$$\left\|\pi_{\mathbb{R}^d}(\bar{h}_{\vec{\alpha}} a_{\mathbf{t}} v)\right\| = \left\|\pi_{\mathbb{R}^d}(\bar{h}_{\vec{\alpha}} a_{\mathbf{t}} w)\right\|, \quad \text{where } a_{\mathbf{t}} \overset{\mathrm{def}}{=} \mathrm{diag}\left(e^{t_1}, \ldots, e^{t_d}, e^{-\sum_{j=1}^d t_j}\right).$$

Using (58) and (63), we obtain that for some $\beta_v, \beta_w \in \mathbb{K}$, which are linearly independent over $\mathbb{Q}$, we have

$$\left\|\begin{pmatrix} \sum_{j=1}^d e^{t_j}(\sigma_j(\alpha_1) - \alpha_1)\sigma_j(\beta_v) \\ \vdots \\ \sum_{j=1}^d e^{t_j}(\sigma_j(\alpha_d) - \alpha_d)\sigma_j(\beta_v) \end{pmatrix}\right\| = \left\|\begin{pmatrix} \sum_{j=1}^d e^{t_j}(\sigma_j(\alpha_1) - \alpha_1)\sigma_j(\beta_w) \\ \vdots \\ \sum_{j=1}^d e^{t_j}(\sigma_j(\alpha_d) - \alpha_d)\sigma_j(\beta_w) \end{pmatrix}\right\|.$$

Since we are working with the sup-norm, this implies that there are indices $1 \le k_1, k_2 \le d$ and $\omega = \pm 1$ such that for a set of positive measure of $\mathbf{t}$,

$$\sum_{j=1}^d e^{t_j}(\sigma_j(\alpha_{k_1}) - \alpha_{k_1})\sigma_j(\beta_v) = \omega \sum_{j=1}^d e^{t_j}(\sigma_j(\alpha_{k_2}) - \alpha_{k_2})\sigma_j(\beta_w).$$

This is an equality of analytic expressions, which holds for $\mathbf{t}$ in a set of positive measure, and thus it must hold for all $\mathbf{t}$. In particular, we can take partial derivatives $\frac{\partial}{\partial t_j}|_{\mathbf{t}=0}$ to obtain that for any $j = 1, \ldots, d$ we have

$$(\sigma_j(\alpha_{k_1}) - \alpha_{k_1})\sigma_j(\beta_v) = \omega(\sigma_j(\alpha_{k_2}) - \alpha_{k_2})\sigma_j(\beta_w). \tag{108}$$

If $k_1 = k_2$ it follows that $\beta_v = \omega\beta_w$, which is a contradiction. Assume therefore that $k_1 \neq k_2$. Multiplying (108) by $\sigma_j(\beta_w^{-1})$ and letting $\beta \overset{\mathrm{def}}{=} \beta_v/\beta_w$ we get

$$(\sigma_j(\alpha_{k_1}) - \alpha_{k_1})\sigma_j(\beta) = \omega(\sigma_j(\alpha_{k_2}) - \alpha_{k_2}). \tag{109}$$

Recall our convention $\sigma_n = \mathrm{Id}$, which implies that for $\gamma \in \mathbb{K}$, the trace Tr satisfies

$$\sum_{j=1}^d \sigma_j(\gamma) = \mathrm{Tr}(\gamma) - \gamma.$$

Summing (109) over j gives

$$\mathrm{Tr}(\alpha_{k_1}\beta) - \alpha_{k_1}\beta - \alpha_{k_1}(\mathrm{Tr}(\beta) - \beta) = \omega(\mathrm{Tr}(\alpha_{k_2}) - \alpha_{k_2} - d\alpha_{k_2}),$$

and thus

$$\mathrm{Tr}(\alpha_{k_1}\beta - \omega\alpha_{k_2}) - \mathrm{Tr}(\beta)\alpha_{k_1} + n\alpha_{k_2} = 0$$

and this is a nontrivial linear dependence over $\mathbb{Q}$ between $1, \alpha_{k_1}, \alpha_{k_2}$. This is a contradiction. $\square$

Remark 9.7. There are other norms for which the conclusion of Lemmas 9.5 and 9.6 fails. Here is a sketch of how one can build an example. Let $d = 2$. We define a norm, a lattice Λ, two vectors $v, w \in \Lambda$, and positive r, ε, so that:

(a) v, w are primitive in Λ and linearly independent;
(b) Λ arises from a number field as in §7.3;
(c) v, w are both in ∂C_r, with $C_r^\circ \cap \Lambda = \{0\}$;
(d) for all $t \in (-\varepsilon, \varepsilon)$,

$$\|g_t \pi_{\mathbb{R}^2}(v)\| = \|g_t \pi_{\mathbb{R}^2}(w)\|, \quad \text{where } g_t \stackrel{\text{def}}{=} \operatorname{diag}\left(e^t, e^{-t}\right).$$

One can then see using from Proposition 8.13(c), that in this case the boundary of $\mathcal{B}$ must have positive $\mu_{\mathcal{S}_{r_0}}$-measure.

To this end, we first construct a norm $\|\cdot\|$, two linearly independent vectors u_1, u_2 in $\mathbb{R}^2$, and positive ε, r, such that $r = \|g_t u_1\| = \|g_t u_2\|$ for all $t \in (-\varepsilon, \varepsilon)$. Let

$$u_1 = \begin{pmatrix} 1 \\ 1/4 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}.$$

We will define the norm in $\mathbb{R}^2$ by specifying a symmetric convex body $\mathbf{B}$ which is its unit ball. The boundary of $\mathbf{B}$ consists of four small smooth arcs $\pm\gamma_i, i = 1, 2$, where γ_i passes through u_i, and four line segments connecting the ends of these arcs. See Figure 3. The γ_i are carefully chosen so that (d)

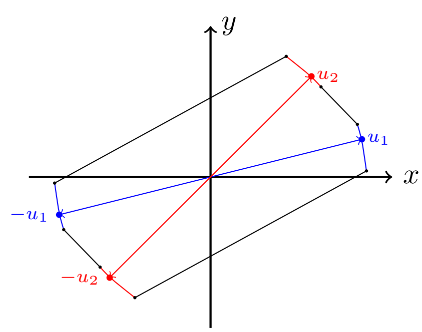


FIGURE 3. Defining the norm by a carefully chosen convex set in $\mathbb{R}^2$.

is satisfied, for some small $\varepsilon > 0$, and for $u_1 = \pi_{\mathbb{R}^2}(v),\ u_2 = \pi_{\mathbb{R}^2}(w)$. Since γ_i passes through u_i and determines the norm, the requirement (d) implies that γ_1 uniquely determines γ_2, and we have to choose γ_1 so that the resulting figure is convex. This can be shown using an explicit computation in polar coordinates. Moreover the computation shows that in this construction, we have freedom to vary u_1, u_2 in some open set in $\mathbb{R}^2 \times \mathbb{R}^2$.

It is not hard to construct a lattice $\Lambda \subset \mathbb{R}^3$ containing two primitive linearly independent vectors v, w such that $\pi_{\mathbb{R}^2}(v) = u_1, \pi_{\mathbb{R}^2}(w) = u_2$. Moreover it is not hard to choose Λ so that (c) holds, and again this can be carried out for u_1, u_2 in some open set in $\mathbb{R}^2 \times \mathbb{R}^2$. Using the fact that number field lattices are dense (up to a rescaling) in the space of lattices, one sees that one can also arrange that (b) holds.

Recall Definition 4.7. The following temperedness result will be important for our analysis. For related results, see [Lag82], [CC] and [Che13].

Proposition 9.8. *The set $\mathcal{B}$ is tempered.*

Proof. Assume otherwise. Recalling the notation (75), let M be large enough so that any set of cardinality $M+1$ in $C_1(e) = \{\left(\begin{smallmatrix} v \\ c \end{smallmatrix}\right) \in \mathbb{R}^n : ||v|| \leq 1, |c| \leq e\}$ contains distinct points $\left(\begin{smallmatrix} v \\ c \end{smallmatrix}\right), \left(\begin{smallmatrix} v' \\ c' \end{smallmatrix}\right)$ such that $||v - v'|| < \frac{1}{3}$ and $|c - c'| < 1$. Such a number M exists by the compactness of $C_1(e)$. By applying a linear transformation which dilates the horizontal subspace, one sees that for any $r > 0$, in any subset of $C_r(e)$ of cardinality $M+1$, there are distinct points $\left(\begin{smallmatrix} v \\ c \end{smallmatrix}\right), \left(\begin{smallmatrix} v' \\ c' \end{smallmatrix}\right)$ such that $||v - v'|| < \frac{r}{3}$ and $|c - c'| < 1$.

We claim that $\mathcal{B}$ is M-tempered. Indeed, suppose by contradiction that one can find $\Lambda \in \mathcal{B}$ and $0 = t_0 < t_1 < \cdots < t_M \leq 1$ such that $a_{t_j}\Lambda \in \mathcal{B}$ for $0 \leq j \leq M$. Then, for each j the vector $w_j \stackrel{\text{def}}{=} v(a_{t_j}\Lambda)$ (see (74)) satisfies

$$w_j \in a_{t_j}\Lambda_{\text{prim}} \cap D_{r_0} \quad \text{and } a_{t_j}\Lambda \cap C^{\circ}_{\pi_{\mathbb{R}^d}(w_j)} = \{0\}.$$

Applying a_{-t_j}, we find vectors $a_{-t_j}w_j = \begin{pmatrix} v_j \\ e^{t_j} \end{pmatrix} \in \Lambda$ such that

$$\Lambda \cap C^{(j)} = \{0\}, \quad \text{where } C^{(j)} \stackrel{\text{def}}{=} C^{\circ}_{||v_j||}(e^{t_j}).$$

In particular, this implies $||v_0|| \geq ||v_1|| \geq \cdots \geq ||v_M||$, and thus

$$\begin{pmatrix} v_j \\ e^{t_j} \end{pmatrix} \in C_{\|v_0\|}(e) \quad \text{for } j = 0, \ldots, M.$$

By the property of M, there are indices $j_1 < j_2$ such that $||v_{j_1} - v_{j_2}|| < \frac{\|v_0\|}{3}$ and $|e^{t_{j_2}} - e^{t_{j_1}}| < 1$. The difference

$$w = \begin{pmatrix} v_{j_2} \\ e^{t_{j_2}} \end{pmatrix} - \begin{pmatrix} v_{j_1} \\ e^{t_{j_1}} \end{pmatrix}$$

then belongs to the lattice Λ but also to the interior of the cylinder $C_{||v_0||}(1)$, which contradicts the assumption $\Lambda \in \mathcal{B}$. $\square$

9.2. **The set $\mathcal{S}_\varepsilon$ for ε-approximations.**

Lemma 9.9. *In both Cases I and II, for any $0 < r \leq r_0$, the sets $\mathcal{S}_r$ are $\mu_{\mathcal{S}_{r_0}}$-JM.*

Proof. By Lemma 8.1, $\mathscr{X}_n(D_r)$ is closed in $\mathscr{X}_n$ and hence in $\mathscr{X}_n(D_{r_0})$. Again by Lemma 8.1, $\mathcal{S}^{\sharp}_{r_0} \cap \mathscr{X}_n(D^{\circ}_r)$ is open in $\mathscr{X}_n(D_{r_0})$. It follows that

$$\partial_{\mathcal{S}_{r_0}}\mathcal{S}_r \subset \mathscr{X}_n(D_r) \smallsetminus \left(\mathcal{S}^{\sharp}_{r_0} \cap \mathscr{X}_n(D^{\circ}_r)\right) \subset \mathscr{X}_n(D_{r_0}, 2) \cup \mathscr{X}_n(D_r \smallsetminus D^{\circ}_r).$$

By Lemma 8.4, $\mu_{\mathcal{S}_{r_0}}(\mathscr{X}_n(D_{r_0}, 2)) = 0$, and hence, by Corollary 4.4, it suffices to show that

$$\mu(\mathscr{X}_n(M_r)) = 0,$$

where $M_r = (D_r \smallsetminus D_r^\circ)^{\mathbb{R}}$ is as in (91). In Case I, this follows from Lemma 8.8 as in the proof of Lemma 8.9; in Case II, this follows from Lemma 8.14. □

Proposition 9.10. *For any $d > 1$, for any $\varepsilon > 0$ and any norm, $\mathcal{S}_\varepsilon$ is not tempered. For $d = 1$, and any $\varepsilon > 0$, $\mathcal{S}_\varepsilon$ is tempered.*

Proof. Suppose $d > 1$. Given M and ε, let $\Lambda \in \mathscr{X}_n$ be a lattice containing the primitive vectors $\mathbf{u} \overset{\mathrm{def}}{=} \frac{1}{2M}\mathbf{e}_n$ and $\mathbf{v} \overset{\mathrm{def}}{=} (v, 1)^{\mathrm{t}}$, where $v \in \mathbb{R}^d$ satisfies $\|v\| < \frac{\varepsilon}{2}$. Such a lattice exists because $d > 1$. Then Λ contains the vectors

$$\mathbf{v}_j \overset{\mathrm{def}}{=} \mathbf{v} + j\mathbf{u} = \begin{pmatrix} v \\ 1 + \frac{j}{2M} \end{pmatrix}, \quad j = 0, \dots, M.$$

Let $t_j \overset{\mathrm{def}}{=} \log\left(1 + \frac{j}{2M}\right) \in [0, 1)$. This choice ensures that $a_{t_j}\Lambda$ contains the vector

$$a_{t_j}\mathbf{v}_j = \begin{pmatrix} e^{\frac{t_j}{2}} v \\ 1 \end{pmatrix}$$

which shows that $a_{t_j}\Lambda \in \mathcal{S}_\varepsilon$. Since M is arbitrary, $\mathcal{S}_\varepsilon$ is not tempered.

Now suppose $d = 1$. By Minkowski's second theorem, there is $\kappa > 0$ satisfying the following. For any lattice $\Lambda \in \mathscr{X}_2$, denote by v_1 a shortest nonzero vector of Λ, and by v_2 a shortest vector such that v_1, v_2 are linearly independent. Then $\|v_2\| \geq \frac{\kappa}{\|v_1\|}$. Thus, for any $C > 0$ there is $c > 0$ so that if $\|v_1\| < c$, then there is no $v_2 \in \Lambda_{\mathrm{prim}} \smallsetminus \{\pm v_1\}$ with $\|v_2\| < C$. Now, given $\varepsilon > 0$, choose C large enough so that the ball of radius C around the origin contains the rectangle $\mathbf{R} \overset{\mathrm{def}}{=} [-\varepsilon, \varepsilon] \times [1, e]$. Let c be the corresponding constant, and choose M large enough so that any M points in $\mathbf{R}$ contain a pair of distinct points of distance less than c. With this choice, $\mathcal{S}_\varepsilon$ is M-tempered. Indeed, if this were not the case, there would be a lattice $\Lambda \in \mathscr{X}_2$ and $0 = t_0 \leq t_1 < \cdots < t_M \leq 1$ such that $a_{t_j}\Lambda \in \mathcal{S}_\varepsilon$. This implies that Λ contains $M + 1$ primitive vectors in $\mathbf{R}$, and hence one of their differences, which is a nonzero vector of Λ, has length less than c. But by the choice of c and C we get that $\mathbf{R} \cap \Lambda_{\mathrm{prim}}$ contains only two vectors $\pm v_1$. □

9.3. **The adelic space, cross-section and cross-section measure.** We now lift the cross-section $\mathcal{S}_{r_0}$ to the adelic space, and, using the theory developed so far, derive properties crucial to our discussion.

