BADLY APPROXIMABLE VECTORS ON FRACTALS

DMITRY KLEINBOCK AND BARAK WEISS

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Abstract. For a large class of closed subsets $\mathcal{C}$ of $\mathbb{R}^n$, we show that the intersection of $\mathcal{C}$ with the set of badly approximable vectors has the same Hausdorff dimension as $\mathcal{C}$. The sets are described in terms of measures they support. Examples include (but are not limited to) self-similar sets such as Cantor's ternary sets or attractors for irreducible systems of similarities satisfying Hutchinson's open set condition.

1. Introduction

We say that $\mathbf{x} \in \mathbb{R}^n$ is *badly approximable* if there is $c > 0$ such that for any $\mathbf{p} \in \mathbb{Z}^n$, $q \in \mathbb{N}$ one has

$$\|q\mathbf{x} - \mathbf{p}\| \geq \frac{c}{q^{1/n}}. \tag{1.1}$$

We denote the set of all badly approximable vectors in $\mathbb{R}^n$ by $\mathbf{BA}$. It is well known that Lebesgue measure of $\mathbf{BA}$ is zero, but nevertheless this set is quite large, namely its Hausdorff dimension is equal to n [S2]. When $n = 1$, a number is badly approximable if and only if its continued fraction coefficients are bounded. For $n > 1$ there is no analogous description, and very few explicit examples of badly approximable vectors are known.

The goal of the present paper is to describe a large class of subsets of $\mathbb{R}^n$ which contain many badly approximable vectors. These sets will be described in terms of geometric properties of measures which they support. Thus we will show that whenever a measure μ on $\mathbb{R}^n$ satisfies certain conditions, the intersection of its support with $\mathbf{BA}$ has Hausdorff dimension equal to that of $\operatorname{supp}\mu$. The results are new even in the case $n = 1$.

Let us introduce some notation and terminology. For $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$, $B(\mathbf{x}, r)$ stands for the open ball of radius r centered at $\mathbf{x}$. For a ball $B = B(\mathbf{x}, r)$ and $a > 0$, we denote $B(\mathbf{x}, ar)$ by aB. For an affine subspace $\mathcal{L} \subset \mathbb{R}^n$ we denote by $\mathcal{L}^{(\varepsilon)}$ the ε-neighborhood of $\mathcal{L}$ (with respect to the Euclidean metric). Hausdorff dimension will be denoted by 'dim'.

In what follows, μ will be a locally finite Borel measure on $\mathbb{R}^n$. Following [KLW], given $C, \alpha > 0$ and $U \subset \mathbb{R}^n$ we say that μ is *absolutely* (C,α)*-decaying on* U if for any non-empty open ball $B \subset U$ of radius r centered in $\operatorname{supp}\mu$, any affine hyperplane $\mathcal{L} \subset \mathbb{R}^n$, and any $\varepsilon > 0$ one has

$$\mu\left(B \cap \mathcal{L}^{(\varepsilon)}\right) \le C\left(\frac{\varepsilon}{r}\right)^{\alpha}\mu(B)\,. \tag{1.2}$$

Given $D \ge 1$, say that μ is D*-Federer on* U if one has

$$\mu\,(3B) \le D\mu\,(B)$$

for every ball B centered in $\operatorname{supp}\mu$ with $3B \subset U$. In some papers this condition (more precisely, its stronger form with $U = \mathbb{R}^n$) is referred to as the 'doubling property'.

We will say that a measure is *absolutely decaying* (resp., *Federer*) if for μ-a.e. point of $\mathbb{R}^n$ there exist a neighborhood U of this point and $C, \alpha > 0$ (resp., $D > 0$) such that μ is absolutely (C,α)-decaying (resp., D*-Federer*) on U. Measures which are absolutely decaying and Federer form a subclass of the class of *friendly* measures, defined and studied in [KLW]. We refer the reader to §2 and §6 for more on absolutely decaying measures, and in particular to Remark 6.2 for a discussion of related conditions on measures considered recently in [PV2, KTV, U2, U3].

Let us also define the *lower pointwise dimension* of μ at $\mathbf{x}$ by

$$\underline{d}_\mu(\mathbf{x}) \stackrel{\text{def}}{=} \liminf_{r\to 0}\frac{\log\mu\big(B(\mathbf{x},r)\big)}{\log r}\,,$$

and for $B \subset \mathbb{R}^n$ put

$$\underline{d}_\mu(B) \stackrel{\text{def}}{=} \inf_{\mathbf{x}\in B}\underline{d}_\mu(\mathbf{x})\,.$$

The following is our main result:

Theorem 1.1. *Let μ be an absolutely decaying and Federer measure on $\mathbb{R}^n$. Then for any open ball B with $\mu(B) > 0$ one has*

$$\dim(\mathbf{BA} \cap \operatorname{supp}\mu \cap B) \ge \underline{d}_\mu(B)\,.$$

We remark that is well known, see e.g. [P, Theorem 7.1], that if $\underline{d}_\mu(\mathbf{x}) \ge \beta$ for μ-a.e. $\mathbf{x}$, then $\dim(\operatorname{supp}\mu) \ge \beta$. Consequently,

$$\dim(\operatorname{supp}\mu \cap B) \ge \underline{d}_\mu(B) \quad \forall \text{ open } B \text{ with } \mu(B) > 0\,. \tag{1.3}$$

Furthermore, equality in (1.3) holds for many natural measures. For example, take $\beta > 0$ and say that μ *satisfies a β-power law* on an open subset U of $\mathbb{R}^n$ if there are constants c_1, c_2 such that

$$c_1 r^\beta \le \mu\big(B(\mathbf{x}, r)\big) \le c_2 r^\beta \text{ whenever } \mathbf{x} \in \operatorname{supp}\mu \text{ and } B(\mathbf{x}, r) \subset U\,. \tag{1.4}$$

We will say that μ *satisfies a β-power law* (or sometimes simply *satisfies a power law*) if μ-a.e. point of $\mathbb{R}^n$ has a neighborhood U such that μ satisfies a β-power law on U.

This condition is well studied. A set for which the restriction of the Hausdorff measure, in the appropriate dimension, satisfies a power law is sometimes called *regular* or *Ahlfors–David regular* (see e.g. [Mat, Chaps. 4–6] and references therein). A measure μ with a β-power law is obviously Federer, and also satisfies

$$\beta = \dim(\operatorname{supp}\mu \cap B) = \underline{d}_\mu(B) = \underline{d}_\mu(\mathbf{x}) \tag{1.5}$$

for any open B with $\mu(B) > 0$ and any $\mathbf{x} \in \operatorname{supp}\mu$, see e.g. [Mat, Thm. 5.7].

Corollary 1.2. *Assume that μ is absolutely decaying and satisfies a power law. Then for any open $B \subset \mathbb{R}^n$ one has*

$$\dim(\mathbf{BA} \cap \operatorname{supp}\mu \cap B) = \dim(\operatorname{supp}\mu \cap B)\,.$$

A different proof of the above statement has recently appeared in [KTV].

It is easy to see that Corollary 1.2 is a special case of Theorem 1.1. Indeed, it is enough to assume that $\operatorname{supp}\mu \cap B \ne \varnothing$, in which case one can write

$$\underline{d}_\mu(B) \overset{\text{Thm. 1.1}}{\le} \dim(\mathbf{BA} \cap \operatorname{supp}\mu \cap B) \le \dim(\operatorname{supp}\mu \cap B) \overset{(1.5)}{=} \underline{d}_\mu(B)\,,$$

and the assertion follows.

Note that the absolute decay condition can be used to estimate $\underline{d}_\mu$ from below: namely, μ being absolutely (C, α)-decaying on U implies that $\underline{d}_\mu(B) \ge \alpha$ for any ball $B \subset U$. However, this estimate is not necessarily optimal. In the case $n = 1$, the absolute decay condition is similar to a condition introduced by W. Veech in [V] (see Remark 6.2). Besides Lebesgue measure, the simplest example of a measure satisfying the conditions of Corollary 1.2 is given by the coin-tossing measure on Cantor's ternary set $\mathcal{C}$; it follows that badly approximable numbers inside $\mathcal{C}$ form a set of Hausdorff dimension $\frac{\log 2}{\log 3} = \dim(\mathcal{C})$. Even this very special case does not appear in the literature, although it is known to experts and is provable by other methods. We are grateful to Yuval Peres for describing a proof to us.

Additional examples of measures which are absolutely decaying and satisfy a power law are Hausdorff measures, in the appropriate dimension, restricted to self-similar (or, more generally, self-conformal) sets satisfying Hutchinson's open set condition. This was proved in [KLW] in the self-similar case, and has been recently generalized by Urbanski [U2, U3]. We remind the reader of Hutchinson's setup in §7, and also describe new examples of measures which are absolutely decaying and satisfy a power law. These measures need not be supported on self-similar/self-conformal sets, and may have arbitrarily small dimension. They provide new examples to which the results of [KLW] and the present paper apply. In §7 we also construct a measure on the real line which is absolutely decaying and Federer, but does not satisfy a power law; thus Theorem 1.1 is applicable to a larger class of measures than Corollary 1.2.

The proof of Theorem 1.1 involves three intermediate steps. First, using a theorem of Dani [D1], it is shown that Theorem 1.1 follows from a dynamical result (Theorem 3.2) about abundance of certain bounded trajectories in the space G/Γ, where $G = \mathrm{SL}_{n+1}(\mathbb{R})$ and $\Gamma = \mathrm{SL}_{n+1}(\mathbb{Z})$. Bounded trajectories are then constructed by an iterative procedure, originally introduced in [KM1], using a result (Proposition 5.3) on uniform return (in terms of a given measure μ) to large compact subsets of G/Γ. This procedure is described in detail in §4. Proposition 5.3 in turn is deduced from quantitative nondivergence estimates recently established in [KLW].

The construction of bounded trajectories using uniform return estimates has several other applications; two of them, to bounded trajectories of the Teichmüller geodesic flow and to Diophantine approximation with weights, are briefly discussed in the last section of the paper.

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2. Preliminaries

In this section we introduce some notation and collect some well-known results which we will need in later sections.

If $\mathcal{G}$ is a collection of sets, we let $\cup\mathcal{G} \stackrel{\mathrm{def}}{=} \bigcup_{B\in\mathcal{G}} B$. If φ is a map defined on $\cup\mathcal{G}$, we let $\varphi(\mathcal{G}) \stackrel{\mathrm{def}}{=} \{\varphi(B) : B \in \mathcal{G}\}$.

Proposition 2.1 (Besicovitch, see e.g. Theorem 2.7 in [Mat])**.** *For any $n \in \mathbb{N}$ there exists $N = N_n \in \mathbb{N}$ (the Besicovitch constant of $\mathbb{R}^n$) with the following property: for any bounded subset A of $\mathbb{R}^n$ and any collection $\mathcal{G}$ of closed balls in $\mathbb{R}^n$ such that each point of A is the center of some ball of $\mathcal{G}$, there are disjoint countable subcollections $\mathcal{G}_1, \dots, \mathcal{G}_N$ such that A is covered by $\bigcup_{i=1}^N \cup\mathcal{G}_i$. Consequently, any such $\mathcal{G}$ contains a countable subcovering $\mathcal{G}'$ of A of multiplicity at most N.*

Throughout the paper $n \in \mathbb{N}$ will be fixed, and the Besicovitch constant of $\mathbb{R}^n$ will be denoted by N.

