FINITE DIMENSIONAL REPRESENTATIONS AND SUBGROUP ACTIONS ON HOMOGENEOUS SPACES

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ABSTRACT. Let H be an $\mathbb{R}$-subgroup of a $\mathbb{Q}$-algebraic group G. We study the connection between the dynamics of the subgroup action of H on $G/G_{\mathbb{Z}}$ and the representation-theoretic properties of H being observable and epimorphic in G. We show that if H is a $\mathbb{Q}$-subgroup then H is observable in G if and only if a certain H orbit is closed in $G/G_{\mathbb{Z}}$; that if H is epimorphic in G then the action of H on $G/G_{\mathbb{Z}}$ is minimal, and that the converse holds when H is a $\mathbb{Q}$-subgroup of G; and that if H is a $\mathbb{Q}$-subgroup of G then the closure of the orbit under H of the identity coset image in $G/G_{\mathbb{Z}}$ is the orbit of the same point under the observable envelope of H in G. Thus in subgroup actions on homogeneous spaces, closures of 'rational orbits' (orbits in which everything which can be defined over $\mathbb{Q}$, is defined over $\mathbb{Q}$) are always submanifolds.

1. Introduction

If G is a Lie group and H and Γ are subgroups, there is a natural subgroup action of H on the homogeneous space G/Γ. The dynamics of these actions have been extensively studied. We will be concerned with such questions as when is an H-orbit closed, when is the H action minimal (all orbits are dense), and when is the H action uniquely ergodic (there is a unique invariant probability measure). These questions turn out to be related to some questions in the theory of finite dimensional representations of algebraic groups.

If $\mathbf{G}$ is an algebraic group (in this paper algebraic groups will be denoted by boldface letters, to distinguish them from Lie groups) and $\mathbf{H}$ is an algebraic subgroup, $\mathbf{H}$ is said to be *observable* in $\mathbf{G}$ if it is the stabilizer of a vector in a finite dimensional algebraic representation, and *epimorphic* in $\mathbf{G}$ if any $\mathbf{H}$-invariant vector is already $\mathbf{G}$-invariant. Observable subgroups were introduced by Bialinicki-Birula, Hochschild and Mostow in 1963 ([BHM]). They wanted to know which $\mathbf{H}$ have the property that every representation of $\mathbf{H}$ is the restriction of a $\mathbf{G}$ representation. In characteristic zero, this is equivalent to observability, as well as to the property that the homogeneous space $\mathbf{G}/\mathbf{H}$ is quasi-affine. Epimorphic subgroups were introduced by Bergman in 1970, and their study was recently taken up by Bien and Borel in [BB1, BB2, BB3].

The first result linking observability and subgroup actions is a so-called 'bounded denominators argument'. For example, Raghunathan ([Ra], Prop. 10.15), proves that if H and G are the connected component of the identity in $\mathbb{Q}$-groups $\mathbf{H}$ and $\mathbf{G}$, and $\mathbf{H}$ has no nontrivial $\mathbb{Q}$ characters, and $\Gamma = \mathbf{G}_{\mathbb{Z}}$, then the H orbit $H\Gamma$ is closed. This argument extends to the case when $\mathbf{H}$ is observable in $\mathbf{G}$. Observability is used to ensure that the image of Γ in the quasi-affine $H\backslash G$ be contained in a discrete set. Since observability and epimorphicity are opposing properties, this argument can also be used to derive epimorphicity from the density of a certain H orbit (see Proposition 3).

Using epimorphicity to derive conclusions in the dynamics of subgroup actions is considerably more difficult. Mozes ([Mo]) was able to prove that under certain conditions, if $\mathbf{H}$ is epimorphic in $\mathbf{G}$ and Γ is a lattice in G, then the action of H on G/Γ is uniquely ergodic. His proof utilized the fundamental results of Ratner (see [R3] for a survey) on the actions of unipotent subgroups of G on G/Γ.

Although unique ergodicity and minimality are independent conditions, they often appear together. Thus it is natural to ask whether, when H is epimorphic in G, the action of H on G/Γ is minimal. That this is so when H is $\mathbb{R}$-algebraic and Γ is arithmetic (Theorem 9) is the main result of this paper. Raghunathan has given an example of an epimorphic Lie subgroup (not an algebraic subgroup) whose action on $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ is not minimal (but is uniquely ergodic). The example, which was reported by Ratner in [R3], but never published, is included for the sake of completeness.

The idea of the proof is as follows: by Mozes' theorem, it is enough to show that every H-orbit-closure supports an H-invariant measure, and by a theorem of Bien and Borel we assume that H is solvable, and write $H = TU$ with T a torus normalizing the unipotent U. To produce an H-invariant measure we take a U-invariant measure and show that when translated by some elements of T, the measure does not escape to infinity. To keep the measure from escaping we use some results of Dani and Margulis giving representation-theoretic conditions which ensure that unipotent subgroups orbits remain bounded away from infinity with probability close to 1 (see [DM], and [Sh], [EMS] for generalizations).

The proof also requires some more information on the structure of epimorphic subgroups. Particularly, we need a result (Theorem 4) saying that if $\mathbf{G}$ is an algebraic group and $\mathbf{H}$ is epimorphic in $\mathbf{G}$, then the intersection of $\mathbf{H}$ with $\mathbf{G}_0$, the subgroup of $\mathbf{G}$ generated by unipotent elements in $\mathbf{G}$, is epimorphic in $\mathbf{G}_0$. We also need some results about the structure of epimorphic subgroups of real groups with compact factors. These are given in Section 3.

In the final section we present several applications. First we show that rational orbits (that is, $\mathbf{H}$ and $\mathbf{G}$ are $\mathbb{Q}$-groups, Γ is the integer points of G, and the orbit contains a point which is the projection of a $\mathbb{Q}$-point in G) have closures which are orbits of larger groups. We also show that the density of a rational orbit implies the density of all orbits. Finally we obtain necessary and sufficient conditions for $\mathbf{H}$ to be observable/epimorphic in $\mathbf{G}$ (when both H and G are defined over $\mathbb{Q}$) in terms of the subgroup action of H on $G/G_{\mathbb{Z}}$.

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2. Preliminaries

2.1. **Subgroup Actions on Homogeneous Spaces.** Let G be a Lie Group and Γ a closed subgroup. The homogeneous space G/Γ is a manifold. Any subgroup $H < G$ acts on G/Γ by left translations: $h(g\Gamma) = hg\Gamma$. Any continuous transitive G action can be identified with such a space. For $H < G, x \in G, \Lambda = H \cap x\Gamma x^{-1}$, the orbit $Hx\Gamma$ is closed if and only if the orbit map $H/H \cap \Lambda \to G/\Gamma$, defined by $h\Lambda \mapsto hx\Gamma$, is a proper map (a homeomorphism onto its image).

Γ is called a *lattice* if Γ is discrete and there is a G invariant regular Borel probability measure on the σ-algebra of Borel sets on G/Γ. This measure is then smooth, and positive on open sets. Γ is called cocompact if G/Γ is compact. The set of probability measures on G/Γ, equipped with the weak-$*$ topology, is a convex topological G-space, with G acting by

$$\int_{G/\Gamma} f(x) dg\mu(x) = \int_{G/\Gamma} f(gx) d\mu(x),$$

where f is any continuous function on G/Γ with compact support, and $g \in G$. If Γ is cocompact then this G-space is compact.

We say that two subgroups Γ and Λ are *commensurable* if $\Gamma \cap \Lambda$ is of finite index in both. The action of a subgroup H is called *uniquely ergodic* if the only H-invariant probability measure on G/Γ is the G-invariant one. The action of H is called *minimal* if every orbit in G/Γ is dense. If G is connected, minimality is a property which depends only on the conjugacy class of H and the commensurability class of Γ.

