# CHARACTERIZATIONS OF LATTICE SURFACES

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ABSTRACT. We answer a question of Vorobets by showing that the lattice property for flat surfaces is equivalent to the existence of a positive lower bound for the areas of affine triangles. We show that the set of affine equivalence classes of lattice surfaces with a fixed positive lower bound for the areas of triangles is finite and we obtain explicit bounds on its cardinality. We deduce several other characterizations of the lattice property.

## 1. INTRODUCTION

Our objects of study are translation and half-translation surfaces and their affine automorphism groups. These structures arise in the study of rational polygonal billiards. They also arise in Thurston's classification of surface diffeomorphisms in connection with measured foliations. Isomorphic structures arise in complex analysis where they are called respectively abelian differentials and quadratic differentials. We will use the term flat surface for both translation and half translation surfaces when it is not important to distinguish between the two types. For more details on flat surfaces see [Vo, MaTa, Zo].

Let Aff($M$) denote the affine automorphism group of a flat surface $M$. For a typical flat surface this group is trivial; however surfaces with non-trivial automorphism groups are quite interesting. Taking the differential of the automorphism yields a homomorphism $D : \text{Aff}(M) \to G$ with finite kernel, where $G$ is either $\text{SL}(2, \mathbb{R})$ or $\text{PSL}(2, \mathbb{R})$ depending on whether $M$ is a translation or half-translation surface. The image $\Gamma_M$ of this homomorphism is called the *Veech group* of $M$. We say that $M$ is a *lattice surface* if $\Gamma_M$ is a lattice, i.e. has finite covolume in $G$. In a celebrated paper [Ve1], Veech constructed a family of lattice surfaces, and showed that lattice surfaces have striking dynamical properties, in particular they satisfy the 'Veech dichotomy' which will be discussed below. Other examples have been constructed by several authors and classifications of lattice surfaces are known in special cases (see [GuJu, KeSm, Pu, Cal, McM]), but an overall classification does not yet exist.

Vorobets [Vo] found a connection between the lattice property and the collection of areas of triangles in the surface. A flat surface is equipped with a finite set of distinguished points $\Sigma = \Sigma_M$ which contains all cone points (those with cone angle not equal to $2\pi$) and may contain other points. We always assume $\Sigma \neq \varnothing$, and that elements of $\mathrm{Aff}(M)$ preserve $\Sigma$. A *triangle in M* is the image of an affine map from a triangle in the plane to $M$, which takes the vertices of the triangle to points in $\Sigma$, is injective on the interior of the triangle, and such that interior points do not map to $\Sigma$. In particular the vertices of the triangle need not be distinct. The collection of areas of triangles will be denoted by $\mathcal{T}(M)$. Vorobets proved that $M$ is a lattice surface if and only if $\mathcal{T}(M)$ is finite. Say that *M has no small triangles* if $\inf \mathcal{T}(M) > 0$. Vorobets also showed that if $M$ has no small triangles then it satisfies the Veech dichotomy, and raised the question of whether the no small triangles property is equivalent to the lattice property.

**Theorem 1.1.** *A flat surface has the lattice property if and only if it has no small triangles.*

Let us consider only surfaces $M$ which have total area 1. For $\alpha > 0$, let
$$\mathrm{NST}(\alpha) = \{M : \inf \mathcal{T}(M) \geq \alpha\}.$$
Say that $M$ and $M'$ are *affinely equivalent* if there is an affine homeomorphism between them; note that for surfaces of area 1, $\mathcal{T}(M)$ depends only on the affine equivalence class of $M$.

**Theorem 1.2.** *For any $\alpha > 0$, $\mathrm{NST}(\alpha)$ contains a finite number of affine equivalence classes.*

See Proposition 7.2 for an explicit bound on this number, and see Proposition 7.3 for a bound on the sum, over all $M \in \mathrm{NST}(\alpha)$, of the co-areas of $\Gamma_M$.

Besides Theorem 1.1, our results yield several different characterizations of lattice surfaces, which we collect in Theorem 1.3 below. To state them we introduce some terminology. A *saddle connection* on $M$ is a straight segment on $M$ connecting points of $\Sigma$ (which need not be distinct), with no points of $\Sigma$ in its interior. Let $\mathcal{L} = \mathcal{L}_M$ denote the set of all saddle connections on $M$, and let $\mathcal{L}(\theta) = \mathcal{L}_M(\theta)$ denote the set of saddle connections in direction $\theta$. We say that $\theta$ is a *periodic direction* if each connected component of $M \smallsetminus \mathcal{L}(\theta)$ is a cylinder with a waist curve in direction $\theta$. Veech [Ve1] showed that a lattice surface satisfies the following dichotomy:

I. If $\mathcal{L}(\theta) \neq \varnothing$ then $\theta$ is a periodic direction.

II. If $\mathcal{L}(\theta) = \varnothing$ then $\mathcal{F}_\theta$ is uniquely ergodic.

This dichotomy does not characterize the lattice property (see [SmWe1]), but a modification of property I introduced by Vorobets does. Namely, say that $M$ is *uniformly completely periodic* if there is $s > 0$ such that each $\theta$ for which $\mathcal{L}(\theta) \neq \varnothing$ is a periodic direction, and the ratio of lengths of any two segments in $\mathcal{L}(\theta)$ does not exceed $s$. We will show that this property characterizes the lattice property. Moreover, for a lattice surface in $\mathrm{NST}(\alpha)$ we will provide an effective estimate for $s$ in terms of $\alpha$, see Proposition 3.2.

A periodic direction $\theta$ is called *parabolic* if the moduli of all the cylinders are commensurable, and $M$ is called *uniformly completely parabolic* if it is uniformly completely periodic, with all periodic directions parabolic.

Another characterization involves the set of holonomy vectors of saddle connections. Associated with $\delta \in \mathcal{L}$ is a vector $\mathrm{hol}(\delta)$ in the plane of the same length and direction as $\delta$. We set

$$\mathrm{hol}(M) = \{\mathrm{hol}(\delta) : \delta \in \mathcal{L}\}.$$

This is a discrete $\Gamma_M$-invariant subset of the plane which has attracted considerable attention. For $\beta > 0$, we let

$$\mathrm{NSVT}(\beta) = \{M : \inf\{|v_1 \wedge v_2| : v_i \in \mathrm{hol}(M), v_1 \wedge v_2 \neq 0\} \geq \beta\},$$

and say that $M$ has *no small virtual triangles* if it belongs to some $\mathrm{NSVT}(\beta)$. We will show that having no small virtual triangles is also equivalent to the lattice property.

Two other characterizations involve the dynamics of the $G$-action on the space of flat surfaces. Associated with $M$ is the topological data consisting of the underlying surface, the number of points in $\Sigma$ and the associated cone angles, and whether or not the corresponding foliations are orientable. The set of all $M$ sharing this data is a noncompact orbifold called a *stratum*. The restriction of the action of $G$ to the subgroup $\{g_t\}$, where

$$g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

is called the *geodesic flow*.

**Theorem 1.3.** *The following are equivalent for a flat surface $M$:*
- (i) *$M$ is a lattice surface.*
- (ii) *$M$ is uniformly completely periodic.*
- (iii) *$M$ is uniformly completely parabolic.*
- (iv) *$|\mathcal{T}(M)| < \infty$.*

(v) *The set of triangles for $M$ consists of finitely many $\mathrm{Aff}(M)$-orbits.*

(vi) *$M$ has no small triangles.*

(vii) *$\{u \wedge v : u, v \in \mathrm{hol}(M)\}$ is a discrete set of numbers.*

(viii) *For any $T > 0$, the set*

$$\{(\xi, \eta) \in \mathcal{L}_M \times \mathcal{L}_M : |\mathrm{hol}(\xi) \wedge \mathrm{hol}(\eta)| < T\}$$

*contains finitely many $\mathrm{Aff}(M)$-orbits.*

(ix) *$M$ has no small virtual triangles.*

(x) *The $G$-orbit of $M$ is closed.*

(xi) *There is a compact subset $K$ of the stratum containing $M$ such that for any $g \in G$, the geodesic orbit of $gM$ intersects $K$.*

Many of these implications are due to Vorobets [Vo, §6]. The main new implication is (vi) $\implies$ (i). Our first proof of this implication is sketched in [SmWe2].