For a measure μ on $\mathbb{R}^n$ and a measurable map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, the *pushforward* $\varphi_*\mu$ of μ under φ is defined by $\varphi_*\mu(A) \stackrel{\mathrm{def}}{=} \mu(\varphi^{-1}(A))$. It will be also convenient to introduce the following notation: for $\mathbf{y} \in \mathbb{R}^n$ and $c \geq 1$ let $\varphi_{\mathbf{y},c}$ be the affine transformation of $\mathbb{R}^n$ defined by

$$\varphi_{\mathbf{y},c}(\mathbf{x}) \stackrel{\mathrm{def}}{=} c(\mathbf{x} - \mathbf{y})\,. \tag{2.1}$$

The following immediately follows from the definitions:

Lemma 2.2. *Let μ be absolutely (C, α)-decaying (resp. D-Federer) on U. Then for any $\mathbf{y} \in \mathbb{R}^n$ and any $c \geq 1$, $(\varphi_{\mathbf{y},c})_*\mu$ is absolutely (C, α)-decaying (resp. D-Federer) on $\varphi_{\mathbf{y},c}(U)$.*

The maps $\varphi_{\mathbf{y},c}$ will be repeatedly used for 'zooming in' on smaller parts of sets and measures: observe that $\varphi_{\mathbf{y},c}$ maps a ball $B(\mathbf{y}, r/c)$ onto $B(0, r)$. It will be important for our purposes, given a measure μ and a ball $B = B(0, r) \subset \mathbb{R}^n$, to consider all measures obtained from μ by zooming in on sub-balls of B. More precisely, for any $a > 1$ we define

$$\mathcal{M}_{\mu,B,a} \stackrel{\mathrm{def}}{=} \{(\varphi_{\mathbf{y},a^k})_*\mu : \mathbf{y} \in \operatorname{supp}\mu,\ k \in \mathbb{Z}_+,\ B(\mathbf{y}, r/a^k) \subset B\}\,. \tag{2.2}$$

If B is a subset of $\mathbb{R}^n$ and f is a real-valued function on $\mathbb{R}^n$, let

$$\|f\|_B \stackrel{\mathrm{def}}{=} \sup_{x\in B} |f(x)|\,;$$

and if μ is a measure on $\mathbb{R}^n$ such that $\mu(B) > 0$, we define $\|f\|_{\mu,B}$ to be equal to $\|f\|_{B\cap\operatorname{supp}\mu}$. Given C, $\alpha > 0$, a subset U of $\mathbb{R}^n$, a measure μ on U and a real-valued function f on U, say that f is (C, α)-*good on U with respect to μ* if for any open ball $B \subset U$ centered in $\operatorname{supp}\mu$ and any $\varepsilon > 0$ one has

$$\mu\big(\{x \in B : |f(x)| < \varepsilon\}\big) \le C\left(\frac{\varepsilon}{\|f\|_{\mu,B}}\right)^{\alpha} \mu(B)\,. \tag{2.3}$$

See [KM2], [BKM] and [KLW] for various properties and examples. Here is one of them, a modified version of [BKM, Lemma 3.3]:

Lemma 2.3. *Let U be a subset of $\mathbb{R}^n$, μ a measure on U, $m \in \mathbb{N}$, $C, \alpha > 0$, and let $\mathbf{f} = (f_1, \dots, f_m)$ be a map $U \to \mathbb{R}^m$ such that each f_i is (C,α)-good on U with respect to μ. Then the function $\mathbf{x} \mapsto \|\mathbf{f}(\mathbf{x})\|$, where $\|\cdot\|$ is the standard Euclidean norm, is $(\sqrt{m}C, \alpha)$-good on U with respect to μ.*

We will also need the following facts:

Lemma 2.4. *Let μ be absolutely (C,α)-decaying on $U \subset \mathbb{R}^n$. Then any affine function f on $\mathbb{R}^n$:*

(i) *is (C', α)-good on U with respect to μ, where*
$$C' = C(C^{-1/\alpha} + 2)^{\alpha}\,; \tag{2.4}$$

(ii) *satisfies*
$$\|f\|_B \le (1 + 2C^{1/\alpha})\|f\|_{\mu,B} \tag{2.5}$$
for any ball $B \subset U$ centered in $\operatorname{supp}\mu$.

Proof. Without loss of generality we can assume that f is nonzero, and, after suitable scaling and taking absolute value, replace it by the distance function from some hyperplane $\mathcal{L}$ in $\mathbb{R}^n$. Then the left hand side of (2.3) coincides with that of (1.2), so, if we denote $\|f\|_{\mu,B}$ by δ, to establish (i) we need to prove that for any ball $B \subset U$ centered in $\operatorname{supp}\mu$ radius r one has

$$\mu\left(B \cap \mathcal{L}^{(\varepsilon)}\right) \le C'\left(\frac{\varepsilon}{\delta}\right)^{\alpha} \mu(B)\,. \tag{2.6}$$

Denoting by r the radius of B, one has

$$\frac{\mu\left(B \cap \mathcal{L}^{(\varepsilon)}\right)}{\mu(B)} \overset{\text{by (1.2)}}{\le} C\left(\frac{\varepsilon}{r}\right)^{\alpha} = C\left(\frac{\delta}{r}\right)^{\alpha}\left(\frac{\varepsilon}{\delta}\right)^{\alpha}\,. \tag{2.7}$$

On the other hand, $B \cap \mathcal{L}^{(\varepsilon)} = \varnothing$ when $\varepsilon < \delta - 2r$, and otherwise one has

$$\frac{\mu\left(B \cap \mathcal{L}^{(\varepsilon)}\right)}{\mu(B)} \le \left(\frac{\delta}{\delta - 2r}\right)^{\alpha}\left(\frac{\delta - 2r}{\delta}\right)^{\alpha} \le \left(\frac{\delta}{\delta - 2r}\right)^{\alpha}\left(\frac{\varepsilon}{\delta}\right)^{\alpha}\,. \tag{2.8}$$

The minimum of the right hand sides of (2.7) and (2.8) achieves its biggest value when $\delta/r = C^{-\alpha} + 2$, hence one has (2.6) with C' as in (2.4).

To demonstrate (ii), note that for any $\varepsilon > \delta$ one has

$$1 = \frac{\mu\left(B \cap \mathcal{L}^{(\varepsilon)}\right)}{\mu(B)} \le C\left(\frac{\varepsilon}{r}\right)^{\alpha},$$

hence $\delta \ge C^{-1/\alpha} r$. On the other hand one clearly has $r \ge \frac{\|f\|_B - \delta}{2}$, which immediately yields (2.5). $\square$

In the remaining part of this section we describe an elementary construction of compact subsets of $\mathbb{R}^n$. Let A_0 be a compact subset of $\mathbb{R}^n$, and let μ be a finite measure on A_0. Say that a countable family $\mathcal{A}$ of compact subsets of A_0 is *tree-like*[1] *relative to* μ if $\mathcal{A}$ is the union of finite subcollections $\mathcal{A}_k$, $k \in \mathbb{N}$, such that $\mathcal{A}_0 = \{A_0\}$ and the following four conditions are satisfied:

(TL0) $\mu(A) > 0$ for any $A \in \mathcal{A}$;

(TL1) $\forall\, k \in \mathbb{N} \quad \forall\, A, B \in \mathcal{A}_k$ either $A = B$ or $\mu(A \cap B) = 0$;

(TL2) $\forall\, k \in \mathbb{N} \quad \forall\, B \in \mathcal{A}_k \quad \exists\, A \in \mathcal{A}_{k-1}$ such that $B \subset A$;

(TL3) $\forall\, k \in \mathbb{N} \quad \forall\, B \in \mathcal{A}_{k-1} \quad \mathcal{A}_k(B) \ne \varnothing$, where

$$\mathcal{A}_k(B) \stackrel{\text{def}}{=} \{A \in \mathcal{A}_k : A \subset B\}.$$

The reason for this terminology is quite clear: every member of the family corresponds to a node of a certain tree, A_0 being the root, and sets from $\mathcal{A}_k$ correspond to vertices of the kth generation. Conditions (TL1–3) say that every vertex of the tree has at least one child and (except for the root) a unique parent, and sets corresponding to nodes of the same generation are μ-essentially disjoint.

Let $\mathcal{A}$ be a tree-like collection of sets relative to a measure μ. For each $k \in \mathbb{N}$, the sets $\cup\mathcal{A}_k$ are nonempty and compact, and from (TL2) it follows that $\cup\mathcal{A}_k$ is contained in $\cup\mathcal{A}_{k-1}$ for any $k \in \mathbb{N}$. Therefore one can define the (nonempty) *limit set* of $\mathcal{A}$ to be

$$\mathbf{A}_\infty = \bigcap_{k \in \mathbb{N}} \cup\mathcal{A}_k.$$

Note that $\mathbf{A}_\infty \subset \operatorname{supp}\mu$ in view of (TL0).

In many cases it is important that, as $k \to \infty$, the sets from $\mathcal{A}_k$ become smaller. We will formalize it by defining the *kth stage diameter* $d_k(\mathcal{A})$ of $\mathcal{A}$:

$$d_k(\mathcal{A}) \stackrel{\text{def}}{=} \max_{A \in \mathcal{A}_k} \operatorname{diam}(A),$$

and saying that $\mathcal{A}$ is *strongly tree-like* if it is tree-like and in addition

[1]The terminology is borrowed from [KM1], but the definition is slightly changed for the sake of better exposition.

(STL) $\lim_{k\to\infty} d_k(\mathcal{A}) = 0$.

Note that any compact subset of $\mathbb{R}^n$ is a limit set of a strongly tree-like collection of sets; for example, for $K \subset [0,1]^n$ we may take for $\mathcal{A}_k$ the dyadic cubes of sidelength 2^{-k} in $[0,1]^n$ whose intersection with K is nonempty, and take Lebesgue measure for μ. On the other hand, a representation of a compact set as a limit set of a strongly tree-like collection often helps to estimate its Hausdorff dimension. To state the desired estimate requires some more terminology.

For $k \in \mathbb{Z}_+$ and $B \in \mathcal{A}_k$ let us define the 'density of children' of B in $\mathcal{A}$ by

$$\delta(B,\mathcal{A}) \stackrel{\text{def}}{=} \frac{\mu\big(\cup\,\mathcal{A}_{k+1}(B)\big)}{\mu(B)}\,,$$

and then let

$$\Delta_k(\mathcal{A}) \stackrel{\text{def}}{=} \min_{B\in\mathcal{A}_k} \delta(B,\mathcal{A})\,;$$

note that the latter is always positive due to (TL3).

The following lemma generalizes results of C. McMullen [Mc, Proposition 2.2] and M. Urbanski [U1, Lemma 2.1].

Lemma 2.5. *Let $\mathcal{A}$ be a strongly tree-like (relative to μ) collection of subsets of A_0. Then there exists a measure ν with $\mathbf{A}_\infty = \operatorname{supp}\nu$ such that for any $\mathbf{x} \in \mathbf{A}_\infty$,*

$$\underline{d}_\nu(\mathbf{x}) \geq \underline{d}_\mu(\mathbf{x}) - \limsup_{k\to\infty} \frac{\sum_{i=0}^k \log \Delta_i(\mathcal{A})}{\log\, d_k(\mathcal{A})}\,. \tag{2.9}$$

Consequently, for any open ball B intersecting $\mathbf{A}_\infty$ one has

$$\dim(\mathbf{A}_\infty \cap B) \geq \underline{d}_\mu(B) - \limsup_{k\to\infty} \frac{\sum_{i=0}^k \log \Delta_i(\mathcal{A})}{\log\, d_k(\mathcal{A})}\,.$$

Proof. We basically follow the argument of [U1]. Define measures ν_k inductively as follows: put $\nu_0 = \mu|_{A_0}$, and, given ν_{k-1}, set

$$\nu_k(A) \stackrel{\text{def}}{=} \sum_{B\in\mathcal{A}_{k-1}} \frac{\mu\big(\cup\,\mathcal{A}_k(B)\cap A\big)}{\mu\big(\cup\,\mathcal{A}_k(B)\big)}\nu_{k-1}(B)\,. \tag{2.10}$$

This definition makes sense since by (TL0) and (TL3) one knows that $\mu\big(\cup\mathcal{A}_k(B)\big) > 0$ for any $B \in \mathcal{A}_{k-1}$. The countable additivity of ν_k can be easily shown using (TL1) and (TL2), and one can see by induction that

$$\nu_k(\cup\mathcal{A}_k) = \mu(A_0) \quad \text{for every } k \in \mathbb{N}\,.$$

Further, in view of (TL1) one has $\nu_{k+1}(A) = \nu_k(A)$ for each $A \in \mathcal{A}_k$. Hence one can conclude, using induction, that $\nu_\ell(A) = \nu_k(A)$ for each

$A \in \mathcal{A}_k$ and $\ell \geq k$. It follows then from (STL) that the sequence $\{\nu_k\}$ has a unique weak limit, a finite measure ν with $\operatorname{supp} \nu = \mathbf{A}_\infty$ such that

$$\nu(A) = \nu_k(A) \quad \text{for any } A \in \mathcal{A}_k\,. \tag{2.11}$$

Making use of (2.10) and (2.11), one inductively computes that

$$\nu(A) \leq \frac{\mu(A)}{\prod_{i=0}^{k-1} \Delta_i(\mathcal{A})} \quad \text{for any } A \in \mathcal{A}_k\,. \tag{2.12}$$