2.2. **Algebraic Groups.** Throughout this paper, $\mathbf{G}$ will denote an (affine) algebraic group over $\mathbb{C}$. That is, $\mathbf{G}$ is a subgroup of $GL(n, \mathbb{C})$ which is the set of simultaneous zeros of some set of polynomials in the matrix elements X_{ij} and $(\det(X_{ij}))^{-1}$. For a subfield $k \subset \mathbb{C}$, we say that $\mathbf{G}$ is defined over k, or a k-group, if these polynomials can be chosen to have coefficients in k. The ring of regular functions on $\mathbf{G}$, denoted by $\mathbb{C}[\mathbf{G}]$, is the set of functions on $\mathbf{G}$ which are polynomials in the X_{ij} and in $(det(X_{ij}))^{-1}$. The subring of polynomials with coefficients in k is denoted by $k[\mathbf{G}]$. $\mathbf{G}$ acts on $\mathbb{C}[\mathbf{G}]$ on the left by $(gf)(x) = f(xg)$. If $\mathbf{H}$ is an algebraic subgroup of the algebraic group $\mathbf{G}$ (notation: $\mathbf{H} < \mathbf{G}$), we denote the regular functions invariant under elements of $\mathbf{H}$ by $\mathbb{C}[\mathbf{G}]^{\mathbf{H}}$.

For any subring R of $\mathbb{C}$ (e.g., $R = \mathbb{Z}$ or $R = \mathbb{Q}$) we will let $\mathbf{G}_R$ denote $\mathbf{G} \cap GL(n, R) = \{(X_{ij}) \in \mathbf{G} : X_{ij} \in R, \det(X_{ij}) \text{ is a unit of } R\}$. (Note that $\mathbf{G}_{\mathbb{Z}}$ depends on the realization of $\mathbf{G}$ as a subgroup of $GL(n, \mathbb{C})$, i.e., is not invariant under an isomorphism defined over $\mathbb{Q}$. Still, its commensurability class is well defined).

Notation: Throughout this paper, we will write $G = \mathbf{G}^0_{\mathbb{R}}$ - the connected component of the identity of the real points of $\mathbf{G}$.

If k is a subfield of $\mathbb{C}$ we say that $\mathbf{H}$ is a k-subgroup of $\mathbf{G}$ if there are polynomials $\{f_i\}_{i \in \mathcal{I}}$ and $\{g_j\}_{j \in \mathcal{J}}$, with coefficients in k, such that $\mathbf{G} = \{A \in M_n(\mathbb{C}) : f_i(A) = 0, \forall i \in \mathcal{I}\}$ and $\mathbf{H} = \{A \in \mathbf{G} : g_j(A) = 0, \forall j \in \mathcal{J}\}$. The notation $\mathbf{H} \overset{\mathbb{Q}}{<} \mathbf{G}$ means that $\mathbf{H}$ is a $\mathbb{Q}$-subgroup of $\mathbf{G}$. If V is a d-dimensional vector space over $\mathbb{C}$, then $GL(V)$ denotes the group of linear automorphisms of V; a choice of basis in V identifies $GL(V)$ with $GL(d, \mathbb{C})$. A representation $\rho : \mathbf{G} \to GL(V)$ is a homomorphism defined by regular functions, i.e., in any identification of $GL(V)$ with $GL(d, \mathbb{C})$, the matrix elements $\rho(g)_{ij}$ are in $\mathbb{C}[\mathbf{G}]$ for $1 \leq i, j \leq d$. A representation is said to be defined over k, or a k-representation, if by choosing a suitable basis for V, all the matrix elements $\rho(g)_{ij}$ are in $k[G]$. A vector $v \in V$ is said to be in V_k if v is in the k-span of this basis. Note that all representations considered in this paper are finite-dimensional.

A (k-) character of $\mathbf{G}$ is a (k-) representation of $\mathbf{G}$ in which V is one dimensional. The set of all characters (resp., all k-characters) on $\mathbf{G}$ is denoted by $X(G)$ (resp., $X(G)_k$) and carries the structure of an abelian group.

A theorem of Chevalley says that whenever $\mathbf{H}$ is an algebraic subgroup of $\mathbf{G}$, there exists a representation $\rho : \mathbf{G} \to GL(V)$, and a vector $v \in V$ such that $\mathbf{H} = \{g \in \mathbf{G} : \rho(g)v \in \mathbb{C}v\}$ (where $\mathbb{C}v$ is the line through v). In this case, $\mathbf{H}$ acts on the line through v via a character, i.e., there exists $\chi \in X(H)$ such that $\rho(h)v = \chi(h)v$ for $h \in \mathbf{H}$.

If $\mathbf{H} \overset{\mathbb{Q}}{<} \mathbf{G}$ then ρ, χ, v can be chosen so that ρ is a $\mathbb{Q}$-representation, $\chi \in X(H)_{\mathbb{Q}}$, and $v \in V_{\mathbb{Q}}$. Chevalley's theorem is used to endow the homogeneous space $\mathbf{G}/\mathbf{H}$ with the structure of a quasi-projective algebraic variety (an open subset of a projective variety).

For some $\mathbf{H}$ and $\mathbf{G}$, more can be said.

Definition 1. $\mathbf{H}$ *is said to be* observable *in* $\mathbf{G}$ *if there exists a representation* $\rho : \mathbf{G} \to GL(V)$ *and a vector* $v \in V$ *such that* $\mathbf{H} = \{g \in \mathbf{G} : \rho(g)v = v\}$. *This will be denoted by* $\mathbf{H} <_o \mathbf{G}$. *If* $\mathbf{G}$ *is a* $\mathbb{Q}$*-group and* ρ *can be chosen to be a* $\mathbb{Q}$*-representation with* $v \in V_{\mathbb{Q}}$*, then* $\mathbf{H}$ *is said to be* $\mathbb{Q}$*-observable.*

The following theorem is due to [BHM] and [G]:

Theorem 1. *The following are equivalent:*

1. $\mathbf{H} <_o \mathbf{G}$.
2. *Every representation* $\rho : \mathbf{H} \to GL(V)$ *is a restriction of a representation* $\bar{\rho} : \mathbf{G} \to GL(W)$*, that is,* W *is a vector space containing the* $\bar{\rho}(H)$ *invariant subspace* V*, and* ρ *is the restriction of* $\bar{\rho}$ *to* $\mathbf{H}$ *and to* V*.*
3. *The homogeneous space* $\mathbf{G}/\mathbf{H}$ *is a quasi-affine variety (i.e., can be embedded in an affine algebraic variety).*
4. $\mathbf{H} = \{g \in \mathbf{G} : \forall f \in \mathbb{C}[\mathbf{G}]^{\mathbf{H}}, gf = f\}$ *(the pointwise fixer of* $\mathbb{C}[\mathbf{G}]^{\mathbf{H}}$*).*

Also, $\mathbf{H} <_o \mathbf{G} \Leftrightarrow \mathbf{H}^0 <_o \mathbf{G}$*, where* $\mathbf{H}^0$ *is the connected component of the identity in* $\mathbf{H}$.

The equivalent conditions in Theorem 1 above can be relativized to $\mathbb{Q}$. For example, we may substitute $\mathbb{Q}$-observable for observable, $\mathbb{Q}$ representations for representations, $\mathbb{Q}[\mathbf{G}]$ for $\mathbb{C}[\mathbf{G}]$, and so on. The proof given in [BHM] can be "$\mathbb{Q}$-relativized" to show that all the relativized notions are equivalent. For our purposes, however, the following will suffice:

Proposition 1. *If* $\mathbf{H} \overset{\mathbb{Q}}{<} \mathbf{G}$*, with* $\mathbf{H}$ *connected, then* $\mathbf{H}$ *is observable in* $\mathbf{G}$ *if and only if* $\mathbf{H}$ *is* $\mathbb{Q}$*-observable in* $\mathbf{G}$.

Proof (see [G]): It is clear that if $\mathbf{H}$ is $\mathbb{Q}$-observable in $\mathbf{G}$, then it is observable.

