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DISTRIBUTION OF LATTICE ORBITS ON HOMOGENEOUS VARIETIES

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Abstract. Given a lattice Γ in a locally compact group G and a closed subgroup H of G, one has a natural action of Γ on the homogeneous space $V = H\backslash G$. For an increasing family of finite subsets $\{\Gamma_T : T > 0\}$, a dense orbit $v\cdot\Gamma$, $v \in V$ and compactly supported function φ on V, we consider the sums $S_{\varphi,v}(T) = \sum_{\gamma\in\Gamma_T} \varphi(v\gamma)$. Understanding the asymptotic behavior of $S_{\varphi,v}(T)$ is a delicate problem which has only been considered for certain very special choices of H, G and $\{\Gamma_T\}$. We develop a general abstract approach to the problem, and apply it to the case when G is a Lie group and either H or G is semisimple. When G is a group of matrices equipped with a norm, we have $S_{\varphi,v}(T) \sim \int_{G_T} \varphi(vg)dg$, where $G_T = \{g \in G : \|g\| < T\}$ and $\Gamma_T = G_T \cap \Gamma$. We also show that the asymptotics of $S_{\varphi,v}(T)$ is governed by $\int_V \varphi\, d\nu$, where ν is an explicit limiting density depending on the choice of v and $\|\cdot\|$.

1 Introduction

Let V be a manifold equipped with a transitive right action of a Lie group G, so that V is identified with $H\backslash G$ for some subgroup H of G stabilizing a point of V. Let Γ be a lattice in G, that is, a discrete subgroup of finite co-volume. In this paper we study the asymptotic distribution of Γ-orbits. More precisely, we fix a proper function $D : G \to [0,\infty)$, set

$$G_T = \left\{g \in G : D(g) < T\right\},$$

and study, for an arbitrary $\varphi \in C_c(V)$ and $v \in V$, the asymptotic behavior of the sum

$$S_{\varphi,v}(T) = \sum_{\gamma\in\Gamma\cap G_T} \varphi(v\cdot\gamma)$$

as $T \to \infty$.

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We have not assumed that there is an invariant measure on V, and certainly not a finite one. Thus this problem does not belong to the classical framework of ergodic theory, and has many surprising features.

1.1 A simple case. Perhaps the simplest nontrivial case is when

$$V = \mathbb{R}^2 \setminus \{0\}, \quad G = \mathrm{SL}(2,\mathbb{R}), \quad D(g) = \|g\|,$$

where

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \left(|a|^p + |b|^p + |c|^p + |d|^p\right)^{1/p}, \quad 1 \le p < \infty.$$

Then H is (up to conjugation) the one-parameter group of upper-triangular unipotent matrices. The distribution of $\mathrm{SL}(2,\mathbb{Z})$-orbits in V was considered by Nogueira [N], and the case of a general lattice by Ledrappier [L]. Ledrappier showed that for $\varphi \in C_c(V)$ and $v \in V$ such that $v \cdot \Gamma$ is dense in V, one has

$$S_{\varphi,v}(T) \sim \left(c_\Gamma \int \frac{\varphi(Y)}{\|v\| \cdot \|Y\|} dY \right) T,$$

where dY denotes Lebesgue measure on $\mathbb{R}^2$, $\|\cdot\|$ is the p-norm on $\mathbb{R}^2$, and c_Γ is an explicit constant depending on the lattice Γ (here and throughout the paper, the notation $A(T) \sim B(T)$ means that $A(T)/B(T) \to_{T\to\infty} 1$).

In this example, one already notices that the limiting measure $\frac{1}{\|v\|}\frac{dY}{\|Y\|}$ is not Γ-invariant and depends essentially on the choice of v and the choice of norm used on G. A computation with the Haar measure of G shows that in this case the asymptotics of $S_{\varphi,v}(T)$ is the same as that of the corresponding average along the (unique) G-orbit, namely

$$S_{\varphi,v}(T) \sim \widetilde{S}_{\varphi,v}(T), \quad \text{where } \widetilde{S}_{\varphi,v}(T) = \int_{G_T} \varphi(v \cdot g)\, dm(g),$$

and m is a suitably normalized Haar measure on G.

We reconsider and generalize this example in §12.4.

1.2 The general problem. It is natural to try to extend this result to a general case. Namely, given an arbitrary locally compact second countable topological group G, a lattice Γ in G, a transitive G-space V and a proper function $D : G \to [0,\infty)$, one would like to:

- Obtain an explicit expression for the asymptotics of $S_{\varphi,v}(T)$.
- Show that $S_{\varphi,v}(T) \sim \widetilde{S}_{\varphi,v}(T)$.