Now take $\mathbf{x} \in \mathbf{A}_\infty$ and $0 < r < \sup_k d_k(\mathcal{A})$. Then there exists $k = k(r)$ such that $d_{k+1}(\mathcal{A}) \leq r < d_k(\mathcal{A})$, and one can write

$$\begin{aligned} \nu\big(B(\mathbf{x}, r)\big) &\leq \nu\big(\cup \{A \in \mathcal{A}_{k+1} : A \cap B(\mathbf{x}, r) \neq \varnothing\}\big) \\ &\stackrel{(2.12)}{\leq} \frac{\mu\big(\cup \{A \in \mathcal{A}_{k+1} : A \cap B(\mathbf{x}, r) \neq \varnothing\}\big)}{\prod_{i=0}^{k} \Delta_i(\mathcal{A})} \leq \frac{\mu\big(B(\mathbf{x}, 2r)\big)}{\prod_{i=0}^{k} \Delta_i(\mathcal{A})}\,. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\log \nu\big(B(\mathbf{x}, r)\big)}{\log r} &\geq \frac{\log \mu\big(B(\mathbf{x}, 2r)\big) - \sum_{i=0}^{k} \log \Delta_i(\mathcal{A})}{\log r} \\ &\geq \frac{\log \mu\big(B(\mathbf{x}, 2r)\big)}{\log(2r)} \left(1 + \frac{\log 2}{\log r}\right) - \frac{\sum_{i=0}^{k} \log \Delta_i(\mathcal{A})}{\log\, d_k(\mathcal{A})}\,. \end{aligned}$$

Since $\lim_{r\to 0} k(r) = 0$ due to (STL), taking $\liminf_{r\to 0}$ of both sides of the above inequality yields (2.9). It remains to mention that the second part of the lemma follows immediately from (1.3). $\square$

3. From bounded trajectories to badly approximable vectors

Let $G = \mathrm{SL}_{n+1}(\mathbb{R})$, $\Gamma = \mathrm{SL}_{n+1}(\mathbb{Z})$, and denote by $\pi : G \to G/\Gamma$, $g \mapsto g\Gamma$, the natural projection map. G acts on G/Γ by left translations via the rule $g\pi(h) = \pi(gh)$, $g, h \in G$. Equivalently one can describe G/Γ as the space of unimodular lattices in $\mathbb{R}^{n+1}$, with $\pi(g)$ corresponding to the lattice $g\mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}$, and the action of G on G/Γ coming from the linear action of G on $\mathbb{R}^{n+1}$. We will be interested in the action of the one-parameter subsemigroup

$$F \stackrel{\text{def}}{=} \{g_t : t \geq 0\}$$

of G on G/Γ, where

$$g_t \stackrel{\text{def}}{=} \operatorname{diag}(e^{t/n}, \ldots, e^{t/n}, e^{-t})\,. \tag{3.1}$$

Note that the action of elements of F on a lattice Λ contracts the last component of every vector of Λ and expands the remaining components.

Fix a norm $\|\cdot\|$ on $\mathbb{R}^{n+1}$, and for $\varepsilon > 0$ let

$$K_\varepsilon \stackrel{\text{def}}{=} \pi\big(\{g \in G : \|g\mathbf{v}\| \geq \varepsilon \quad \forall\, \mathbf{v} \in \mathbb{Z}^{n+1} \smallsetminus \{0\}\}\big)\,, \tag{3.2}$$

i.e., K_ε is the collection of all unimodular lattices in $\mathbb{R}^{n+1}$ which contain no nonzero vector smaller than ε. Recall that G/Γ is noncompact and has finite G-invariant measure. Each K_ε, however, is compact (Mahler's Compactness Criterion, see e.g. [R, Chapter 10]), and $\{K_\varepsilon\}_{\varepsilon>0}$ is an exhaustion of G/Γ.

Let us also define the following map from $\mathbb{R}^n$ to G:

$$\tau(\mathbf{x}) \stackrel{\text{def}}{=} \begin{pmatrix} I_n & \mathbf{x} \\ 0 & 1 \end{pmatrix} \tag{3.3}$$

(here I_n stands for the $n \times n$ identity matrix). Note that the lattice $\tau(\mathbf{x})\mathbb{Z}^{n+1}$ is given by

$$\tau(\mathbf{x})\mathbb{Z}^{n+1} = \left\{ \begin{pmatrix} q\mathbf{x} - \mathbf{p} \\ q \end{pmatrix} : q \in \mathbb{Z},\ \mathbf{p} \in \mathbb{Z}^n \right\},$$

thus serving as a connecting tool between the two sides of the inequality (1.1). From this observation it is not hard to interpret badly approximable points of $\mathbb{R}^n$ in terms of bounded F-trajectories on G/Γ as follows:

Proposition 3.1 (Dani [D1])**.** $\mathbf{x} \in \mathbf{BA}$ *iff* $F\tau(\mathbf{x})\mathbb{Z}^{n+1} \subset K_\varepsilon$ *for some* $\varepsilon > 0$*.*

It is clear from the above proposition that Theorem 1.1 will immediately follow from

Theorem 3.2. *Let μ and B be as in Theorem 1.1. Then for any $\Lambda \in G/\Gamma$, one has*

$$\dim\left(\{\mathbf{x} \in \operatorname{supp}\mu \cap B : F\tau(\mathbf{x})\Lambda \text{ is bounded}\}\right) \geq \underline{d}_\mu(B)\,. \tag{3.4}$$

It is important that the group

$$H \stackrel{\text{def}}{=} \{\tau(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$$

is the so-called *expanding horoshperical subgroup* of F; in other words, H-orbits on G/Γ are exactly the unstable leaves with respect to the F-action. More precisely, for any fixed $t > 0$ the conjugation by g_t gives rise to an expanding homothety of H of the form

$$g_t\tau(\mathbf{x})g_{-t} = \tau\big(e^{(1+1/n)t}\mathbf{x}\big)\,. \tag{3.5}$$

Let us observe that taking μ equal to Lebesgue measure in Theorem 3.2 (or equivalently, Haar measure on H) one can deduce that

$$\begin{gathered}\text{for every } \Lambda \in G/\Gamma\,, \text{ the set } \big\{h \in H : Fh\Lambda \text{ is bounded}\big\} \\ \text{has full Hausdorff dimension at any point of } H.\end{gathered} \tag{3.6}$$

As a historical remark, let us point out that Dani in [D1] used the correspondence of Proposition 3.1 and Schmidt's result on the full Hausdorff dimension of the set **BA** to derive (3.6); and that later a dynamical proof of a generalization of (3.6) appeared in [KM1]. See [K4, §3] for a more detailed historical account.

One of the main technical tools used in [KM1] was an iterative procedure of constructing points with bounded trajectories based on Lemma 2.5, which can produce many of them assuming that a certain 'uniform return' condition is satisfied. In the present paper we follow a similar strategy. Namely, in the next section we describe a modified version of the aforementioned procedure, thereby reducing Theorem 3.2 to verifying a uniform return property (Corollary 5.4). The latter is deduced in §5 from the quantitative nondivergence estimates of [KLW].

4. From uniform return to bounded trajectories

The goal of this section is to describe an abstract scheme for constructing bounded trajectories of certain actions, which, in particular, will be applicable in the context of the previous section, that is, the action of F and H on G/Γ. Namely, for $n \in \mathbb{N}$ and $a > 0$ let us denote by $S_{n,a}$ the semidirect product $\mathbb{R}^n \rtimes \mathbb{Z}$ of $\mathbb{R}^n$ and $\mathbb{Z} \cong \{g^k : k \in \mathbb{Z}\}$ given by

$$S_{n,a} \stackrel{\text{def}}{=} \langle \mathbb{R}^n, g \mid g\mathbf{x}g^{-1} = a\mathbf{x} \quad \forall\, \mathbf{x} \in \mathbb{R}^n \rangle\,.$$

Note that, in view of (3.5), for any $t > 0$ the subgroup of G as in the previous section generated by g_t and H is isomorphic to $S_{n,e^{(1+1/n)t}}$.

Let us assume now that we are given an action of the group $S_{n,a}$ on an abstract set Y, which we will simply denote by $(\gamma, y) \mapsto \gamma y$, $\gamma \in S_{n,a}$, $y \in Y$. Suppose also that we are given a family $\mathcal{M}$ of measures on $\mathbb{R}^n$, a ball $B \subset \mathbb{R}^n$, two subsets K, Q of Y, and a positive η. Say that $\mathcal{M}$ has the *η-uniform return property* with respect to the triple (B, K, Q) if for any $\nu \in \mathcal{M}$ and any $y \in Q$ one has

$$\nu\big(\{\mathbf{x} \in B : (g\mathbf{x})y \in K\}\big) \geq (1 - \eta)\nu(B)\,.$$

The following theorem is the main result of this section:

Theorem 4.1. *Given $a \geq 2$, $D > 0$ and a neighborhood U of 0 in $\mathbb{R}^n$, let μ be a D-Federer measure on U with $0 \in \operatorname{supp}\mu$, and let a closed* *ball $B \subset U$ centered at 0, two subsets K, Q of an $S_{n,a}$-space Y and $0 < \eta < 1/DN$ satisfy:*

(i) $(2B)K \subset Q$

(ii) *the family $\mathcal{M}_{\mu,B,a}$ has the η-uniform return property with respect to (B, K, Q).*

Then for any $y \in Q$ one has

$$\dim\left(\left\{\mathbf{x} \in \operatorname{supp}\mu \cap B : (g^k\mathbf{x})y \in Q \quad \forall\, k \in \mathbb{N}\right\}\right) \geq \underline{d}_\mu(B) - \frac{\log(\frac{1}{1/DN-\eta})}{\log a}\,. \tag{4.1}$$

Proof. First note that it is enough to prove the theorem for $Y = S_{n,a}$ (that is, for the left action of $S_{n,a}$ on itself). Indeed, if for $y \in Y$ one denotes by π_y the ($S_{n,a}$-equivariant) map $S_{n,a} \to Y$, $\gamma \mapsto \gamma y$, it is not hard to see that (ii) is equivalent to saying that for any $y \in Q$

$$\begin{gathered}\text{the family } \mathcal{M}_{\mu,B,a} \text{ has the } \eta\text{-uniform return}\\ \text{property with respect to } (B, \mathcal{K}, \mathcal{Q})\,,\end{gathered} \tag{4.2}$$

where $\mathcal{K} = \mathcal{K}(y) \stackrel{\text{def}}{=} \pi_y^{-1}(K)$ and $\mathcal{Q} = \mathcal{Q}(y) \stackrel{\text{def}}{=} \pi_y^{-1}(Q)$. Likewise, (i) is equivalent to saying that for any $y \in Q$ one has

$$(2B)\mathcal{K} \subset \mathcal{Q}\,. \tag{4.3}$$

On the other hand, if, given $y \in Q$, one knows that

$$\dim\left(\left\{\mathbf{x} \in \operatorname{supp}\mu \cap B : g^k\mathbf{x}\gamma \in \mathcal{Q} \quad \forall\, k \in \mathbb{N}\right\}\right) \geq \underline{d}_\mu(B) - \frac{\log(\frac{1}{1/DN-\eta})}{\log a} \tag{4.4}$$

for any $\gamma \in \mathcal{Q}$, one can let γ be the identity element in $S_{n,a}$ (which belongs to $\mathcal{Q}$ as long as $y \in Q$), and deduce (4.1). Thus it is enough to start with two subsets $\mathcal{K}, \mathcal{Q}$ of $S_{n,a}$, assume (4.3) and (4.2), and demonstrate (4.4).