Conversely, if $\mathbf{H} <_o \mathbf{G}$ then $\mathbf{H}$ is the pointwise fixer of $\mathbb{C}[\mathbf{G}]^{\mathbf{H}}$. Since $\mathbf{G}$ is a $\mathbb{Q}$-group, $\mathbb{C}[\mathbf{G}] = \mathbb{Q}[\mathbf{G}] \otimes_{\mathbb{Q}} \mathbb{C}$ (see [Bo], AG 14.2). Since $\mathbf{H}$ is a connected $\mathbb{Q}$-group, $\mathbf{H}_{\mathbb{Q}}$ is Zariski dense in $\mathbf{H}$ and therefore $\mathbb{C}[\mathbf{G}]^{\mathbf{H}} = \mathbb{Q}[\mathbf{G}]^{\mathbf{H}_{\mathbb{Q}}} \otimes_{\mathbb{Q}} \mathbb{C}$. By the Noetherian property of algebraic varieties, there are $f_1, \ldots, f_n \in \mathbb{Q}[\mathbf{G}]^{\mathbf{H}}$ such that $\mathbf{H} = \{g \in \mathbf{G} : gf_i = f_i, i = 1, \ldots, n\}$.

Each f_i is contained in a finite dimensional $\mathbf{G}$-invariant subspace V_i of $\mathbb{C}[\mathbf{G}]$, defined over $\mathbb{Q}$, namely $V_i = \text{span}(\{gf_i : g \in \mathbf{G}\})$. Thus $\mathbf{H}$ is the stabilizer of the $\mathbb{Q}$-vector $(f_1, \dots, f_n)$ in the $\mathbb{Q}$-representation $V_1 \bigoplus \dots \bigoplus V_n$ of $\mathbf{G}$. □

In [BHM] it is shown that $\mathbf{H} <_o \mathbf{G}$ if $\mathbf{H}$ is normal in $\mathbf{G}$, unipotent, or reductive. It is obvious that observability is transitive:

$$\mathbf{G}_1 <_o \mathbf{G}_2, \ \mathbf{G}_2 <_o \mathbf{G}_3 \Rightarrow \mathbf{G}_1 <_o \mathbf{G}_3.$$

Also,

$$\mathbf{G}_1 <_o \mathbf{G}_3, \mathbf{G}_1 < \mathbf{G}_2 < \mathbf{G}_3 \Rightarrow \mathbf{G}_1 <_o \mathbf{G}_2.$$

Sukhanov [Su] has shown that $\mathbf{H} <_o \mathbf{G}$ if and only if $\text{Rad}(\mathbf{H}) <_o \mathbf{G}$, where $\text{Rad}(\mathbf{H})$ is the radical of $\mathbf{H}$. Sukhanov has also given representation-theoretic necessary and sufficient conditions for observability.

The following is essentially Proposition 10.15 in [Ra]; the quasi-affine structure of $\mathbf{G}/\mathbf{H}$ ensures that the image of $G_{\mathbb{Z}}$ is contained in a discrete set.

Proposition 2. *Let* $\mathbf{H}$ *be* $\mathbb{Q}$*-observable in* $\mathbf{G}$*, and let* $\Gamma = \mathbf{G}_{\mathbb{Z}}$*. If* $x \in G_{\mathbb{Q}}$ *and* $\pi : G \to G/\Gamma$ *is the projection map, then the orbit* $H\pi(x)$ *is closed in* G/Γ*.*

Proof: Since the identity component $\mathbf{H}_0 < \mathbf{H}$ is a normal $\mathbb{Q}$-subgroup of finite index, we may assume that $\mathbf{H}$ is connected. We may also assume that $x = e$ is the identity, since $H\pi(x)$ is closed if and only if $x^{-1}Hx\pi(e)$ is closed, and $x^{-1}\mathbf{H}x$ is also a $\mathbb{Q}$-observable $\mathbb{Q}$-subgroup. Let $\rho : \mathbf{G} \to GL(V)$ be a $\mathbb{Q}$- representation, and $v \in V_{\mathbb{Q}}$ be such that $\mathbf{H} = \{g \in \mathbf{G} : \rho(g)v = v\}$.

We want to show that $H\Gamma$ is closed in G. Suppose $h_n\gamma_n \to y, \ h_n \in H, \ \gamma_n \in \Gamma$. Then $\rho(y^{-1})v \leftarrow \rho(\gamma_n^{-1}h_n^{-1})v = \rho(\gamma_n^{-1})v$.

The sequence $\rho(\gamma_n^{-1})v$ is contained in a discrete subset of V, defined by a common denominator for the rational coefficients appearing in ρ and in v. Therefore for some n_0, $\rho(y^{-1})v = \rho(\gamma_{n_0}^{-1})v$, which means that $\rho(y\gamma_{n_0}^{-1})$ fixes v and thus $y \in H\Gamma$.

□

Definition 2. *Let* $\mathbf{H} < \mathbf{G}$*. We say that* $\mathbf{H}$ *is* epimorphic *in* $\mathbf{G}$ *if for any representation* $\rho : \mathbf{G} \to GL(V)$*, if* $v \in V$ *is* $\rho(\mathbf{H})$*-invariant then* v *is* $\rho(\mathbf{G})$*-invariant. We denote this by* $\mathbf{H} <_e \mathbf{G}$*. We say that* $\mathbf{H}$ *is* $\mathbb{Q}$-epimorphic *if the same holds for all* $\mathbb{Q}$ *representations* $\rho : \mathbf{G} \to GL(V)$ *and all* $v \in V_{\mathbb{Q}}$*.*

Epimorphic subgroups were introduced by Bergman in [Ber] in the context of category theory, and studied by Bien and Borel in [BB1, BB2, BB3].

Theorem 2. [BB1] *Let* $\mathbf{G}$ *be connected, and* $\mathbf{H} < \mathbf{G}$*. Then the following are equivalent:*

1. $\mathbf{H} <_e \mathbf{G}$.
2. $\mathbb{C}[\mathbf{G}]^{\mathbf{H}}$ *is one dimensional (i.e., an* $\mathbf{H}$ *invariant regular function on* $\mathbf{G}$ *is constant).*
3. *If* $\rho : \mathbf{G} \to GL(V)$*, and* $V = V_1 \bigoplus V_2$*, where each* V_i *is* $\rho(H)$ *invariant, then each* V_i *is* $\rho(G)$ *invariant.*

Suppose $\mathbf{G}$ is connected. If $\mathbf{H} \overset{\mathbb{Q}}{<} \mathbf{G}$ and $\mathbf{H}$ is $\mathbb{Q}$-epimorphic in $\mathbf{G}$ then $\mathbf{H} <_e \mathbf{G}$ (see [BB3]). Parabolic subgroups (i.e., $\mathbf{H} < \mathbf{G}$ such that $\mathbf{G}/\mathbf{H}$ is complete) are epimorphic. There are no epimorphic proper subgroups of solvable groups. If $\mathbf{G}_1 <_e \mathbf{G}_2$ and $\mathbf{G}_2 <_e \mathbf{G}_3$ then $\mathbf{G}_1 <_e \mathbf{G}_3$. If $\mathbf{G}$ is generated by subgroups $G_i,\ i \in \mathcal{I}$, and for each $i \in \mathcal{I},\ \mathbf{H} \cap \mathbf{G}_i <_e \mathbf{G}_i$, then $\mathbf{H} <_e \mathbf{G}$. If ϕ is a homomorphism of algebraic groups and $\mathbf{H} <_e \mathbf{G}$ then $\phi(\mathbf{H}) <_e \phi(\mathbf{G})$. If $\mathbf{H} <_e \mathbf{G}$ then $\mathbf{H}$ contains a solvable subgroup $\mathbf{H}_0$ such that $\mathbf{H}_0 <_e \mathbf{G}$. Moreover, according to a theorem of Bien and Borel (see [BB1], Theorem 2) if $\mathbf{H}$ and $\mathbf{G}$ are defined over $\mathbb{R}$, G is generated by its unipotent elements and $\mathbf{H} <_e \mathbf{G}$ then there is a solvable subgroup $\mathbf{H}_0$ of $\mathbf{H}$, which splits over $\mathbb{R}$, such that $\mathbf{H}_0 <_e \mathbf{G}$. If $\mathbf{G}$ is simple, it contains a 3 dimensional epimorphic subgroup (see [BB1, BB3, Mo]).