Let $\mathscr{X}_n^{\mathbb{A}}$ be the adelic space, let $\pi : \mathscr{X}_n^{\mathbb{A}} \to \mathscr{X}_n$ be the projection, and let $m_{\mathscr{X}_n^{\mathbb{A}}}$ and $m_{\tilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}}}$ be the measures introduced in §7.4 and §7.4.1 respectively. Let

$$\widetilde{\mathcal{S}}_{r_0} \overset{\mathrm{def}}{=} \pi^{-1}(\mathcal{S}_{r_0}), \quad \widetilde{\mathcal{B}} \overset{\mathrm{def}}{=} \pi^{-1}(\mathcal{B}), \quad \text{and } \widetilde{\mathcal{S}}_\varepsilon \overset{\mathrm{def}}{=} \pi^{-1}(\mathcal{S}_\varepsilon). \tag{110}$$

Theorem 9.11. *Let μ be $m_{\mathscr{X}_n^{\mathbb{A}}}$ (Case I) or $\mu = m_{\tilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}}}$ (Case II). Then, $\widetilde{\mathcal{S}}_{r_0}$ is a μ reasonable cross-section. The sets $\widetilde{\mathcal{S}}_\varepsilon$ are $\mu_{\widetilde{\mathcal{S}}_{r_0}}$-JM, and the set $\widetilde{\mathcal{B}}$*

is tempered. In Case I, $\widetilde{\mathcal{B}}$ is $\mu_{\widetilde{\mathcal{S}}_{r_0}}$-JM, and the same statement holds in Case II provided the norm on $\mathbb{R}^d$ is either the sup-norm or an $\bar{A}_{\vec{\alpha}}$-analytic norm (in particular, if it is the Euclidean norm). In Case I the measure $\mu_{\widetilde{\mathcal{S}}_{r_0}}$ is K_f-invariant (where K_f is as in (68)), and in Case II, $\mu_{\widetilde{\mathcal{S}}_{r_0}}$ is $M_{\vec{\alpha}}$-invariant (where $M_{\vec{\alpha}}$ is as in §7.4.1).

Proof. The fact that $\widetilde{\mathcal{S}}_{r_0}$ is a μ-cross-section, its μ-reasonability, and the statements about Jordan measurability and temperedness are immediate consequences of Proposition 6.1, the fact that π is open, and the results in §8 and §9 proved above.

For the invariance of $\mu_{\widetilde{\mathcal{S}}_{r_0}}$ under K_f in Case I, we use Theorem 4.4, item (vii), the fact that K_f preserves $m_{\mathscr{X}_n^{\mathbb{A}}}$, and the fact that K_f acts transitively on the fibers of π and hence leaves $\widetilde{\mathcal{S}}_{r_0}$ invariant (see the discussion of π in §7.4). For the invariance of $\mu_{\widetilde{\mathcal{S}}_{r_0}}$ under $M_{\vec{\alpha}}$ in Case II, we use the same argument, noting that μ is $M_{\vec{\alpha}}$-invariant and $M_{\vec{\alpha}}$ preserves the fibers of π. □

Given $\theta \in \mathbb{R}^d$ we define $\Lambda_\theta = u(-\theta)\mathbb{Z}^n$ as in (62) and (15), and define its lift to the adelic space by

$$\widetilde{\Lambda}_\theta \stackrel{\text{def}}{=} (u(-\theta), e_f)\,\mathrm{SL}_n(\mathbb{Q}) \in \mathscr{X}_n^{\mathbb{A}}. \tag{111}$$

Proposition 9.12. *Let $\vec{\alpha}$ be as in Case II, and let $\mu = m_{\tilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}}}$ (see Proposition 7.7 for notation). Suppose the norm on $\mathbb{R}^d$ is either the sup norm or $\bar{A}_{\vec{\alpha}}$-analytic (see Definition 9.4). Let $\varepsilon \in (0, r_0)$ be large enough so that $\mu_{\mathcal{S}_{r_0}}(\mathcal{S}_\varepsilon) > 0$, and let μ' be equal to any one of $\mu_{\widetilde{\mathcal{S}}_{r_0}}$, $\mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{S}}_\varepsilon}$, $\mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}}$. Then the point $\widetilde{\Lambda}_{\vec{\alpha}}$ is (a_t, μ')-generic (see Definition 5.1). Moreover, if $t_k \to \infty$ are such that $a_{t_k}\widetilde{\Lambda}_{\vec{\alpha}} \in \widetilde{\mathcal{S}}_{r_0}$ and $\lim_k a_{t_k}\widetilde{\Lambda}_{\vec{\alpha}} = x$, then $x \in \operatorname{supp}\mu_{\widetilde{\mathcal{S}}_{r_0}}$.*

Proof. By Proposition 7.7, any point in $\tilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}}$ is (a_t, μ)-generic, and thus, by Propositions 7.2 and 7.5, $\widetilde{\Lambda}_{\vec{\alpha}}$ is (a_t, μ)-generic.

The statement will follow by applying Theorem 5.11. We verify its conditions. First observe that $\widetilde{\Lambda}_{\vec{\alpha}} \notin \Delta^{\mathbb{R}}_{\widetilde{\mathcal{S}}_{r_0}}$. To see this note that if $\widetilde{\Lambda}_{\vec{\alpha}} \in \Delta^{\mathbb{R}}_{\widetilde{\mathcal{S}}_{r_0}}$, the orbit $\left\{a_t\widetilde{\Lambda}_{\vec{\alpha}} : t > 0\right\}$ intersects $\widetilde{\mathcal{S}}_{r_0,<\varepsilon'}$ for arbitrarily small $\varepsilon' > 0$. This leads to a contradiction using a similar argument to the one giving the proof of Proposition 8.13(d).

Next, we apply Theorem 5.11 to $\mathcal{S}'$ being one of the three sets under consideration, $\widetilde{\mathcal{S}}_{r_0}, \widetilde{\mathcal{S}}_\varepsilon, \widetilde{\mathcal{B}}$. We need to check that $\mathcal{S}'$ is $\mu_{\widetilde{\mathcal{S}}_{r_0}}$-JM and that $\mu_{\widetilde{\mathcal{S}}_{r_0}}(\mathcal{S}') > 0$. The Jordan measurability of $\mathcal{S}'$ follows from Lemmas 9.5, 9.6, 9.9. The positivity of $\mu_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{S}}_\varepsilon)$ holds by choice of ε and the positivity of $\mu_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{B}})$ follows from Lemma 9.3 and the fact that $\mu_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{B}}) = \mu_{\mathcal{S}_{r_0}}(\mathcal{B})$ (by Proposition 6.1(b) and Proposition 7.7).

We prove the last assertion in the statement. Let x be an accumulation point in positive time of $a_t\widetilde{\Lambda}_{\vec{\alpha}}$. Note that by (111), (67), (69), and (71) we have that $\widetilde{\Lambda}_{\vec{\alpha}} = q\widetilde{\Lambda}$ where q satisfies $a_t q a_{-t} \to e_\infty$. Thus, the accumulation points of $a_t\widetilde{\Lambda}_{\vec{\alpha}}$ in positive time are equal to those of $a_t\widetilde{\Lambda}$. By Proposition 7.7, a_t acts uniquely ergodically on the orbit $\tilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}}$, and hence $x \in \tilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}} = \operatorname{supp}\mu$. It is clear that $\operatorname{supp}\mu_{\widetilde{\mathcal{S}}_{r_0}} = \operatorname{supp}\mu \cap \widetilde{\mathcal{S}}_{r_0}$ and so $x \in \operatorname{supp}\mu_{\widetilde{\mathcal{S}}_{r_0}}$ as claimed.

□

9.4. **The special case $d = 1$.** When $d = 1$ we can fully describe the cross-section $\mathcal{S}_{r_0}$ and the sets $\mathcal{S}_\varepsilon$ and $\mathcal{B}$ (see Figure 4).

Proposition 9.13. *Let $d = 1$ and let*

$$u_x \stackrel{\text{def}}{=} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad h_y \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \quad \Lambda_{x,y} \stackrel{\text{def}}{=} u_x h_y \mathbb{Z}^2.$$

Then we can choose $r_0 = 1$, and for $\varepsilon \in (0,1)$,

$$\mathcal{S}_\varepsilon = \left\{\Lambda_{x,y} : |x| \le \varepsilon,\ y \in [0,1)\right\}. \tag{112}$$

Setting

$$f_1(t) \stackrel{\text{def}}{=} -\frac{1}{1+t} \qquad \text{and} \qquad f_2(t) \stackrel{\text{def}}{=} \frac{1}{2-t},$$

we have

$$\mathcal{B} = \left\{\Lambda_{x,y} \in \mathcal{S}^\sharp_{r_0} : y \in [0,1),\ f_1(y) \le x \le f_2(y)\right\}, \tag{113}$$

and the map $(x,y) \mapsto \Lambda_{x,y}$ is injective on $\{(x,y) : y \in (0,1),\ |x| < 1\}$.

Proof. In case $d = 1$ the choice $r_0 = 1$ satisfies (76), and the group $\{h_y : y \in \mathbb{R}\}$ coincides with the group H in (82). The orbit $H\mathbb{Z}^2$ is a periodic orbit consisting of all lattices that contain $\mathbf{e}_2$ as a primitive vector, and $\{h_y : y \in [0,1)\}$ is a fundamental domain for the quotient $H/H(\mathbb{Z}) \cong H\mathbb{Z}^2$. Clearly $(x,1)^{\mathrm{t}} \in \Lambda_{x,y} \cap D_1$ when $|x| \le \varepsilon$, and conversely, for every $\Lambda \in \mathcal{S}_\varepsilon$ there is $x \in [-\varepsilon, \varepsilon]$ such that $(x,1)^{\mathrm{t}} \in \Lambda_{\mathrm{prim}}$, and hence $u_{-x} \in H\mathbb{Z}^2$. This proves (112).

To see the injectivity, note that

$$\Lambda_{x,y} = \begin{pmatrix} 1+xy & x \\ y & 1 \end{pmatrix}\mathbb{Z}^2 = \left\{\begin{pmatrix} m(1+xy)+nx \\ my+n \end{pmatrix} : m,n \in \mathbb{Z}\right\}. \tag{114}$$

Thus, when $\Lambda_{x,y} \in \mathcal{S}^\sharp_{r_0}$, the vector $(x,1)^{\mathrm{t}}$ is the unique vector in $\Lambda_{x,y} \cap D_1$. That is, for $\Lambda_{x,y} \in \mathcal{S}^\sharp_{r_0}$, x is uniquely determined, and since $u_{-x}\Lambda \in H\mathbb{Z}^2$, it follows that $y \in [0,1)$ is also uniquely determined. When $\Lambda_{x,y} \notin \mathcal{S}^\sharp_{r_0}$ there must be a horizontal vector in $\Lambda_{x,y}$ of length less than 2, that is, integers (m,n) in (114) satisfying

$$my + n = 0 \quad \text{and} \quad |m(1+xy)+nx| < 2.$$

Plugging the first of these equations into the second implies $m \in \{-1,0,1\}$. If $y \in (0,1)$ and $(m,n) \ne (0,0)$, $my+n = 0$ is now impossible. This proves the injectivity. We note in passing that the map we have defined is essentially the map ψ of (86).

We now determine the set of $(x, y) \in (-1, 1) \times [0, 1)$ for which $\Lambda_{x,y} \in \mathcal{B}$. Using (114) and the definition of $\mathcal{B}$ we see that $\Lambda_{x,y} \notin \mathcal{B}$ if and only if $\Lambda_{x,y} \notin \mathcal{S}^{\sharp}_{r_0}$ or there is $(m, n) \notin \{(0, 0), (0, 1)\}$ such that

$$my + n \in [0, 1) \tag{115}$$

and

$$|m(1 + xy) + nx| \leq |x|. \tag{116}$$

The condition (115) implies that $n = -\lfloor my \rfloor$. Suppose first that $x \in [0, 1)$. In this case, (116) becomes $|x(my + n) + m| \leq x$, and by plugging (115) into (116) and examining the possibilities for m, n one sees that the only possibility is $m = -1, n = 1$. This gives the inequality $1 + xy - x \leq x$, which is equivalent to $x \geq f_2(y)$. In the second case $x \in (-1, 0]$ the only possibilities become $m = 1, n = 0$, leading to $1+xy \leq -x$ or $x \leq -f_1(y)$. $\square$

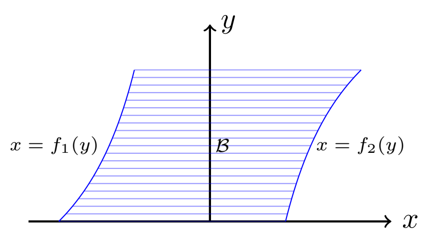


Figure 4. The set $\{(x, y) : y \in [0, 1),\ f_1(y) \leq x \leq f_2(y)\}$ parameterizing $\mathcal{B}$.

Remark 9.14. For $d > 1$, our description of $\mathcal{S}_{r_0}$ is not as explicit since it depends on the choice of an explicit fundamental domain for $H\mathbb{Z}^n$ in H. It would be interesting to completely describe the set $\mathcal{B}$, for $d = 2$ and some fixed norm on $\mathbb{R}^2$.

10. Interpreting the visits to the cross-section

Given $\theta \in \mathbb{R}^d$, let $\widetilde{\Lambda}_\theta$ be as in (111). The goal of this section is to read off Diophantine properties of θ, from the successive times t_i for which $a_{t_i}\widetilde{\Lambda}_\theta \in \widetilde{\mathcal{S}_{r_0}}$. More precisely, for $\widetilde{\mathcal{B}}$, $\widetilde{\mathcal{S}}_\varepsilon$ as in (110), we will relate the successive visits to $\widetilde{\mathcal{B}}$ to best approximations, and the successive visits to $\widetilde{\mathcal{S}}_\varepsilon$, to ε-approximations.

10.1. **The adelic cross-section as a Cartesian product.** Let $\psi : \mathcal{S}^{\sharp}_{r_0} \to \mathscr{E}_n \times \bar{B}_{r_0}$ be as in (86), and $\widetilde{\mathcal{S}}^{\sharp}_{r_0} \overset{\mathrm{def}}{=} \pi^{-1}(\mathcal{S}^{\sharp}_{r_0})$. We now augment ψ and define a map

$$\widetilde{\psi} : \widetilde{\mathcal{S}}^{\sharp}_{r_0} \to \mathscr{E}_n \times \bar{B}_{r_0} \times \widehat{\mathbb{Z}}^n \quad \text{by} \quad \widetilde{\psi} = (\psi \circ \pi, \psi_f); \tag{117}$$

that is, the first two coordinates of $\widetilde{\psi}$ are given by $\psi \circ \pi$, and the third coordinate is given by a map $\psi_f : \widetilde{\mathcal{S}}^{\sharp}_{r_0} \to \widehat{\mathbb{Z}}^n$, which we now define.