The proofs of Theorems 1.1 and 1.2 are similar. Vorobets showed that if $M$ has no small triangles any saddle connection direction is parabolic, so associated with two non-parallel saddle connections are a pair of cylinder decompositions of $M$, each with cylinders of commensurable moduli. Moreover a bound on triangle areas leads to an upper bound on certain combinatorial data associated with two cylinder decompositions. Arguments of Thurston and Veech show that such combinatorial data determine a flat surface up to affine equivalence. This immediately yields Theorem 1.2.

To derive Theorem 1.1 we introduce the *spine* $\Pi$ of a translation surface $M$; this is a $\Gamma_M$-invariant tree in $\mathbb{H}$, whose edges are labeled by two saddle connections on $M$, and correspond to the surfaces affinely equivalent to $M$ for which these saddle connections are simultaneously shortest. There is a retraction $\rho : \mathbb{H} \to \Pi$ taking a surface in which all shortest saddle connections are parallel, to an affinely equivalent surface which has shortest saddle connections in two or more directions. For each edge $e$ of $\Pi$, $\rho^{-1}(e)$ is a finite-area domain in $\mathbb{H}$, and $e$ is associated with a pair of cylinder decompositions. A bound on triangle areas bounds the combinatorial data and thus bounds the number of edges in $\Pi/\Gamma_M$, giving a bound on the area of $\mathbb{H}/\Gamma_M$.

Theorems 1.1 and 1.2 suggest a natural ordering on the set of lattice surfaces. For a lattice surface $M$, let $\alpha(M)$ denote the largest $\alpha$ for which $M \in \mathrm{NST}(\alpha)$. Then all (affine equivalence classes of) lattice surfaces may be written as

$$M_1, M_2, \ldots, \quad \text{with} \quad \alpha(M_1) \geq \alpha(M_2) \geq \cdots, \tag{1}$$

and a similar ordering exists with the NSVT condition instead of the NST condition.

It would be desirable to have an algorithm which, given $\alpha > 0$, lists all surfaces in NST($\alpha$) or NSVT($\alpha$). Our analysis yields an explicit finite set of surfaces, presented in terms of Thurston-Veech combinatorial data, which contains all the above lattice surfaces, and implicitly, an algorithm for distinguishing the lattice from the non-lattice surfaces. This is our motivation for providing effective proofs and explicit estimates for the cardinality of certain finite sets. We suspect our explicit estimates are far from optimal and believe additional theoretical work would be required in order to create an algorithm which is practically feasible.

In the interest of presenting the simplest formulae, our estimates may be presented in terms of either triangle or virtual triangle areas. We do not present estimates involving the genus of $M$ or the stratum containing $M$, both because these are harder to obtain, and because the same effective search will produce the lattice surfaces in all genera.

The relation between the no small triangles and no small virtual triangles properties is described in the following:

**Theorem 1.4.** NSVT($\alpha/2$) $\subset$ NST($\alpha$) $\subset$ NSVT($2\alpha e^{-1/(2\alpha e)}$).

Without conducting computer searches, we are able to determine $M_1, \ldots, M_7$. They are all arithmetic, and comprise NST $\left(\frac{1}{6}\right)$.

## 2. BASICS

We review some definitions here. For more details we refer the reader to [MaTa, Vo, Zo]. Throughout this paper, $M$ denotes a connected oriented surface with a flat structure. When there is no danger of confusion, we will also denote by $M$ the underlying topological surface. We denote the genus of $M$ by $g$. A half-translation structure may be thought of as an equivalence class of atlases of charts $(U_\alpha, \varphi_\alpha)$ covering all but a non-empty finite set $\Sigma = \Sigma_M$, such that the transition functions $\mathbb{R}^2 \to \mathbb{R}^2$ are of the form $\vec{x} \mapsto \pm\vec{x} + \vec{c}$, and such that around each $\sigma \in \Sigma$ the charts glue together to form a cone point with cone angle $2\pi(r + 1)$, where $r = r_\sigma \in \left\{-\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$. A translation structure is similarly defined, with the requirement that the transition functions

are of the form $\vec{x} \mapsto \vec{x} + \vec{c}$. Points in $\Sigma$ are called *singularities*. Several authors permit singularities $\sigma$ for which $r_\sigma = 0$; such points are called *removable singularities* or *marked points*. Except in §9, we will always assume that our singularities are not removable.

An orientation-preserving homeomorphism $\varphi : M_1 \to M_2$ which is affine in each chart is called an *affine isomorphism*, or an *affine automorphism* if $M_1 = M_2$. We denote the set of affine automorphisms of $M$ by $\mathrm{Aff}(M)$. An affine isomorphism whose linear part is $\pm\mathrm{Id}$ is called an *translation equivalence*. The map $D : \mathrm{Aff}(M) \to G$ which assigns to $\varphi$ its linear part has a finite kernel, consisting of translation equivalences of $M$. Two atlases are considered equivalent if there is a translation equivalence betweeen them.

There is a standard *orientation double cover* construction which associates to each half-translation surface $M$ a translation surface $M'$ with a branched degree two translation cover $M' \to M$; thus many statements about flat surfaces can be reduced to statements about translation surfaces.

A standard analogue of the Gauss-Bonnet formula (see [Vo]) says that

$$\sum r_\sigma = 2g - 2. \tag{2}$$

Summing the total angle at all the singularities we obtain the number

$$\tau = \tau(M) = \sum_\sigma 2\pi(r_\sigma + 1) = 2\pi \left(2g - 2 + |\Sigma|\right). \tag{3}$$

Since each triangle in a triangulation of $M$ with vertices in $\Sigma$ contributes a total angle of $\pi$, the number of triangles in such a triangulation is exactly $\tau/\pi$.

A flat surface inherits a Euclidean area form from the plane and we normalize our surfaces by assuming that each surface has unit area. An affine automorphism of $M$ is a self-homeomorphism which is affine in each chart.

For a flat surface $M$ let $\vec{r}_M = (r_\sigma)_{\sigma \in \Sigma_M}$. The set of all translation equivalence classes of (half-) translation surfaces of a given combinatorial type, namely those for which the data $\vec{r}_M$ is fixed, is called a *stratum*. Each stratum is equipped with a structure of an affine orbifold, which is locally modeled on $H^1(M, \Sigma; \mathbb{R}^2)$ (in the case of translation surfaces), or an appropriate subspace of the $H^1(\widetilde{M}, \widetilde{\Sigma}; \mathbb{R}^2)$ (in the case of half-translation surfaces, where $\widetilde{M}, \widetilde{\Sigma}$ are the orientation double cover of $M, \Sigma$).

There is an action of $G$ on $\mathcal{H}$ by post-composition on each chart in an atlas. It follows from the above that $\Gamma_M = \{g \in G : gM = M\}$. We

define

$$
r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \qquad \widetilde{h}_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},
$$

which we view as one-parameter subgroups of $G$.

By a *direction* we mean an element of $P(\mathbb{R}^2)$ (resp. $S^1$) in the case of half-translation (resp. translation) surfaces. There is a natural action of $G$ on the set of directions.