Observability and epimorphicity are opposite properties. More precisely, we have:

Theorem 3. [G, BB1, BB3] *Let* $\mathbf{H} < \mathbf{G}$*. Then there exists a unique algebraic subgroup* $\mathbf{E}$ *such that* $\mathbf{H} <_e \mathbf{E} <_o \mathbf{G}$*.* $\mathbf{E}$ *is the largest subgroup of* $\mathbf{G}$ *in which* $\mathbf{H}$ *is epimorphic, and the smallest subgroup of* $\mathbf{G}$ *containing* $\mathbf{H}$ *which is observable in* $\mathbf{G}$*.* $\mathbf{E}$ *is the pointwise fixer of* $\mathbb{C}[\mathbf{G}]^{\mathbf{H}}$*. If* $\mathbf{H}$ *is connected then so is* $\mathbf{E}$*. If* $\mathbf{H}$ *is a* k*-subgroup of* $\mathbf{G}$ *for some subfield* k *of* $\mathbb{C}$ *then* $\mathbf{E}$ *is also a* k*-subgroup of* $\mathbf{G}$*.*

$\mathbf{E}$ is called the *observable envelope* of $\mathbf{H}$ in $\mathbf{G}$.

Proposition 3. *Let* $\mathbf{H} \overset{\mathbb{Q}}{<} \mathbf{G}$*, with* $\mathbf{G}$ *connected. Assume that the orbit* $H\pi(e)$ *is dense in* G/Γ*, where* $\Gamma = \mathbf{G}_{\mathbb{Z}}$ *and* $\pi : G \to G/\Gamma$ *is the projection. Then* $\mathbf{H} <_e \mathbf{G}$*.*

Proof: Assume otherwise; then $\mathbf{E}$, the observable envelope of $\mathbf{H}$ in $\mathbf{G}$, is a proper $\mathbb{Q}$-subgroup of $\mathbf{G}$. Since $\mathbf{G}$ is connected, $\mathbf{E}$ has strictly lower dimension. By Proposition 2, $E\pi(e)$ is a proper closed subset of G/Γ, containing $H\pi(e)$.

□

3. Epimorphic Subgroups

This section presents several auxilliary results about the structure of epimorphic groups.

It is often useful to know for which subgroups $\mathbf{E} < \mathbf{G}$, $\mathbf{H} <_e \mathbf{G}$ implies $\mathbf{H} \cap \mathbf{E} <_e \mathbf{E}$. In this context, we mention that Bien and Borel have constructed examples of $\mathbf{H} <_e \mathbf{G}$ such that $\mathbf{H}$ does not contain the unipotent radical of $\mathbf{G}$, and of $\mathbf{H} <_e \mathbf{G}$, $\mathbf{G}$ reductive, such that $\mathbf{H}$ does not contain the center of $\mathbf{G}$. The following is also worth noting:

Example: Let $\mathbf{G} = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$, and let

$$\mathbf{H} = \left\{ \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \right) : a, b, c \in \mathbb{C} \right\}$$

Then $\mathbf{H}$ is epimorphic in $\mathbf{G}$ but $\mathbf{H} \cap SL(2, \mathbb{C}) \times \{e\}$ is not epimorphic in $SL(2, \mathbb{C}) \times \{e\}$.

Theorem 4. *Let $\mathbf{G}$ be an algebraic group, and let $X_0 \subset X(G)$. Let $\mathbf{G}_0 = \bigcap_{\chi \in X_0} \ker(\chi), \mathbf{H}_0 = \mathbf{H} \cap \mathbf{G}_0$. Then $\mathbf{H} <_e \mathbf{G}$ if and only if $\mathbf{H}_0 <_e \mathbf{G}_0,\ \mathbf{G} = \mathbf{H}\mathbf{G}_0$.*

Proof: The 'if' part is clear.

Conversely, let $\mathbf{H} <_e \mathbf{G}$. Let $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{G}_0$ be the quotient map. Then $\pi(\mathbf{H})$ is epimorphic in the torus $\mathbf{G}/\mathbf{G}_0$, and is therefore equal to it. This implies that $\mathbf{G} = \mathbf{H}\mathbf{G}_0$.

We may assume that $\mathbf{H}$ is solvable, and write $\mathbf{H} = \mathbf{T}\mathbf{U}$, where $\mathbf{T}$ is a torus, $\mathbf{U}$ is the unipotent radical of $\mathbf{H}$, and $\mathbf{T}$ normalizes $\mathbf{U}$. Since $\pi|\mathbf{U} : \mathbf{U} \to \mathbf{G}/\mathbf{G}_0$ is a homomorphism from a unipotent group to a torus, it is trivial and $\mathbf{U} < \mathbf{H}_0$. Therefore we may write $\mathbf{H} = \mathbf{T}_1\mathbf{T}_0\mathbf{U}$, where $\mathbf{H}_0 = \mathbf{T}_0\mathbf{U}$ and $\pi|\mathbf{T}_1$ is an isomorphism between $\mathbf{T}_1$ and $\mathbf{G}/\mathbf{G}_0$. This isomorphism induces an isomorphism $X(\mathbf{T}_1) \cong \langle X_0 \rangle$, where $\langle X_0 \rangle$ is the subgroup of $X(G)$ generated by X_0.

Let $\rho : \mathbf{G}_0 \to GL(V)$. Since $\mathbf{G}_0 <_o \mathbf{G}$, ρ is the restriction to $(\mathbf{G}_0,\ V)$ of a representation $\bar{\rho} : \mathbf{G} \to GL(\bar{V})$. For any $\chi \in \langle X_0 \rangle$, ρ is also the restriction to $(\mathbf{G}_0,\ V)$ of the representation $g \mapsto \chi(g)\bar{\rho}(g)$.

Let W be the subspace of $\bar{V}$ consisting of $\bar{\rho}(\mathbf{H}_0)$ invariant vectors. $\mathbf{T}_1$ normalizes $\mathbf{H}_0$ and therefore $\bar{\rho}(\mathbf{T}_1)$ leaves W invariant. Thus we may write $W = \bigoplus_{\chi \in \langle X_0 \rangle} W_\chi$, where for each χ, and each $w \in W_\chi,\ t \in T_1,\ \rho(t)w = \chi(t)w$. Multiplying $\bar{\rho}$ by χ^{-1}, we get a representation of $\mathbf{G}$ in which each $w \in W_\chi$ is $\bar{\rho}(\mathbf{H})$ invariant, and therefore $\bar{\rho}(\mathbf{G})$ invariant. This means that each $w \in W_\chi$ is $\rho(\mathbf{G}_0)$ invariant, and thus every $w \in W$ is $\rho(\mathbf{G}_0)$ invariant.

□

Corollary 1. *If* $\mathbf{G}$ *is reductive,* $G = S \times Z$*, where* S *is semisimple and* Z *is the center of* G*, and if* $\mathbf{H} <_e \mathbf{G}$*, then* $\mathbf{H} \cap S <_e S$*.*

□

The following propositions are useful when dealing with epimorphic subgroups of real algebraic groups with compact factors.