A number of special cases of this program were carried out in work of the first-named author, all involving $G = \mathrm{SL}(n,\mathbb{R})$, $D(g) = \sqrt{\mathrm{tr}\,({}^t g\, g)}$ and a general lattice Γ: in [G1], the case in which V is the Furstenberg boundary of G, and in [G2], the case in which V is the space of k-frames in $\mathbb{R}^n$, $1 \leq k < n$. Additional cases were studied in [LP], [Ma], [O], [GM], [GO].

Our goal in this paper is twofold. First we abstract some ideas of the above-mentioned papers and develop a general axiomatic framework for studying the problem. We specify certain explicit conditions on G, H, D, Γ under which the above problems can be solved. Our axiomatic framework can be applied in principle to investigate previously known equidistribution results (as we illustrate by the example from section 1.1), but verification of the conditions on G, H, D, Γ may be quite involved. In the second part of the paper, we check these conditions in two important new cases: when H is semisimple, or when G is semisimple and the Γ-action admits a finite invariant measure.

1.3 Our two main theorems. For convenience of the reader, in order to introduce our results, we begin with two special cases and some concrete examples. We postpone the formulation of our results in their most general form until section 2. Let G be a connected Lie group, Γ a lattice in G, and H a connected simple noncompact subgroup of G. We take the function D to be one of the following:

- Let $\Psi : G \to \mathrm{GL}(d, \mathbb{R})$ be a representation with compact kernel, $\|\cdot\|$ any norm on $\mathrm{Mat}_d(\mathbb{R})$, and
$$D(g) = \max\left\{1, \|\Psi(g)\|\right\}. \tag{1}$$
- Let G be closed subgroup of a semisimple group L, d the Cartan–Killing metric on the symmetric space Ω of L, $P : L \to \Omega$ the natural projection, and
$$D(g) = d\big(P(g), P(e)\big). \tag{2}$$

In the celebrated paper [EsMS], the asymptotics of discrete Γ-orbits in $H\backslash G$ were determined for a specific choice of D as in (1). Here we give the complete answer for the case of *dense* Γ-orbits:

Theorem 1.1. *For every $\varphi \in C_c(H\backslash G)$ and $x \in H\backslash G$ such that $\overline{x\Gamma} = H\backslash G$,*
$$S_{\varphi,x}(T) \sim \widetilde{S}_{\varphi,x}(T) \quad \text{as } T \to \infty\,.$$
Moreover,
$$\lim_{T\to\infty} \frac{S_{\varphi,x}(T)}{\lambda(H \cap G_T)} = \int_{H\backslash G} \varphi \, d\nu_x$$
where λ is a Haar measure on H and ν_x is an explicit smooth measure on $H\backslash G$.

Now we consider an action of a lattice Γ in a simple noncompact Lie group G, acting by measure preserving transformations on a probability measure space (X, μ), which we assume is *algebraic*. Namely, we suppose that $X = L/\Lambda$ where L is a Lie group, Λ is a lattice in L, μ is the normalized Haar measure, and one of the following holds:

- The group G is a closed subgroup of L and Γ acts on $X = L/\Lambda$ by left multiplication.
- The group G is a closed subgroup of $\mathrm{Aut}(L)$ and Γ acts on $X = L/\Lambda$ by automorphisms (stabilizing Λ).

For the sets G_T defined by either (1) or (2), we have the following equidistribution result:

Theorem 1.2. *For every $\varphi \in C_c(X)$ and $x_0 \in X$ such that $\overline{\Gamma x_0} = X$,*

$$\frac{1}{|\Gamma \cap G_T|} \sum_{\gamma \in \Gamma \cap G_T} \varphi(\gamma^{-1} x_0) \to \int_X \varphi \, d\mu$$

as $T \to \infty$.

1.4 Methods. We use two strategies to prove our results. First, in the presence of a Γ-invariant probability measure as in Theorem 1.2, the standard construction of an induced action gives a G-action, and one may understand the asymptotic behavior of the Γ-orbits by studying the corresponding G-orbits. Secondly, when this assumption is not satisfied (as in Theorem 1.1) one may use 'duality' to replace the question about Γ-orbits on $H \backslash G$ with a problem about H-orbits on G/Γ (a similar approach was introduced in [DuRS] and employed in [EsMS]). Although more general, the second strategy is more difficult to implement because the question about H-orbits involves averaging along certain 'skew-balls' with respect to the function D.