A *cylinder* for $M$ is a topological annulus on the surface which is isometric to $\mathbb{R}/c\mathbb{Z} \times (0, h)$, where $c$ is the *circumference* of the cylinder and $h$ is its *height*. We say that the cylinder is *horizontal (vertical)* if the angle between the horizontal direction and the direction of the waist curve is 0 (resp. $\pi/2$). Note that if the cylinder is vertical then the height actually measures its horizontal width. A cylinder is *maximal* if it is not contained in a larger cylinder, and this implies that both of its boundary components contain singularities. We define $\mathcal{L}(\theta)$ to be the collection of closed saddle connections in direction $\theta$. Recall that if $\theta$ is a periodic direction then $M$ has a *cylinder decomposition in direction $\theta$* i.e., the complement of $\mathcal{L}(\theta)$ is a union of cylinders. An automorphism $\varphi \in \mathrm{Aff}(M)$ is called *parabolic* if $D\varphi \in \Gamma_M$ is a parabolic matrix. The *inverse modulus* of a cylinder as above is $\mu = w/h$. We say that real numbers $\mu_1, \ldots, \mu_k \in \mathbb{R}$ are *commensurable* if $\mu_i/\mu_j \in \mathbb{Q}$ for all $i, j \in \{1, \ldots, k\}$. If this holds, we denote by $\mu = \mathrm{LCM}(\mu_1, \ldots, \mu_k)$ the smallest positive number which is an integer multiple of all the $\mu_i$, and we call $(n_1, \ldots, n_k)$, where $n_i = \mu/\mu_i$, the *Dehn twist vector* corresponding to the cylinder decomposition. Note that by definition $\gcd(n_1, \ldots, n_k) = 1$. This terminology is motivated by the fact if $\varphi \in \mathrm{Aff}(M)$ is such that $D\varphi$ is parabolic and fixes the direction $\theta$, then there is a cylinder decomposition in direction $\theta$ with the inverse moduli of the cylinders commensurable. Conversely, given a decomposition of $M$ into cylinders for which the $\mu_i$ are commensurable, one can construct an associated parabolic affine automorphism. We say that $\varphi$ is *simple* if $\varphi$ fixes the saddle connections $\mathcal{L}(\theta)$ pointwise. This implies that each cylinder is taken to itself and that the map on each cylinder is topologically a certain number of Dehn twists. Let $n_1, \ldots, n_k$ be a collection of natural numbers and let $\mu \neq 0$ satisfy the equation $n_j \mu_j = \mu$. Then there is a simple parabolic automorphism $\varphi$ with derivative $D\varphi = r_\theta h_\mu r_{-\theta}$ which induces $n_j$ Dehn twists in the cylinder $C_j$.

## 3. Cylinder decompositions

Vorobets showed that cylinder decompositions arise in connection with the no small triangles condition:

**Proposition 3.1** (Vorobets). *If $M$ has no small triangles then any direction $\theta$ for which $\mathcal{L}(\theta) \neq \varnothing$ is parabolic.*

In addition Vorobets obtained bounds on the ratios of lengths of saddle connections in $\mathcal{L}(\theta)$ and the Dehn twist numbers $n_j$. We will review Vorobets' arguments here and use them to relate the no small triangles and the no small virtual triangle conditions. Where possible we will improve the constants that arise.

**Proposition 3.2.** *If $M \in \mathrm{NST}(\alpha)$ then $M$ is $s$-uniformly periodic for $s = \min \left\{ e^{1/(2\alpha e)}, (2(r-1)\alpha)^{1-r} \right\}$, where $r$ is the maximal number of cylinders in a cylinder decomposition on $M$.*

*Proof.* Suppose $\theta$ is a saddle connection direction for $M \in \mathrm{NST}(\alpha)$. We know from Proposition 3.1 that $\theta$ is a periodic direction on $M$. Let $\sigma, \sigma' \in \mathcal{L}_M(\theta)$. Suppose first that $\sigma, \sigma'$ are on the boundary of the same cylinder $C$, of area $A$. Let $h$ be the height of $C$, and let $x, x'$ be the lengths of $\sigma, \sigma'$. Then $xh \leq A$ since the circumference of $C$ is at least $x$, and $x'h \geq 2\alpha$ since $C$ contains a triangle of area $x'h/2$. This implies that

$$\frac{x}{x'} = \frac{xh}{x'h} \leq \frac{A}{2\alpha}. \tag{4}$$

Now suppose $\sigma, \sigma'$ are not on the boundary of the same cylinder. Since $M$ is connected, there is a chain of cylinders $C_1, \ldots C_j$ such that $\sigma = \sigma_1$ is on the boundary of $C_1$ and $\sigma' = \sigma_{j+1}$ is on the boundary of $C_j$, and $C_i$ and $C_{i+1}$ meet along a segment $\sigma_i$ of length $x_i$ for $i = 1, \ldots, j$. Applying (4) $j$ times we have

$$\frac{x}{x'} = \prod_{i=1}^{j} \frac{x_i}{x_{i+1}} \leq \prod_{i=1}^{j} \frac{A_i}{2\alpha}. \tag{5}$$

Since $M$ has area 1 and the cylinders $C_i$ are disjoint,

$$\sum A_i \leq 1 \tag{6}$$

The maximum of the right hand side of (5) subject to the constraint (6) occurs when when $A_i = 1/j$, so that

$$\frac{x}{x'} \leq (2j\alpha)^{-j}.$$

The maximum of $(2j\alpha)^{-j}$ as a function of $j$ occurs when $j = 1/(2\alpha e)$. If $1/(2\alpha e) \geq r - 1$ then the maximum value of $2j\alpha^{-j}$ occurs when

$j = r - 1$. In this case the maximum value of $2j\alpha^{-j}$ is $(2(r-1)\alpha)^{1-r}$. If $1/(2\alpha e) \leq r - 1$ then the maximum value of $(2j\alpha)^{-j}$ occurs when $j = r - 1$. In this case the maximum value of $(2j\alpha)^{-j}$ is $e^{1/(2\alpha e)}$. $\qquad\square$

We deduce the following statement, which implies Theorem 1.4.

**Proposition 3.3.** *If $M \in \mathrm{NST}(\alpha)$ and the number of cylinders in a decomposition of $M$ is at most $r$, then $M \in \mathrm{NSVT}(\beta)$ where*

$$\beta = 2\alpha \max \left\{ e^{-1/(2\alpha e)}, (2(r-1)\alpha)^{r-1} \right\}.$$

*Proof.* Let $M \in \mathrm{NST}(\alpha)$. Let $\xi$ and $\eta$ be nonparallel saddle connections on $M$. By applying an element of $G$ we may assume that $\xi$ is horizontal and $\eta$ vertical. Since $M \in \mathrm{NST}(\alpha)$ there is a decomposition of $M$ into horizontal cylinders, so there is a saddle connection $\xi'$ parallel to $\xi$ making an angle of $\pi/2$ with $\eta$ at one of the endpoints of $\eta$. By Proposition 3.2 we have $x' \leq sx$, where $x, x'$ are the lengths of $\xi, \xi'$ and $s = \min \left\{ e^{1/(2\alpha e)}, (2(r-1)\alpha)^{1-r} \right\}$.

Consider the right triangle $\Delta$ in the plane with horizontal and vertical sides the same length as $\xi', \eta$. If there is an isometric copy of $\Delta$ embedded in $M$ under an affine mapping sending its edges to $\xi', \eta$, we would have $|\xi' \wedge \eta|/2 = \mathrm{area}\,\Delta \geq \alpha$. If there is no such isometric copy of $\Delta$ then $M$ contains an isometric copy of a triangle contained in $\Delta$, and again $|\xi' \wedge \eta| \geq 2\alpha$. Therefore

$$|\xi \wedge \eta| \geq \frac{1}{s}|\xi' \wedge \eta| \geq \frac{2\alpha}{s} = \beta.$$

$\qquad\square$

**Remark 3.4.** *When applying Propositions 3.2, 3.3 it is useful to have a bound for the maximal number of cylinders in a cylinder decomposition on $M$. A standard bound of $2g+|\Sigma|-2$ can be found in e.g. [KeMaSm, proof of Lemma, p. 302]. A better bound is $g + i + \left\lfloor \frac{j}{2} \right\rfloor - 1$, where $g$ is the genus of $M$, and $i$ (resp. $j$) is the number of even (resp. odd)-angled singularities of $M$. A proof of this bound, due to the first-named author, is given in [Na] for the case of translation surfaces; the same proof works for half-translation surfaces.*

For the cylinder decompositions that arise via Proposition 3.1, Vorobets bounded the number and size of the Dehn twist numbers in terms of the lower bound for areas of triangles. We give our version here.

**Proposition 3.5.** *The number of Dehn twist vectors for a cylinder decomposition of a surface in $\mathrm{NST}(\alpha)$ into $r$ cylinders is bounded above by:*

$$(8\alpha)^{2(1-r)}(1 - 2\log 8\alpha)^{r-1}.$$

*Proof.* Let $M = C_1 \cup \cdots \cup C_r$ be a cylinder decomposition, where the cylinders have inverse moduli $\mu_1, \ldots, \mu_r$. We can recover the Dehn twist vector $\vec{n}$ from the numbers $\mu_i/\mu_j$.