Given $\widehat{\Lambda} \in \widetilde{\mathcal{S}}^{\sharp}_{r_0}$ we may write $\widehat{\Lambda} = (g_\infty, g_f)\Gamma_{\mathbb{A}}$ with $g_f \in K_f$, and then $\Lambda \overset{\text{def}}{=} \pi(\widehat{\Lambda}) = g_\infty \mathbb{Z}^n$. Since $\Lambda \in \mathcal{S}^{\sharp}_{r_0}$, the vector $v(\Lambda)$ defined in (87) is the unique primitive vector of the lattice Λ in D_{r_0}, and the columns of g_∞ are a basis of Λ. Replacing g_∞ by another representative of the coset $g_\infty\Gamma$, we may assume that the nth column of g_∞ is $v(\Lambda)$. Moreover, the uniqueness of $v(\Lambda)$ implies that if $\gamma \in \Gamma$ satisfies that $g_\infty\gamma$ is another representative of Λ having this property, then

$$g_\infty \mathbf{e}_n = v(\Lambda) = g_\infty \gamma \mathbf{e}_n,$$

and hence $\gamma \in H(\mathbb{Z})$. With these choices we define

$$\psi_f(\widehat{\Lambda}) \overset{\text{def}}{=} g_f \mathbf{e}_n; \tag{118}$$

that is, if we write $g_f = (g_p)_{p \in \mathbf{P}}$, then $\psi_f(\widehat{\Lambda})$ is the element of $\widehat{\mathbb{Z}}^n$ whose p coordinate is the n-th column of g_p. The above discussion implies that ψ_f is well-defined.

Lemma 10.1. *The map $\widetilde{\psi}$ is K_f-equivariant and continuous.*

Proof. The fact that $\widetilde{\psi}$ commutes the K_f-actions on $\widetilde{\mathcal{S}}^{\sharp}_{r_0}$ and on $\hat{\mathbb{Z}}^n_{\text{prim}}$ follows directly from the procedure defining $\widetilde{\psi}$ discussed above.

We prove its continuity. We have already seen in Lemma 8.5 that $\psi \circ \pi$ is continuous, and it remains to establish the continuity of ψ_f. Assume $\widehat{\Lambda}_k \to \widehat{\Lambda}$ in $\widetilde{\mathcal{S}}^{\sharp}_{r_0}$ and write $\widehat{\Lambda}_k = \left(g_\infty^{(k)}, g_f^{(k)}\right)\Gamma_{\mathbb{A}}$, $\widehat{\Lambda} = (g_\infty, g_f)\Gamma_{\mathbb{A}}$, where $\Gamma_{\mathbb{A}} \overset{\text{def}}{=} \mathrm{SL}_n(\mathbb{Q})$ and where the representatives are chosen so that $g_f, g_f^{(k)} \in K_f$ and $g_\infty \mathbf{e}_n, g_\infty^{(k)} \mathbf{e}_n \in D_{r_0}$. There are $\gamma_k \in \Gamma_{\mathbb{A}}$ such that

$$\left(g_\infty^{(k)}\gamma_k, g_f^{(k)}\gamma_k\right) \to (g_\infty, g_f),$$

and we need to show that $g_f^{(k)}\mathbf{e}_n \to g_f \mathbf{e}_n$. It follows from our choice of representatives that

$$\gamma_k \in \Gamma_{\mathbb{A}} \cap K_f = \mathrm{SL}_n(\mathbb{Z}).$$

We claim that $\gamma_k \mathbf{e}_n = \mathbf{e_n}$ for all large enough k; this will imply $g_f^{(k)}\mathbf{e}_n = g_f^{(k)}\gamma_k \mathbf{e}_n \to g_f \mathbf{e}_n$ and conclude the proof. To prove the claim, suppose by contradiction that $\gamma_k \mathbf{e}_n \neq \mathbf{e}_n$ for infinitely many k, then along a subsequence, the lattices $\Lambda_k = \pi(\widehat{\Lambda}_k)$ contain the two distinct primitive vectors $v_k = g_\infty^{(k)}\mathbf{e}_n$, $w_k = g_\infty^{(k)}\gamma_k \mathbf{e}_n$, and the sequence Λ_k converges in $\mathscr{X}_n$. Note that by assumption $v_k, w_k \in D_{r_0}$. By passing to a further subsequence we may assume that the sequences (v_k) and (w_k) converge to limits in D_{r_0}. Since the sequence of lattices (Λ_k) is bounded in $\mathscr{X}_n$, and since v_k, w_k are distinct, the limits $\lim v_k, \lim w_k$ are distinct vectors in D_{r_0} which belong to the limit lattice $\pi(\widehat{\Lambda}) = g_\infty \mathbb{Z}^n$. This contradicts the assumption that $\widehat{\Lambda} \in \widetilde{\mathcal{S}}^\sharp_{r_0}$. $\square$

We now describe the image of the cross-section measure under $\widetilde{\psi}$. We first set up some notation. Denote the natural projections by

$$\begin{aligned} P_1 &: \mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n_{\mathrm{prim}} \to \mathscr{E}_n \\ P_2 &: \mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n_{\mathrm{prim}} \to \mathbb{R}^d \\ P_3 &: \mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n_{\mathrm{prim}} \to \widehat{\mathbb{Z}}^n_{\mathrm{prim}} \\ P_{12} &: \mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n_{\mathrm{prim}} \to \mathscr{E}_n \times \mathbb{R}^d. \end{aligned} \tag{119}$$

Also let $\mathrm{Proj} : \mathbb{R}^d \to \mathbb{S}^{d-1}$ be the radial projection as in (26). Given an $\{a_t\}$-invariant measure μ on $\mathscr{X}_n^{\mathbb{A}}$, let $\mu_{\widetilde{\mathcal{S}}_{r_0}}$ be the cross-section measure, and define a measure on $\mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n_{\mathrm{prim}}$ by

$$\nu \stackrel{\mathrm{def}}{=} \widetilde{\psi}_* \mu_{\widetilde{\mathcal{S}}_{r_0}}. \tag{120}$$

Let $\nu^{(\mathscr{E}_n)}$, $\nu^{(\mathbb{R}^d)}, \nu^{(f)}, \nu^{(\infty)}$, $\nu^{(\mathbb{S}^{d-1})}, \nu^{(\mathscr{X}_d)}$ denote the projection of ν under the maps $P_1, P_2, P_3, P_{12}, \mathrm{Proj} \circ P_2, \pi_{\mathscr{X}_d} \circ P_1$ respectively. Let $\bar{A}^{(1)}_{\vec{\alpha}}, M_{\vec{\alpha}}$ be the groups in (95) and §7.4.1 respectively, and note that $\bar{A}^{(1)}_{\vec{\alpha}}$ and acts linearly on $\mathbb{R}^d$ and on $\mathscr{E}_n$ via the embedding $\bar{A}^{(1)}_{\vec{\alpha}} \subset H$.

We have:

Proposition 10.2. *In Case I, with $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$, we have*

$$\nu = \frac{1}{\zeta(n)} \left(m_{\mathscr{E}_n} \times m_{\mathbb{R}^d}|_{\bar{B}_{r_0}} \times m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}} \right); \tag{121}$$

in particular, the measures $\nu^{(\mathscr{E}_n)}$, $\nu^{(\mathbb{R}^d)}$, $\nu^{(f)}$, $\nu^{(\mathscr{X}_d)}$ are scalar multiples of the measures $m_{\mathscr{E}_n}$, $m_{\mathbb{R}^d}|_{B_{r_0}}$, $m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}$, $m_{\mathscr{X}_d}$, and the measures $\nu^{(\mathbb{R}^d)}$ and $\nu^{(\mathbb{S}^{d-1})}$ are invariant under any linear transformations of $\mathbb{R}^d$ preserving the norm $\|\cdot\|$.

In Case II, with $\mu = m_{\tilde{L}_{\vec{\alpha}} \tilde{y}_{\vec{\alpha}}}$, we have $\nu = \nu^{(\infty)} \times \nu^{(f)}$, and there exist finitely many subsets $\mathcal{O}_i \subset \bar{A}_1$ homeomorphic to closed balls, lattices $\bar{\Lambda}_i \in \bar{A}_{\vec{\alpha}} y_{\vec{\alpha}} \cap \mathscr{E}_n$, and vectors $v_i \in \bar{\Lambda}_i$, $i = 1, \dots, k$, such that the following hold:

1. *The measure $\nu^{(\mathscr{E}_n)}$ is the sum of the push-forwards of the restriction to each $\mathcal{O}_i$ of a Haar measure on $\bar{A}_1$, via the orbit map $\bar{a} \mapsto \bar{a}\bar{\Lambda}_i$. In particular, $\nu^{(\mathscr{E}_n)}$ is supported on a $(d-1)$-dimensional submanifold of $\mathscr{E}_n$.*
2. *The measure $\nu^{(\mathbb{R}^d)}$ is the sum of the pushforwards of the restriction to each $\mathcal{O}_i$ of a Haar measure on $\bar{A}_1$, via the orbit map $\bar{a} \mapsto \bar{a}v_i$. In particular, $\nu^{(\mathbb{R}^d)}$ is supported on a $(d-1)$-dimensional submanifold of $\mathbb{R}^d$.*

(3) *The measure $\nu^{(f)}$ is the unique $M_{\vec{\alpha}}$-invariant probability measure supported on the orbit $M_{\vec{\alpha}}\mathbf{e}_n \subset \hat{\mathbb{Z}}_{\mathrm{prim}}$.*
(4) *The measure $\nu^{(\infty)}$ is the sum of the pushforwards of the restriction to each $\mathcal{O}_i$ of a Haar measure on $\bar{A}_1$, via the orbit map $\bar{a} \mapsto \bar{a}(\bar{\Lambda}_i, v_i)$. In particular, it is supported on a $(d-1)$ dimensional submanifold and is singular with respect to $\nu^{(\mathscr{E}_n)} \times \nu^{(\mathbb{R}^d)}$.*
(5) *The measure $\nu^{(\mathbb{S}^{d-1})}$ is supported on a proper subset of $\mathbb{S}^{d-1}$, and in particular, is not invariant under the group of orthogonal transformations of $\mathbb{R}^d$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{X}_n^{\mathbb{A}} \supset \widetilde{\mathcal{S}}_{r_0}^{\sharp} & \xrightarrow{\widetilde{\psi}} & \mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}_{\mathrm{prim}}^n \\ \downarrow \pi & & \downarrow P_{12} \\ \mathscr{X}_n \supset \mathcal{S}_{r_0}^{\sharp} & \xrightarrow[\psi]{} & \mathscr{E}_n \times \mathbb{R}^d \end{array} \tag{122}$$

Recall from Lemma 10.1, that $\widetilde{\psi}$ is K_f-equivariant. The map ψ has the following weak equivariance property: Let $H_0 \stackrel{\mathrm{def}}{=} \left\{ \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix} \right) : g \in \mathrm{SL}_d(\mathbb{R}) \right\} \subset H$. Note that $\bar{A}_1 \subset H_0$ and that H_0 acts diagonally on $\mathscr{E}_n \times \mathbb{R}^d$. If $\Lambda \in \mathcal{S}_{r_0}^{\sharp}$ and $h \in H_0$ is such that $h\Lambda \in \mathcal{S}_{r_0}^{\sharp}$ and $hv_\Lambda = v_{h\Lambda}$, then $\psi(h\Lambda) = h\psi(\Lambda)$.

Suppose we are in Case I. By Theorem 9.11, the measure $\mu_{\widetilde{\mathcal{S}}_{r_0}}$ is K_f-invariant. By the K_f-equivariance of $\widetilde{\psi}$, ν is K_f-invariant as well. Since K_f acts transitively on $\hat{\mathbb{Z}}_{\mathrm{prim}}$ and $m_{\hat{\mathbb{Z}}_{\mathrm{prim}}}$ is the unique K_f-invariant probability measure there, we see that ν must be the product of $\nu^{(\infty)}$ and $m_{\hat{\mathbb{Z}}_{\mathrm{prim}}}$. Also, by Proposition 8.7 and the commutativity of the diagram, $(P_{12})_*\nu = \frac{1}{\zeta(n)} \left(m_{\mathscr{E}_n} \times m_{\mathbb{R}^d}|_{\bar{B}_{r_0}} \right)$, and (121) follows. The fact that $\nu^{(\mathscr{X}_d)}$ is proportional to $m_{\mathscr{X}_d}$ is immediate from (14). Finally, the additional assertion about the invariance properties of $\nu^{(\mathbb{R}^d)}$ and $\nu^{(\mathbb{S}^{d-1})}$ follows from the fact that a linear transformation preserving $\|\cdot\|$ also preserves $\mathcal{S}_{r_0}$ (which can be readily seen from (77) and (78)), Theorem 4.4(vii), and the weak equivariance property of ψ discussed above.

Now suppose we are in Case II. By Theorem 4.4(vii) the group $M_{\vec{\alpha}}$ preserves $\mu_{\widetilde{\mathcal{S}}_{r_0}}$. It also preserves the fibers of P_{12} and using Proposition 7.7 and the fact that $\widetilde{\mathcal{S}}_{r_0}$ is $M_{\vec{\alpha}}$-invariant we conclude that it acts transitively on the intersection of each fiber with $\operatorname{supp} \mu_{\widetilde{\mathcal{S}}_{r_0}}$. Therefore, $\nu = \nu^{(\infty)} \times \nu^{(f)}$. Assertion (3) now follows from equations (118) and (71). Assertion (4), and consequently (1) and (2), follows from Proposition 8.13 and the weak equivariance property of ψ discussed above. Assertion (5) follows from Proposition 7.4; indeed, if $\operatorname{supp} \nu^{(\mathbb{S}^{d-1})}$ contained an eigendirection for the group

$\bar{A}$ then some $\Lambda \in A\Lambda_0$ would contain a vector whose horizontal component was in the direction of one of the axes. $\square$

10.2. **Hitting the subsets $\mathcal{B}$ and $\mathcal{S}_\varepsilon$.**

Definition 10.3. We say that two infinite sequences $(a_k)_{k=1}^{\infty}$, $(b_k)_{k=1}^{\infty}$ are *prefix-equivalent* if there exist k_0, ℓ_0 such that $a_{k_0+i} = b_{\ell_0+i}$ for all $i \geq 0$. By convention, any two finite sequences are prefix-equivalent, and a finite sequence is not prefix-equivalent to an infinite sequence.