In order to do this, given a D-Federer measure ν on U and an element γ of $S_{n,a}$, we are going to define a (possibly empty) collection $\mathcal{H} = \mathcal{H}(\gamma, \nu)$ of disjoint closed balls A of radius r/a each contained in $B = B(0, r)$ and centered in $\operatorname{supp}\nu$ such that

$$gA\gamma \cap \mathcal{K} \neq \varnothing \tag{4.5}$$

for any $A \in \mathcal{H}(\gamma, \nu)$, and

$$\nu\big(\cup\, \mathcal{H}(\gamma, \nu)\big) \geq \frac{\nu(B)}{DN} - \nu\big(\{\mathbf{x} \in B : g\mathbf{x}\gamma \notin \mathcal{K}\}\big)\,. \tag{4.6}$$

Indeed, first consider the collection $\mathcal{G}$ of all balls of radius r/a centered in $B(0, r/3) \cap \operatorname{supp}\nu$ (note that all those balls are contained in B since we have assumed that $a \geq 2$). Using Proposition 2.1 one can choose a disjoint subcollection $\mathcal{G}'$ such that

$$\nu\left(\cup\mathcal{G}'\right) \geq \frac{\nu\big(B(0,r/3)\big)}{N} \overset{\text{Federer}}{\geq} \frac{\nu(B)}{DN}\,. \tag{4.7}$$

Now define $\mathcal{H}$ to be the set of balls A in $\mathcal{G}'$ satisfying (4.5). Then (4.6) follows from (4.7) and

$$\nu\left(\cup\mathcal{H}\right) \geq \nu\left(\cup\mathcal{G}'\right) - \nu\big(\{\mathbf{x} \in B : g\mathbf{x}\gamma \notin \mathcal{K}\}\big)\,.$$

Note that if in addition $\gamma \in \mathcal{Q}$ and $\nu \in \mathcal{M}_{\mu,B,a}$, it follows from (4.6) and (4.2) that

$$\frac{\nu\big(\cup\mathcal{H}(\gamma,\nu)\big)}{\nu(B)} \geq 1/DN - \eta\,; \tag{4.8}$$

in particular, the collection $\mathcal{H}$ is non-empty as long as $\eta < 1/DN$.

Next, let us fix γ and construct a certain collection $\mathcal{A}$ of subsets of B. Here is the inductive construction. First let $\mathcal{A}_0 \overset{\text{def}}{=} \{B\}$, then define

$$\mathcal{A}_1 \overset{\text{def}}{=} \mathcal{H}(\gamma,\mu)\,;$$

and, more generally, if $\mathcal{A}_i$ is defined for all $i \leq k$, we let

$$\mathcal{A}_{k+1} \overset{\text{def}}{=} \bigcup_{A\in\mathcal{A}_k} \mathcal{A}_{k+1}(A)\,,$$

where for $A = \overline{B(\mathbf{y}, r_0/a^k)} = \varphi^{-1}_{\mathbf{y},a^k}(B)$ we define

$$\mathcal{A}_{k+1}(A) \overset{\text{def}}{=} \varphi^{-1}_{\mathbf{y},a^k}\Big(\mathcal{H}\left(g^k\mathbf{y}\gamma, (\varphi_{\mathbf{y},a^k})_*\mu\right)\Big)\,.$$

Lemma 4.2. *For any $k \in \mathbb{Z}_+$ and $A \in \mathcal{A}_{k+1}$, one has*

(a) $g^{k+1}A\gamma \cap \mathcal{K} \neq \varnothing$;
(b) $g^{k+1}A\gamma \subset \mathcal{Q}$;
(c) *A is centered in* $\operatorname{supp}\mu$.

Proof. By definition, $A \in \mathcal{A}_{k+1}$ if and only if

$$A = \varphi^{-1}_{\mathbf{y},a^k}(A') = g^{-k}A'g^k\mathbf{y}\,, \tag{4.9}$$

where $A' \in \mathcal{H}\left(g^k\mathbf{y}\gamma, (\varphi_{\mathbf{y},a^k})_*\mu\right)$. Therefore $g^{k+1}A\gamma = gA'g^k\mathbf{y}\gamma$, which has non-empty intersection with $\mathcal{K}$ in view of (4.5), hence (a). As for (b), it is an immediate consequence of (a) and (4.3). For (c), note that A' is centered in $\operatorname{supp}(\varphi_{\mathbf{y},a^k})_*\mu$, and apply (4.9). □

Note that if one in addition assumes that $\gamma \in \mathcal{Q}$, then part (b) of the above lemma implies that

$$g^k\mathbf{y}\gamma \in \mathcal{Q} \quad \text{whenever } k \in \mathbb{Z}_+ \text{ and } \overline{B(\mathbf{y}, r/a^k)} \in \mathcal{A}_k\,. \tag{4.10}$$

We now claim that $\mathcal{A}$ is strongly tree-like relative to μ as long as γ is chosen to lie in $\mathcal{Q}$. Indeed, properties (TL1), (TL2) and (STL), with $d_k(\mathcal{A}) = 2r/a^k$, are immediate from the construction. As for (TL3), it follows from the choice of $\eta < 1/DN$ and

Lemma 4.3. *For any $k \in \mathbb{Z}_+$ and $A \in \mathcal{A}_k$, one has*

$$\delta(A, \mathcal{A}) \geq 1/DN - \eta\,.$$

Proof. Let $A = \overline{B(\mathbf{y}, r/a^k)} = \varphi_{\mathbf{y},a^k}^{-1}(B)$, denote $(\varphi_{\mathbf{y},a^k})_*\mu$ by ν, and write

$$\delta(A, \mathcal{A}) = \frac{\mu\big(\cup \mathcal{A}_{k+1}(A)\big)}{\mu(A)} = \frac{\mu\Big(\cup\, \varphi_{\mathbf{y},a^k}^{-1}\,\big(\mathcal{H}(g^k\mathbf{y}\gamma, \nu)\big)\Big)}{\mu\big(\varphi_{\mathbf{y},a^k}^{-1}(B)\big)} = \frac{\nu\,\big(\cup\mathcal{H}(g^k\mathbf{y}\gamma, \nu)\big)}{\nu(B)}\,,$$

which is not less than $1/DN - \eta$ in view of (4.8) and (4.10). □

Applying Lemma 2.5, we conclude that the dimension of the limit set $\mathbf{A}_\infty$ of $\mathcal{A}$ is at least

$$\begin{aligned}\dim(\mathbf{A}_\infty) &\geq \underline{d}_\mu(B) - \limsup_{k\to\infty} \frac{\sum_{i=0}^k \log(1/DN - \eta)}{\log(2r/a^k)} \\ &= \underline{d}_\mu(B) - \limsup_{k\to\infty} \frac{k+1}{k} \cdot \frac{\log(1/DN - \eta)}{\frac{\log(2r)}{k} - \log a} = \underline{d}_\mu(B) - \frac{\log\big(\frac{1}{1/DN-\eta}\big)}{\log a}\,.\end{aligned}$$

It remains to observe that parts (b) and (c) of Lemma 4.2 imply that $\mathbf{A}_\infty$ is contained in the set in the left hand side of (4.4). □

5. From quantitative nondivergence to uniform return

We now return to the setup of §3. That is, let $G = \mathrm{SL}_{n+1}(\mathbb{R})$, $\Gamma = \mathrm{SL}_{n+1}(\mathbb{Z})$, $\pi : G \to G/\Gamma$ the projection map, and K_ε defined as in (3.2). We also define

$$\mathcal{W} \stackrel{\mathrm{def}}{=} \text{ the set of nonzero rational subspaces of } \mathbb{R}^{n+1}\,.$$

Fix a Euclidean structure on $\mathbb{R}^{n+1}$, and for $g \in G$ and $V \in \mathcal{W}$ define $\ell_V(g)$ to be the covolume of $gV \cap g\mathbb{Z}^{n+1}$ in gV. Equivalently, one can extend the Euclidean norm $\|\cdot\|$ from $\mathbb{R}^{n+1}$ to its exterior algebra, and set

$$\ell_V(g) \stackrel{\mathrm{def}}{=} \|g(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k)\|\,,$$

where $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a generating set for $\mathbb{Z}^{n+1} \cap V$; note that the above quantity does not depend on the choice of $\{\mathbf{v}_i\}$.

The following 'abstract nondivergence' theorem is a special case of [KLW, Theorem 4.3].

Theorem 5.1. *Given $n \in \mathbb{N}$, and positive constants C, D, α, there exists $C_1 = C_1(n, C, D, \alpha) > 0$ with the following property. Suppose $B \subset \mathbb{R}^n$ is a ball, μ is measure on $\mathbb{R}^n$ such that B is centered at $\operatorname{supp}\mu$ and μ is D-Federer on $\tilde{B} \stackrel{\mathrm{def}}{=} 3^{n+1}B$, $h : \tilde{B} \to G$ is a continuous map, ϱ is a positive number not greater than 1; and suppose also that for each $V \in \mathcal{W}$*

(i) *the function $\ell_V \circ h$ is (C, α)-good on $\tilde{B}$ with respect to μ,*

and

(ii) $\|\ell_V \circ h\|_{\mu,B} \ge \varrho$.

Then for any $0 < \varepsilon \le \varrho$,

$$\mu\big(\{\mathbf{x} \in B : \pi\big(h(\mathbf{x})\big) \notin K_\varepsilon\}\big) \le C_1 \left(\frac{\varepsilon}{\varrho}\right)^\alpha \mu(B)\,.$$

We remark that this theorem generalizes [KM2, Theorem 5.2], which, in its turn, builds on quantitative estimates for non-divergence of unipotent flows due to Dani and Margulis [Mar, D2]. A more general version can be found in [KT], where in particular an explicit value of $C_1 = (n+1)C(D^2N)^{n+1}$ is given.

We are going to apply the above theorem choosing h of a special form. Namely, using g_t as defined in (3.1) and τ as defined in (3.3), and taking an arbitrary $u \in G$ and $t > 0$, let

$$h_{t,u}(\mathbf{x}) \stackrel{\mathrm{def}}{=} g_t \tau(\mathbf{x}) u\,. \tag{5.1}$$

The following has been essentially proved in [KM2]:

Lemma 5.2. *Let $h_{t,u}$ be defined as in* (5.1).

(a) *For any $u \in G$, $t > 0$ and $\mathbf{w} \in \bigwedge(\mathbb{R}^{n+1})$, the map $\mathbf{x} \mapsto h_{t,u}(\mathbf{x})\mathbf{w}$ is affine.*

(b) *For any compact subset Q of G/Γ and any nonempty ball $B \subset \mathbb{R}^n$ there exists $t_0 = t_0(Q, B) > 0$ such that*

$$\|\ell_V \circ h_{t,u}\|_B \ge 1 \tag{5.2}$$

for all $u \in \pi^{-1}(Q)$, $t \ge t_0$ and $V \in \mathcal{W}$.

Proof. Let us fix a basis $\mathbf{e}_1, \dots, \mathbf{e}_{n+1}$ of $\mathbb{R}^{n+1}$, and for $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n+1\}$, $i_1 < i_2 < \cdots < i_k$, we let $\mathbf{e}_I \stackrel{\mathrm{def}}{=} \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} \in \bigwedge^k(\mathbb{R}^{n+1})$, with the convention $\mathbf{e}_\varnothing = 1$. Expand $u\mathbf{w}$ with respect to the corresponding basis of $\bigwedge(\mathbb{R}^{n+1})$:

$$u\mathbf{w} = \sum_{I \subset \{1,\dots,n+1\}} w_I \mathbf{e}_I\,. \tag{5.3}$$

To prove (a), it suffices to show that the map $\mathbf{x} \mapsto h_{t,u}(\mathbf{x})\mathbf{e}_I$ is affine for each I, which is easily verified. Indeed, the action of $\tau(\mathbf{x})$ leaves $\mathbf{e}_1, \dots, \mathbf{e}_n$ invariant and sends $\mathbf{e}_{n+1}$ to $\mathbf{e}_{n+1} + \sum_{i=1}^n x_i \mathbf{e}_i$, therefore

$$\tau(\mathbf{x})\mathbf{e}_I = \begin{cases} \mathbf{e}_I & \text{if } n+1 \notin I \\ \mathbf{e}_I + \sum_{i=1,\, i\notin I}^n \pm x_i \mathbf{e}_{I\cup\{i\}\smallsetminus\{n+1\}} & \text{otherwise}, \end{cases} \tag{5.4}$$

and an application of g_t clearly does not make things any worse.