We first recall Vorobets' argument which gives a restriction on the ratio of moduli of neighboring cylinders. Say that we have a pair of cylinders $C_i$ and $C_j$ with an edge in common. Let $h$ and $c$ be the height and circumference of a cylinder and let $A$ be its area. As shown in [Vo, proof of Prop. 6.2], applying the $G$-action and Dehn twists in the cylinders one can find a triangle in $C_i \cup C_j$ with base $h_i$ and height in $w_0 + \mathbb{Z}\mu_i h_j + \mathbb{Z}w_j = w_j \left( \dfrac{\mu_i}{\mu_j}\mathbb{Z} + \mathbb{Z} \right)$ for some $w_0$. Since for $\mu_i/\mu_j = p/q$ we have that $\mathbb{Z} + \dfrac{\mu_i}{\mu_j}\mathbb{Z}$ is $1/2q$-dense in $\mathbb{R}$, we have by our hypothesis

$$q \leq \frac{h_i w_j}{4\alpha}.$$

If we interchange the roles of the cylinders we get $p \leq h_j w_i/(4\alpha)$, and multiplying the equations gives

$$pq \leq \frac{h_i w_i h_j w_j}{16\alpha^2} = \frac{A_i A_j}{16\alpha^2}. \tag{7}$$

The maximum of the right hand side of (7) under the constraint $A_i + A_j \leq 1$ is attained when $A_i = A_j = 1/2$ so we get

$$pq \leq \frac{1}{64\alpha^2}.$$

To count the possible values we note that the number of pairs of natural numbers $(x, y)$ that satisfy $xy \leq K$ is bounded by the area under the graph of $y = \min(K, \log x)$ from $x = 0$ to $x = K$ and this area is $K(1 + \log K)$. Thus the number of possible ratios of moduli of neighboring cylinders is bounded by $\dfrac{1}{64\alpha^2}\left( 1 + \log\dfrac{1}{64\alpha^2} \right)$.

The connections between cylinders can be recorded in a graph whose vertices are cylinders and whose edges correspond to cylinders which share a segment. Since $M$ is connected this graph is connected. In order to recover $\vec{n}$ it suffices to know the ratios of moduli for a collection of edges that span a maximal tree in this graph. The number of edges in a maximal tree is $r - 1$, and we obtain our bound of

$$\prod_{edges} \frac{1}{64\alpha^2}\left( 1 + \log\frac{1}{64\alpha^2} \right) = (8\alpha)^{2(1-r)}(1 - 2\log 8\alpha)^{r-1}.$$

$\square$

We will also need a similar bound in terms of $\beta$, when $M \in \mathrm{NSVT}(\beta)$. Note that this bound is independent of $r$.

**Proposition 3.6.** *If $M \in \mathrm{NSVT}(\beta)$ and $\vec{n} = (n_1, \ldots, n_r)$ is a Dehn twist vector for a parabolic direction on $M$ with a decomposition into $r$ cylinders, then for any $i, j$:*

$$pq \leq \frac{1}{64\beta^2}, \qquad \text{where} \quad \frac{n_i}{n_j} = \frac{p}{q} \quad \text{and} \quad \gcd(p, q) = 1. \qquad (8)$$

*Proof.* By renumbering assume $i = 1$, $j = 2$. Let $C_1, C_2$ be the corresponding cylinders and let $\delta_1, \delta_2$ be saddle connections passing across the cylinders, so that $\mathrm{hol}(\delta_i) = (x_i, h_i)$, where $|h_i|$ is the height of $C_i$ and $|x_i|$ is no greater than the circumference $c_i$ of $C_i$. By applying Dehn twists in $C_1, C_2$ and the $G$-action, we find an affinely equivalent surface for which $\mathrm{hol}(\delta_1) = (0, h_1)$ and $\mathrm{hol}(\delta_2) = (x, h_2)$ with $|x| \leq c_1/2q$. This implies

$$\beta \leq |\mathrm{hol}(\delta_1) \wedge \mathrm{hol}(\delta_2)| = |xh_2| \leq \frac{c_1 h_2}{2q}.$$

Interchanging the roles of $C_1, C_2$ we obtain $\beta \leq \frac{c_2 h_1}{p}$ so that

$$pq \leq \frac{A_1 A_2}{4\beta^2}, \qquad (9)$$

where $A_i$ is the area of $C_i$. The maximum of the right hand side of (9) subject to the requirement $A_1 + A_2 \leq 1$ is obtained when $A_1 = A_2 = 1/2$, and we obtain (8). $\square$

## 4. The spine and a criterion for finite covolume

Given a flat surface $M$ of area 1 we can consider the collection of all area 1 surfaces $M'$ affinely equivalent to $M$. An affine equivalence $f : M \to M'$ determines an element $g = Df \in G$, whose coset $g\Gamma_M$ in $G/\Gamma_M$ depends only on $M'$. This establishes an identification of $G/\Gamma_M$ with its affine equivalence class. The Veech group $\Gamma = \Gamma_M$ is the isotropy group of $M$ so we can identify this family with $G/\Gamma$. Identifying two flat surfaces which are isometric, we find that the collection of isometry classes of surfaces affinely equivalent to $M$ can be identified with $\mathrm{PSO}(2, \mathbb{R}) \backslash G/\Gamma$ or $\mathbb{H}/\Gamma$ where $\mathbb{H}$ is the hyperbolic plane. The image of this quotient in moduli space is also called the Teichmüller disk associated to $M$.

We define $\lambda(M)$ to be the infimum of lengths of saddle connections in $M$. Since the collection of lengths of saddle connections is discrete this infimum is achieved and is positive. The function $\lambda$ depends only

on the isometry class of $M$ so it is well defined on $\mathbb{H}/\Gamma$. It is also useful to view it as a $\Gamma$-invariant function on $\mathbb{H}$.

The function $\lambda$ is bounded above.

**Proposition 4.1.** *If $\tau = \tau(M)$ is as in (3) then $\lambda(M) \leq \sqrt{2/\tau}$.*

*Proof.* Write $\lambda$ for $\lambda(M)$. Let $M_0 \subset M$ consist of points $p \in M$ for which the distance from $p$ to a point of $\Sigma$ is less than $\lambda/2$, so that $p$ is closest to a unique $q \in \Sigma$. If we fix a $q \in \Sigma$ then the set of points in $M_0$ closest to $q$ has area $c_q \lambda^2/2$ where $c_q$ is the cone angle at $q$. By (3), the area of $M_0$ is $\tau \lambda^2/2$. Since the total area of $M$ is 1 we have $\lambda \leq \sqrt{2/\tau}$.                                                      $\square$

We say a saddle connection $\sigma$ *has minimal length* if $\ell(\sigma) = \lambda(M)$. The number of minimal length saddle connections in $M$ is finite and is generically one. Let $\Pi$ be the subset of $\mathbb{H}$ corresponding to surfaces affinely isomorphic to $M$ which have at least two non-parallel minimal length saddle connections. This is a locally finite geodesic subcomplex of $\mathbb{H}$ which we call the *spine* of $M$. A related but distinct construction is the tesselation studied in Veech [Ve3] and Bowman [Bo].

The edges of $\Pi$ correspond to surfaces with minimal length saddle connections in exactly two directions. These edges are geodesic segments in $\mathbb{H}$. Indeed, if $\xi$ and $\eta$ are minimal length saddle connections in distinct directions on $M$ then the vectors $v = \mathrm{hol}(\xi)$ and $w = \mathrm{hol}(\eta)$ have the same length. We can rotate the translation structure so that the vector $v + w$ is horizontal and the vector $v - w$ is vertical. In these coordinates $v = (x, y)$ and $w = (x, -y)$ for some $x, y \in \mathbb{R}$. If we apply the geodesic flow to $M$ the holonomy vectors of these saddle connections with respect to $g_t(M)$ are $(e^{t/2}x, e^{-t/2}y)$ and $(e^{t/2}x, -e^{-t/2}y)$, so the vectors continue to have the same length. In fact this geodesic in $\mathbb{H}$ is exactly the set of translation structures for which $\xi$ and $\eta$ have the same length. The edge in $\Pi$ that contains $M$ is a segment contained in this geodesic.