Following either strategy, one is led to a question about the distribution of an orbit of a connected Lie group on a finite measure homogeneous space. Such questions have been thoroughly studied in recent years, building on Ratner's classification of measures invariant under unipotent flows and subsequent work of many authors. The precise result we require follows from work of Nimish Shah [S]. We refer the reader to [KSS] for a recent and detailed account of this theory.

Our strategy also requires a precise understanding of the asymptotics of volumes of certain 'skew balls' in H, with respect to a norm in a linear representation, or with respect to the natural G-invariant metric on the symmetric space of G. For example, we determine the asymptotics of the volume of the sets $\{g \in G : \|g\| < T\}$, where G is a semisimple Lie group realized as a matrix group in $\mathrm{Mat}_d(\mathbb{R})$ and $\|\cdot\|$ is an arbitrary linear norm on $\mathrm{Mat}_d(\mathbb{R})$. Even this very natural question has only been settled in the literature in certain special cases.

1.5 Applications. In the next section we establish notation and state our results in more detail, and at the end of the section, describe the organization of the paper. To conclude this introduction, we list some applications which illustrate the scope of our methods. The results of this section are proved in section 11.

1.5.1 Values of indefinite irrational quadratic forms on frames. Let Q be a nondegenerate indefinite quadratic form in $d \geq 3$ variables, which is not a scalar multiple of a rational form. Recall that the Oppenheim conjecture, proved by Margulis [M], states that the set $Q(\mathbb{Z}^d)$ of its values at integer points is dense in $\mathbb{R}$. A strengthening of this result was obtained by Dani and Margulis and by Borel and Prasad. Denote by $\mathcal{F}_d$ the space of unimodular frames in $\mathbb{R}^d$, i.e.

$$\mathcal{F}_d = \big\{(f_1, \dots, f_d) : f_i \in \mathbb{R}^d\,,\ \mathrm{Vol}(\mathbb{R}^d/(\mathbb{Z}f_1 \oplus \cdots \oplus \mathbb{Z}f_d)) = 1\big\}\,.$$

Let $\mathcal{F}_d(\mathbb{Z})$ be the space of integer unimodular frames. For a frame $f \in \mathcal{F}_d$, denote by $\bar{Q}(f)$ the corresponding Gram matrix:

$$\bar{Q}(f) = \big(Q(f_i, f_j)\big)_{i,j=1,\dots,d} \in \mathrm{Mat}_d(\mathbb{R})$$

(note that we denote by $Q(\,\cdot\,,\cdot\,)$ the bilinear form corresponding to $Q(\,\cdot\,)$). Then it can be shown (see [DM] and [BoP]) that the set $\bar{Q}(\mathcal{F}_d(\mathbb{Z}))$ is dense in $\bar{Q}(\mathcal{F}_d)$. Our results imply a quantitative strengthening of this fact. Let

$$G = \big\{g \in \mathrm{GL}(d, \mathbb{R}) : \det g = \pm 1\big\}\,.$$

Recall that the map

$$g \in G \mapsto g\,\mathbf{e} \in \mathcal{F}_d\,, \tag{3}$$

where $\mathbf{e}$ is the frame consisting of the standard basis vectors, is a diffeomorphism. This endows $\mathcal{F}_d$ with a natural measure μ coming from Haar measure m on G, which we normalize by requiring that $m\,(G/G(\mathbb{Z})) = 1$. Note that $\bar{Q}(\mathcal{F}_d)$ is a submanifold of $\mathrm{Mat}_d(\mathbb{R})$, consisting of symmetric matrices of a given signature. We have

COROLLARY 1.3. *Let Q be as above. Fix any norm $\|\cdot\|$ on $\mathbb{R}^d$ and let $A \subset \bar{Q}(\mathcal{F}_d)$ be a bounded set whose boundary has measure zero. Then*

$$\#\big\{f \in \mathcal{F}_d(\mathbb{Z}) : \|f\| < T\,,\ \bar{Q}(f) \in A\big\} \sim \mu\big(\{f \in \mathcal{F}_d : \|f\| < T\,,\ \bar{Q}(f) \in A\}\big)$$

(here $\|(f_1, \dots, f_d)\| = \max_{i=1,\dots,d} \|f_i\|$). Explicitly, these quantities are asymptotic to

$$\begin{array}{ll} CT^{p(q-1)} & \text{if } p < q \\ C \log T\, T^{p(p-1)} & \text{if } p = q \end{array}$$

where (p, q), $1 \leq p \leq q$, is the signature of Q and C is an explicitly computable constant depending on A, Q and $\|\cdot\|$.

Note that in contrast with the quantitative version of the Oppenheim conjecture, proved by Eskin, Margulis and Mozes [EsMM], in this result there are no difficulties arising in signatures $(1,2)$ and $(2,2)$.