For (b), take $V \in \mathcal{W}$ of dimension k, $1 \le k \le n+1$, let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a generating set for $\mathbb{Z}^{n+1} \cap V$, and denote $\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k$ by $\mathbf{w}$. It follows from Minkowski's Convex Body Theorem, see e.g. [S3, Chapter II, Theorem 2B], that $uV \cap u\mathbb{Z}^{n+1}$ contains a nonzero vector of norm at most $2v_k^{-1/k}\ell_V(u)^{1/k}$, where v_k is the volume of the unit ball in $\mathbb{R}^k$. Now recall that u is chosen so that lengths of all nonzero vectors of $u\mathbb{Z}^{n+1}$ are uniformly bounded away from zero. Therefore there exists ϱ depending only on Q and n such that at least one of the coefficients w_I in (5.3) has absolute value not less than ϱ.

Note that $\ell_V \circ h_{t,u}(\mathbf{x}) = \|h_{t,u}(\mathbf{x})\mathbf{w}\|$, and that (5.2) holds trivially for any $t > 0$ if $k = n+1$, so let us assume $1 \le k \le n$. Using (5.3), (5.4) and (3.1), write $h_{t,u}(\mathbf{x})\mathbf{w}$ as a sum of two terms: $e^{-(1-\frac{k-1}{n})t}\sum_{n+1\in I} w_I \mathbf{e}_I$ and

$$e^{\frac{k}{n}t} \sum_{n+1\notin I} \left(w_I + \sum_{i\in I} \pm w_{I\cup\{n+1\}\smallsetminus\{i\}} x_i \right) \mathbf{e}_I\,. \tag{5.5}$$

Then observe that every component w_I of $\mathbf{w}$ appears in the sum (5.5), which implies that for some I the projection of $h_{t,u}(\mathbf{x})\mathbf{w}$ onto $\mathbf{e}_I$ is an affine function with at least one coefficient of absolute value not less than $\varrho e^{\frac{k}{n}t} \ge \varrho e^{t/n}$. Therefore (5.2) holds whenever e^t is not less than $(\varrho \cdot \operatorname{diam}(B)/2)^{-n}$. $\square$

Proposition 5.3. *Given $n \in \mathbb{N}$ and $C, D, \alpha, \eta > 0$, there exists a compact subset K of G/Γ with the following property: for any compact $Q \subset G/\Gamma$ and any nonempty ball $B \subset \mathbb{R}^n$ centered at 0, there exists $t_0 > 0$ such that $\forall\, t > t_0$, $\forall\, u \in \pi^{-1}(Q)$, and for any measure μ on $\mathbb{R}^n$ with $0 \in \operatorname{supp}\mu$ which is absolutely (C,α)-decaying and D-Federer on $3^{n+1}B$, one has*

$$\mu\big(\{\mathbf{x} \in B : \pi\big(h_{t,u}(\mathbf{x})\big) \in K\}\big) \ge (1-\eta)\mu(B)\,. \tag{5.6}$$

Proof. It follows from Lemma 5.2(a), Lemma 2.4(i) and Lemma 2.3 that for any $u \in G$, $t > 0$ and $\mathbf{w} \in \bigwedge(\mathbb{R}^{n+1})$, the function $\ell_V \circ h_{t,u}$ is $(2^{n/2}C', \alpha)$-good on $3^{n+1}B$ with respect to μ, where C' is as in (2.4). Choosing $u \in \pi^{-1}(Q)$ and $t > t_0$, with t_0 as in Lemma 5.2(b), one deduces from Lemma 2.4(ii) that

$$\|\ell_V \circ h_{t,u}\|_{\mu,B} \geq (1+2C^{1/\alpha})^{-1}.$$

Thus one can take $\varrho = (1+2C^{1/\alpha})^{-1}$ and

$$C_2 = C_1\big(n, 2^{n/2}C(C^{-1/\alpha}+2)^\alpha, D, \alpha\big)(1+2C^{1/\alpha})^\alpha,$$

and apply Theorem 5.1 with $h = h_{t,u}$ to establish that for any $0 < \varepsilon < (1+2C^{1/\alpha})^{-1}$ one has

$$\mu\big(\{\mathbf{x} \in B : \pi\big(h_{t,u}(\mathbf{x})\big) \in K_\varepsilon\}\big) \geq (1 - C_2\varepsilon^\alpha)\mu(B).$$

To deduce (5.6), it remains to take $K = K_\varepsilon$, where ε is small enough so that $C_2\varepsilon^\alpha < \eta$. □

Corollary 5.4. *Given $n \in \mathbb{N}$ and $C, D, \alpha, \eta > 0$ with $\eta < 1/DN$, there exists a compact subset Q of G/Γ with the following property: for any $\Lambda \in G/\Gamma$ and any ball $B \subset \mathbb{R}^n$ centered at 0 there exists $t_0 > 0$ such that for any measure μ on $\mathbb{R}^n$ with $0 \in \operatorname{supp}\mu$ which is absolutely (C,α)-decaying and D-Federer on $3^{n+1}B$ one has*

$$\dim\left(\{\mathbf{x} \in \operatorname{supp}\mu \cap B : g_{kt}\tau(\mathbf{x})\Lambda \in Q \quad \forall\, k \in \mathbb{N}\}\right) \geq \underline{d}_\mu(B) - \frac{\log\big(\frac{1}{1/DN-\eta}\big)}{(1+\frac{1}{n})t} \tag{5.7}$$

for any $t > t_0$.

Proof. As was mentioned before, for any fixed $t > 0$ the subgroup of G generated by g_t and H is isomorphic to $S_{n,a}$ where $a = e^{(1+\frac{1}{n})t}$. With some abuse of notation, let us identify $S_{n,a}$ with its image under the isomorphism sending $\mathbf{x} \in \mathbb{R}^n$ to $\tau(\mathbf{x})$ and g to g_t.

We claim that K as in Proposition 5.3 has the following property: for any compact $Q \subset G/\Gamma$, any ball $B \subset \mathbb{R}^n$ centered at 0 and any $a > e^{(1+\frac{1}{n})t_0}$, where t_0 is as in Proposition 5.3, the family

$$\mathcal{M} \stackrel{\text{def}}{=} \left\{ \begin{array}{c} \text{all measures } \nu \text{ on } \mathbb{R}^n \subset S_{n,a} \text{ with } 0 \in \operatorname{supp}\nu \text{ which are} \\ \text{absolutely } (C,\alpha)\text{-decaying and } D\text{-Federer on } 3^{n+1}B \end{array} \right\} \tag{5.8}$$

has the η-uniform return property with respect to (B, K, Q). Indeed, if $\Lambda \in G/\Gamma$ is given by $\Lambda = u\mathbb{Z}^{n+1}$, it is clear that $\pi\big(h_{t,u}(\mathbf{x})\big)$ coincides with $g_t\tau(\mathbf{x})\Lambda$, and $\Lambda \in Q \iff u \in \pi^{-1}(Q)$ for any subset Q of G/Γ. Letting $a = e^{(1+\frac{1}{n})t}$ with $t > t_0$, one observes that the η-uniform return property of $\mathcal{M}$ with respect to (B, K, Q) is an immediate consequence of (5.6).

To establish (5.7), note that it follows from Lemma 2.2 that for any μ as in the statement of the Corollary and any $a > 1$, $\mathcal{M}_{\mu,B,a}$ is a subfamily of $\mathcal{M}$ as in (5.8). Thus it remains to choose a compact subset Q of G/Γ containing both Λ and $\overline{(2B)}K$, if needed increase t_0 so that $e^{(1+\frac{1}{n})t_0} \geq 2$, and apply Theorem 4.1 to conclude that (5.7) holds for any $t > t_0$. □

Proof of Theorem 3.2. One knows that there exists $\mathbf{x}_0 \in \operatorname{supp}\mu \cap B$ and constants C, D, α such that μ is absolutely (C, α)-decaying and D-Federer on a neighborhood of $\mathbf{x}_0$. Thus, by shrinking the size of B and changing coordinates, one can assume that B is centered at $0 \in \operatorname{supp}\mu$, and μ is absolutely (C, α)-decaying and D-Federer on $3^{n+1}B$. Hence Corollary 5.4 applies, resulting in estimate (5.7) for any $t > t_0$. It remains to observe that $\bigcup_{k\in\mathbb{N}} g_{kt}\tau(\mathbf{x})\Lambda \subset Q$ implies that $F\tau(\mathbf{x})\Lambda$ is contained in a compact set $\bigcup_{0\leq s\leq t} g_{-s}Q$, and that the right hand side of (5.7) tends to $\underline{d}_\mu(B)$ as $t \to \infty$. □

6. Conditions on measures

In this section we discuss the conditions on measures which we have used. First we show that, under the assumption that μ is Federer, the absolute decay of μ can be expressed in several equivalent ways, in particular, it suffices to consider hyperplanes $\mathcal{L}$ passing through centers of balls.

We will use the following notation: for two open subsets U_1, U_2 of $\mathbb{R}^n$, say that $U_1 \subset_3 U_2$ if for any ball $B \subset U_1$, $3B$ is contained in U_2.

Proposition 6.1. *For a Federer measure μ on $\mathbb{R}^n$, the following conditions are equivalent:*

(1) *μ is absolutely decaying.*
(2) *For μ-a.e. point of $\mathbb{R}^n$ there exist a neighborhood U of this point and $C, \alpha > 0$ such that for all affine hyperplanes $\mathcal{L}$ and all balls $B = B(\mathbf{x}, r) \subset U$ with $\mathbf{x} \in \operatorname{supp}\mu \cap \mathcal{L}$, (1.2) holds.*
(3) *μ-a.e. point of $\mathbb{R}^n$ has a neighborhood U for which*

$$\sup_{\mathbf{x}\in\operatorname{supp}\mu,\, B(\mathbf{x},r)\subset U,\, \mathcal{L}\ni\mathbf{x}} \frac{\mu\left(B(\mathbf{x}, r) \cap \mathcal{L}^{(\delta r)}\right)}{\mu\big(B(\mathbf{x}, r)\big)} \longrightarrow_{\delta\to 0} 0\,. \tag{6.1}$$

(4) *μ-a.e. point of $\mathbb{R}^n$ has a neighborhood $\hat{U}$ for which*

$$\sup_{\mathbf{x}\in\operatorname{supp}\mu,\, B(\mathbf{x},r)\subset\hat{U},\, \text{any } \mathcal{L}} \frac{\mu\left(B(\mathbf{x}, r) \cap \mathcal{L}^{(\delta r)}\right)}{\mu\big(B(\mathbf{x}, r)\big)} \longrightarrow_{\delta\to 0} 0\,. \tag{6.2}$$

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are immediate. Suppose (3) holds; for μ-a.e. $\mathbf{x}_0 \in \mathbb{R}^n$ choose $U \ni \mathbf{x}_0$ as in (3), and also assume, as we may, that μ is D-Federer on U for some $D > 0$. Then take a neighborhood $\hat{U} \subset_3 U$ of $\mathbf{x}_0$. After that, given a ball $B = B(\mathbf{x}, r) \subset \hat{U}$ centered in $\operatorname{supp}\mu$ and any affine hyperplane $\mathcal{L}$, choose $\mathbf{y} \in B \cap \operatorname{supp}\mu \cap \mathcal{L}^{(\delta r)}$ (if there is no such $\mathbf{y}$, then there is nothing to worry about). Let $\mathcal{L}_{\mathbf{y}}$ be the affine hyperplane parallel to $\mathcal{L}$ passing through $\mathbf{y}$. Then for any $\delta > 0$,

$$\mathcal{L}^{(\delta r)} \subset \mathcal{L}_{\mathbf{y}}^{(2\delta r)} \quad \text{and} \quad B \subset B_{\mathbf{y}} \stackrel{\text{def}}{=} B(\mathbf{y}, 2r) \subset 3B \subset U\,.$$

Hence

$$\frac{\mu\left(B \cap \mathcal{L}^{(\delta r)}\right)}{\mu(B)} \le \frac{\mu\left(B_{\mathbf{y}} \cap \mathcal{L}_{\mathbf{y}}^{(2\delta r)}\right)}{\mu(B_{\mathbf{y}})} \frac{\mu(B_{\mathbf{y}})}{\mu(B)} \stackrel{\text{Federer}}{\le} D \frac{\mu\left(B_{\mathbf{y}} \cap \mathcal{L}_{\mathbf{y}}^{(2\delta r)}\right)}{\mu(B_{\mathbf{y}})}\,,$$

which shows that (6.2) is a consequence of (6.1).