For $\theta \in \mathbb{R}^d$, let

$$\mathcal{Y}_\theta \stackrel{\text{def}}{=} \{t \geq 0 : a_t\Lambda_\theta \in \mathcal{S}_{r_0}\} \quad \text{ and } \quad \mathcal{Y}_\theta^\sharp \stackrel{\text{def}}{=} \left\{t \in \mathcal{Y}_\theta : a_t\Lambda_\theta \in \mathcal{S}_{r_0}^\sharp\right\}.$$

If $t \in \mathcal{Y}_\theta \smallsetminus \mathcal{Y}_\theta^\sharp$ then Λ_θ contains two vectors with the same vertical component e^t, and with horizontal components differing by a vector of size $O\left(e^{-t/d}\right)$. Thus, by discreteness of Λ_θ, $\mathcal{Y}_\theta \smallsetminus \mathcal{Y}_\theta^\sharp$ is finite, and thus $\mathcal{Y}_\theta$ and $\mathcal{Y}_\theta^\sharp$ are prefix-equivalent. Observe that in the notation of (87), for each $t \in \mathcal{Y}_\theta^\sharp$, there is a unique primitive vector $v(a_t\Lambda_\theta) \in (a_t\Lambda_\theta)_{\text{prim}} \cap D_{r_0}$. In particular,

$$u(\theta)a_{-t}v(a_t\Lambda_\theta) \in \mathbb{Z}^n_{\text{prim}}.$$

The next proposition is key to our analysis:

Proposition 10.4. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$, $\varepsilon > 0$, and $\theta \in \mathbb{R}^d$.*

(1) *Let $\{t_k\}$ be the ordering of $\left\{t \in \mathcal{Y}_\theta^\sharp : a_t\Lambda_\theta \in \mathcal{B}\right\}$ as an increasing sequence. Then the sequence $\mathbf{v}_k = (\mathbf{p}_k, q_k) \stackrel{\text{def}}{=} u(\theta)a_{-t_k}v(a_t\Lambda_\theta) \in \mathbb{Z}^n_{\text{prim}}$ is prefix-equivalent to the sequence of best-approximations of θ.*
(2) *Let $\{t_k\}$ be the ordering of $\left\{t \in \mathcal{Y}_\theta^\sharp : a_t\Lambda_\theta \in \mathcal{S}_\varepsilon\right\}$ as an increasing sequence. Then the sequence $\mathbf{w}_k = (\mathbf{p}_k, q_k) \stackrel{\text{def}}{=} u(\theta)a_{-t_k}v(a_t\Lambda_\theta) \in \mathbb{Z}^n_{\text{prim}}$ is prefix-equivalent to the sequence of ε-approximations of θ.*
(3) *For $t \in \mathcal{Y}_\theta^\sharp$, if we let $\mathbf{u} = u(\theta)a_{-t}v(a_t\Lambda_\theta) \in \mathbb{Z}^n_{\text{prim}}$, then (with the notation (17), (2)),*

$$\widetilde{\psi}\left(a_t\widetilde{\Lambda}_\theta\right) = \left(\rho_{\mathscr{E}_n}(\mathbf{u}), \operatorname{disp}(\theta, \mathbf{u}), \mathbf{u}\right). \tag{123}$$

Proof. Since $\mathcal{Y}_\theta \smallsetminus \mathcal{Y}_\theta^\sharp$ is finite, we can restrict attention to $t \in \mathcal{Y}_\theta^\sharp$. We first treat ε-approximations. Note that for $\mathbf{u} = \binom{\mathbf{p}}{q} \in \mathbb{Z}^n$,

$$\mathbf{u} \text{ is an } \varepsilon\text{-approximation of } \theta \iff q^{1/d}\|\mathbf{p} - q\theta\| \leq \varepsilon \text{ and } \mathbf{u} \in \mathbb{Z}^n_{\text{prim}}.$$

This in turn implies that for $t = \frac{1}{d}\log q$, the lattice $a_t\Lambda_\theta$ satisfies

$$\begin{aligned}(a_t\Lambda_\theta)_{\text{prim}} &\ni \operatorname{diag}\left(q^{1/d}, \dots, q^{1/d}, q^{-1}\right) u(-\theta) \begin{pmatrix}\mathbf{p}\\ q\end{pmatrix} \\ &= \begin{pmatrix} q^{1/d}(\mathbf{p} - q\theta) \\ 1\end{pmatrix} = \begin{pmatrix}\operatorname{disp}(\theta, \mathbf{u}) \\ 1\end{pmatrix} \in D_\varepsilon.\end{aligned} \tag{124}$$

Reversing this computation shows that if $t \geq 0$ is such that $a_t\Lambda_\theta \in \mathcal{S}_\varepsilon$, i.e. $(a_t\Lambda_\theta)_{\mathrm{prim}} \cap D_\varepsilon$ is not empty, then there exist $q \in \mathbb{N}$ and $\mathbf{p} \in \mathbb{Z}^d$ such that $\mathbf{u} = (\mathbf{p}, q)^{\mathrm{t}} \in \mathbb{Z}^n_{\mathrm{prim}}$ and for $t = \frac{1}{d}\log q$ we have $a_t u(-\theta) \left(\begin{smallmatrix}\mathbf{p}\\ q\end{smallmatrix}\right) \in D_\varepsilon$. In other words, $\mathbf{u}$ is an ε-approximation of θ.

The bijection between visits of $a_t\Lambda_\theta$ to $\mathcal{B}$ and best approximations described in the statement follow the same computation coupled with the following observation: a primitive vector $\left(\begin{smallmatrix}\mathbf{p}\\ q\end{smallmatrix}\right) \in \mathbb{Z}^n_{\mathrm{prim}}$ with $q \in \mathbb{N}$ is a best approximation of θ if and only if

$$C_{||\mathbf{p}-q\theta||}(q) \cap \Lambda_\theta = \left\{0, \pm u(-\theta)\begin{pmatrix}\mathbf{p}\\ q\end{pmatrix}\right\}$$

where

$$C_{||\mathbf{p}-q\theta||}(q) = \left\{\begin{pmatrix}\mathbf{x}\\ y\end{pmatrix} \in \mathbb{R}^n : ||\mathbf{x}|| \leq ||\mathbf{p} - q\theta||;\ |y| \leq q\right\}.$$

Acting with a_t on $\Lambda_\theta = u(-\theta)\mathbb{Z}^n$ with $t = \frac{1}{d}\log q$ to bring $u(-\theta)\left(\begin{smallmatrix}\mathbf{p}\\ q\end{smallmatrix}\right)$ to D_{r_0}, we see that $\mathbf{u}$ is a best approximation for θ if and only if $a_t\Lambda_\theta \in \mathcal{B}$ with t as above and $v(a_t\Lambda_\theta) = a_t u(-\theta)\mathbf{u}$.

The third assertion is a straightforward computation combining (124) with (2), (17), (86) and (117). □

Using Proposition 10.4, we now show that for $\varepsilon > \varepsilon_0$ which appears in (7) the condition appearing in Proposition 9.12 is satisfied.

Lemma 10.5. *Let $\vec{\alpha}$ be as in Case II and let $\mu = m_{\tilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}}}$ (see Proposition 7.7 for notation). Let ε_0 be as in (7). Then, for any $\varepsilon > \varepsilon_0$ we have that $\mu_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{S}}_\varepsilon) > 0$.*

Proof. Let $\varepsilon > \varepsilon' > \varepsilon_0$. By definition of ε_0 and by Proposition 10.4, there exists a sequence $t_k \to \infty$ such that $a_{t_k}\widetilde{\Lambda}_{\vec{\alpha}} \in \widetilde{\mathcal{S}}_{\varepsilon'}$. Since the orbit $\left\{a_t\widetilde{\Lambda}_{\vec{\alpha}}\right\}_{t>0}$ is bounded in $\mathscr{X}_n^{\mathbb{A}}$ and since $\widetilde{\mathcal{S}}_{\varepsilon'}$ is closed, we may assume without loss of generality that $a_{t_k}\widetilde{\Lambda}_{\vec{\alpha}}$ converges to some point x. By Proposition 9.12 we deduce that $x \in \widetilde{\mathcal{S}}_{\varepsilon'} \cap \operatorname{supp}\mu_{\widetilde{\mathcal{S}}_{r_0}}$. Furthermore, since x is an accumulation point of a bounded forward orbit of a_t, it follows that the two-sided orbit of x is bounded. Therefore $x \in \widetilde{\mathcal{S}}^\sharp_{r_0}$ (since if $\pi(x)$ contains two vectors with the same vertical coordinate, the orbit $a_t x$ diverges in negative time). By Lemma 8.1, $\mathscr{X}_n(D^\circ_\varepsilon) \cap \mathcal{S}^\sharp_{r_0}$ is open in $\mathcal{S}_{r_0}$, and hence its preimage in $\mathscr{X}_n^{\mathbb{A}}$ is open in $\widetilde{\mathcal{S}}_{r_0}$. By choice of ε' we have $\mathcal{S}_{\varepsilon'} \subset \mathscr{X}_n(D^\circ_\varepsilon)$, therefore

$$x \in \pi^{-1}(\mathscr{X}_n(D^\circ_\varepsilon) \cap \mathcal{S}^\sharp_{r_0}) \subset \widetilde{\mathcal{S}}_\varepsilon,$$

and we deduce that $\mu_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{S}}_\varepsilon) > 0$. □

11. Properties of the cross-section measures

In this section we will describe some properties of measures on the product space $\mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$. The measures we will consider arise as follows: in Case I let $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$, and in Case II let $\mu = m_{\tilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}}}$ be as in §7.4.1. Let $\mu_{\widetilde{\mathcal{S}}_{r_0}}$ be the cross-section measure, let $\widetilde{\psi}, \widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{S}}_\varepsilon$ be as in (117) and (110), and let

$$\lambda \stackrel{\text{def}}{=} \begin{cases} \widetilde{\psi}_* \left(\mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{S}}_\varepsilon}\right) & (\varepsilon\text{-approximations}) \\ \widetilde{\psi}_* \left(\mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}}\right) & (\text{best approximations}). \end{cases} \tag{125}$$

Note that although our notation does not reflect this, both measures depend on the norm, and in the case of ε-approximations, λ depends on ε. For such a measure λ, let $\lambda^{(\mathscr{E}_n)}, \lambda^{(\mathbb{R}^d)}, \lambda^{(f)}, \lambda^{(\infty)}$ denote the image of λ under the projections P_1, P_2, P_3, P_{12} as in (119), and let $\lambda^{(\mathscr{X}_d)}$, $\lambda^{(\mathbb{S}^{d-1})}$ denote the image under the maps $\pi_{\mathscr{X}_d} \circ P_1$ and $\mathrm{Proj} \circ P_2$ (see (13), (26)). We will go through the different cases and give some properties of these measures.

The case of ε-approximations is much simpler. The following two propositions are immediate from (84) and Proposition 10.2.

Proposition 11.1 (Case I, ε-approximations)**.** *Let $\varepsilon \in (0, r_0)$, where r_0 is large enough to satisfy (76). Let $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$, and let λ be as in (125) (ε-approximations). Let B_ε denote the ball of radius ε around the origin in $\mathbb{R}^d$ and let $V_d = m_{\mathbb{R}^d}(B_1)$. Then*

$$\lambda^{(\mathscr{E}_n)} = \frac{\varepsilon^d V_d}{\zeta(n)} m_{\mathscr{E}_n}, \qquad \lambda^{(\mathbb{R}^d)} = \frac{1}{\zeta(n)} m_{\mathbb{R}^d}|_{B_\varepsilon},$$

$$\lambda^{(f)} = \frac{\varepsilon^d V_d}{\zeta(n)} m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}, \qquad \lambda^{(\mathscr{X}_d)} = \frac{\varepsilon^d V_d}{\zeta(n)} m_{\mathscr{X}_d}.$$

The measures $\lambda^{(\mathbb{R}^d)}$, $\lambda^{(\mathbb{S}^{d-1})}$ are preserved by any linear transformations of $\mathbb{R}^d$ preserving the norm $\|\cdot\|$.

Proposition 11.2 (Case II, ε-approximations)**.** *Let $d \geq 2$ and let $\varepsilon \in (0, r_0)$, where r_0 is large enough to satisfy (76). Let $\mu = m_{\tilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}}}$ be as in §7.4.1 and let λ be as in (125). Then λ is proportional to $\lambda^{(\infty)} \times \lambda^{(f)}$, $\lambda^{(\infty)}$ is not proportional to $\lambda^{(\mathscr{E}_n)} \times \lambda^{(\mathbb{R}^d)}$, and the projections $\lambda^{(\mathscr{E}_n)}$, $\lambda^{(\mathbb{R}^d)}$ satisfy statements (1), (2), (3) of Proposition 10.2. The measures $\lambda^{(\mathbb{R}^d)}$, $\lambda^{(\mathbb{S}^{d-1})}$ are not invariant under the group of orthogonal transformations.*

We now turn to best approximations. Let

$$\widehat{\mathcal{B}} \stackrel{\text{def}}{=} \psi(\mathcal{B}) \subset \mathscr{E}_n \times \mathbb{R}^d,$$

so that

$$(\Lambda, v) \in \widehat{\mathcal{B}} \iff \varphi(\Lambda, v) \in \mathcal{B}, \tag{126}$$

and denote the indicator function of $\widehat{\mathcal{B}}$ by $\mathbf{1}_{\widehat{\mathcal{B}}}$.

Proposition 11.3. *Let λ be the measure as in (125) (best approximations). Then $\lambda = \lambda^{(\infty)} \times \lambda^{(f)}$, where in Case I, $\lambda^{(f)}$ is a multiple of $m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}$, and in Case II, using the notation of §7.4.1, $\lambda^{(f)}$ is a multiple of the $M_{\vec{\alpha}}$-invariant measure on an $M_{\vec{\alpha}}$-orbit on $\widehat{\mathbb{Z}}^n_{\mathrm{prim}}$. The measure $\lambda^{(\infty)}$ is absolutely continuous with respect to the measure $\nu^{(\infty)}$ as in (120), and the Radon-Nikodym derivative is given by*

$$\frac{d\lambda^{(\infty)}}{d\nu^{(\infty)}}(\Lambda, v) = \mathbf{1}_{\widehat{\mathcal{B}}}(\Lambda, v). \tag{127}$$

Proof. The proof that λ can be written as a product $\lambda^{(\infty)} \times \lambda^{(f)}$, and that $\lambda^{(f)}$ is either the unique (up to scaling) K_f-invariant measure, or an $M_{\vec{\alpha}}$-invariant measure on an $M_{\vec{\alpha}}$-orbit, is identical to the one used for ν, in the proof of Proposition 10.2. Formula (127) is clear from (110), (122) and (125). $\square$

Formula (127) is an explicit formula for $\lambda^{(\infty)}$. We will now use it to describe $\lambda^{(\mathscr{E}_n)}$, $\lambda^{(\mathscr{X}_d)}$, $\lambda^{(\mathbb{R}^d)}$, $\lambda^{(\mathbb{S}^{d-1})}$ in more detail. For this we need to consider Case I and Case II separately. We begin with the simpler Case II.