Assume in Propositions 4,5,6 and Corollaries 2 and 3 that $\mathbf{G}$ is an algebraic connected $\mathbb{R}$-group, $\mathbf{H}$ a connected $\mathbb{R}$-subgroup with $\mathbf{H} <_e \mathbf{G}$. If $\mathbf{G}_0$ is a normal $\mathbb{R}$-subgroup of $\mathbf{G}$ with G/G_0 compact, we say that $\mathbf{G}_0$ is *cocompact* in $\mathbf{G}$. If $\mathbf{G}$ is such that for any $\mathbb{R}$-homomorphism $\phi : \mathbf{G} \to \mathbf{G}'$ with $\phi(G)$ compact, $\phi(\mathbf{G})$ is trivial, we say that $\mathbf{G}$ *has no compact factors.*

Proposition 4. *If* $\rho : \mathbf{G} \to GL(V)$ *is an* $\mathbb{R}$*-representation with* $\rho(H)$ *compact, then any* $\rho(\mathbf{H})$*-invariant subspace defined over* $\mathbb{R}$ *is* $\rho(\mathbf{G})$*-invariant.*

Proof: Let W be a $\rho(\mathbf{H})$ invariant subspace, defined over $\mathbb{R}$, and let $w \in \bigwedge_1^d W$ be a nonzero vector, where d is the dimension of W. Then $\rho(\mathbf{H})$ leaves the line through w invariant, acting on it via a character defined over $\mathbb{R}$. Since $\rho(H)$ is compact, the connected component of unity in $\rho(H)$ leaves w invariant, and since this component is Zariski dense in $\rho(\mathbf{H})$, so does $\rho(\mathbf{H})$. By epimorphicity, so does $\rho(\mathbf{G})$. □

Proposition 5. *Let* $\phi : \mathbf{G} \to \mathbf{G}''$ *be an* $\mathbb{R}$*-homomorphism with* $\phi(H)$ *compact. Then* $\phi(\mathbf{H}) = \phi(\mathbf{G})$*.*

Proof: Let $\mathbf{G}' = \phi(\mathbf{G})$ and $\mathbf{H}' = \phi(\mathbf{H})$. Then $\mathbf{H}' <_e \mathbf{G}'$ and H' is compact. Consider the adjoint representation of $\mathbf{G}'$ on its Lie algebra $\mathfrak{G}$. This is an $\mathbb{R}$-representation in which $\mathbf{H}'$ stabilizes its Lie algebra $\mathfrak{H}$, an $\mathbb{R}$-subspace. By Proposition 4, $\mathbf{G}'$ also stabilizes $\mathfrak{H}$, so $\mathbf{H}'$ is a normal subgroup of $\mathbf{G}'$. But a proper normal subgroup is never epimorphic, and therefore $\mathbf{H}' = \mathbf{G}'$.

□

Proposition 6. *Let* $\mathbf{H}_0$ *be a normal cocompact* $\mathbb{R}$*-subgroup of* $\mathbf{H}$*, and let* $\mathbf{G}_0$ *be the observable envelope of* $\mathbf{H}_0$ *in* $\mathbf{G}$*. Then* $\mathbf{G}_0$ *is a normal subgroup of* $\mathbf{G}$*, and* $\mathbf{G}/\mathbf{G}_0 \cong \mathbf{H}/\mathbf{H} \cap \mathbf{G}_0$*. In particular,* $\mathbf{G}_0$ *is cocompact, contains the unipotent radical of* $\mathbf{G}$*, and* $\mathbf{G}/\mathbf{G}_0$ *is a factor of* $\mathbf{H}/\mathbf{H}_0$*.*

Proof: The last three assertions follow from the fact that $\mathbf{H} \cap \mathbf{G}_0$ contains $\mathbf{H}_0$ and is therefore cocompact, and from the fact that a cocompact $\mathbb{R}$-subgroup of $\mathbf{G}$ contains all the $\mathbb{R}$-unipotent elements of $\mathbf{G}$.

Let $\rho : \mathbf{G} \to GL(V)$ be an $\mathbb{R}$-representation and $v \in V_{\mathbb{R}}$ be such that $\mathbf{G}_0 = \{g \in \mathbf{G} : \rho(g)v = v\}$. Let W denote the subspace of $\rho(\mathbf{H}_0)$-invariant vectors. Then any element of W is also $\rho(\mathbf{G}_0)$-invariant, and

W is a $\rho(\mathbf{H})$-invariant subspace since $\mathbf{H}_0$ is normal in $\mathbf{H}$. By Proposition 4, W is also $\rho(\mathbf{G})$ invariant, and we restrict ρ to W (retaining the name ρ).

Then $\mathbf{G}_0$ is the kernel of ρ. This implies that $\mathbf{G}_0$ is normal in $\mathbf{G}$. Also $\rho(\mathbf{G}) = \rho(\mathbf{H})$ by Proposition 5. We have therefore $\mathbf{G}/\mathbf{G}_0 \cong \rho(\mathbf{G}) = \rho(\mathbf{H}) \cong \mathbf{H}/\mathbf{H} \cap \mathbf{G}_0$.

□

Corollary 2. *Suppose* $\mathbf{G}_0$ *is a connected cocompact normal* $\mathbb{R}$*-subgroup of* $\mathbf{G}$ *with no compact factors, and let* $\mathbf{H}_0 = \mathbf{G}_0 \cap \mathbf{H}$. *Then* $\mathbf{H}_0 <_e \mathbf{G}_0$.

Corollary 3. *Let* $\mathbf{G}$ *be semisimple, and let* $\mathbf{H}_0$ *be a normal cocompact* $\mathbb{R}$*-subgroup of* $\mathbf{H}$. *If* H_0 *has a nontrivial projection on every compact factor of* G *then* $\mathbf{H}_0 <_e \mathbf{G}$.

We conclude our discussion of the structure of epimorphic groups with the following:

Theorem 5. *Let* $\mathbf{G}$ *be an* $\mathbb{R}$*-algebraic group and* $\mathbf{H}$ *an* $\mathbb{R}$*-subgroup. Let* $\mathbf{G}_0$ *be the subgroup of* $\mathbf{G}$ *generated by* $\mathbb{R}$*-unipotent elements in* $\mathbf{G}$ *and let* $\mathbf{H}_0 = \mathbf{H} \cap \mathbf{G}_0$. *Then* $\mathbf{H} <_e \mathbf{G}$ *if and only if* $\mathbf{H}_0 <_e \mathbf{G}_0$ *and* $\mathbf{G} = \mathbf{H}\mathbf{G}_0$.

Proof: The 'if' part is clear.

For the converse, let $\mathbf{G}_1 = \bigcap_{\chi \in X(\mathbf{G})} \ker \chi$. By Theorem 4, $\mathbf{H}_1 = \mathbf{H} \cap \mathbf{G}_1$ is epimorphic in $\mathbf{G}_1$ and $\mathbf{G} = \mathbf{H}\mathbf{G}_1$. Also $\mathbf{G}_0 < \mathbf{G}_1$. So in proving the theorem we can replace $\mathbf{G}$ with $\mathbf{G}_1$, i.e., assume that $\mathbf{G}$ has no characters.

Now $\mathbf{G}_0$ has no compact factors. We claim that since $\mathbf{G}$ has no characters, $\mathbf{G}_0$ is cocompact in $\mathbf{G}$. To see this, write $\mathbf{G} = \mathbf{TSN}$, where $\mathbf{N}$ is the unipotent radical of $\mathbf{G}$, $\mathbf{S}$ is semisimple, and $\mathbf{T}$ is a torus centralizing $\mathbf{S}$. It is clear that the restriction map $X(\mathbf{G}) \to X(\mathbf{T})$ is an isomorphism, and therefore $\mathbf{G} = \mathbf{SN}$. Now $\mathbf{G}_0 = \mathbf{S}_0\mathbf{N}$, where $\mathbf{S}_0$ is the product of all noncompact simple factors of $\mathbf{S}$. This proves our claim.

By Corollary 2, $\mathbf{H}_0 <_e \mathbf{G}_0$ and by Proposition 5, $\mathbf{H}$ projects onto $\mathbf{G}/\mathbf{G}_0$, that is, $\mathbf{G} = \mathbf{H}\mathbf{G}_0$.

□

The following Lemma shows, loosely speaking, that if $\mathbf{H} <_e \mathbf{G}$ and we are given a finite number of $\mathbf{H}$-characters arising as subrepresentations of $\mathbf{G}$-representations, then there is a sector in $\mathbf{H}$ along which all the characters increase simultaneously (compare with 4.7 of [BB3]).

Let $\mathbf{G}$ be an $\mathbb{R}$-algebraic group generated by the $\mathbb{R}$-unipotent elements contained in G, and let $\mathbf{H}$ be an $\mathbb{R}$-split solvable algebraic subgroup with $\mathbf{H} <_e \mathbf{G}$. Let $\mathcal{C}$ denote the subset of $X(\mathbf{H})$ consisting of characters χ for which there exists a representation $\rho : \mathbf{G} \to GL(V)$ and $v \in V$ such that v is not $\rho(\mathbf{G})$- invariant and for all $h \in \mathbf{H}$, $\rho(h)v = \chi(h)v$. Write $\mathbf{H} = \mathbf{TU}$, where $\mathbf{T}$ is an $\mathbb{R}$-split torus and $\mathbf{U}$ is unipotent. Identifying $T = \mathbf{T}_{\mathbb{R}}$ with its Lie algebra, and characters with their differentials at the identity, $X(\mathbf{H}) \cong X(\mathbf{T}) = X(\mathbf{T})_{\mathbb{R}}$ is just a subgroup of T^*, the space of linear functionals on the real vector space T. We let $\langle t, \chi \rangle, t \in T, \chi \in X(\mathbf{T})$, denote the duality.