Now suppose (4) holds. For μ-a.e. $\mathbf{x}_0 \in \mathbb{R}^n$ choose $\hat{U} \ni \mathbf{x}_0$ as in (4), and again assume that μ is D-Federer on $\hat{U}$ for some $D > 0$. Then take a neighborhood $U \subset_3 \hat{U}$ of $\mathbf{x}_0$. After that take an arbitrary $0 < \eta < 1/DN$, and choose δ so that

$$\sup_{\mathbf{x} \in \operatorname{supp}\mu,\, B(\mathbf{x}, r) \subset \hat{U},\, \text{any } \mathcal{L}} \frac{\mu\left(B(\mathbf{x}, r) \cap \mathcal{L}^{(\delta r)}\right)}{\mu(B(\mathbf{x}, r))} < \eta\,. \tag{6.3}$$

Let $c \stackrel{\text{def}}{=} \delta/2$. We will show by induction on k that for any ball $B \subset U$ centered at $\operatorname{supp}\mu$ of radius r and any hyperplane $\mathcal{L}$,

$$\frac{\mu\left(B \cap \mathcal{L}^{(c^k r)}\right)}{\mu(B)} \le (DN\eta)^k. \tag{6.4}$$

The case $k = 1$ follows from (6.3). For $k > 1$, let

$$\mathcal{B} \stackrel{\text{def}}{=} \left\{ B\left(\mathbf{x}, \frac{c^k r}{2}\right) : \mathbf{x} \in B \cap \operatorname{supp}\mu \cap \mathcal{L}^{(c^{k+1} r)} \right\},$$

and, using Proposition 2.1, choose $\mathcal{B}'$ to be a subcollection of multiplicity at most N covering $B \cap \operatorname{supp}\mu \cap \mathcal{L}^{(c^{k+1} r)}$. Note that

$$\cup\mathcal{B}' \subset 3B \cap \mathcal{L}^{(c^k r)}. \tag{6.5}$$

We therefore have:

$$\begin{aligned}
\mu\left(B\cap\mathcal{L}^{(c^{k+1}r)}\right) &\le \mu\left(\bigcup_{B'\in\mathcal{B}'} B'\cap\mathcal{L}^{(c^{k+1}r)}\right) \le \sum_{B'\in\mathcal{B}'}\mu\left(B'\cap\mathcal{L}^{(\delta c^k r/2)}\right)\\
&\overset{(6.3)}{\le} \eta\sum_{B'\in\mathcal{B}'}\mu(B') \overset{\text{Prop. 2.1}}{\le} \eta N\,\mu\left(\cup\mathcal{B}'\right)\\
&\overset{(6.5)}{\le} \eta N\,\mu\left(3B\cap\mathcal{L}^{(c^k r)}\right) \overset{(6.4)}{\le} \eta N(\eta ND)^k\mu(3B)\\
&\overset{\text{Federer}}{\le} (\eta ND)^{k+1}\mu(B).
\end{aligned}$$

This proves (6.4).

Now let $\alpha \overset{\text{def}}{=} \frac{\log\eta_0}{\log c}$, where $\eta_0 \overset{\text{def}}{=} DN\eta$, let $C \overset{\text{def}}{=} \frac{1}{\eta_0}$, let $r>0$ and let $\varepsilon>0$ with $\frac{\varepsilon}{r}\le c$. Suppose $B\subset U$ is a ball of radius r centered in $\operatorname{supp}\mu$. Choose k so that

$$c^{k+1} < \frac{\varepsilon}{r} \le c^k\,. \tag{6.6}$$

We obtain:

$$\begin{aligned}
\mu\left(B\cap\mathcal{L}^{(\varepsilon)}\right) = \mu\left(B\cap\mathcal{L}^{(\frac{\varepsilon}{r}r)}\right) &\overset{(6.6)}{\le} \mu\left(B\cap\mathcal{L}^{(c^k r)}\right)\\
&\overset{(6.4)}{\le} \eta_0^k\,\mu(B) = \frac{\eta_0^{k+1}}{\eta_0}\,\mu(B)\\
&\overset{(6.6)}{\le} \frac{\eta_0^{\frac{\log\varepsilon/r}{\log c}}}{\eta_0}\,\mu(B) = C\left(\frac{\varepsilon}{r}\right)^{\alpha}\mu(B)\,.
\end{aligned}$$

Enlarging C if necessary to account for the case $\frac{\varepsilon}{r}>c$, we obtain (1). □

Note that the above proof shows that, if one in addition assumes that μ is D-Federer on $\mathbb{R}^n$, all the above conditions are equivalent to

(5) For μ-a.e. point of $\mathbb{R}^n$ there exist a neighborhood $\hat{U}$ of this point, $\delta>0$ and $0<\eta<1/DN$ such that (6.3) holds.

Remark 6.2. Several recent papers, such as [PV2, KTV, U2, U3], deal with a more uniform version of the absolute decay condition. Let us say that a measure μ on $\mathbb{R}^n$ is *uniformly absolutely decaying* if there exist positive C,α,r_0 such that (1.2) holds for any affine hyperplane $\mathcal{L}\subset\mathbb{R}^n$, any $\varepsilon>0$, and any open ball B centered in $\operatorname{supp}\mu$ of radius $r<r_0$. Clearly this is a property which implies absolute decay. On the other hand, the difference is not significant, as in all naturally arising absolutely decaying measures (see e.g. the subsequent section) the uniform

property can be established as well. Note however that our condition has an advantage of being invariant with respect to restrictions to open subsets of $\mathbb{R}^n$.

Arguing as in the proof of Proposition 6.1, one can easily show that μ is uniformly absolutely decaying if and only if for some $r_0 > 0$ one has

$$\sup_{\mathbf{x}\in\operatorname{supp}\mu,\,0<r<r_0,\,\mathcal{L}\ni\mathbf{x}} \frac{\mu\left(B(\mathbf{x},r)\cap\mathcal{L}^{(\delta r)}\right)}{\mu\big(B(\mathbf{x},r)\big)} \longrightarrow_{\delta\to 0} 0\,. \tag{6.7}$$

Note that in the case $n = 1$, that is, if μ is a measure on the real line, condition (6.7) may be restated as follows:

$$\sup_{x\in\operatorname{supp}\mu,\,0<r<r_0} \frac{\mu\big(B(x,\delta r)\big)}{\mu\big(B(x,r)\big)} \longrightarrow_{\delta\to 0} 0\,.$$

An a-priori stronger requirement, introduced by Veech [V] in the study of quadratic differentials, is:

$$\sup_{x\in\mathbb{R},\,0<r<1} \frac{\mu\big(B(x,\delta r)\big)}{\mu\big(B(x,r)\big)} \longrightarrow_{\delta\to 0} 0.$$

Later, in [W], the following condition was used:

There are C, $\alpha > 0$ such that $\forall\, x \in \mathbb{R}$, $\delta > 0$, $0 < r < 1$,

$$\mu\big(B(x,\delta r)\big) \le C\delta^{\alpha}\mu\big(B(x,r)\big)$$

Still later, Urbanski [U2] used the following condition:

There are $0 < \eta < 1$, $\delta > 0$, $r_0 > 0$ such that $\forall\, x \in \mathbb{R}$, $0 < r < r_0$,

$$\mu\big(B(x,\delta r)\big) \le \eta\mu\big(B(x,r)\big).$$

Using arguments as in the proofs of Proposition 6.1 and [U2, Prop. 3.1], it may be shown that these four conditions on μ are actually equivalent (and imply absolute decay). In this case one need not impose the condition that μ is Federer. Details are left to the reader.

We now observe that measures obeying a β-power law for large enough β are absolutely decaying.

Proposition 6.3. *Suppose μ satisfies a β-power law, with $\beta > n - 1$. Then μ is absolutely decaying, with $\alpha = \beta + 1 - n$.*

This is simple and well known, see e.g. [PV2, U2]. We include a proof for completeness.

Proof. For μ-a.e. $\mathbf{x}_0 \in \mathbb{R}^n$ choose $U \ni \mathbf{x}_0$ and c_1, $c_2 > 0$ such that (1.4) holds, and take a neighborhood $\hat{U} \subset_3 U$ of $\mathbf{x}_0$. Let $B = B(\mathbf{x}, r) \subset \hat{U}$

with $\mathbf{x} \in \operatorname{supp} \mu$, let $\mathcal{L}$ be an affine hyperplane, and let $0 < \varepsilon < r$. Let

$$\mathcal{G} \stackrel{\text{def}}{=} \left\{ B(\mathbf{y}, 2\varepsilon) : \mathbf{y} \in B \cap \mathcal{L}^{(\varepsilon)} \cap \operatorname{supp} \mu \right\} .$$

Note that all the above balls are contained in U. Using Proposition 2.1, take $\mathcal{G}_1$ to be a disjoint subcollection such that

$$\mu(\cup \mathcal{G}_1) \geq \frac{\mu\left(B \cap \mathcal{L}^{(\varepsilon)}\right)}{N}. \tag{6.8}$$

For each $A \in \mathcal{G}_1$, $A \cap \mathcal{L}$ is a $(n-1)$-dimensional ball of radius at least ε, contained in $\mathcal{L} \cap 3B$, and these balls are disjoint. Also $\mathcal{L} \cap 3B$ is a $(n-1)$-dimensional ball of radius at most $3r$. Considering the $(n-1)$-dimensional volume we find that there is a constant c, depending only on n, such that

$$\#\mathcal{G}_1 \leq c \left(\frac{r}{\varepsilon}\right)^{n-1}. \tag{6.9}$$

We therefore have:

$$\begin{aligned} \frac{\mu\left(\mathcal{L}^{(\varepsilon)} \cap B\right)}{\mu(B)} &\stackrel{(1.4),\ (6.8)}{\leq} \frac{N}{c_1 r^\beta}\, \mu\left(\cup \mathcal{G}_1\right) \\ &\stackrel{(1.4)}{\leq} \frac{N}{c_1 r^\beta}\, \#\mathcal{G}_1 c_2 \varepsilon^\beta \stackrel{(6.9)}{\leq} \frac{Ncc_2}{c_1} \left(\frac{\varepsilon}{r}\right)^{\beta+1-n}. \end{aligned}$$

Enlarging the constant $\frac{Ncc_2}{c_1}$ if necessary to account for the case $\varepsilon \geq r$, we conclude that for some $C > 0$, μ is absolutely $(C, \beta + 1 - n)$-decaying. $\square$

To conclude this section, we describe the absolute decay condition in terms of limits of the 'zooming in' process. Namely, suppose U is a bounded open subset of $\mathbb{R}^n$, μ a measure on $\mathbb{R}^n$,

$$R \stackrel{\text{def}}{=} \max\{r > 0 : B(\mathbf{x}, r) \subset U \text{ for some } \mathbf{x} \in \operatorname{supp} \mu\},$$

and $\widetilde{B} \stackrel{\text{def}}{=} B(0, R)$. A measure ν on $\widetilde{B}$ is called a *U-mini-measure for μ* if there are $\mathbf{x} \in \operatorname{supp} \mu$ and $a \geq 1$ such that $B(\mathbf{x}, R/a) \subset U$ and

$$\nu = \frac{1}{\mu(B(\mathbf{x}, R/a))} \left((\varphi_{\mathbf{x},a})_* \mu\right)|_{\widetilde{B}}$$

(where $\varphi_{\mathbf{x},a}$ is defined by (2.1)); that is, ν is obtained from μ by 'zooming in' on $B(\mathbf{x}, R/a)$ and renormalizing. We say that ν is a *U-micro-measure for μ* if it is an accumulation point of U-mini-measures for μ, with respect to the weak-$*$ topology on measures on $\widetilde{B}$. A measure μ is called *nonplanar* if for any affine hyperplane $\mathcal{L}$, $\mu(\mathcal{L}) = 0$.

The terminology of micro-measures, which was introduced by Furstenberg, enables us to formulate another characterization of absolute decay. Since we will not use it, we leave the proof (based on Proposition 6.1) as an exercise.

Proposition 6.4. *A Federer measure μ on $\mathbb{R}^n$ is absolutely decaying if and only if μ-a.e. point of $\mathbb{R}^n$ has a neighborhood U such that all of U-micro-measures for μ are nonplanar.*

7. Examples

In this section we construct measures μ which are absolutely decaying and satisfy a power law, that is, measures to which Corollary 1.2 applies. Note that such measures are also 'friendly' in the sense of [KLW]. We also exhibit examples of measures which satisfy the assumptions of Theorem 1.1 but not those of Corollary 1.2.