Proposition 11.4. *Let $d \geq 2$, and suppose the norm on $\mathbb{R}^d$ is either $\bar{A}_{\vec{\alpha}}$-analytic (see Definition 9.4), or is the sup-norm. Let $\mu = m_{\tilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}}}$ be as in Proposition 7.7, let λ be given by (125) (best approximations), and let $\lambda^{(\mathscr{E}_n)}, \lambda^{(\mathbb{R}^d)}, \lambda^{(f)}, \lambda^{(\infty)}, \lambda^{(\mathscr{X}_d)}$ be as above.*

We have

(1) *The measures $\lambda^{(\mathscr{E}_n)}$, $\lambda^{(\mathbb{R}^d)}$, $\lambda^{(\mathscr{X}_d)}$ are singular with respect to the measures $m_{\mathscr{E}_n}$, $m_{\mathbb{R}^d}$, $m_{\mathscr{X}_d}$.*
(2) *The measure $\lambda^{(\mathbb{S}^{d-1})}$ is not globally supported. In particular $\lambda^{(\mathbb{R}^d)}$ is not invariant under all orthogonal transformations.*
(3) *The measure $\lambda^{(\infty)}$ is not a scalar multiple of $\lambda^{(\mathscr{E}_n)} \times \lambda^{(\mathbb{R}^d)}$.*

Proof. Assertions (1) and (2) follow from Propositions 10.2 and 11.3. Furthermore, suppose that $(\Lambda, v) \in \mathrm{supp}\ \lambda^{(\infty)}$. In particular $\Lambda_1 \stackrel{\mathrm{def}}{=} \varphi(\Lambda, v) \in \mathcal{B}$. Since $\mathcal{B}$ is open in $\mathcal{S}_{r_0}$ (see Lemma 9.1) and ψ is a continuous inverse of φ on $\mathcal{B}$, there is a neighborhood $\mathcal{U}$ of the identity in $\bar{A}^{(1)}_{\vec{\alpha}}$, and a neighborhood $\mathcal{V}$ of $(\Lambda, v) \in \mathscr{E}_n \times \mathbb{R}^d$, such that

$$\mathrm{supp}\ \lambda^{(\infty)} \cap \mathcal{V} = \{\psi(\bar{a}\Lambda_1) : \bar{a} \in \mathcal{U}\}.$$

Restricting the measures $\lambda^{(\infty)}, \lambda^{(\mathscr{E}_n)}, \lambda^{(\mathbb{R}^d)}$ to the image of $\mathcal{U}$ under each of the maps

$$\bar{a} \mapsto \psi(\bar{a}\Lambda_1), \qquad \bar{a} \mapsto \bar{a}\Lambda, \qquad \bar{a} \mapsto \bar{a}v,$$

we see that in these open sets, each of these measures is supported on a $d-1$ dimensional manifold, and this implies (3). $\square$

11.1. Further properties of the measures for best approximations, Case I.

Proposition 11.5. *In Case I, for best approximations, the measures*

$$\lambda^{(\mathscr{E}_n)},\ \lambda^{(\mathscr{X}_d)},\ \lambda^{(\mathbb{R}^d)},\ \lambda^{(\mathbb{S}^{d-1})}$$

are absolutely continuous with respect to $m_{\mathscr{E}_n}, m_{\mathscr{X}_d}, m_{\mathbb{R}^d}, m_{\mathbb{S}^{d-1}}$*, and we have the following formulae for the Radon-Nikodym derivatives:*

$$\frac{d\lambda^{(\mathscr{E}_n)}}{dm_{\mathscr{E}_n}}(\Lambda) = \frac{1}{\zeta(n)} \int_{B_{r_0}} \mathbf{1}_{\widehat{\mathcal{B}}}(\Lambda, v)\, dm_{\mathbb{R}^d}(v) \tag{128}$$

$$\frac{d\lambda^{(\mathbb{R}^d)}}{dm_{\mathbb{R}^d}}(v) = \frac{1}{\zeta(n)} \int_{\mathscr{E}_n} \mathbf{1}_{\widehat{\mathcal{B}}}(\Lambda, v)\, dm_{\mathscr{E}_n}(\Lambda) \tag{129}$$

$$\frac{d\lambda^{(\mathscr{X}_d)}}{dm_{\mathscr{X}_d}}(\Lambda') = \frac{1}{\zeta(n)} \int_{B_{r_0}} \int_{\mathbb{T}_{\Lambda'}} \mathbf{1}_{\widehat{\mathcal{B}}}(\Lambda(\Lambda', f, \mathbf{e}_n), v)\, dm_{\mathbb{T}_{\Lambda'}}(f)\, dm_{\mathbb{R}^d}(v) \tag{130}$$

where $\mathbb{T}_{\Lambda'}$ *is as in* (23) *and* $\Lambda(\cdot)$ *is as in* (24)*, and, for some* $c > 0$*,*

$$\frac{d\lambda^{(\mathbb{S}^{d-1})}}{dm_{\mathbb{S}^{d-1}}}(\omega) = c \int_0^{r_0} t^{d-1} \int_{\mathscr{E}_n} \mathbf{1}_{\widehat{\mathcal{B}}}(\Lambda, t\omega)\, dm_{\mathscr{E}_n}(\Lambda)\, dt. \tag{131}$$

Proof. Equations (128), (129) both follow immediately from (121) and (127). For (130), we give an expression for $m_{\mathscr{X}_d}$ in terms of $m_{\mathscr{E}_n}$. Let H be the group as in (12), let G_0 and W denote respectively the subgroup of H obtained by setting $\mathbf{x} = 0$ and $A = 0$ in (12). Thus $G_0 \cong \mathrm{SL}_d(\mathbb{R})$, $W \cong \mathbb{R}^d$, and $H = G_0 \ltimes W$. By Lemma 8.15, the Haar measure on H can be written as $dm_H(A\mathbf{x}) = dm_{G_0}(A)dm_W(\mathbf{x})$. The discussion in the last paragraph of §2 shows that the matrix A gives the projected lattice $\Lambda' = A\mathbb{Z}^d$, and the vector $\mathbf{x}$ gives the lift functional $f \in \mathbb{T}_{\Lambda'}$. Therefore, for fixed $\Lambda' \in \mathscr{X}_d$,

$$dm_{\mathscr{X}_d}(\Lambda') = \int_{\mathbb{T}_{\Lambda'}} dm_{\mathscr{E}_n}(\Lambda(\Lambda', f, \mathbf{e}_n))\, dm_{\mathbb{T}_{\Lambda'}}(f),$$

and using this, in combination with (128) and the uniqueness of disintegration of measures, we obtain (130). By polar coordinates, there is some $c > 0$ such that $dm_{\mathbb{R}^d}(t\omega) = ct^{d-1}dm_{\mathbb{S}^{d-1}}(\omega)$, and thus (131) follows from (129). □

Proposition 11.6. *In Case I, for best approximations, the measures*

$$\lambda^{(\mathscr{E}_n)},\ \lambda^{(\mathbb{R}^d)},\ \lambda^{(\mathscr{X}_d)},\ \lambda^{(\mathbb{S}^{d-1})}$$

satisfy:

(a) $\lambda^{(\infty)}$ *is not a scalar multiple of* $\lambda^{(\mathscr{E}_n)} \times \lambda^{(\mathbb{R}^d)}$*.*
(b) The measures $\lambda^{(\mathscr{E}_n)}, \lambda^{(\mathscr{X}_d)}, \lambda^{(\mathbb{S}^{d-1})}$ *have full support, and the support of* $\lambda^{(\mathbb{R}^d)}$ *contains a neighborhood of the origin.*

(c) *For $d > 1$, there is $c > 0$ such that for any $\Lambda \in \mathscr{E}_n$ and any $\Lambda' \in \mathscr{X}_d$,*

$$\frac{d\lambda^{(\mathscr{E}_n)}}{dm_{\mathscr{E}_n}}(\Lambda) \leq c \cdot \mathrm{sys}(\pi_{\mathbb{R}^d}(\Lambda))^d \quad \text{and} \quad \frac{d\lambda^{(\mathscr{X}_d)}}{dm_{\mathscr{X}_d}}(\Lambda') \leq c \cdot \mathrm{sys}(\Lambda')^d, \tag{132}$$

where $\mathrm{sys}(\Lambda')$ is the length of the shortest nonzero vector of $\Lambda' \in \mathscr{X}_d$. In particular $\lambda^{(\mathscr{E}_n)}$ and $\lambda^{(\mathscr{X}_d)}$ are not scalar multiples of $m_{\mathscr{E}_n}$ and $m_{\mathscr{X}_d}$.

(d) *For any $\Lambda \in \mathscr{E}_n$ with no nonzero horizontal vectors, and any $\omega \in \mathbb{S}^{d-1}$, the set*

$$\{t \in \mathbb{R} : \varphi(\Lambda, t\omega) \in \mathcal{B}\} \tag{133}$$

is an interval containing 0. In particular, for any $\omega \in \mathbb{S}^{d-1}$, the function

$$t \mapsto \frac{d\lambda^{(\mathbb{R}^d)}}{dm_{\mathbb{R}^d}}(t\omega) \tag{134}$$

is monotone non-increasing, is not a.e. an indicator function, and $\mathrm{supp}\, \lambda^{(\mathbb{R}^d)}$ is star-shaped around the origin.

(e) *The measures $\lambda^{(\mathbb{R}^d)}$, $\lambda^{(\mathbb{S}^{d-1})}$ are invariant under any linear transformation of $\mathbb{R}^d$ preserving the norm $\|\cdot\|$. In particular, for the Euclidean norm, $\lambda^{(\mathbb{R}^d)}$ and $\lambda^{(\mathbb{S}^{d-1})}$ are $\mathrm{SO}_d(\mathbb{R})$-invariant.*

Proof. Assume first that $d = 1$. In this case, the assertions all follow easily from Proposition 9.13. Indeed, in this case there is only one norm on $\mathbb{R}^d = \mathbb{R}$, the bundle $\mathscr{E}_n$ is isomorphic to $\mathbb{R}/\mathbb{Z}$ via the map $y \mapsto h_y\mathbb{Z}^2$, and the measures $\lambda^{(\mathscr{E}_n)}$, $\lambda^{(\mathbb{R}^d)}$ are simply the pushforwards to the x and y axes, of the set in (113).

We now assume that $d > 1$, and note that for any $\Lambda \in \mathscr{E}_n$ there is $\varepsilon = \varepsilon(\Lambda) > 0$ such that for all $v \in \mathbb{R}^d$ with $\|v\| < \varepsilon$, we have $\varphi(\Lambda, v) \in \mathcal{B}$. Indeed, this follows from the discreteness of Λ and the fact that the cylinders C_r get smaller and smaller as $r \to 0$. This implies that the integrands in (128), (129), (130), (131) are positive on sets of positive measure, and (b) follows.

For (c), let $\Lambda \in \mathscr{E}_n$, let $\Lambda' = \pi_{\mathbb{R}^d}(\Lambda) \in \mathscr{X}_d$, and let $\mathbf{x}' \in \Lambda'$ with $\|\mathbf{x}'\| = \mathrm{sys}(\Lambda')$. Then Λ contains a vector $\mathbf{x} \in \pi_{\mathbb{R}^d}^{-1}(\mathbf{x}')$ whose horizontal component is $\mathbf{x}'$ and whose vertical component is in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Let $v \in \mathbb{R}^d$ with $\|v\| > 2\,\mathrm{sys}(\Lambda')$ and let $u(v)$ be as in (15). By the triangle inequality,

$$\|\pi_{\mathbb{R}^d}(u(v)\mathbf{x})\| \leq \|\mathbf{x}'\| + \frac{\|v\|}{2} < \|v\| = \|\pi_{\mathbb{R}^d}(u(v)\mathbf{e}_n)\|,$$

and hence $u(v)\Lambda \notin \mathcal{B}$. Using (128) we obtain

$$\frac{d\lambda^{(\mathscr{E}_n)}}{dm_{\mathscr{E}_n}}(\Lambda') \leq \frac{1}{\zeta(n)}\, m_{\mathbb{R}^d}(B(0, 2\,\mathrm{sys}(\Lambda'))),$$

and the first inequality in (132) follows. For the second inequality, repeat this argument for every Λ of the form $\Lambda = \Lambda(\Lambda', f, \mathbf{e}_n)$ as in (24), and use (130).

For assertion (a), note from Proposition 10.2 that $\nu^{(\infty)}$ is a scalar multiple of $m_{\mathscr{E}_n} \times m_{\mathbb{R}^d}|_{B_{r_0}}$. By (127) and the fact that $\lambda^{(\mathscr{E}_n)}$ and $\lambda^{(\mathbb{R}^d)}$ are of full support, if $\lambda^{(\infty)}$ were a scalar multiple of $\lambda^{(\mathscr{E}_n)} \times \lambda^{(\mathbb{R}^d)}$ then its Radon-Nikodym derivative would be constant on $\mathscr{E}_n \times B_{r_0}$. This contradicts the first inequality in (132).

For (d), let $\Lambda \in \mathscr{E}_n$ have no horizontal vectors and let $\omega \in \mathbb{R}^d$, $\|\omega\| = 1$. Then $\varphi(\Lambda, v) \in \mathcal{B}$ if and only if for every nonzero $\mathbf{x} \in \Lambda$ with vertical component $x_n \in (-1, 1)$, we have $\|v\| < \|\mathbf{x}' + x_n v\|$, where $\mathbf{x}' \stackrel{\text{def}}{=} \pi_{\mathbb{R}^d}(\mathbf{x}) \neq 0$. In other words, the set in (133), which we denote by $\bar{I}(\Lambda, \omega)$, can be written as

$$\bigcap_{\substack{\mathbf{x} \in \Lambda \smallsetminus \{0\} \\ x_n \in (-1,1)}} I(\mathbf{x}, \omega), \quad \text{where } I(\mathbf{x}, \omega) \stackrel{\text{def}}{=} \left\{ t \in \mathbb{R} : \|t\omega\| < \|\mathbf{x}' + x_n t\omega\| \right\}.$$

In order to show that $\operatorname{supp} \lambda^{(\mathbb{R}^d)}$ is star-shaped, it suffices to show that $\bar{I}(\Lambda, \omega)$ is an interval containing 0 for each Λ, and for this it is enough to show that $I(\mathbf{x}, \omega)$ is an interval containing 0 for every $\mathbf{x}$. Clearly $I(\mathbf{x}, \omega)$ is bounded and contains 0, and if it were not an interval, there would be $0 < t_1 < t_2$ such that $\|t_i\omega\| = \|\mathbf{x}' + x_n t_i \omega\|$ for $i = 1, 2$. This implies

$$\begin{aligned} t_2 - t_1 &= \|t_2\omega\| - \|t_1\omega\| = \|\mathbf{x}' + x_n t_2 \omega\| - \|\mathbf{x}' + x_n t_1 \omega\| \\ &= \|\mathbf{x}' + x_n t_2 \omega\| - \|\mathbf{x}' + x_n t_2 \omega + x_n(t_1 - t_2)\omega\| \leq \|x_n(t_1 - t_2)\omega\| < t_2 - t_1, \end{aligned}$$

a contradiction. The assertion about monotonicity in (d) now follows using (129). If the function in (134) were an indicator function for some ω, by (129), the interval $\bar{I}(\Lambda, \omega)$ would actually be the same for almost all Λ. To see that this cannot be the case, let Λ be some lattice, and let $\mathbf{x} \in \Lambda \smallsetminus \{0\}$ so that $\sup I(\mathbf{x}, \omega) = \sup \bar{I}(\Lambda, \omega)$. For almost every Λ, $\mathbf{x}$ is unique with this property. Now for almost all nearby lattices $\Lambda' \in \mathscr{E}_n$, containing a small perturbation $\mathbf{x}'$ of $\mathbf{x}$, we will have $\sup \bar{I}(\Lambda', \omega) = \sup I(\mathbf{x}', \omega) \neq \sup I(\mathbf{x}, \omega) = \sup \bar{I}(\Lambda, \omega)$.

Assertion (e) follows immediately from Proposition 10.2 and the fact that $\mathcal{B}$ is invariant under any linear transformation of the horizontal plane preserving the norm. □

11.2. **Cut-and-project structure of approximations to algebraic vectors.** This section is not needed for the proof of our main results. Its purpose is to highlight a certain structure that the set of approximations to algebraic vectors posses.

A *cut-and-project set* is a subset $X_0 \subset \mathbb{R}^r$ for which there are $s \in \mathbb{N}$, a lattice $\Delta \subset \mathbb{R}^{r+s}$, and $W \subset \mathbb{R}^s$ for which

$$X_0 = \{\mathbf{x} \in \mathbb{R}^r : \exists \mathbf{y} \in W \text{ such that } (\mathbf{x}, \mathbf{y})^{\mathrm{t}} \in \Delta\}.$$

Here $(\mathbf{x}, \mathbf{y})^{\mathrm{t}}$ denotes the vector whose first r entries are those of $\mathbf{x}$ and whose last s entries are those of $\mathbf{y}$. In some cases the vector spaces $\mathbb{R}^r, \mathbb{R}^s, \mathbb{R}^{r+s}$ appearing in the above definition are taken to be more general locally compact abelian groups. We will refer to the more general case of cut-and-project sets arising in this way as *generalized cut-and-project sets*. Cut-and-project sets are widely studied in the field of mathematical quasicrystals, see [BG13] for a comprehensive introduction.