Clearly $\Pi$ is $\Gamma_M$-invariant. The vertices of $\Pi$ correspond to surfaces with minimal length saddle connections in three or more directions. We will use $\Pi$ to get a condition for the lattice property.

**Proposition 4.2.** *The area of $\mathbb{H}/\Gamma_M$ is bounded above by $2\pi N$ where $N$ is the number of edges in $\Pi/\Gamma_M$.*

*Proof.* Write $\Gamma = \Gamma_M$. We define a retraction $\rho : \mathbb{H}/\Gamma \to \Pi/\Gamma$. We will describe $\rho$ as a $\Gamma$-invariant map from $\mathbb{H}$ to $\Pi$. Let $z \in \mathbb{H}$. It corresponds to a set of isometric translation surfaces, i.e. to $\mathrm{SO}(2, \mathbb{R}) \cdot M'$ for some $M'$ in the $G$-orbit of $M$. If $M'$ has minimal length saddle connections

in two or more non-parallel directions then $M'$ represents a point in $\Pi$ and we define $\rho(z) = z$. If all minimal length saddle connections on $M$ are in the same direction, by rotating the coordinate system on the surface we can assume that this direction is horizontal. Now apply the geodesic flow $g_t$ to the rotated surface. The length of the shortest saddle connection $\sigma$ in the surface $g_t(M')$ is $e^{t/2}$ times the original length of $\sigma$. As long as $g_t(M')$ is in the complement of $\Pi$, $\sigma$ remains a minimal length segment. Since the length of the shortest curve in $g_t(M')$ is bounded above, for some $t_0$ we have $g_{t_0}(M') \in \Pi$, and we define $\rho(z)$ to be the isometry class of $g_{t_0}(M')$.

We now show that the set of all points which retract to a given point corresponding to the surface $M_0$ in $\Pi$ is a union of geodesic rays, one for each direction of a minimal length saddle connection in $M_0$. Pick a saddle connection $\sigma$ of minimal length in $M_0$. By rotating the coordinate system assume that $\sigma$ is horizontal. Consider the collection of surfaces that we get by flowing by $g_{-t}$ for $t > 0$. The matrix for $g_{-t}$ contracts the horizontal direction more than any other direction so in each of these surfaces $g_{-t}M_0$ the saddle connection $\sigma$ still has minimal length and is shorter than all saddle connections in other directions. It follows that for each of these surfaces the retraction takes $g_{-t}M_0$ to $M_0$.

Pick an edge $e$ of $\Pi$. There are minimal length saddle connections in two directions. Pick one direction and let $\sigma$ be a saddle connection in this direction. The collection of surfaces which retract to $e$ and correspond to the direction of $\sigma$ in $\Pi$ is a union of geodesic rays all of which are asymptotic to the same point on the boundary of $\mathbb{H}$ (which corresponds to the direction of $\sigma$). It follows that the set of points that retract to $e$ consists of two geodesic triangles each with one vertex at infinity. Since the area of a geodesic triangle is bounded above by $\pi$ the total area of $\mathbb{H}/\Gamma$ is bounded above by $2\pi N$. $\square$

**Remark 4.3.** *The existence of the retraction $\rho : \mathbb{H} \to \Pi$ implies that $\Pi$ is homotopy-equivalent to $\mathbb{H}$, i.e. a tree.*

As we explained each edge $e$ in $\Pi$ is a geodesic segment corresponding to the surfaces for which two saddle connections, say $\xi$ and $\eta$ have the same length. There is a unique point on the geodesic containing $e$ corresponding to a surface $M_0$ where the saddle connections $\xi$ and $\eta$ have equal length and are perpendicular. It is conceivable that $M_0$ is not in $e$, but nevertheless we have:

**Lemma 4.4.** *In the surface $M_0$ the common length of $\xi$ and $\eta$ is no more than $\sqrt{2/\tau}$, where $\tau$ is as in (3).*

*Proof.* Let $M'$ correspond to a point in $e$. By Proposition 4.1, in this surface the lengths of $\xi$ and $\eta$ are bounded above by $\sqrt{2/\tau}$. Since the $g_t$-action preserves $|\xi \wedge \eta| = \ell(\xi)\ell(\eta)\sin\theta$, where $\theta$ is the angle between $\xi$ and $\eta$, the minimum of $\ell(\xi)$ is attained when $\xi$ and $\eta$ are perpendicular. Thus the lengths at $M_0$ are no greater than the lengths at $M'$, and we can use Proposition 4.1. $\qquad\square$

We say that a translation surface $M$ is in *standard form* if it has vertical and horizontal saddle connections, and if the shortest horizontal and vertical saddle connections have the same length and this length is bounded by $\sqrt{2/\tau(M)}$.

**Corollary 4.5.** *The area of $\mathbb{H}/\Gamma_M$ is at most $2\pi E$ where $E$ is the number of standard form surfaces affinely equivalent to $M$.*

$\square$

## 5. The complexity of pairs of cylinder decompositions

A pair of cylinder decompositions in distinct directions determine a decomposition of the surface into parallelograms. We measure the complexity of the pair by counting the number of parallelograms in this decomposition. We will show that a lower bound on the area of virtual triangles allows us to find a pair of cylinder decompositions for which we can bound the complexity.

**Proposition 5.1.** *Suppose $M \in \mathrm{NSVT}(\beta_0)$ and $\xi$ and $\eta$ are saddle connections on $M$ with $|\mathrm{hol}(\xi)\wedge\mathrm{hol}(\eta)| = \beta \geq \beta_0$. Then the number of rectangles in the pair of cylinder decompositions determined by $\xi$ and $\eta$ is no more than $\beta/\beta_0^2$.*

*Proof.* By changing coordinates using an element of $G$ we may assume that $\xi$ is horizontal and $\eta$ is vertical, and they have the same length. The product of the lengths is $\beta$ so both lengths are $\sqrt{\beta}$. Let $h$ be the height of a horizontal cylinder. There are singularities on the top and bottom of this cylinder and a saddle connection $\sigma$ between them. The quantity $|\mathrm{hol}(\sigma)\wedge\mathrm{hol}(\xi)|$ is just $h\sqrt{\beta}$. By assumption $|\mathrm{hol}(\sigma)\wedge\mathrm{hol}(\xi)| \geq \beta_0$ so $h \geq \beta_0/\sqrt{\beta}$. A similar calculation for the width of the vertical cylinders gives $w \geq \beta_0/\sqrt{\beta}$. It follows that the area of any rectangle is at least $\beta_0^2/\beta$. Since the total area of $M$ is 1, the number of cylinders is at most $\beta/\beta_0^2$. $\qquad\square$

**Corollary 5.2.** *A surface in $\mathrm{NSVT}(\beta)$ is affinely equivalent to a standard form surface with at most $1/\beta$ rectangles.*

*Proof.* This follows from Proposition 5.1 with $\beta = \beta_0$. $\qquad\square$

**Corollary 5.3.** *A standard form surface in* $\mathrm{NSVT}(\beta)$ *has a decomposition into at most* $2/(\tau\beta^2)$ *rectangles.*

*Proof.* Let $\xi$ and $\eta$ be horizontal and vertical saddle connections of common length at most $\sqrt{2/\tau}$. Then $|\mathrm{hol}(\xi) \wedge \mathrm{hol}(\eta)| \leq 2/\tau$. By Proposition 5.1, the number of rectangles in the corresponding pair of cylinders decomposition is bounded above by $2/(\tau\beta^2)$. $\square$

## 6. Pairs of cylinder decompositions

In this section we review ideas of Thurston [Th] and Veech [Ve1] that make it possible to bound the number of standard form surfaces in $\mathrm{NSVT}(\beta)$. To this end assume $M$ is a flat surface for which the horizontal and vertical directions are periodic. The intersections of the horizontal and vertical cylinders divide the surface into rectangles. Let us label the rectangles with numbers $1, \ldots, \ell$ and call the resulting surface a *labeled surface*. Let $S_\ell$ denote the permutation group on $\ell$ symbols. Assume first that $M$ is a translation surface, so that each rectangle has a well defined top, bottom, left and right edge. For $k \in \{1, \ldots, \ell\}$ let $\sigma_1(k) = k'$ if the $k'$th rectangle is attached to the right side of the $k$th rectangle. Let $\sigma_2(k) = k'$ if the $k'$th rectangle is attached to the top side of the $k$th rectangle. As in [EsOk] we observe that connectedness of $M$ is equivalent to the condition that group generated by $\sigma_1$ and $\sigma_2$ acts transitively on $\{1, \ldots, \ell\}$. The collection of horizontal cylinders corresponds to the set of cycles of $\sigma_1$ and the collection of vertical cylinders corresponds to the set of cycles of $\sigma_2$. The orders of the singular points of $M$ is determined by the cycle structure of the commutator of $\sigma_1$ and $\sigma_2$. If $M$ is a half-translation surface, a gluing pattern can be defined in a similar way. For example this can be defined using the orientation double cover. We will omit the details.