Lemma 1. *Let $\mathbf{G}, \mathbf{H}, \mathbf{T}, \mathbf{U}, \mathcal{C}$ be as above. $\mathcal{C}$ is closed under addition. For any finite $\mathcal{C}_0 \subset \mathcal{C}$, the convex hull of $\mathcal{C}_0$ does not contain zero, and there exists an open subsemigroup T^+ of T, generating T as a group, such that for all $\chi \in \mathcal{C}_0, t \in T^+, \langle t, \chi \rangle > 0$.*

Proof: For $i = 1, 2$, let $\rho_i : \mathbf{G} \to GL(V_i)$ be two representations, and $v_i \in V_i$ two vectors on which $\mathbf{H}$ acts via χ_i. Then $\mathbf{H}$ acts on $v_1 \otimes v_2$ (in the representation $\rho_1 \otimes \rho_2 : \mathbf{G} \to GL(V_1 \otimes V_2)$) via the character $\chi_1 + \chi_2$. Since $\mathbf{G}$ has no characters, $\rho_i(\mathbf{G})$ doesn't leave any of the lines $\mathbb{C}v_i$ invariant. This implies that $(\rho_1 \otimes \rho_2)(\mathbf{G})$ does not leave the line through $v_1 \otimes v_2$ invariant, and thus $v_1 \otimes v_2$ isn't $\rho(\mathbf{G})$-invariant. Therefore $\chi_1 + \chi_2 \in \mathcal{C}$.

$X(\mathbf{T})$ is a discrete cocompact subgroup of T^*, and so contains a basis (over $\mathbb{R}$) of T^* which $\mathbb{Z}$-spans $X(\mathbf{T})$.

Let $\mathcal{D}$ be the convex hull of $\mathcal{C}_0$. Suppose, by way of contradiction, that $\mathcal{D}$ contains zero. Then $0 = \sum c_\chi \chi$, where the c_χ are non-negative real numbers, not all zero, and the sum is taken over $\chi \in \mathcal{C}_0$. Let $\mathcal{C}_1 \subset \mathcal{C}_0$ be the set of χ for which $c_\chi \neq 0$.

We may assume that all the c_χ are rational numbers. Indeed, the space of all vectors $(b_\chi)_{\chi \in \mathcal{C}_1}$ such that $\sum b_\chi \chi = 0$ is a linear subspace in $\mathbb{R}^{|\mathcal{C}_1|}$, defined over $\mathbb{Q}$, and we know it contains a vector $(c_\chi)_{\chi \in \mathcal{C}_1}$ in the positive sector $\{(v_\chi) : \forall \chi \in \mathcal{C}_1, v_\chi > 0\}$. By continuity, this sector contains a vector with rational coefficients. Multiplying by a common denominator, we get that $0 = \sum a_\chi \chi$, where a_χ is a non-negative integer for all $\chi \in \mathcal{C}_0$. This means that $0 \in \mathcal{C}$, contradicting the assumption that $\mathbf{H} <_e \mathbf{G}$.

Thus $\mathcal{D}$ is a closed convex subset of $\mathbb{R}^d$ (where d is the dimension of T) not containing zero, and therefore there is some $t \in T$ such that $\langle t, \chi \rangle > 0$ for all $\chi \in \mathcal{D}$. Therefore the subsemigroup $T^+ = \{t \in T : \langle t, \chi \rangle > 0, \forall \chi \in \mathcal{C}_0\}$ is nonempty, open, and satisfies the required conditions.

□

4. Minimality

We now quote some theorems we will be using.

Theorem 6. [Mo]

Let $\mathbf{G}$ *be an algebraic* $\mathbb{R}$*-group, and suppose* $\mathbf{H}$ *and* $\mathbf{E}$ *are* $\mathbb{R}$*-subgroups with* $\mathbf{H} <_e \mathbf{E}$*. Let* Γ *be a lattice in* G*. Then any Borel probability measure on* G/Γ *which is* H*-invariant is also* E*-invariant. In particular, if* $\mathbf{H} <_e \mathbf{G}$ *then the action of* H *on* G/Γ *is uniquely ergodic.*

Remarks:

1. Combining Theorem 6 with Proposition 3, we see that if $\mathbf{H}$ and $\mathbf{G}$ are $\mathbb{Q}$-groups, satisfying the assumptions of the theorem, $\Gamma = G_{\mathbb{Z}}$ is a lattice and the action of H on G/Γ is minimal, then the action of H on G/Γ is uniquely ergodic.
2. The theorem is stated differently in [Mo]. The assumption that $\mathbf{H}$ is algebraic is not stated, though it is used. Also, Mozes assumes that E is generated by the unipotent elements contained in it. To show that the latter assumption is unnecessary, we argue as follows: Let $\mathbf{E}_0$ be the subgroup of $\mathbf{E}$ generated by $\mathbb{R}$-unipotent elements contained in it, and let $\mathbf{H}_0 = \mathbf{H} \cap \mathbf{E}_0$. By Theorem 5, $\mathbf{H}_0 <_e \mathbf{E}_0$ and $\mathbf{H}\mathbf{E}_0 = \mathbf{E}$. An H-invariant measure is obviously H_0-invariant, and by Mozes' theorem (as stated in [Mo]), is also invariant under E_0. Since $E = HE_0$, such a measure is also E-invariant.

Theorem 7. [DM] *Let* $\mathbf{G}$ *be an algebraic group defined over* $\mathbb{Q}$*, without* $\mathbb{Q}$*-characters, let* $\Gamma = \mathbf{G}_{\mathbb{Z}}$*, a lattice in* G*, and let* $\pi : G \to G/\Gamma$ *be the projection. Let* $\mathbf{G} = \mathbf{S}\mathbf{N}$ *denote the algebraic Levi decomposition of* $\mathbf{G}$*, where* $\mathbf{S}$ *is a reductive* $\mathbb{Q}$*-subgroup and* $\mathbf{N}$ *is the unipotent radical of* $\mathbf{G}$*, and let* $\tau : \mathbf{G} \to \mathbf{S}$ *be the quotient map. Let* $\Gamma_0 = \mathbf{S}_{\mathbb{Z}}$*, a lattice in* S*, with* $\pi_0 : S \to S/\Gamma_0$ *the projection (see diagram accompanying Remark 2 below).*

Then for $i = 1, \ldots, r$*, there are* $\mathbb{Q}$*-representations* $\rho_i : \mathbf{S} \to GL(V_i)$*,* $p_i \in (V_i)_{\mathbb{Q}}$*, and a finite subset* F *of* $\mathbf{S}_{\mathbb{Q}}$*, such that for any* $\epsilon > 0, \theta > 0$*, there is a compact subset* $K \subset G/\Gamma$*, such that for any one-parameter unipotent subgroup* $V = \{u_t : t \in \mathbb{R}\}$ *of* G*, and any* $x_0 \in G$*, one of the following holds:*

1. *For all* T *large enough,*
$$\frac{1}{T}|\{t \in [0, T] : u_t\pi(x_0) \in K\}| \geq 1 - \epsilon,$$
where $|\cdot|$ *denotes Lebesgue measure on* $\mathbb{R}$*.*
2. *There is some* $1 \leq i \leq r$ *and some* $\lambda \in \Gamma_0 F$ *such that* $\rho_i(\bar{x}_0\lambda)p_i$ *is a* $\rho_i(\bar{V})$*-invariant vector and* $\|\rho_i(\bar{x}_0\lambda)(p_i)\| < \theta$*, where* $\bar{V} = \tau(V)$ *and* $\bar{x}_0 = \tau(x_0)$*.*

Remarks:

1. Since all norms on the V_i are equivalent, we may choose the norm appearing in the second case above as we please.
2. Dani and Margulis actually proved Theorem 7 for G a *reductive* $\mathbb{Q}$-group without $\mathbb{Q}$-characters; however, the fiber of the map $\bar{\tau}$ induced from τ (replacing Γ, if necessary, by a commensurable lattice Γ') as in the following diagram, has compact fiber $N/N_{\mathbb{Z}}$. This shows that the theorem as stated reduces immediately to the case in which $\mathbf{G}$ is reductive.