For discrete sets $X' \subset X \subset \mathbb{R}$, we say that X' *has full density in* X if $\frac{\#(X'\cap[0,T])}{\#(X\cap[0,T])} \to_{T\to\infty} 1$, and say that discrete $X, Y \subset \mathbb{R}$ are *strongly asymptotic* if there are subsets X', Y' of full density and a bijection $\tau : X' \to Y'$ such that

$$|x' - \tau(x')| \longrightarrow_{x'\to\infty} 0 \quad (x' \in X').$$

Let K be a compact abelian group, let $\mathbb{G} = K \times \mathbb{R}$, let $\pi_{\mathbb{R}} : \mathbb{G} \to \mathbb{R}$ be the projection, and let $d_{\mathbb{G}}$ be a translation invariant metric on $\mathbb{G}$. For $X \subset \mathbb{G}$ and $T \in \mathbb{R}$ we set $X^{\geq T} \stackrel{\text{def}}{=} \{(k, x) \in \mathbb{G} : x \geq T\}$, and say that $X, Y \subset \mathbb{G}$ are *strongly asymptotic* if there are subsets X', Y' of X and Y, such that $\pi_{\mathbb{R}}(X'), \pi_{\mathbb{R}}(Y')$ are of full density in $\pi_{\mathbb{R}}(X)$ and $\pi_{\mathbb{R}}(Y)$, and there is a bijection $\tau : X' \to Y'$ such that $d_{\mathbb{G}}(x, \tau(x)) \to 0$ as $\pi_{\mathbb{R}}(x) \to \infty$.

Our analysis yields the following:

Proposition 11.7. *Let $\vec{\alpha}$ be as in Case II, let $\|\cdot\|$ be some norm on $\mathbb{R}^d$, let $\varepsilon > 0$, and let $\mathbf{u}_k = (\mathbf{p}_k, q_k) \in \mathbb{Z}^n$ be either the sequence of best approximation of $\vec{\alpha}$, or its sequence of ε-approximations. Then the sequence $(\log(q_k))_{k\in\mathbb{N}} \subset \mathbb{R}$ is strongly asymptotic to a one-dimensional cut-and-project set, and the sequence $(\mathbf{p}_k, \log(q_k)) \subset \widehat{\mathbb{Z}}^n \times \mathbb{R}$, is strongly asymptotic to a generalized cut-and-project set.*

Sketch of proof. We will not be using this statement in the sequel, and we only sketch the proof. Let $\mathbb{T} = \mathbb{R}^d/\Delta$ be a torus, where Δ is a lattice in $\mathbb{R}^d$, let $\pi_{\mathbb{T}} : \mathbb{R}^d \to \mathbb{T}$ be the projection, let $\mathbf{z} \in \mathbb{R}^d \smallsetminus \{0\}$, and let $\alpha_t \curvearrowright \mathbb{T}$ be the corresponding *straightline flow* defined by $\alpha_t \pi_{\mathbb{T}}(\mathbf{x}) = \pi_{\mathbb{T}}(\mathbf{x} + t\mathbf{z})$. A cut-and-project set in $\mathbb{R}$ can also be described as the set of visit times as in (30), to a section $\mathcal{S}$ which is a *bounded linear section*; i.e., the image under $\pi_{\mathbb{T}}$ of a bounded subset of an affine subspace of dimension $d-1$; see [ASW22, Prop. 2.3] for a proof.

Let $\bar{A}_{\vec{\alpha}}, y_{\vec{\alpha}}$ be as in (69), let $\bar{A}_{y_{\vec{\alpha}}}$ be the stabilizer group as in (70), let $\bar{\mathfrak{a}} \cong \mathbb{R}^d$ be the Lie algebra of $\bar{A}_{\vec{\alpha}}$, let $\exp : \bar{\mathfrak{a}} \to \bar{A}_{\vec{\alpha}}$ be the exponential map and denote its inverse by log. Then the map exp induces an isomorphism between the compact orbit $\bar{A}_{\vec{\alpha}} y_{\vec{\alpha}}$ and the torus $\mathbb{T} \stackrel{\text{def}}{=} \bar{\mathfrak{a}}/\log(\bar{A}_{y_{\vec{\alpha}}})$, and this map conjugates the a_t-action on $\bar{A}_{\vec{\alpha}} y_{\vec{\alpha}}$ to a straightline flow on $\mathbb{T}$. Moreover, as in Proposition 8.13, this isomorphism maps the cross-section $\mathcal{S}_{r_0}$ to a bounded linear section. Therefore, for $\mathcal{S}' = \mathcal{B}$ or $\mathcal{S}' = \mathcal{S}_\varepsilon$, $\{t \in \mathbb{R} : a_t y_{\vec{\alpha}} \in \mathcal{S}'\}$ is a cut-and-project set. Recall from Proposition 10.4 that the set of denominators q_k of convergents to $\vec{\alpha}$, and the set of visit times $\{t \geq 0 : a_t \Lambda_{\vec{\alpha}} \in \mathcal{S}_{r_0}\}$ are related by $t_k = \frac{1}{d}\log(q_k)$, and from Proposition 7.5 that $y_{\vec{\alpha}} = q\Lambda_{\vec{\alpha}}$ for some $q \in H^-$, the contracting horospherical subgroup of $\{a_t\}$. Using this, and the fact that $\mathcal{S}'$ is $\mu_{\mathcal{S}_{r_0}}$-JM, one can prove that for any $\varepsilon > 0$, for all large enough t, there is a bijection between the visit times of the two trajectories

$\{a_t y_{\vec{\alpha}}\}$, $\{a_t \Lambda_{\vec{\alpha}}\}$ to points which are of distance at least ε from the boundary of $\mathcal{S}'$; and from this, one can deduce that these sets are strongly asymptotic.

For the second assertion, we use the notation of §7.4.1. We let $\Delta \stackrel{\text{def}}{=} \mathrm{Stab}_{\tilde{L}_{\vec{\alpha}}}(\tilde{y}_{\vec{\alpha}})$. In the cut-and-project construction, we now replace $\mathbb{R}^r$ with $\{a_t\} \times M_{\vec{\alpha}}$, and R^s with $\bar{A}^{(1)}_{\vec{\alpha}}$, and use Proposition 10.4 and a generalization of [ASW22, Prop. 2.3]. □

12. Invariance under the weak stable foliation

For $\theta \in \mathbb{R}^d$, let $\widetilde{\Lambda}_\theta$ be as in (111). Let $\mathscr{X}_n^{\mathbb{A}}$ and $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$ be as in §7.4, let $\widetilde{\mathcal{S}}_{r_0}$ be as in (110), and let $\mu_{\widetilde{\mathcal{S}}_{r_0}}$ be the cross-section measure. The goal of this section is the following:

Proposition 12.1. *For Lebesgue a.e. $\theta \in \mathbb{R}^d$, $\widetilde{\Lambda}_\theta$ is $(a_t, \mu_{\widetilde{\mathcal{S}}_{r_0}})$-generic. Moreover, it is $(a_t, \mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}})$-generic as well as $(a_t, \mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{S}}_\varepsilon})$-generic for any $\varepsilon \in (0, r_0)$.*

Recalling the distinction in Definition 5.1, we define

$$\Omega_1 \stackrel{\text{def}}{=} \left\{ \widetilde{\Lambda} \in \mathscr{X}_n^{\mathbb{A}} : \Lambda \text{ is } (a_t, \mu)\text{-generic} \right\}$$

and

$$\Omega_2 \stackrel{\text{def}}{=} \left\{ \widetilde{\Lambda} \in \mathscr{X}_n^{\mathbb{A}} : \Lambda \text{ is } (a_t, \mu_{\widetilde{\mathcal{S}}_{r_0}})\text{-generic} \right\}.$$

Our goal will be to establish that $\Omega_1 \smallsetminus \Omega_2$ is negligible.

Remark 12.2. Recall from Proposition 5.9 that if $\mathcal{S}$ were tempered and reasonable, or if we were interested in $\mu_{\mathcal{S}}|_E$ for some tempered subset, then we would have $\Omega_1 = \Omega_2$, and the arguments of this section could be avoided.

Let $G = \mathrm{SL}_n(\mathbb{A})$ and let H^- and H^0 be the groups defined in (54) and (55), and set $H^{\leq} \stackrel{\text{def}}{=} H^- H^0$. Note that H^0 normalizes H^- and therefore $H^{\leq}$ is a group. We refer to $H^{\leq}$ as the *weak stable subgroup* corresponding to $\{a_t\}$. Also write

$$H^+ \stackrel{\text{def}}{=} \{g \in G : a_t g a_{-t} \to_{t \to -\infty} \mathrm{Id}\} = \left\{ u(v) : v \in \mathbb{R}^d \right\},$$

where $u(v)$ is as in (15). Note that $H^+ \cong \mathbb{R}^d$ is unimodular. We denote its Haar measure by m_{H^+}. This is simply the image of Lebesgue measure under the map $\mathbf{x} \mapsto h_{\mathbf{x}}$.

We will derive Proposition 12.1 from the following three statements.

Proposition 12.3. *For any open subgroup $W \subset H^{\leq}$, and any W-invariant subset $\Omega \subset \mathscr{X}_n^{\mathbb{A}}$ of full μ-measure, and any $\widetilde{\Lambda} \in \mathscr{X}_n^{\mathbb{A}}$, $\left\{ h \in H^+ : h\widetilde{\Lambda} \in \Omega \right\}$ is of full m_{H^+}-measure.*

Proposition 12.4. *The set Ω_1 is $H^{\leq}$-invariant and of full μ-measure.*

Proposition 12.5. *There is an open subgroup $W \subset H^{\leq}$ and a W-invariant set $\Omega_3 \subset \mathscr{X}_n^{\mathbb{A}}$ such that the set $\Delta^{\mathbb{R}}_{\widetilde{\mathcal{S}}_{r_0}}$ defined via (40) and (34) is contained in Ω_3, and $\mu(\Omega_3) = 0$.*

Proof of Proposition 12.1 assuming Propositions 12.3, 12.4, 12.5. Let W and Ω_3 be as in Proposition 12.5. By Proposition 5.7,

$$\Omega_1 \smallsetminus \Omega_3 \subset \Omega_1 \smallsetminus \Delta^{\mathbb{R}}_{\widetilde{\mathcal{S}}} \subset \Omega_2.$$

By Proposition 12.4, $\Omega_1 \smallsetminus \Omega_3$ is a W-invariant set of full measure, and thus by Proposition 12.3, $\{\theta \in \mathbb{R}^d : \widetilde{\Lambda}_\theta \in \Omega_1 \smallsetminus \Omega_3\}$ is a set of full measure, proving the first assertion. Moreover, applying Theorem 5.11 and taking into account the fact that Ω_3 contains $\Delta_{\widetilde{\mathcal{S}}_{r_0}}$, we see that any θ in this set is also $(a_t, \mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}})$-generic as well as $(a_t, \mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{S}}_\varepsilon})$-generic. □

Proof of Proposition 12.3. Let $G_f = \mathrm{SL}_n(\mathbb{A}_f), G_\infty = \mathrm{SL}_n(\mathbb{R})$ be as in §7.4, and let

$$H^{\leq}_\infty \overset{\mathrm{def}}{=} H^{\leq} \cap G_\infty. \tag{135}$$

The groups H^+ and $H^{\leq}$ are complementary, in the following sense:

$$H^{\leq} \cap H^+ = \{e\}, \quad H^{\leq} = H^{\leq}_\infty \times G_f, \quad \dim H^{\leq}_\infty + \dim H^+ = \dim G_\infty.$$

It follow that there are neighborhoods $\mathcal{U}_1, \mathcal{U}_2$ of the identity in $H^{\leq}$ and H^+ respectively, such that the map

$$\mathcal{U}_1 \times \mathcal{U}_2 \to G, \qquad (h^{\leq}, h^+) \mapsto h^{\leq} h^+$$

is a homeomorphism onto its image in G. For any any $\widetilde{\Lambda} \in \mathscr{X}_n^{\mathbb{A}}$, using the fact that W is open in $H^{\leq}$ and the stabilizer of $\widetilde{\Lambda}$ in G is discrete, we can replace $\mathcal{U}_1, \mathcal{U}_2$ with smaller open sets around the identity, to obtain that $\mathcal{U}_1 \subset W$ and the map

$$\mathcal{U}_1 \times \mathcal{U}_2 \to \mathscr{X}_n^{\mathbb{A}}, \qquad (w, h^+) \mapsto wh^+\widetilde{\Lambda}$$

is a homeomorphism onto its image, which is a neighborhood of $\widetilde{\Lambda}$ in $\mathscr{X}_n^{\mathbb{A}}$. Furthermore, by Lemma 8.15, in this neighborhood of $\widetilde{\Lambda}$, the measure μ can be written as $d\mu(wh^+\widetilde{\Lambda}) = dm_W^{\mathrm{left}}(w) dm_{H^+}(h^+)$. Since Ω is W-invariant, this implies that m_{H^+}-a.e. $h^+ \in H^+$ satisfies $h^+\widetilde{\Lambda} \in \Omega$. □

Proof of Proposition 12.4. By Lemma 7.6, the flow $\{a_t\} \curvearrowright \mathscr{X}_n^{\mathbb{A}}$ is ergodic and hence Ω_1 is of full μ-measure. To see that Ω_1 is $H^{\leq}$-invariant, use Proposition 7.2 and the fact that μ is H^0-invariant. □

Proof of Proposition 12.5. Take a sequence $r_i \nearrow \infty$ and define $\mathcal{S}_{r_i}$ as in (78). Using the projection $\pi : \mathscr{X}_n^{\mathbb{A}} \to \mathscr{X}_n$, let

$$\widetilde{\mathcal{S}}_i \overset{\mathrm{def}}{=} \pi^{-1}(\mathcal{S}_{r_i}), \qquad \Delta_i \overset{\mathrm{def}}{=} \Delta^{\mathbb{R}}_{\widetilde{\mathcal{S}}_i}.$$

The sets $\widetilde{\mathcal{S}}_i$ are reasonable cross-sections by Theorem 8.6 and Proposition 6.1, and the sets Δ_i satisfy $\mu(\Delta_i) = 0$ by Lemma 5.8. Let

$$\mathcal{D} \stackrel{\text{def}}{=} \left\{ \widetilde{\Lambda} \in \mathscr{X}_n^{\mathbb{A}} : \pi(\widetilde{\Lambda}) \cap \mathbb{R}^d \neq \{0\} \right\},$$

where $\pi : \mathscr{X}_n^{\mathbb{A}} \to \mathscr{X}_n$ is the projection. Then for $\widetilde{\Lambda} \in \mathcal{D}$ we have $a_t\widetilde{\Lambda} \to_{t\to-\infty} \infty$, and thus by Poincaré recurrence, $\mu(\mathcal{D}) = 0$. Finally let

$$\Omega_3 \stackrel{\text{def}}{=} \mathcal{D} \cup \bigcup_{i=1}^{\infty} \Delta_i.$$

It is clear that $\mu(\Omega_3) = 0$. Let K_f be as in (68) , let $H_\infty^{\lneqq}$ be as in (135), and let

$$W \stackrel{\text{def}}{=} \left\{ (g_\infty, g_f) \in G : g_\infty \in H_\infty^{\lneqq}, g_f \in K_f \right\}. \tag{136}$$

Our goal is to prove that Ω_3 is W-invariant. Since elements of $H^{\leq}$ act on $\mathbb{R}^d$ by multiplication by scalars, we have that $\mathcal{D}$ is $H^{\leq}$-invariant; in particular, $\mathcal{D}$ is W-invariant. Thus it suffices to show that for each i and each $\widetilde{\Lambda} \in \Delta_i \smallsetminus \mathcal{D}$, and each $g \in W$, we have $g\widetilde{\Lambda} \in \Delta_j$ for all j large enough.