We can also reverse this process. Say that $\sigma_1$ and $\sigma_2$ are elements of $S_\ell$, for which the group generated by $\sigma_1$ and $\sigma_2$ acts transitively on $\{1, \ldots, \ell\}$. Let $\vec{a}^{(1)}, \vec{a}^{(2)}$ be vectors with positive real entries, indexed by the cycles of $\sigma_1, \sigma_2$. Then there is a labeled translation surface $M$ where the heights of the horizontal (vertical) cylinders are recorded in $\vec{a}^{(1)}$ (resp. $\vec{a}^{(2)}$); indeed, let $R_1, \ldots, R_\ell$ be (labeled) rectangles where, if $k$ belongs to the $i$th (resp. $j$th) cycle of $\sigma_1$ ($\sigma_2$) then the height (width) of $R_k$ is $a_i^{(1)}$ (resp $a_j^{(2)}$). We see that this surface is determined uniquely as a labeled surface.

We say that a labeled surface is *normalized* if the horizontal and vertical directions are parabolic, and the shortest saddle connections in the horizontal and vertical direction have equal length. Associated

with such a surface is a pair of Dehn twist vectors corresponding to the horizontal and vertical cylinder decompositions. We have:

**Proposition 6.1** (Veech). *The pair of permutations $(\sigma_1, \sigma_2)$ and the Dehn twist vectors uniquely determine $M$ as a normalized labeled translation surface.*

*Proof.* By the preceding discussion it suffices to show that the permutations and Dehn twist vectors uniquely determine the vectors $\vec{a}^{(d)}$, $d = 1, 2$. The Dehn twist vectors prescribe the ratios of inverse moduli of cylinders in a labeled surface; we will show that determining these ratios uniquely determines $\vec{a}^{(d)}$ up to two free variables. The assumptions that $M$ is normalized and has area 1 will be used to get a unique solution.

Let $B = (b_{ij})$ be an integer matrix where $i$ ranges over the cycles of $\sigma_1$ and $j$ over the cycles of $\sigma_2$, and $b_{ij}$ is the number of elements in the intersection of the corresponding cycles. Define

$$\vec{c}^{(1)} = B\vec{a}^{(2)} \quad \text{and} \quad \vec{c}^{(2)} = B^t\vec{a}^{(1)}.$$

Note that on a labeled surface, the circumference of the $j$th horizontal cylinder is the sum of the widths of the rectangles contained in it. This is just $\sum b_{ij} a_j^{(2)}$. Thus, if $\vec{a}^{(1)}$ contains the heights of horizontal cylinders on a labeled surface, then $\vec{c}^{(2)}$ contains the circumferences of vertical cylinders and vice versa.

Now let $A_d = \operatorname{diag}(\vec{n}^{(d)})$, i.e. an integer diagonal matrix, indexed by cycles of $\sigma_d$, containing the Dehn twist data. In order for the surface determined by $\vec{a}^{(1)}, \vec{a}^{(2)}$ to be parabolic in the horizontal and vertical directions, with the prescribed Dehn twist vectors, the inverse modulus $\mu_i^{(d)}$ of the $i$th cylinder must satisfy $n_i^{(d)} \mu_i^{(d)} = \mu^{(d)}$. Here $\mu^{(1)}$ and $\mu^{(2)}$ are variables. The relationship between inverse moduli and Dehn twist numbers gives:

$$\mu^{(d)} \vec{a}^{(d)} = A_d \vec{c}^{(d)}.$$

Putting these matrix equations together gives

$$\mu^{(1)} \mu^{(2)} \vec{a}^{(1)} = A_1 B A_2 B^t \vec{a}^{(1)},$$

so $\vec{a}^{(1)}$ is an eigenvector of $E_1 = A_1 B A_2 B^t$ corresponding to the eigenvalue $\mu^{(1)} \mu^{(2)}$. The assumption that $\langle \sigma_1, \sigma_2 \rangle$ acts transitively on $\{1, \ldots, \ell\}$ implies that $E_1$ is an irreducible non-negative matrix. By the uniqueness of a positive eigenvector for an irreducible non-negative matrix [Ga], $\vec{a}^{(1)}$ is determined, up to a positive scalar, by the matrix $E_1$. Similarly $\vec{a}^{(2)}$ is uniquely determined up to scaling by $E_2 = A_2 B^t A_1 B$.

The shortest horizontal (vertical) saddle connection is a sum of some of the entries in $\vec{a}^{(2)}$ (resp. $\vec{a}^{(1)}$). Requiring these sums to be equal

implies that the scaling of $\vec{a}^{(1)}$ determines the scaling of $\vec{a}^{(2)}$. Rescaling both vectors by the same factor $\gamma$ changes the area of the surface by $\gamma^2$. There is a unique positive value of $\gamma$ which produces a surface of area 1. $\qquad\square$

**Corollary 6.2.** *If $M \in \mathrm{NSVT}(\beta)$ then condition (viii) of Theorem 1.1 holds.*

*Proof.* Consider two saddle connections $\xi$ and $\eta$ on $M$ with $|\mathrm{hol}(\xi) \wedge \mathrm{hol}(\eta)| = \beta_0 < T$. By Propositions 3.1 the directions of $\xi$ and $\eta$ are parabolic directions on $M$, by Proposition 3.5 the Dehn twist vectors are bounded, and by Proposition 5.1 the number of rectangles in the corresponding pair of cylinder decompositions is bounded, hence so are the possibilities for the corresponding permutations. For any two such pairs $(\xi_i, \eta_i)$, $i = 1, 2$, there is $g \in G$ such that $g\xi_1 = \xi_2$ and $g\eta_1 = \eta_2$. If the corresponding Dehn twist vectors and permutations are the same, then $gM$ is affinely equivalent to $M$ by Proposition 6.1, i.e. $g \in \Gamma_M$. Thus the numbers of $\Gamma_M$-orbits of such pairs $(\xi, \eta)$ is finite. $\qquad\square$

## 7. Counting cylinder intersection patterns

In this section we will obtain a combinatorial formula which will bound the number of affine equivalence classes of translation surfaces in $\mathrm{NSVT}(\beta)$ and the sums of the co-areas of the Veech groups of surfaces in $\mathrm{NSVT}(\beta)$. The first bound will imply Theorem 1.2 and the second will imply Theorem 1.1, since the case of half-translation surfaces follows via the orientation double cover.

Let $X_\ell$ be the set of pairs of permutations $(\sigma_1, \sigma_2)$ in $S_\ell$ for which the group $\langle \sigma_1, \sigma_2 \rangle$ acts transitively on the set $\{1, \ldots, \ell\}$. As we have seen such a pair together with the Dehn twist vectors determine a normalized labeled surface decomposed into $\ell$ rectangles. Since the horizontal and vertical cylinders of the surface correspond to cycles of $\sigma_1$ and $\sigma_2$, the Dehn twist vector $\vec{n}_j$ may be viewed as a function from the cycles of $\sigma_j$ to the natural numbers. Let $\mathcal{Y}(\ell, \beta)$ denote the set of quadruples $(\sigma_1, \sigma_2, \vec{n}_1, \vec{n}_2)$ where $(\sigma_1, \sigma_2) \in X_\ell$ and $\vec{n}_j$ are Dehn twist vectors satisfying the bound in (8), in particular, compatible with $M$ being in $\mathrm{NSVT}(\beta)$.