7.1. **Hutchinson's construction and its generalizations.** A map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a *similarity* if it can be written as $\varphi(\mathbf{x}) = \varrho\Theta(\mathbf{x}) + \mathbf{y}$, where $\varrho \in \mathbb{R}_+$, $\Theta \in \mathrm{O}(n)$ and $\mathbf{y} \in \mathbb{R}^n$. It is said to be *contracting* if $\varrho < 1$. It is known (see [H] for a more general statement) that for any finite family $\varphi_1, \ldots, \varphi_m$ of contracting similarities there exists a unique nonempty compact set K, called the *limit set* of the family, such that

$$K = \bigcup_{i=1}^{m} \varphi_i(K).$$

Say that $\varphi_1, \ldots, \varphi_m$ as above satisfy the *open set condition* if there exists an open subset $U \subset \mathbb{R}^n$ such that

$$\varphi_i(U) \subset U \text{ for all } \ i = 1, \ldots, m\,,$$

and

$$i \neq j \Longrightarrow \varphi_i(U) \cap \varphi_j(U) = \varnothing\,.$$

The family $\{\varphi_i\}$ is called *irreducible* if there is no finite collection of proper affine subspaces which is invariant under each φ_i. Well-known self-similar sets, like Cantor's ternary set, Koch's curve or Sierpinski's gasket, are all examples of limit sets of irreducible families of contracting similarities satisfying the open set condition.

Suppose $\{\varphi_i\}_{i=1}^m$ is a family of contracting similarities of $\mathbb{R}^n$ satisfying the open set condition, let K be its limit set, β the Hausdorff dimension of K, and μ the restriction of the β-dimensional Hausdorff measure to K. J. Hutchinson [H] gave a simple formula for β and proved that μ is positive, finite, and satisfies a β-power law. Assuming that $\{\varphi_i\}$ is irreducible, it was proved in [KLW, §8] that μ is absolutely decaying, and announced without proof in [KLW, §10] that $\dim(\mathbf{BA} \cap K) = \beta$. The latter is now clearly seen to be a consequence of Corollary 1.2.

In a recent preprint [U3], Urbanski extended the results of [KLW] to a larger class of measures. In particular, he proved that if $\{\varphi_i\}_{i=1}^m$ is a conformal irreducible iterated function system in $\mathbb{R}^n$, $n \ge 2$, K is its limit set (see [U3, §1] for definitions) and $\beta = \dim(K)$, then the β-dimensional Hausdorff measure restricted to K is absolutely decaying. The fact that it satisfies a power law was known before, see [B] or [MU, Lemma 3.14]. Hence, by Corollary 1.2, the intersection of K with $\mathbf{BA}$ has full Hausdorff dimension. Another preprint [U2] of Urbanski contains examples of absolutely decaying and Federer measures on the real line, in particular those coming from certain infinite iterated function systems.

7.2. **More tree-like collections.** We now look more closely at limit measures constructed in the proof of Lemma 2.5. Let $A_0 \stackrel{\text{def}}{=} [0,1]^n$ be the unit cube. Fix

$$M \in \mathbb{N},\ 0 < \lambda < 1 \quad \text{such that} \quad \lambda M^n \in \mathbb{N}$$

(in particular $\lambda \ge 1/M^n$), and let μ be Lebesgue measure on A_0. Define a tree-like (relative to μ) family $\mathcal{A}$ of subsets of A_0 inductively as follows. Set $\mathcal{A}_0 \stackrel{\text{def}}{=} \{A_0\}$, and, given a cube $A \in \mathcal{A}_k$, subdivide A into M^n subcubes of equal size, with edges parallel to the coordinate axes, and let $\mathcal{A}_{k+1}(A)$ consist of λM^n of them, chosen arbitrarily. Then by an easy induction, the following properties hold for all $k \in \mathbb{N}$:

- Each $A \in \mathcal{A}_k$ is a cube with sidelength equal to M^{-k};
- $\#\mathcal{A}_k = (\lambda M^n)^k$.

From this it is easy to see that the union $\mathcal{A}$ of $\mathcal{A}_k$ over $k \in \mathbb{Z}_+$ is strongly tree-like relative to μ. Furthermore, the limit measure ν, defined as the unique weak limit of sequence $\{\nu_k\}$ given by (2.10), satisfies

$$\nu(A) = (\lambda M^n)^{-k} \quad \forall\, A \in \mathcal{A}_k\,. \tag{7.1}$$

Proposition 7.1. *The limit measure ν satisfies a β-power law, where $\beta = \dim(\mathbf{A}_\infty) = n - \frac{\log(1/\lambda)}{\log M}$.*

Proof. It is immediate from the construction that $\operatorname{supp}\nu_k = \cup\mathcal{A}_k$ for all $k \in \mathbb{N}$, and that the diameter of $A \in \mathcal{A}_k$ is $\frac{\sqrt{n}}{M^k}$. Let $\mathbf{x} \in \operatorname{supp}\nu$, $0 < r \le \sqrt{n}/M$, $B = B(\mathbf{x}, r)$, and let $k \in \mathbb{N}$ be such that

$$\frac{\sqrt{n}}{M^{k+1}} < r \le \frac{\sqrt{n}}{M^k} \tag{7.2}$$

Since $\mathbf{x} \in \operatorname{supp} \nu \subset \operatorname{supp} \nu_{k+1}$, there exists $A \in \mathcal{A}_{k+1}$ with $\mathbf{x} \in A$. By (7.2) we see that $A \subset B$, hence

$$\nu(B) \geq \nu(A) = 1/(\lambda M^n)^{k+1}. \tag{7.3}$$

On the other hand, by (7.2), $\operatorname{diam}(B) = 2r \leq 2\sqrt{n} M^{-k}$ and hence there exists C, depending only on M and n, such that

$$\#\mathcal{E} < C, \quad \text{where } \mathcal{E} \stackrel{\text{def}}{=} \{A \in \mathcal{A}_{k+1} : A \cap B \neq \varnothing\}. \tag{7.4}$$

This implies that

$$\nu(B) \leq C/(\lambda M^n)^{k+1}. \tag{7.5}$$

Reworking (7.2), we have

$$\frac{\log \sqrt{n} - \log r}{\log M} < k + 1 \leq \frac{\log M + \log \sqrt{n} - \log r}{\log M}, \tag{7.6}$$

and putting together (7.3), (7.5) and (7.6) gives the required inequalities. □

Example 7.2. Take $\lambda > 1/M$ in the above construction. Then, by Propositions 6.3 and 7.1, the limit measure ν is absolutely decaying and satisfies a power law, hence satisfies the assumptions of Corollary 1.2.

Proposition 7.3. *Suppose that in the above construction there is a constant $c > 0$ such that for each $k \in \mathbb{N}$ and each cube $A \in \mathcal{A}_k$ the following condition holds:*

$$\begin{gathered} \textit{for every affine hyperplane } \mathcal{L} \subset \mathbb{R}^n \\ \#\{B \in \mathcal{A}_{k+1}(A) : B \cap \mathcal{L}^{(c/M^k)} = \varnothing\} \geq 1\,. \end{gathered} \tag{7.7}$$

Then the limit measure ν is absolutely decaying.

Proof. Note that (7.7) implies that λM^n is at least $n+1$, so, by Proposition 7.1, ν satisfies a power law and hence is D-Federer for some D. So, in view of the remark after the proof of Proposition 6.1 is enough to find $\delta > 0$ and $0 < \eta < 1/DN$ such that

$$\frac{\nu\left(B(\mathbf{x}, r) \cap \mathcal{L}^{(\delta r)}\right)}{\nu(B(\mathbf{x}, r))} < \eta \tag{7.8}$$

for any $\mathbf{x} \in \operatorname{supp} \nu$, any affine hyperplane $\mathcal{L}$ and small enough r.

Take $r \leq \sqrt{n}/M$, define k by (7.2), put $B = B(\mathbf{x}, r)$, and let C, $\mathcal{E}$ be as in (7.4). Suppose $A \in \mathcal{E}$. Then by (7.3),

$$\nu(A) \leq \nu(B)\,. \tag{7.9}$$

Note that for any $\ell \in \mathbb{N}$,

$$\#\mathcal{A}_{k+1+\ell}(A) = (\lambda M^n)^{\ell}\,. \tag{7.10}$$

Also, (7.7) and a straightforward induction on ℓ show that for any $\ell \in \mathbb{N}$,

$$\#\mathcal{G}_\ell \le (\lambda M^n - 1)^\ell \,, \tag{7.11}$$

where

$$\mathcal{G}_\ell \stackrel{\text{def}}{=} \{A' \in \mathcal{A}_{k+1+\ell}(A) : A' \cap \mathcal{L}^{(c/M^{k+\ell})} \neq \varnothing\}.$$

Let $\varrho \stackrel{\text{def}}{=} \frac{\lambda M^n - 1}{\lambda M^n}$. Then one has

$$\frac{\nu(A \cap \mathcal{L}^{(c/M^{k+\ell})})}{\nu(A)} \stackrel{(7.1)}{\le} \frac{\#\mathcal{G}_\ell}{\#\mathcal{A}_{k+\ell}(A)} \stackrel{(7.10),(7.11)}{\le} \varrho^\ell \,. \tag{7.12}$$

Choose $\ell \in \mathbb{N}$ large enough so that $\eta \stackrel{\text{def}}{=} C\varrho^\ell < \frac{1}{DN}$, and set

$$\delta \stackrel{\text{def}}{=} \frac{c}{\sqrt{n} M^\ell} \,. \tag{7.13}$$

Putting together previous computations, one has

$$\begin{aligned} \frac{\nu(B \cap \mathcal{L}^{(\delta r)})}{\nu(B)} &\stackrel{(7.2),(7.13)}{\le} \frac{\nu(B \cap \mathcal{L}^{(c/M^{k+\ell})})}{\nu(B)} \\ &\stackrel{(7.9)}{\le} \sum_{A \in \mathcal{E}} \frac{\nu(A \cap \mathcal{L}^{(c/M^{k+\ell})})}{\nu(A)} \stackrel{(7.4),(7.12)}{\le} C\varrho^\ell < \eta \,, \end{aligned}$$

proving (7.8). $\square$

Example 7.4. It is clear that one can keep choosing as few as $n+1$ subcubes at each stage, and not necessarily in the self-similar way, and still satisfy (7.7). See Figure 1 for examples. This gives a way to construct limit sets $\mathbf{A}_\infty$ of tree-like families of subsets of A_0 with $\dim(\mathbf{A}_\infty) = \frac{\log(n+1)}{\log M}$ arbitrarily small, which are not limit sets of families of contracting similarities, and such that $\dim(\mathbf{BA} \cap \mathbf{A}_\infty) = \dim(\mathbf{A}_\infty)$.

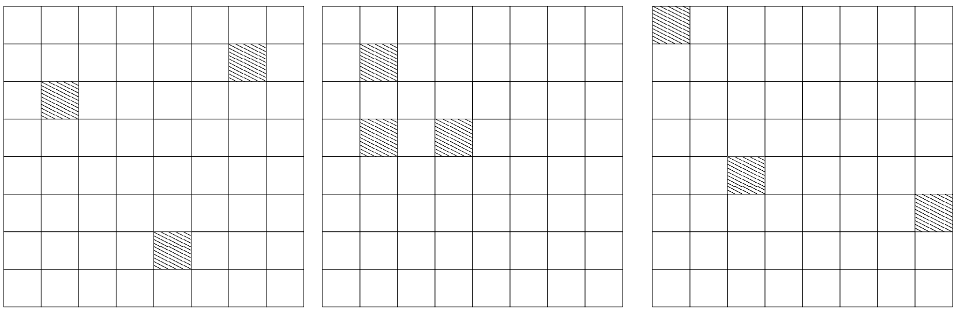

FIGURE 1. Three collections satisfying (7.7). Here $n = 2$ and $M = 8$.