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/\Gamma' \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\bar{\tau}} \\ S & \xrightarrow[\pi_0]{} & S/\Gamma_0 \end{array}$$

Let U be a Lie group. We say that a condition C is satisfied by *almost all one parameter subgroups of* U, if the set $\{X \in \mathfrak{U} : \{\exp(tX)\}$ satisfies $C\}$ is conull in the Lie algebra $\mathfrak{U}$ of U, with respect to the Lebesgue measure class.

Let G be a Lie group and Γ a closed subgroup. Suppose μ is a probability measure on G/Γ, $x_0 \in G/\Gamma$. We say that a one-parameter subgroup $\{g(t) : t \in \mathbb{R}\}$ is (μ, x_0) *generic* if for any $f \in C_c(G/\Gamma)$ (a continuous function with compact support on G/Γ) we have

$$\lim_{T\to\infty} 1/T \int_0^T f(g(t)x_0)dt = \int_{G/\Gamma} f d\mu.$$

We say that an element $u \in G$ is *Ad-unipotent* if its image $Ad(u) \in GL(\mathfrak{G})$ is unipotent. If $\mathbf{G}$ is an algebraic group, unipotent elements are Ad-unipotent.

Theorem 8. [R1, R2]

Let G be a connected Lie group, Γ a lattice in G, $\pi : G \to G/\Gamma$ the projection, $x_0 \in G$, and $U < G$ a connected Lie subgroup generated by Ad-unipotent elements. Then there is a U-invariant U-ergodic probability measure μ supported on the closure of $U\pi(x_0)$. If all the elements of U are Ad-unipotent then almost every one parameter subgroup of U is (x_0, μ) generic.

Remark: See [R3] for a survey of Ratner's results. The second conclusion stated here follows from [R3], Theorems 4, 5 and 6, since the one parameter subgroups not satisfying the conclusion are contained in a countable union of subgroups of U of strictly lower dimension.

We now state and prove our main theorem.

Theorem 9. *Let $\mathbf{G}$ be an algebraic $\mathbb{Q}$-group, let $\mathbf{H}$ be an $\mathbb{R}$-subgroup with $\mathbf{H} <_e \mathbf{G}$, and let $\Gamma = \mathbf{G}_{\mathbb{Z}}$. Then the action of H on G/Γ is minimal.*

Proof: We first show that we may assume that $\mathbf{G}$ has no $\mathbb{Q}$-characters. Let $\bar{\mathbf{G}} = \bigcap_{\chi \in X(\mathbf{G})} \ker(\chi)$ (a $\mathbb{Q}$-group), $\bar{\Gamma} = \Gamma \cap \bar{\mathbf{G}}, \bar{\mathbf{H}} = \mathbf{H} \cap \bar{\mathbf{G}}$. Then, by Theorem 4, $\bar{\mathbf{H}} <_e \bar{\mathbf{G}}$, and $\mathbf{G} = \mathbf{H}\bar{\mathbf{G}}$. The closure of $H\Gamma$ is an H-invariant closed set containing the closure of $\bar{H}\bar{\Gamma}$; once we have proved that the last set is $\bar{G}$, it will follow that the closure of $H\Gamma$ must be G. To prove that $Hx\Gamma$ is dense in G for any $x \in G$, replace $\mathbf{H}$ with $x^{-1}\mathbf{H}x$, and repeat the argument.

If $\mathbf{G}$ is a $\mathbb{Q}$-group without $\mathbb{Q}$-characters, it satisfies the hypotheses of Theorem 7. Let $\mathbf{G}_0$ be the subgroup of $\mathbf{G}$ generated by $\mathbb{R}$-unipotent elements contained in $\mathbf{G}$, and let $\mathbf{H}_0 = \mathbf{H} \cap \mathbf{G}_0$. By Theorem 5, $\mathbf{H}_0 <_e \mathbf{G}_0$ and $\mathbf{G} = \mathbf{H}\mathbf{G}_0$. By Theorem 2 of [BB1], $\mathbf{H}_0$ contains an $\mathbb{R}$-split solvable subgroup $\mathbf{H}_1$, with $\mathbf{H}_1 <_e \mathbf{G}_0$.

Let $x_0 \in G$. We claim that in order to prove that $H\pi(x_0)$ is dense in G/Γ, it suffices to produce a Borel probability measure supported on $\overline{H\pi(x_0)}$ (the closure of $H\pi(x_0)$ in G/Γ) which is H_1-invariant. Indeed, such a measure will be G_0-invariant by Theorem 6, and so $\overline{H\pi(x_0)}$ will contain the support of the measure, a nonempty G_0-invariant set. Since $G = HG_0$, this will imply that $\overline{H\pi(x_0)} = G/\Gamma$.

We will write $\mathbf{H}_1 = \mathbf{TU}$, with $\mathbf{T}$ a torus and $\mathbf{U}$ unipotent. Let $X = G/\Gamma \cup \{\infty\}$ be the one point compactification, and $\mathcal{P}$ the set of probability measures on X, a compact convex G-space. Let M be the closure of $H\pi(x_0)$ in X and let $\mathcal{M} \subset \mathcal{P}$ be the set of measures which are U-invariant and supported on M. We know from Theorem 8 that $\mathcal{M}$ contains the U-ergodic U-invariant measure ν supported on $\overline{U\pi(x_0)}$. We have $\nu(\infty) = 0$. Let T^+ be a subsemigroup of T, generating T as a group, and satisfying the conclusions of Lemma 1 with respect to a set of characters $\mathcal{C}_0$, which will be specified later. $\mathcal{M}_0$, the convex closure of $T^+\nu$ in $\mathcal{P}$, is a T^+ invariant compact convex set which is contained in $\mathcal{M}$. We will show that $\mathcal{M}_0$ consists of measures μ such that $\mu(\infty) = 0$. By the amenability of T^+, there will be a T^+-invariant measure in $\mathcal{M}_0$, and since T^+ generates T, this measure will be invariant under both T and U, giving us what we are after.

We need to show that for any $\epsilon > 0$, there is a compact subset $K \subset G/\Gamma$, such that for any $t \in T^+, t\nu(K) \geq 1 - \epsilon$. It suffices to show that for some $\theta > 0$ and any $t \in T^+$, there is a one-parameter unipotent subgroup $V = \{u_s : s \in \mathbb{R}\}$ which is $(t\nu, t\pi(x_0))$ generic, and does not satisfy the second condition in Theorem 7.

Assume the notation of Theorem 7. For $i = 1, \ldots, r$, let W_i denote the vector subspace of $\rho_i(U)$-invariant vectors in V_i. Since T normalizes U, the W_i are $\rho_i(T)$-invariant spaces, and therefore each W_i splits as a sum $\bigoplus_{\chi \in X(\mathbf{H})} W_\chi$, with $\rho_i(h)w = \chi(h)w$ for all $w \in W_\chi, h \in H$. Let $\mathcal{C}_0$

denote all the nontrivial characters in $X(\mathbf{H})$ appearing in these sums, and note that they satisfy the conditions of Lemma 1. On each W_i define a norm by stipulating a basis, which is the union of bases of the various W_χ, to be orthonormal, and extend this to a norm $\|\cdot\|_i$ on V_i. Then for $t \in T^+, w \in W_i$, since $d\chi(t) \geq 0$ for $t \in T^+$, we have

$$\|\rho_i(t)w\|_i \geq \|w\|_i. \tag{1}$$

Let θ be

$$\min_{\lambda \in \Gamma F} \|\rho_i(x_0\lambda)p_i\|_i.$$

This minimum exists because the set $\{\rho_i(\lambda)p_i : \lambda \in \Gamma F\}$ is contained in a discrete subset of $(V_i)_\mathbb{Q}$ with 'bounded denominators' , as in the argument of Proposition 2.