A permutation $\lambda \in S_\ell$ acts on the set of labels $\{1, \ldots, \ell\}$, and this induces an action on $(\sigma_1, \sigma_2, \vec{n}_1, \vec{n}_2)$ by simultaneously conjugating $\sigma_j$ and changing the domains of $\vec{n}_j$. If $\lambda$ fixes the quadruple $(\sigma_1, \sigma_2, \vec{n}_1, \vec{n}_2)$ then re-arranging the rectangles of the corresponding cylinder decompositions gives rise to a translation equivalence of $M$. Moreover the relabeling operation gives the entire translation equivalence class, since any

translation equivalence of a surface preserves the horizontal and vertical directions, thus preserves the cylinder decompositions and takes rectangles to rectangles.

This gives:

**Proposition 7.1.** *The cardinality of any $S_\ell$-orbit in $\mathcal{Y}(\ell, \beta)$ is at least $(\ell - 1)!$*

*Proof.* For $y = (\sigma_1, \sigma_2, \vec{n}_1, \vec{n}_2) \in \mathcal{Y}(\ell, \beta)$, let $M$ be the corresponding normalized labeled surface as in Proposition 6.1. By the above discussion, the stabilizer in $S_\ell$ of $y$ is isomorphic to the kernel $K_M$ of the map $D : \mathrm{Aff}(M) \to G$. Since any translation equivalence permutes the rectangles on $M$, and any equivalence fixing a rectangle must fix all rectangles, the order of $K_M$ is at most $\ell$. □

**Proposition 7.2.** *Let $\widetilde{\mathrm{NSVT}}(\beta)$ be the set of affine equivalence classes in $\mathrm{NSVT}(\beta)$. Then*

$$\left| \widetilde{\mathrm{NSVT}}(\beta) \right| \leq \sum_{\ell \leq 1/\beta} \frac{1}{(\ell - 1)!} \sum_{(\sigma_1, \sigma_2) \in X_\ell} \eta(|\sigma_1|, \beta) \eta(|\sigma_2|, \beta),$$

*where*

$$\eta(r, \beta) = (8\beta)^{2(1-r)} (1 - 2\log 8\beta)^{r-1}$$

*and $|\sigma|$ is the number of cycles for $\sigma$.*

*Proof.* Every class in $\widetilde{\mathrm{NSVT}}(\beta)$ contains a standard form surface $M$, with a decomposition into $\ell \leq 1/\beta$ rectangles, by Corollary 5.2. The corresponding combinatorial data gives rise to a quadruple in $\mathcal{Y}(\ell, \beta)$, and distinct classes in $\widetilde{\mathrm{NSVT}}(\beta)$ will give rise to quadruples in distinct $S_\ell$-orbits. According to Proposition 3.5, for each $(\sigma_1, \sigma_2) \in X_\ell$, the number of Dehn twist vectors $\vec{n}_j$ for which $(\sigma_1, \sigma_2, \vec{n}_1, \vec{n}_2) \in \mathcal{Y}(\ell, \beta)$ is at most $\eta(|\sigma_j|, \beta)$. Thus the cardinality of $\mathcal{Y}(\ell, \beta)$ is bounded by

$$\Phi(\ell, \beta) = \sum_{(\sigma_1, \sigma_2) \in X_\ell} \eta(|\sigma_1|, \beta) \eta(|\sigma_2|, \beta).$$

The size of an $S_\ell$-orbit in $y$ is at least $(\ell - 1)!$ by Proposition 7.1. So the number of orbits in $\mathcal{Y}(\ell, \beta)$ is at most $\Phi(\ell, \beta)/(\ell - 1)!$ The claim follows by summing over $\ell$. □

This explicit estimate, along with a similar bound for half-translation surfaces, proves Theorem 1.2. We define $\mathrm{areaNSVT}(\beta)$ to be the sum over the affine equivalence classes in $\widetilde{\mathrm{NSVT}}(\beta)$ of the areas of the Teichmüller curves $\mathbb{H}/\Gamma_M$.

**Proposition 7.3.** *We have*

$$\text{areaNSVT}(\beta) \leq 2\pi \sum_{\ell \leq 2/\beta^2} \frac{\Phi(\ell, \beta)}{(\ell - 1)!}.$$

*Proof.* Every $M \in \text{NSVT}(\beta)$ which is in standard form contains at most $2/\beta^2$ rectangles by Corollary 5.3. Thus $\sum_{\ell \leq 2/\beta^2} \Phi(\ell, \beta)/(\ell - 1)!$ is an upper bound for the number of standard form surfaces which are affinely isomorphic to some $M \in \text{NSVT}(\beta)$. Now the bound follows from Corollary 4.5. □

Since the sum of the areas is finite each individual area is finite and we get a proof of Theorem 1.1.

**Remark 7.4.** *Our formulae do not distinguish surfaces on different strata, nor do they distinguish connected surfaces from disconnected ones.*

## 8. Proof of Theorem 1.3

The following characterization of lattice surfaces is part of Theorem 1.3.

**Theorem 8.1.** *Let $\mathcal{H}$ be the stratum containing the flat surface $M$. Then $M$ is a lattice surface if and only if the $G$-orbit of $M$ is closed in $\mathcal{H}$.*

*Proof.* For two proofs of this result, see the sketch in Veech [Ve2] or the proof in [SmWe2, §5]. □

*Proof of Theorem 1.3.* Vorobets [Vo] proved the implications (i) $\Longleftrightarrow$ (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (vi) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii). Theorem 1.1 shows that (i) and (vi) are equivalent, Theorem 1.4 shows that (vi) and (ix) are equivalent, and Theorem 8.1 gives the equivalence of (i) and (x). Clearly that (viii) $\Longleftrightarrow$ (vii) $\Longleftrightarrow$ (ix), and Corollary 6.2 implies (ix) $\Longleftarrow$ (viii). Putting all these together one sees that (i)–(x) are equivalent. To conclude the proof we will prove that (i) $\Longrightarrow$ (xi) $\Longrightarrow$ (vii).

Assume (i). It is well-known that if $\Gamma \subset G$ is a lattice then there is a compact $K_1 \subset G/\Gamma$ such that any geodesic orbit intersects $K$; that is, denoting by $\pi : G \to G/\Gamma$ the projection map, for every $g \in G$ one has

$$K_1 \cap \{g_t \pi(g) : t \in \mathbb{R}\} \neq \varnothing.$$

Now taking $\Gamma = \Gamma_M$, and letting $\varphi : G/\Gamma \to \mathcal{H}$ be the orbit map $\varphi(\pi(g)) = gM$, we see that (xi) holds with $K = \varphi(K_1)$.

Now assume (xi). Let $\eta > 0$ be small enough so that for any $M_0 \in K$, the length of any saddle connection for $M_0$ is at least $\eta$. Let $v_1, v_2 \in$

$\mathrm{hol}(M)$ such that $v_1 \wedge v_2 \neq 0$, and suppose $v_i = \mathrm{hol}(\delta_i)$ where $\delta_i \in \mathcal{L}$. For $h \in G$, let $l_i(h)$ be the length of $\delta_i$ with respect to the Euclidean metric on $hM$.

Let $g \in G$ be a linear map such that $gv_1$ is horizontal, $gv_2$ is vertical and both have the same length, which we denote by $c$. By assumption, there is $t = t(g) \in \mathbb{R}$ with $g_t gM \in K$. If $t \geq 0$ then

$$\eta \leq l_2(g_t g) = e^{-t/2} l_2(g) = e^{-t/2} c,$$

so $c \geq \eta$. If $t < 0$ we apply the same argument with $l_1$ instead of $l_2$ to see that $c \geq \eta$. Since $g$ preserves the two dimensional volume element,

$$|v_1 \wedge v_2| = |gv_1 \wedge gv_2| = c^2 \geq \eta^2,$$

which is a positive constant independent of $v_1, v_2$. □

**Remark 8.2.** *Let $C$ be the number of cusps in $\mathbb{H}/\Gamma_M$. Then the arguments of [Vo] show that the number of $\mathrm{Aff}(M)$-orbits of triangles as in alternative (iv) of Theorem 1.3 is between $C$ and $C(2g + |\Sigma| - 2)$.*

## 9. THE SIMPLEST LATTICE SURFACES

In this section we will list the first few lattice surfaces, ordered as in (1). We will also list surfaces with removable singularities.