Example 7.5. We now construct a measure on the real line which is Federer and absolutely decaying (thus scaling) but does not satisfy a power law. Choose an integer $M \geq 3$ and a sequence $m_\ell \uparrow \infty$, to be specified later. Then perform a tree-like construction using M-adic intervals, as follows. Take $A_0 = [0,1]$ and $\mathcal{A} = \{A_0\}$. Given k, let $\ell = \ell_k$ be such that $k \in \{m_\ell + 1, \ldots, m_{\ell+1}\}$. If ℓ_k is odd and $A \in \mathcal{A}_k$, then $\mathcal{A}_{k+1}(A)$ consists of (say) the leftmost and rightmost subintervals of A; and if ℓ_k is even, then $\mathcal{A}_{k+1}(A)$ contains all the subintervals of A.

It follows immediately from Proposition 7.3 that the limiting measure ν is absolutely decaying. Let us show that ν is Federer. Let $B = B(x,r)$ with $x \in \operatorname{supp}\nu$, and choose k by the requirement

$$1/M^k \geq r > 1/M^{k+1}.$$

Let $A \in \mathcal{A}_k$ with $x \in \overline{A}$, then B contains at least one element of $\mathcal{A}_{k+1}$, and $3B$ intersects at most 2 elements of $\mathcal{A}_{k-1}$. All elements of $\mathcal{A}_{k-1}$ have the same ν-measure, say z, and the ν-measure of any element of $\mathcal{A}_{k+1}$ is at least z/M^2. Therefore

$$\frac{\nu(3B)}{\nu(B)} \leq \frac{2z}{z/M^2} = 2M^2\,,$$

as required.

On the other hand, it follows from the discussion in [Mat, Chapter 5] that, if the sequence m_ℓ increases sufficiently rapidly, then the upper (respectively lower) Minkowski dimension of $\mathbf{A}_\infty$ is equal to 1 (respectively $\log 2/\log M$). In particular they are not equal, and hence, by [Mat, Thm. 5.7], the measure ν does not satisfy a power law.

8. Further results

In this section we discuss two more manifestations of the idea of applying uniform return estimates to produce bounded trajectories.

8.1. **Quadratic differentials.** There are many interesting analogies between the study of the dynamics of flows on homogeneous spaces of Lie groups, and flows on the moduli space of quadratic differentials. In this section we present a result, analogous to Theorem 3.2, in the quadratic differential setup. We refer the reader to [MW] and [KW] for all definitions which will be used in this section.

Let S be a compact orientable surface of genus g with n punctures, where $3g + n \geq 3$, and let $\mathcal{Q}$ be the moduli space of unit-area holomorphic quadratic differentials over complex structures on S. This is a noncompact orbifold on which $G \stackrel{\text{def}}{=} \mathrm{SL}(2,\mathbb{R})$ acts continuously. It is partitioned into finitely many G-invariant suborbifolds called *strata*.

Say that $X \subset \mathcal{Q}$ is *bounded in a stratum* if its closure is a compact subset of a single stratum.

For $t \in \mathbb{R}$, let $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in G$ and $F \stackrel{\mathrm{def}}{=} \{g_t : t \geq 0\}$. For $s, \theta \in \mathbb{R}$ let

$$h_s \stackrel{\mathrm{def}}{=} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad r_\theta \stackrel{\mathrm{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The actions of $\{g_t\}, \{h_s\}, \{r_\theta\}$ are often called the (Teichmüller) *geodesic, horocyclic,* and *circle* flows, respectively.

The following holds:

Theorem 8.1. *Let $q \in \mathcal{Q}$, and let μ be an absolutely decaying and Federer measure on $\mathbb{R}$. Then for any open interval B with $\mu(B) > 0$ one has*

(i) $\dim\big(\{s \in \operatorname{supp}\mu \cap B : Fh_s q \text{ is bounded in a stratum}\}\big) \geq \underline{d}_\mu(B)$.

(ii) $\dim\big(\{\theta \in \operatorname{supp}\mu \cap B : Fr_\theta q \text{ is bounded in a stratum}\}\big) \geq \underline{d}_\mu(B)$.

The case in which μ is Lebesgue measure is the main result of [KW]; its proof is similar to that of Theorem 3.2 of the present paper, and can be modified to yield Theorem 8.1. Specifically, one can view $\mathcal{Q}$ as an $S_{1,a}$-space where $S_{1,a}$ is a subgroup of G generated by $H \stackrel{\mathrm{def}}{=} \{h_s\}$ and $a = e^t$, and repeat the argument as in the proof of Corollary 5.4 to derive the needed Hausdorff dimension estimate from Theorem 4.1 and the following uniform return estimate similar to Proposition 5.3:

Proposition 8.2. *For any positive C, α, D, η there exists a compact $K \subset \mathcal{Q}$ with the following property: for any compact $L \subset \mathcal{Q}$ and any interval $B \subset \mathbb{R}$ centered at 0, one can find $t_0 = t_0(L, B) > 0$ such that whenever $t > t_0$, $q \in L$ and μ is an absolutely (C, α)-decaying and D-Federer measure on $3B$ with $0 \in \operatorname{supp}\mu$, one has*

$$\mu\big(\{s \in B : g_t h_s q \in K\}\big) \geq (1-\eta)\mu(B).$$

Proposition 8.2 can be derived from a quantitative nondivergence result for the horocycle flow on moduli space, in terms of a general measure, which is a variant of [MW, Thm. 6.10]. To state it, we introduce the following notation. Let $\widetilde{\mathcal{Q}}$ be the space of (marked) unit area quadratic differentials over complex structures on S, let $\pi : \widetilde{\mathcal{Q}} \to \mathcal{Q}$ be the natural quotient map, and for $\mathbf{q} \in \widetilde{\mathcal{Q}}$ let $\mathcal{L}_{\mathbf{q}}$ denote the set of saddle connections for $\mathbf{q}$. Note that there is a natural identification of $\mathcal{L}_{\mathbf{q}}$ with $\mathcal{L}_{g\mathbf{q}}$ for any $g \in G$. Now for $\delta \in \mathcal{L}_{\mathbf{q}}$ let $l_\delta(\mathbf{q})$ denote the norm of the holonomy vector of δ with respect to the flat metric determined by $\mathbf{q}$, and let

$$K_\varepsilon \stackrel{\text{def}}{=} \pi\left(\left\{\mathbf{q} \in \widetilde{\mathcal{Q}} : \forall \delta \in \mathcal{L}_{\mathbf{q}}, l_\delta(\mathbf{q}) \geq \varepsilon\right\}\right).$$

Note that, as in the homogeneous space set-up, each K_ε is compact, and for each stratum $\mathcal{M}$, $\{\mathcal{M} \cap K_\varepsilon\}_{\varepsilon>0}$ is an exhaustion of $\mathcal{M}$. Then one has the following analogue of Theorem 5.1:

Proposition 8.3. *There are positive constants γ, ρ_0, depending only on S, such that for any positive C, α, D there is a positive C' such that the following holds for any absolutely (C, α)-decaying and D-Federer measure μ on an interval $B \subset \mathbb{R}$. Suppose $J \subset \mathbb{R}$ is an interval with $3J \subset B$, $0 < \rho \leq \rho_0$, $\mathbf{q} \in \widetilde{\mathcal{Q}}$, and suppose that for any $\delta \in \mathcal{L}_{\mathbf{q}}$, $\sup_{t\in J} l_\delta(h_t\mathbf{q}) \geq \rho$. Then for any $0 < \varepsilon < \rho$:*

$$\mu\left(\{s \in J : h_s\pi(\mathbf{q}) \notin K_\varepsilon\}\right) \leq C'\left(\frac{\varepsilon}{\rho}\right)^{\gamma\alpha}\mu(J).$$

The proof of Proposition 8.3 will appear elsewhere; it is similar to that of [MW, Thm. 6.10], but with the assumption that μ is Federer substituting for condition (36) of that paper.

The deduction of Proposition 8.2 from Proposition 8.3 follows the lines of the argument of §5 and is left to the reader.

8.2. **Diophantine approximation with weights.** Given an n-tuple $\mathbf{r} = (r_1, \ldots, r_n)$ with

$$r_i > 0 \quad \text{and} \quad \sum_{i=1}^{n} r_i = 1\,,$$

say that $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is $\mathbf{r}$-*badly approximable* if there is $c > 0$ such that for any $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$, $q \in \mathbb{N}$ one has

$$\max_i |qx_i - p_i|^{1/r_i} \geq \frac{c}{q}\,.$$

Denote the set of all $\mathbf{r}$-badly approximable vectors in $\mathbb{R}^n$ by $\mathbf{r}$-$\mathbf{BA}$. This definition was originally introduced in [K1] but can be traced back to [S4]. Note that one has $\mathbf{BA} = \mathbf{n}$-$\mathbf{BA}$ where $\mathbf{n} \stackrel{\text{def}}{=} (\frac{1}{n}, \ldots, \frac{1}{n})$, so that one can think of the components of $\mathbf{r}$ as the weights assigned to different coordinates of $\mathbf{x}$, recovering the standard definition in the case of equal weights. It follows from W. Schmidt's general version of the Khintchine-Groshev Theorem [S1] that Lebesgue measure of $\mathbf{r}$-$\mathbf{BA}$ is zero. On the other hand, it is mentioned in [S4] that one can prove the existence of $\mathbf{r}$-badly approximable vectors by a variation of a method due to Davenport. The fact that the set $\mathbf{r}$-$\mathbf{BA}$ has full Hausdorff dimension was conjectured in [K2] and was recently proved in [PV1].

This was further extended in the preprint [KTV], where the following was shown: let $\mu = \mu_1 \times \cdots \times \mu_n$, where each μ_i is a measure on $\mathbb{R}$ satisfying a power law; then $\dim(\mathbf{r}\text{-}\mathbf{BA} \cap \operatorname{supp}\mu) = \dim(\operatorname{supp}\mu)$.

Using the method of the present paper, it is possible to develop an alternative proof of the aforementioned result, and in fact establish a slight generalization:

Theorem 8.4. *Let $\mu = \mu_1 \times \cdots \times \mu_n$, where each μ_i is an absolutely decaying and Federer measure on $\mathbb{R}$, and let $B_1, \ldots, B_n$ be open intervals with $\mu_i(B_i) > 0$. Then*

$$\dim\big(\mathbf{r}\text{-}\mathbf{BA} \cap \operatorname{supp}\mu \cap (B_1 \times \cdots \times B_n)\big) \geq \sum_{i=1}^{n} \underline{d}_{\mu_i}(B_i)\,.$$

The first step of the proof is a reduction to a dynamical result; generalizing Dani's correspondence (Proposition 3.1), one can interpret $\mathbf{r}$-badly approximable points of $\mathbb{R}^n$ in terms of boundedness of certain trajectories on G/Γ as follows:

Proposition 8.5 ([K1])**.** $\mathbf{x} \in \mathbf{r}\text{-}\mathbf{BA}$ *if and only if the trajectory*

$$\{g_t^{(\mathbf{r})}\tau(\mathbf{x})\mathbb{Z}^{n+1} : t > 0\}$$

is bounded in G/Γ, where

$$g_t^{(\mathbf{r})} \stackrel{\mathrm{def}}{=} \operatorname{diag}(e^{r_1 t}, \ldots, e^{r_n t}, e^{-t})\,.$$

To construct many bounded $g_t^{(\mathbf{r})}$-trajectories, one applies Theorem 5.1 with h of the form

$$h_{t,u}(\mathbf{x}) \stackrel{\mathrm{def}}{=} g_t^{(\mathbf{r})}\tau(\mathbf{x})u\,.$$

The proofs of Lemma 5.2 and Proposition 5.3 go through with minor changes, since the $g_t^{(\mathbf{r})}$-action still contracts the last component of vectors and expands the remaining components. However the expansion rates are now different, which in particular replaces (3.5) by a more complicated conjugation relation, namely

$$g_t^{(\mathbf{r})}\tau(\mathbf{x})g_{-t}^{(\mathbf{r})} = \tau\big(A_{\mathbf{r}}\mathbf{x}\big)\,, \quad \text{where} \quad A_{\mathbf{r}} \stackrel{\mathrm{def}}{=} \operatorname{diag}(e^{(1+r_1)t}, \ldots, e^{(1+r_n)t})\,.$$

As a result, the uniform return method of §4 has to be modified, which in particular demands more restrictive assumptions on the measure μ.

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Brandeis University, Waltham MA 02454-9110 kleinboc@brandeis.edu

Ben Gurion University, Be'er Sheva, Israel 84105 barakw@math.bgu.ac.il