Let $t \in T^+$ be given. Since $t\nu$ is the U-ergodic U-invariant measure on the closure of $Ut\pi(x_0)$, we know that almost every one parameter subgroup of U is $(t\nu, t\pi(x_0))$ generic. For $i = 1, \dots, r, \lambda \in \Gamma F$, let $U(i, \lambda)$ be the subset of $X \in \mathfrak{U}$ for which:

(*) if the one parameter subgroup $\{\exp(tX)\}$ is contained in the stabilizer of $\rho_i(tx_0\lambda)(p_i)$, then so is the whole group U.

Since the complement of $U(i, \lambda)$ is a subvariety of U of strictly lower dimesion, the set $\bigcap_{i=1,\dots,r,\lambda\in\Gamma F} U(i,\lambda)$ is conull, and therefore there are one parameter subgroups which are both $(t\nu, t\pi(x_0))$ generic and satisfy condition (*) for each $1 \leq i \leq r$ and each $\lambda \in \Gamma F$. If $V = \{u_s : s \in \mathbb{R}\}$ is one such subgroup, then the second condition in Theorem 7 cannot be satisfied. If it were, for some $1 \leq i \leq r, \lambda \in \Gamma F$, then V would be contained in the stabilizer of $\rho_i(tx_0\lambda)(p_i)$, therefore so would U, and we would have $\rho_i(tx_0\lambda)(p_i) \in W_i$. However, $\|\rho_i(tx_0\lambda)(p_i)\|_i < \theta \leq \|\rho_i(x_0\lambda)(p_i)\|_i$, contradicting (1).

□

Let $\mathbf{G}$ be a semisimple $\mathbb{R}$-group. Recall that a lattice Γ in G is called *arithmetic* if there is a semisimple $\mathbb{Q}$-group $\mathbf{G}_0$ and a homomorphism $\pi : G_0 \to G$ with compact kernel, such that Γ and a conjugate of $\pi(\mathbf{G}_\mathbb{Z})$ are commensurable.

Corollary 4. *Let* $\mathbf{G}$ *be a semisimple algebraic* $\mathbb{R}$*-group, and let* $\mathbf{H}$ *be an* $\mathbb{R}$*-subgroup with* $\mathbf{H} <_e \mathbf{G}$. *Let* Γ *be an arithmetic lattice in* G. *Then the action of* H *on* G/Γ *is minimal.*

Proof: Let the notation be as above. $\mathbf{G}$ can be considered a subgroup of $\mathbf{G}_0$, and π can be extended to a homomorphism $\bar{\pi} : \mathbf{G}_0 \to \mathbf{G}$. Let $K = \ker \pi$ and $\mathbf{K} = \ker(\bar{\pi})$. Then $\mathbf{G}_0$ is generated by $\mathbf{K}$ and $\mathbf{G}$ and therefore $\mathbf{HK} <_e \mathbf{G}_0$. By the previous theorem, the action of HK on $G_0/(G_0)_\mathbb{Z}$, which is the lift of the action of H on G/Γ (see diagram), is minimal.

$$\begin{array}{c} HK \circlearrowleft G_0/(G_0)_{\mathbb{Z}} \\ \big\downarrow \\ H \circlearrowleft G/\Gamma \end{array}$$

Therefore the action of H on G/Γ is also minimal.

$\square$

5. Applications

In this section, we let $\mathbf{H} \overset{\mathbb{Q}}{<} \mathbf{G}, \Gamma = \mathbf{G}_{\mathbb{Z}}, \pi : G \to G/\Gamma$ the quotient map, and $e \in \mathbf{G}$ the identity element.

5.1. Closure of Rational Orbits.

Definition 3. *The orbit $H\pi(x)$ is called a* rational orbit *if $x \in \mathbf{G}_{\mathbb{Q}}$.*

Corollary 5. *Let $\mathbf{H}\pi(x)$ be a rational orbit. Let $\mathbf{E}$ be the observable envelope of $\mathbf{H}$ in $\mathbf{G}$. Then the closure of the orbit $H\pi(x)$ is $E\pi(x)$. In particular, the closure of a rational orbit is a submanifold of G/Γ.*

Proof: By conjugating we may assume that x is the identity in $\mathbf{G}$. $\mathbf{E}$ is a $\mathbb{Q}$-group, so by Proposition 2 the orbit $E\pi(x)$ is closed, and therefore the orbit map $\phi_2 : E/E_{\mathbb{Z}} \to G/\Gamma$ is a proper map. So it remains to show that the image of the map $\phi_1 : H/H_{\mathbb{Z}} \to E/E_{\mathbb{Z}}$ is dense (see diagram).

$$\begin{array}{ccccc} H & \longrightarrow & E & \longrightarrow & G \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ H/H_{\mathbb{Z}} & \xrightarrow[\phi_1]{} & E/E_{\mathbb{Z}} & \xrightarrow[\phi_2]{} & G/\Gamma \end{array}$$

This results immediately from the fact that $\mathbf{H} <_e \mathbf{E}$ and Theorem 9.

$\square$

From Theorem 9 and Proposition 3 we conclude also:

Corollary 6. *If any rational orbit is dense, then all orbits are dense.*

5.2. Necessary and Sufficient Conditions.

Corollary 7.

1. $\mathbf{H} <_e \mathbf{G}$ *if and only if the action of H on G/Γ is minimal.*
2. $\mathbf{H} <_o \mathbf{G}$ *if and only if the orbit $H\pi(e)$ is closed.*

Proof:

1. Proposition 3 and Theorem 9.

2. The 'only if' part is Proposition 2.

   Conversely, if **H** is not observable in **G**, then the dimension of the observable envelope **E** of **H** in **G** is strictly greater than the dimension of **H**, and by Corollary 5, the closure of $H\pi(e)$ is $E\pi(e)$.

□

6. Raghunathan's Example

Here is an example due to Raghunathan, which was mentioned by Ratner in [R3] but never published. The author thanks Hee Oh for help in understanding the example. Let $G = SL(3,\mathbb{R}), \Gamma = SL(3,\mathbb{Z}), \pi : G \to G/\Gamma$ the projection, and let $H = TU$, where U is the subgroup of unipotent upper triangular matrices in G and T is the one parameter subgroup $T(t) = diag(\exp(t), \exp(-\alpha t), \exp((\alpha-1)t))$, with α an irrational number greater than one.

T is a Lie group normalizing U, but since α is irrational, it is not algebraic. In fact it is Zariski dense in the diagonal matrices of G, and therefore H is Zariski dense in the upper triangular matrices in G - a Borel subgroup; this shows that $H <_e G$. Notice that H doesn't satisfy the conclusions of Lemma 1.

We will show that the action of H on G/Γ is not minimal by showing that the orbit $H\pi(e)$ is closed. Since the orbit $U\pi(e)$ is compact, it suffices to show that for any $u \in U$, the sequence $T(t_n)u\pi(e)$ can only converge in G/Γ if t_n is bounded. This can be seen by applying Mahler's criterion: Let $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ denote the standard basis of $\mathbb{R}^3$. As $t_n \to -\infty$, the vector $T(t_n)u\mathbf{e_1}$ tends to zero in $\mathbb{R}^3$, and therefore t_n must be bounded below. The two-dimensional plane spanned by $\mathbf{e_1}$ and $\mathbf{e_2}$ is invariant under H, and as $t_n \to \infty$, the volume element on this subspace tends to zero. Therefore the subspace contains integer vectors whose length goes to zero, and t_n is bounded above.

Although H is not algebraic, Theorem 6 applies to the action of H. This is because in the proof given in [Mo], the fact that H is algebraic is used only to reduce to the case that H is $\mathbb{R}$-split solvable (via a Theorem of Bien and Borel), and in this case H is already $\mathbb{R}$-split solvable. Therefore the action of H on G/Γ is uniquely ergodic, and we obtain an example of a subgroup action which is uniquely ergodic but not minimal.

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