Recall that any triangulation of $M$ has $\tau(M)/\pi$ triangles, implying

$$\alpha(M) \leq \frac{\pi}{\tau(M)}. \tag{10}$$

**Proposition 9.1.** *If equality holds in (10) then $M$ is an arithmetic lattice surface, all triangles on $M$ have the same area, all cylinders in a cylinder decomposition of $M$ have the same height and all parallel saddle connections on $M$ have the same length.*

*Proof.* Since $\alpha(M)$ is both a lower bound and the average of the triangle areas in any triangulation, the area of each triangle for $M$ must be equal to $\alpha(M)$. We will now show that for any $\theta$, the length $|\sigma|$ of any $\sigma \in \mathcal{L}_M(\theta)$ is the same. By Theorem 1.3, $\theta$ is a parabolic direction for $M$, so $\sigma$ is contained in the boundary of a cylinder $C$, and there is a singularity $x$ in the boundary component of $C$ opposite to $\sigma$. If $\sigma' \in \mathcal{L}_M(\theta)$ is in the same boundary component of $C$ as $\sigma$, consider the triangles $\Delta$ and $\Delta'$ with apex $x$ and base $\sigma$ and $\sigma'$ respectively. Since these triangles have the same height and area we find that $|\sigma| = |\sigma'|$. Similarly, if $\sigma''$ is a segment on the boundary component of $C$ opposite to $\sigma$ then a triangle $\Delta''$ with base $\sigma''$ and apex an endpoint of $\sigma$ has the same area and height as $\Delta$ so we find that $|\sigma| = |\sigma''|$.

Now consider another cylinder $C'$ whose boundary contains $\sigma$. Since it contains a triangle with base $\sigma$, its height is the same as that of $C$,

and any saddle connection on either of its boundary components has length $|\sigma|$. Continuing in this fashion and using the connectedness of $M$ we find that all segments in $\mathcal{L}_M(\theta)$ have the same length and all cylinders in the corresponding cylinder decomposition have the same height.

Now consider $C, \sigma$ as before and let $\sigma'$ be a saddle connection passing from one boundary component of $C$ to another. Write $v = \operatorname{hol}(\sigma), v' = \operatorname{hol}(\sigma')$, and consider any saddle connection $\lambda$ on $M$. We can write $\operatorname{hol}(\lambda) = nv + n'v'$, where $n$ (respectively $n'$) is the number of times $\lambda$ passes through a cylinder in the direction of $\sigma'$ (resp. $\sigma$). In particular $\operatorname{hol}(M) \subset \mathbb{Z}v \oplus \mathbb{Z}v'$. It follows by [GuJu, §5] that $M$ is arithmetic. $\square$

We now list some examples of arithmetic lattice surfaces $M$ and calculate $\alpha(M)$. Let $\gamma = \tau/\pi$ denote the number of triangles in a triangulation of $M$, so that by (2) we have

$$\gamma = 2\left(\sum r_\sigma + |\Sigma|\right) = 2\left(2g - 2 + |\Sigma|\right). \tag{11}$$

(1) Let $M_1$ be the standard flat torus $[0,1]^2$ with opposite sides identified, with one marked point at the origin. By (11) $\gamma = 2$ so $\alpha(M_1) \leq 1/2$. On the other hand $\operatorname{hol}(M_1) \subset \mathbb{Z}^2$ so that $|v_1 \wedge v_2| \geq 1$ for any two linearly independent $v_1, v_2 \in \operatorname{hol}(M_1)$. Since the area of a triangle with sides $v_1, v_2$ is $|v_1 \wedge v_2|/2$ we see that $\alpha(M_1) = 1/2$.
(2) Let $M_2$ be the standard pillowcase with 4 singularities of total angle $\pi$. Then $\gamma = 4$ so that $\alpha(M) \leq 1/4$. On the other hand $\operatorname{hol}(M_3) \subset \left(\frac{1}{\sqrt{2}}\mathbb{Z}\right)^2$, so that $\alpha(M_3) = 1/4$.
(3) Let $M_3$ be the standard torus with one marked point at the origin and another at $\left(\frac{1}{2}, 0\right)$. Then $\gamma = 4$ so that $\alpha(M_2) \leq 1/4$. On the other hand $\operatorname{hol}(M_2) \subset \mathbb{Z}\left[\frac{1}{2}\right] \oplus \mathbb{Z}$ so that $\alpha(M_2) = 1/4$.

We now show:

**Proposition 9.2.** NST $\left(\frac{1}{4}\right)$ *consists of the affine equivalence classes of* $M_1, M_2, M_3$.

*Proof.* The above discussion shows $M_1, M_2, M_3 \in$ NST $\left(\frac{1}{4}\right)$ and it remains to show that if $M$ is a flat surface with $\alpha = \alpha(M) \geq 1/4$ then $M$ is affinely equivalent to one of the $M_i$. Let $\gamma$ be as above, so by (10) we have $\gamma \in \{2, 4\}$. If $\gamma = 2$ then either $g = 1$ and $|\Sigma| = 1$ or $g = 0$ and $|\Sigma| = 3$. In the first case, since the moduli space of tori with one marked point is a single $G$-orbit, we find that $M$ is affinely equivalent to $M_1$. The second case does not occur as there is no solution to (2) with three singularities.

Now suppose $\gamma = 4$. The only solutions to (10) are $(g = 0, |\Sigma| = 4)$ and $(g = 1, |\Sigma| = 2)$. In the first case it follows from (2) that the four singularities have $r_\sigma = -1/2$ so $M$ is a pillowcase. Since the moduli space of the pillowcase consists of a single $G$-orbit, we have that $M$ is affinely equivalent to $M_2$. In the second case by (11) the two singularities $\sigma_1, \sigma_2$ satisfy either

(i) $r_{\sigma_1} = -1/2, r_{\sigma_2} = 1/2$.
(ii) $r_{\sigma_1} = r_{\sigma_2} = 0$;

In case (i) we obtain a half-translation structure on a torus, but such a flat surface does not exist (see [MaSm]). In case (ii) $M$ is a torus with two marked points. Applying an element of $G$ we may identify $M$ with the unit square, and there is no loss of generality in assuming that one of the marked points is at the origin. By Proposition 9.1, if $\sigma$ is the saddle connection connecting the two marked points inside the unit square, then there is a parallel segment from the second marked point to a singularity, of the same length. This implies that the second marked point is at either of the points $\left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2},\right), \left(\frac{1}{2}, \frac{1}{2}\right)$. All of these cases are affinely equivalent to $M_3$. □

If $\alpha$ is not too small one can continue applying such arguments to identify NST $(\alpha)$. For example, in addition to $M_1, M_2, M_3$, NST $\left(\frac{1}{6}\right)$ consists of a torus with two marked point, a torus with three marked points, a genus 1 half-translation surface made by gluing a torus and a pillowcase along a slit, and a genus 2 surface (see Figure 1). All these examples are arithmetic.
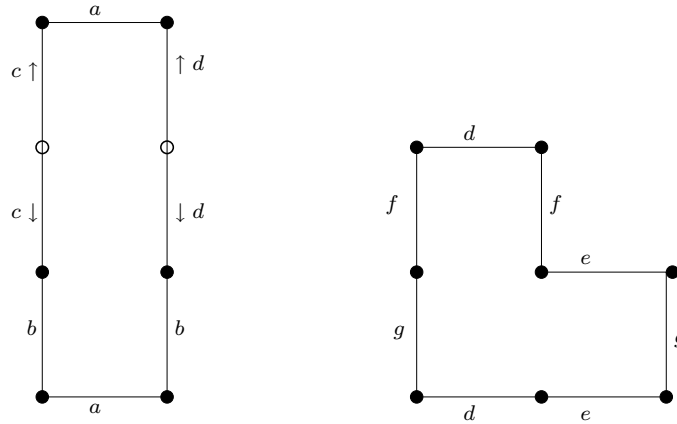


FIGURE 1. Two surfaces in NST $\left(\frac{1}{6}\right)$

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