It is easy to check that

$$H_\infty^{\lneqq} = \left\{ c_\infty(B) h_{\mathbf{x}} a_t : B \in \mathrm{SL}_d(\mathbb{R}),\ \mathbf{x} \in \mathbb{R}^d,\ t \in \mathbb{R} \right\},$$

where

$$h_{\mathbf{x}} \stackrel{\text{def}}{=} \begin{pmatrix} I_d & \mathbf{0} \\ \mathbf{x}^{\mathbf{t}} & 1 \end{pmatrix} \quad \text{and} \quad c_\infty(B) \stackrel{\text{def}}{=} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}.$$

We can write $g = ch_{\mathbf{x}}a_{t_0}$ for $c = (c_\infty, c_f)$, where $c_\infty = c_\infty(B)$ for some B, $\mathbf{x}$ and t_0. A simple matrix computation shows that for all $t \in \mathbb{R}$,

$$a_t g = c h_{\mathbf{x}(t)} a_{t+t_0}, \quad \text{where } \mathbf{x}(t) \stackrel{\text{def}}{=} e^{-(1+\frac{1}{d})t}\mathbf{x}. \tag{137}$$

Since $\widetilde{\Lambda} \in \Delta_i$ there exists $\delta > 0$ such that for all $\varepsilon > 0$, there are arbitrarily large T for which

$$N(\widetilde{\Lambda}, T, \widetilde{\mathcal{S}}_{i,<\varepsilon}) > \delta T.$$

We will show that for all j large enough, and for arbitrarily large T,

$$N(g\widetilde{\Lambda}, T, \widetilde{\mathcal{S}}_{j,<\varepsilon}) > \frac{\delta}{2} T.$$

Let $t_1 < t_2 < \cdots$ be the sequence of visits of $a_t\widetilde{\Lambda}$ to $\widetilde{\mathcal{S}}_{i,<\varepsilon}$ in positive time, so that

$$\#\{k \in \mathbb{N} : t_k \le T\} > \delta T.$$

By definition of $\widetilde{\mathcal{S}}_{i,<\varepsilon}$, for each k there is $0 < \tau_k < \varepsilon$ such that $a_{t_k+\tau_k}\widetilde{\Lambda} \in \widetilde{\mathcal{S}}_i$. Let $t'_k \stackrel{\text{def}}{=} t_k - t_0$, so that $t'_k > 0$ for all but finitely many k. For each k, we have by (137) that

$$a_{t'_k} g\widetilde{\Lambda} = ch_{\mathbf{x}(t_k)} a_{t_k}\widetilde{\Lambda} \in ch_{\mathbf{x}(t'_k)}\widetilde{\mathcal{S}}_i$$
$$a_{t'_k+\tau_k} g\widetilde{\Lambda} = ch_{\mathbf{x}(t'_k)} a_{t_k+\tau_k}\widetilde{\Lambda} \in ch_{\mathbf{x}(t'_k)}\widetilde{\mathcal{S}}_i.$$

Also, $a_t\widetilde{\Lambda} \notin \mathcal{D}$ for any t. Thus it suffices to show that for large enough j, for all k,

$$ch_{\mathbf{x}(t_k)}\left(\widetilde{\mathcal{S}}_i \smallsetminus \mathcal{D}\right) \subset \widetilde{\mathcal{S}}_j.$$

Recall that for any r, $\widetilde{\mathcal{S}}_r = \pi^{-1}\left(\mathscr{X}_n(D_r)\right)$, where D_r is defined in (77), and so

$$\widetilde{\mathcal{S}}_r \smallsetminus \mathcal{D} = \pi^{-1}\left(\mathscr{X}_n(D_r) \cap \mathscr{X}_n^{\sharp}(\mathbb{R}^d)\right).$$

Since $c_f \in K_f$ and $\widetilde{\mathcal{S}}_r$ is K_f-invariant, we can replace c with c_∞. Moreover, $\|\mathbf{x}(t'_k)\| < \|\mathbf{x}\|$ for all but finitely many k. Thus it suffices to show that for any B, any $\mathbf{x}$ and any $r > 0$, for all large enough $r' > 0$ and all $\mathbf{x}'$ with $\|\mathbf{x}'\| \le \|\mathbf{x}\|$, the matrix

$$g_0 \stackrel{\mathrm{def}}{=} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I_d & \mathbf{x}' \\ 0 & 1 \end{pmatrix} \in H_\infty^{\le}$$

satisfies

$$g_0\left(\mathscr{X}_n^{\sharp}(D_r) \cap \mathscr{X}_n^{\sharp}(\mathbb{R}^d)\right) \subset \mathscr{X}_n^{\sharp}(D_{r'}). \tag{138}$$

Let

$$r' > M(M+r), \quad \text{where } M \stackrel{\mathrm{def}}{=} \max(\|B\|_{\mathrm{op}}, \|\mathbf{x}\|).$$

It is easily seen that this choice ensures that

$$g_0 D_r \subset D_{r'}. \tag{139}$$

Because g_0 preserves $\mathbb{R}^d$, the LHS of (138) is equal to $\mathscr{X}_n^{\sharp}(g_0 D_r) \cap \mathscr{X}_n^{\sharp}(\mathbb{R}^d)$. Lattices in $\mathscr{X}_n^{\sharp}(\mathbb{R}^d)$ can contain at most one vector in the affine hyperplane $\mathbf{e}_n + \mathbb{R}^d$ containing D_r. Thus it is enough to show that $\mathscr{X}_n^{\sharp}(g_0 D_r) \subset \mathscr{X}_n(D_{r'})$ which in turn follows from (139). □

13. Concluding the proofs

As discussed in §2, Theorems 1.1, 1.2 and 1.5 all follow from Theorem 2.1.

Proof of Theorem 2.1. In all cases, the measure λ in (125) is finite (see Propositions 8.7, 8.13, 6.1) but need not be a probability measure. We fix a norm on $\mathbb{R}^d$ and $\varepsilon > 0$, and define the following probability measures:

$$\mu^{(\mathbf{e}_n)} \text{ is the normalization of } \lambda \text{ in Case I, best approximations}$$
$$\mu^{(\mathbf{e}_n, \vec{\alpha})} \text{ is the normalization of } \lambda \text{ in Case II, best approximations}$$
$$\nu^{(\mathbf{e}_n)} \text{ is the normalization of } \lambda \text{ in Case I, } \varepsilon\text{-approximations}$$
$$\nu^{(\mathbf{e}_n, \vec{\alpha})} \text{ is the normalization of } \lambda \text{ in Case II, } \varepsilon\text{-approximations,}$$

where in Case II we require that $d \ge 2$ and in Case II for best approximations we also require that the norm is either the sup-norm, or is $\bar{A}_{\vec{\alpha}}$-analytic.

In Case I, let $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$. Then for Lebesgue a.e. θ, by Proposition 12.1, $\widetilde{\Lambda}_\theta$ is $\left(a_t, \mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}}\right)$-generic and $\left(a_t, \mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{S}}_\varepsilon}\right)$-generic. The map $\widetilde{\psi}$ is continuous but is only defined on $\widetilde{\mathcal{S}}_{r_0}^{\sharp}$. Since $\widetilde{\mathcal{S}}_{r_0}^{\sharp}$ is open (by Lemma 8.1) and of full measure (by Lemma 8.4), $\widetilde{\psi}$ coincides with a map on $\widetilde{\mathcal{S}}_{r_0}$ whose set of discontinuities has zero measure with respect to $\mu_{\widetilde{\mathcal{S}}_{r_0}}$. Using Lemma 4.9, we obtain that $\widetilde{\psi}$ maps an equidistributed sequence in $\widetilde{\mathcal{S}}_{r_0}^{\sharp}$ (with respect to either $\frac{1}{\mu_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{B}})}\mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}}$ or $\frac{1}{\mu_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{S}}_{\varepsilon})}\mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{S}}_{\varepsilon}}$) to an equidistributed sequence in $\mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n_{\mathrm{prim}}$ (with respect to the pushed-forward measure). Using Proposition 10.4, we find that the sequences (18), (19) are equidistributed with respect to $\mu^{(\mathbf{e}_n)}$ and $\nu^{(\mathbf{e}_n)}$ respectively. Proposition 11.6 shows that the projected measures in the statement of the theorem have the stated properties.

In Case II, let $\mu = m_{\tilde{L}_{\vec{\alpha}}\tilde{y}_{\vec{\alpha}}}$. By Lemma 10.5 any $\varepsilon > \varepsilon_0$ satisfies $\mu_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{S}}_{\varepsilon}) > 0$. By Proposition 9.12, the lattice $\widetilde{\Lambda}_{\vec{\alpha}}$ is both $\left(a_t, \mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}}\right)$-generic and $\left(a_t, \mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{S}}_{\varepsilon}}\right)$-generic. Using Proposition 10.4, we find that the sequences (18), (19) are equidistributed with respect to the pushforwards $\mu^{(\mathbf{e}_n, \vec{\alpha})}$ and $\nu^{(\mathbf{e}_n, \vec{\alpha})}$ respectively. The desired properties of these measures are given in Propositions 11.4 and 11.2. ☐

Remark 13.1. It is interesting to find other measures on $\mathbb{R}^d$ which give full measure to the set of θ which satisfy the conclusions of Theorems 1.1 and 1.2. In this regard, we note the following. An examination of the proof of Theorem 2.1 shows that its conclusion holds for every θ for which $\tilde{\Lambda}_\theta$ is $\left(a_t, \mu_{\widetilde{\mathcal{S}}_{r_0}}\right)$-generic. Since $\widetilde{\mathcal{B}}$ is tempered, by Proposition 5.9, for best approximations it suffices to show that $\widetilde{\Lambda}_\theta$ is (a_t, μ)-generic for $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$. It is thus an interesting question to provide examples of measures on $\mathbb{R}^d$ satisfying that $\widetilde{\Lambda}_\theta$ is (a_t, μ)-generic for $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$ for a.e. θ. Note that this reduction is not valid in the case of ε-approximations.

Additionally one might want to find measures for which for typical θ, the first two sequences in (3) are equidistributed with respect to $\mu^{(\infty)}$. In this regard, we can work with the real space $\mathscr{X}_n$ instead of $\mathscr{X}_n^{\mathbb{A}}$ and with $\mu = m_{\mathscr{X}_n}$. Using the same strategy, we are led to the question of finding a measure μ satisfying that Λ_θ is $\left(a_t, \mu_{\mathcal{S}_{r_0}}\right)$-generic for a.e. θ. Examples of such measures can be found in [SW19]; they include natural measures on some self-similar fractals, like the middle-thirds Cantor set, Sierpinski triangle or Koch snowflake.

Proof of Corollary 3.1. In light of Theorem 2.1, it suffices to compute the measures $\lambda^{(\mathscr{E}_2)}$ in Case I, for both ε-approximations and best approximations. For ε-approximations we find from Proposition 11.1 that $\lambda^{(\mathscr{E}_2)}$ is a scalar multiple of $m_{\mathscr{E}_2}$, which is just Lebesgue measure on $\left[-\frac{1}{2}, \frac{1}{2}\right] \cong \mathbb{T}_1$. For best approximations we use Propositions 9.13 and (128) to compute the density F. We leave the details to the reader. ☐

Proof of Corollary 3.4. Let $\mu = m_{\mathscr{X}_n}$, let $\mathcal{S}_\varepsilon$, $\mathcal{B}$ be as in (78) and (106), and let

$$\tau^{(\varepsilon)} : \mathcal{S}_\varepsilon \to \mathbb{R}_+ \quad \text{and} \quad \tau^{(\text{best})} : \mathcal{B} \to \mathbb{R}_+$$

denote respectively the functions giving the first return time of the $\{a_t\}$-flow to $\mathcal{S}_\varepsilon$ and $\mathcal{B}$, as in (32). Let $\lambda_\varepsilon^{(KL)}$ and $\lambda_{\text{best}}^{(KL)}$ denote respectively the normalized pushforward of $\mu_{\mathcal{S}_{r_0}}|_{\mathcal{S}_\varepsilon}$ and $\mu_{\mathcal{S}_{r_0}}|_{\mathcal{B}}$ under $\tau^{(\varepsilon)}$ and $\tau^{(\text{best})}$. In both cases, the map τ is Riemann integrable, in the sense that its discontinuity set is a nullset with respect to $\mu|_{\mathcal{S}_{r_0}}$. This can be proved using arguments similar to those used in §8.3, and we omit the proof. Thus (see [Bil68, Thm. 2.7]) the sequences (29) equidistribute with respect to $\lambda_\varepsilon^{(KL)}$ and $\lambda_{\text{best}}^{(KL)}$, showing the first assertion.

To compute the expectation of $\lambda_\varepsilon^{(KL)}$ we use the Kac formula given in Theorem 4.4(vi). The formula implies that when μ is a probability measure, as it is in our case, the expectation of $\tau_{\mathcal{S}}$ is equal to

$$\frac{1}{\mu_{\mathcal{S}_{r_0}}(\mathcal{S}_\varepsilon)} \overset{\text{Prop.8.7}}{=} \frac{\zeta(n)}{m_{\mathscr{X}_n}(\mathscr{X}_n)\, m_{\mathbb{R}^d}(B_\varepsilon)} = \frac{\zeta(n)}{V_{d,\|\cdot\|}\, \varepsilon^d}.$$

□

14. Conclusions regarding equidistributed compact torus orbits

Until now we have dealt with two types of a_t-invariant and ergodic measures: $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$ (Case I) and $\mu_{\widetilde{L}_{\vec{\alpha}}\widetilde{y}_{\vec{\alpha}}}$ (Case II). We proved results regarding the best approximations and ε-approximations of Lebesgue almost any θ (Case I) and the algebraic vector $\vec{\alpha}$ (Case II). The two discussions regarding Case I and Case II were carried out simultaneously but were completely independent to one another. We now wish to establish a connection between the two cases which occurs when one varies the algebraic vector $\vec{\alpha}$ under some assumptions. First we need the following.

Lemma 14.1. *We have that*

$$\left\{\mu \in \mathcal{P}(\mathscr{X}_n^{\mathbb{A}}) : \mu \text{ is } a_1\text{-invariant and } \pi_*\mu = m_{\mathscr{X}_n}\right\} = \left\{m_{\mathscr{X}_n^{\mathbb{A}}}\right\}.$$

As a consequence, if $\mu_k \in \mathcal{P}(\mathscr{X}_n^{\mathbb{A}})$ is a sequence of a_1-invariant probability measures such that $\pi_*\mu_k \to m_{\mathscr{X}_n}$, then $\mu_k \to m_{\mathscr{X}_n^{\mathbb{A}}}$.

Proof. The proof relies on two facts about entropy. The probability measures $m_{\mathscr{X}_n}$ and $m_{\mathscr{X}_n^{\mathbb{A}}}$ are the unique a_1-invariant measures of maximal entropy for the dynamical systems $(\mathscr{X}_n, a_1), (\mathscr{X}_n^{\mathbb{A}}, a_1)$, and in fact they have equal entropies:

$$\mathrm{h}(a_1, m_{\mathscr{X}_n}, \mathscr{X}_n) = \mathrm{h}(a_1, m_{\mathscr{X}_n^{\mathbb{A}}}, \mathscr{X}_n^{\mathbb{A}}) = n.$$

For the space $\mathscr{X}_n$ this follows from [EL10, Corollary 7.10]. For the space $\mathscr{X}_n^{\mathbb{A}}$ this follows from [EL10, Theorem 7.9] together with the unique ergodicity of the $\mathrm{SL}_n(\mathbb{R})$ action on $\mathscr{X}_n^{\mathbb{A}}$.

Assume that $\mu \in \mathcal{P}(\mathscr{X}_n^{\mathbb{A}})$ satisfies $\pi_*\mu = m_{\mathscr{X}_n}$. Then, since π is a factor map,

$$\mathrm{h}(a_1, \mu, \mathscr{X}_n^{\mathbb{A}}) \geq \mathrm{h}(a_1, \mu, \mathscr{X}_n^{\mathbb{A}}) = n.$$

From the uniqueness of measure of maximal entropy we deduce that $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$ as desired.

Now let $\mu_k \in \mathcal{P}(\mathscr{X}_n^{\mathbb{A}})$ be a sequence of a_1-invariant probability measures satisfying that $\pi_*\mu_k \to m_{\mathscr{X}_n}$. It follows that any weak* accumulation point of μ_k must be an a_1-invariant probability measure that projects under π to $m_{\mathscr{X}_n}$. By the first part of lemma, μ_k has only one accumulation point - $m_{\mathscr{X}_n^{\mathbb{A}}}$. We conclude that $\mu_k \to m_{\mathscr{X}_n^{\mathbb{A}}}$ as desired. □

Theorem 14.2. *Assume $d \geq 2$ and that the norm $||\cdot||$ on $\mathbb{R}^d$ we use to define the notion of best approximations is either the Euclidean norm or the sup norm. Let $\vec{\alpha}_k = (\alpha_{k1}, \ldots, \alpha_{kd})^{\mathrm{t}} \in \mathbb{R}^d$ be a sequence of vectors such that for each k, $\{1, \alpha_{k1}, \ldots, \alpha_{kd}\}$ span a totally real number field of degree n over $\mathbb{Q}$. Let $\bar{h}_{\vec{\alpha}_k}$ be as in (63) and $x_{\vec{\alpha}_k} \in \mathscr{X}_n$ be as in (60). If the sequence of periodic probability measures supported on the periodic orbits $\bar{h}^*_{\vec{\alpha}_k} A x_{\vec{\alpha}_k}$ converge weak* to $m_{\mathscr{X}_n}$, then*

$$\mu^{(\mathbf{e}_n, \vec{\alpha}_k)} \longrightarrow \mu^{(\mathbf{e}_n)}, \tag{140}$$

where $\mu^{(\mathbf{e}_n, \vec{\alpha}_k)}, \mu^{(\mathbf{e}_n)} \in \mathcal{P}(\mathscr{E}_n \times \mathbb{R}^d \times \widetilde{\mathbb{Z}}^n)$ are the probability measure corresponding to best approximations of $\vec{\alpha}_k$ (resp. Lebesgue almost any θ) by Theorem 2.1.

Proof. To reduce notational clutter we let $x_k \overset{\mathrm{def}}{=} x_{\vec{\alpha}_k}$ and $\bar{h}_k \overset{\mathrm{def}}{=} \bar{h}_{\vec{\alpha}_k}$. We assume that the sequence of homogeneous measures of the periodic orbits $\bar{h}^*_k A x_k$ converge weak* to $m_{\mathscr{X}_n}$. By applying the involution $x \mapsto x^*$ of $\mathscr{X}_n$ we deduce that the same holds for the sequence of periodic orbits $\bar{h}_k A x^*_k = \bar{A}_{\vec{\alpha}_k} y_{\vec{\alpha}_k}$, where $\bar{A}_{\vec{\alpha}_k}$ and $y_{\vec{\alpha}_k}$ are as defined in (69). That is

$$m_{\bar{A}_{\vec{\alpha}_k} y_{\vec{\alpha}_k}} \longrightarrow m_{\mathscr{X}_n}. \tag{141}$$

Consider the periodic orbits $\tilde{L}_{\vec{\alpha}_k} \tilde{y}_{\vec{\alpha}_k}$ in $\mathscr{X}_n^{\mathbb{A}}$ appearing in Proposition 7.7. Let us denote

$$\mu^{(k)} \overset{\mathrm{def}}{=} m_{\tilde{L}_{\vec{\alpha}_k} \tilde{y}_{\vec{\alpha}_k}}; \quad \mu \overset{\mathrm{def}}{=} m_{\mathscr{X}_n^{\mathbb{A}}}, \tag{142}$$

By Proposition 7.7, $\pi_*\mu^{(k)} = m_{\bar{A}_{\vec{\alpha}_k} y_{\vec{\alpha}_k}}$. Thus, (141) says that $\pi_*\mu^{(k)} \to m_{\mathscr{X}_n}$. Furthermore, $\mu^{(k)}$ is a_t-invariant and so by Lemma 14.1 we deduce that

$$\mu^{(k)} \longrightarrow \mu \tag{143}$$

Let $\mu^{(k)}_{\widetilde{\mathcal{S}}_{r_0}}, \mu_{\widetilde{\mathcal{S}}_{r_0}}$ be the corresponding cross-section measures. We apply Proposition 6.2, where for the tempered subset we take $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{S}}_{r_0}$. Regarding the applicability of Proposition 6.2, we note that if ν denotes any of the measures in (142), then the ν-reasonability of $\widetilde{\mathcal{S}}_{r_0}$, the ν-Jordan measurability of $\mathcal{B}$ and its temperedness follow from Theorem 9.11 (note that this is the point where we use the fact that the norm on $\mathbb{R}^d$ is either the Euclidean norm or the sup norm).

The conclusion of Proposition 6.2 is that $\mu^{(k)}_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}} \longrightarrow \mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}}$, where the convergence is tight convergence. In particular, the convergence holds after renormalizing the restricted measures to be probability measures:

$$\frac{1}{\mu^{(k)}_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{B}})}\mu^{(k)}_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}} \longrightarrow \frac{1}{\mu_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{B}})}\mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}}.$$

Let $\widetilde{\psi}$ be as in (117). By definition (see the proof of Theorem 2.1 in §13),

$$\widetilde{\psi}_*\left(\frac{1}{\mu_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{B}})}\mu_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}}\right) = \mu^{(\mathbf{e}_n)}; \quad \widetilde{\psi}_*\left(\frac{1}{\mu^{(k)}_{\widetilde{\mathcal{S}}_{r_0}}(\widetilde{\mathcal{B}})}\mu^{(k)}_{\widetilde{\mathcal{S}}_{r_0}}|_{\widetilde{\mathcal{B}}}\right) = \mu^{(\mathbf{e}_n,\vec{\alpha}_k)}.$$

By continuity of $\widetilde{\psi}_*$ at $\mu_{\widetilde{\mathcal{S}}_{r_0}}$ (see Lemma 4.9) we obtain (140). □

Remark 14.3. At the moment we do not have a version of Theorem 14.2 for ε-approximations. The reason is that Proposition 6.2 requires temperedness and we only know temperedness for $\mathcal{B}$ and not for $\mathcal{S}_\varepsilon$.

References

[AES16] M. Aka, M. Einsiedler, and U. Shapira, *Integer points on spheres and their orthogonal lattices*, Invent. Math. **206** (2016), no. 2, 379–396. MR3570295

[AGT15] J. S. Athreya, A. Ghosh, and J. Tseng, *Spherical averages of Siegel transforms and spiraling of lattice approximations*, Journal of London Mathematical Society **91** (2015), no. 2, 383–404.

[AK42] W. Ambrose and S. Kakutani, *Structure and continuity of measurable flows*, Duke Math. J. **9** (1942), 25–42. MR5800

[Amb41] W. Ambrose, *Representation of ergodic flows*, Ann. of Math. (2) **42** (1941), 723–739. MR4730

[AN93] P. Arnoux and A. Nogueira, *Mesures de Gauss pour des algorithmes de fractions continues multidimensionnelles*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 6, 645–664. MR1251147

[ASW22] F. Adiceam, Y. Solomon, and B. Weiss, *Cut-and-project quasicrystals, lattices and dense forests*, J. Lond. Math. Soc. (2) **105** (2022), no. 2, 1167–1199. MR4400945

[BG13] M. Baake and U. Grimm, *Aperiodic order. Vol. 1*, Encyclopedia of Mathematics and its Applications, vol. 149, Cambridge University Press, Cambridge, 2013. A mathematical invitation, With a foreword by Roger Penrose. MR3136260

[Bil68] P. Billingsley, *Convergence of probability measures*, John Wiley & Sons, Inc., New York-London-Sydney, 1968. MR0233396

[BJW83] W. Bosma, H. Jager, and F. Wiedijk, *Some metrical observations on the approximation by continued fractions*, Nederl. Akad. Wetensch. Indag. Math. **45** (1983), no. 3, 281–299. MR718069

[BNN06] V. Berthé, H. Nakada, and R. Natsui, *Arithmetic distributions of convergents arising from Jacobi-Perron algorithm*, Indag. Math. (N.S.) **17** (2006), no. 2, 169–185. MR2321379

[Bou04a] N. Bourbaki, *Integration. I. Chapters 1–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2004. Translated from the 1959, 1965 and 1967 French originals by Sterling K. Berberian. MR2018901

[Bou04b] N. Bourbaki, *Integration. II. Chapters 7–9*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2004. Translated from the 1963 and 1969 French originals by Sterling K. Berberian. MR2098271

[CC] Y. Cheung and N. Chevallier, *Lévy-Khintchin theorem for best Diophantine approximations*. preprint, available on arXiv at `https://arxiv.org/abs/1906.11173`.

[Che13] N. Chevallier, *Best simultaneous Diophantine approximations and multidimensional continued fraction expansions*, Mosc. J. Comb. Number Theory **3** (2013), no. 1, 3–56 (English).

[DK02] K. Dajani and C. Kraaikamp, *Ergodic theory of numbers*, Carus Mathematical Monographs, vol. 29, Mathematical Association of America, Washington, DC, 2002. MR1917322

[Doe40] W. Doeblin, *Remarques sur la théorie métrique des fractions continues*, Compositio Math. **7** (1940), 353–371. MR2732

[DS90] E. I. Dinaburg and Ya. G. Sinaĭ, *The statistics of the solutions of the integer equation* $ax - by = \pm 1$, Funktsional. Anal. i Prilozhen. **24** (1990), no. 3, 1–8, 96. MR1082025

[EL10] M. Einsiedler and E. Lindenstrauss, *Diagonal actions on locally homogeneous spaces*, Homogeneous flows, moduli spaces and arithmetic, 2010, pp. 155 - 241.

[EMSS16] M. Einsiedler, S. Mozes, N. Shah, and U. Shapira, *Equidistribution of primitive rational points on expanding horospheres*, Compos. Math. **152** (2016), no. 4, 667–692. MR3484111

[EW11] M. Einsiedler and T. Ward, *Ergodic theory with a view towards number theory*, Graduate Texts in Mathematics, vol. 259, Springer-Verlag London Ltd., London, 2011. MR2723325

[FS14] A. M. Fisher and T. A. Schmidt, *Distribution of approximants and geodesic flows*, Ergodic Theory Dynam. Systems **34** (2014), no. 6, 1832–1848. MR3272774

[Gui14] A. Guilloux, *Equidistribution in S-arithmetic and adelic spaces*, Ann. Fac. Sci. Toulouse Math. (6) **23** (2014), no. 5, 1023–1048. MR3294600

[IK02] M. Iosifescu and C. Kraaikamp, *Metrical theory of continued fractions*, Mathematics and its Applications, vol. 547, Kluwer Academic Publishers, Dordrecht, 2002. MR1960327

[JL88] H. Jager and P. Liardet, *Distributions arithmétiques des dénominateurs de convergents de fractions continues*, Nederl. Akad. Wetensch. Indag. Math. **50** (1988), no. 2, 181–197. MR952514

[Kec95] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR1321597

[Khi36] A. Khintchine, *Zur metrischen Kettenbruchtheorie*, Compositio Math. **3** (1936), 276–285. MR1556944

[Lév36] P. Lévy, *Sur le développement en fraction continue d'un nombre choisi au hasard*, Compositio Math. **3** (1936), 286–303. MR1556945

[Lag82] J. C. Lagarias, *Best simultaneous Diophantine approximations. I. Growth rates of best approximation denominators*, Trans. Amer. Math. Soc. **272** (1982), no. 2, 545–554. MR662052

[LW01] E. Lindenstrauss and B. Weiss, *On sets invariant under the action of the diagonal group*, Ergodic Theory Dynam. Systems **21** (2001), no. 5, 1481–1500. MR1855843 (2002j:22009)

[Mar10] J. Marklof, *The asymptotic distribution of Frobenius numbers*, Invent. Math. **181** (2010), no. 1, 179–207. MR2651383 (2011e:11133)

[Moe82] R. Moeckel, *Geodesics on modular surfaces and continued fractions*, Ergodic Theory Dynam. Systems **2** (1982), no. 1, 69–83. MR684245

[Mos00] N. G. Moshchevitin, *Best simultaneous approximations: norms, signatures, and asymptotic directions*, Math. Notes **67** (2000), no. 5, 618–624 (English).

[Nad13] M. G. Nadkarni, *Basic ergodic theory*, Third, Texts and Readings in Mathematics, vol. 6, Hindustan Book Agency, New Delhi, 2013. MR2963410

[NH16] A. Nevo and T. Horesh, *Horospherical coordinates of lattice points in hyperbolic space: effective counting and equidistribution*, 2016. ArXiv `https://arxiv.org/abs/1612.08215`.

[PR94] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen. MR1278263

[Roe56] W. Roelcke, *über die verteilung der klassen eigentlich assoziierter zweireihiger matrizen, die sich durch eine positiv-definite matrix darstellen lassen*, Math. Ann. **131** (1956), 260–277.

[RR09] M. S Risager and Z. Rudnick, *Equidistribution of the real part of orbits in hyperbolic spaces*, Cont. Math. **484** (2009), 187–194.

[Sch71] W. M. Schmidt, *Approximation to algebraic numbers*, Enseign. Math. (2) **17** (1971), 187–253. MR327672

[Sch98] W. M. Schmidt, *The distribution of sublattices of* $\mathbb{Z}^m$, Monatshefte für Mathematik **125** (1998), 37–81.

[Sie45] C. L. Siegel, *A mean value theorem in geometry of numbers*, Ann. of Math. (2) **46** (1945), 340–347. MR12093

[SW19] D. Simmons and B. Weiss, *Random walks on homogeneous spaces and Diophantine approximation on fractals*, Invent. Math. **216** (2019), no. 2, 337–394. MR3953505

[Szü62] P. Szüsz, *Über die metrische Theorie der diophantischen Approximation. II*, Acta Arith. **8** (1962/63), 225–241. MR153622

[SZ] U. Shapira and C. Zheng, *Translates of* S*-arithmetic periodic orbits and applications.* preprint, available at `https://arxiv.org/abs/2107.05017`.

[Wag88] V. M. Wagh, *A descriptive version of Ambrose's representation theorem for flows*, Proc. Indian Acad. Sci. Math. Sci. **98** (1988), no. 2-3, 101–108. MR994127

[Wei82] A. Weil, *Adeles and algebraic groups*, Progress in Mathematics, vol. 23, Birkhäuser, Boston, Mass., 1982. With appendices by M. Demazure and Takashi Ono. MR670072

[Xie15] S. Xieu, *Lévy's constant*, 2015. SFSU student project, available at `http://math.sfsu.edu/cheung/xieu-amp.pdf`.

Index

Dept. of Mathematics, Technion, Haifa, Israel ushapira@tx.technion.ac.il

Dept. of Mathematics, Tel Aviv University, Tel Aviv, Israel barakw@post.tau.ac.il