1.5.2 Dense projections of lattices. Suppose G is a semisimple non-simple group, and Γ is an irreducible lattice, i.e. the projection of Γ onto each non-cocompact factor of G is dense. Our results imply a quantitative version of this fact. Here we formulate two special cases.

Let G be a Lie group which is a direct product $G = SH$ of its closed subgroups S and H. Suppose Γ is a lattice in G such that $H\Gamma$ is dense in G. Denote by $P : G \to S$ the projection map. Suppose H is simple, and $\Psi_H : H \to \mathrm{GL}(V_H)$, $\Psi_S : S \to \mathrm{GL}(V_S)$ are two irreducible representations with compact kernels.

Let m be a Haar measure on G, normalized by requiring that the co-volume of Γ in G is equal to 1. Let λ and ν be Haar measures on H and S respectively, with $m = \lambda \otimes \nu$ (note that since G and H are unimodular, so is S).

COROLLARY 1.4. *Let the notation be as above.*

(i) *Let $\Psi : G \to \mathrm{GL}(V_H \oplus V_S)$ be the direct sum representation of G, and for a linear norm $\|\cdot\|$ on $V_H \oplus V_S$ let*
$$H_T = \big\{h \in H : \|\Psi(h)\| < T\big\}, \quad \Gamma_T = \big\{\gamma \in \Gamma : \|\Psi(\gamma)\| < T\big\}.$$
For every $\varphi \in C_c(S)$ and every $s_0 \in S$,
$$\frac{1}{\lambda(H_T)} \sum_{\gamma \in \Gamma_T} \varphi\big(s_0 P(\gamma)\big) \underset{T\to\infty}{\longrightarrow} \int_S \varphi \, d\nu\,.$$

(ii) *Let $\Psi : G \to \mathrm{GL}(V_H \otimes V_S)$ be the tensor product representation of G. Choose bases $\{v_1, \dots, v_k\}$, $\{u_1, \dots, u_\ell\}$ of V_H and V_S, and for $1 \le p < \infty$ let $\|\cdot\|$ denote both the p-norm on V_S associated with the basis $\{u_1, \dots, u_\ell\}$ and the p-norm on $V_H \otimes V_S$ associated with the basis $\{v_i \otimes u_j\}$. Then for every $\varphi \in C_c(S)$ and every $s_0 \in S$,*
$$\frac{1}{\lambda(H_T)} \sum_{\gamma \in \Gamma_T} \varphi\big(s_0 P(\gamma)\big) \underset{T\to\infty}{\longrightarrow} c \int_S \varphi(s) \, \frac{d\nu(s)}{\|\Psi_S(s_0 s)\|^m}\,,$$
where m and c are explicitly given positive constants.

1.5.3 Lattice actions by translations. Let L be a noncompact simple connected Lie group, and Λ and Δ lattices in L. The group Λ acts on L/Δ by left translation: if $\pi : L \to L/\Delta$ is the quotient map, then the action is given by
$$\lambda\pi(g) = \pi(\lambda g)\,.$$

It is a simple application of Ratner's theorems that for any $x = \pi(g) \in L/\Delta$, Λx is either finite (in case Λ and $g\Delta g^{-1}$ are commensurable) or dense. Note that a special case of this observation was used recently by Vatsal [V] in his study of Heegner points. A direct application of Theorem 1.2 yields an equidistribution result for the dense Λ-orbits in L/Δ, generalizing recent results of [O], [GO].

Let m be the Haar measure on L, normalized so that $m(L/\Delta) = 1$. We denote the induced finite measures on L/Δ and L/Λ also by m. We assume that L is a closed subgroup of $\mathrm{GL}(d, \mathbb{R})$, fix a norm $\|\cdot\|$ on $\mathrm{Mat}_d(\mathbb{R})$, and set

$$L_T = \left\{\ell \in L : \|\ell\| < T\right\}.$$

Corollary 1.5. *Let $\pi(g_0) \in L/\Delta$ be such that $\overline{\Lambda\pi(g_0)} = L/\Delta$. Then for $\varphi \in C_c(L/\Delta)$,*

$$\frac{1}{m(L_T)} \sum_{\lambda \in \Lambda \cap L_T} \varphi\big(\lambda^{-1}\pi(g_0)\big) \underset{T\to\infty}{\longrightarrow} \frac{1}{m(L/\Lambda)} \int_{L/\Delta} \varphi \, dm\,.$$

1.5.4 Toral automorphisms. Consider $\Gamma = \mathrm{SL}(d, \mathbb{Z})$ acting by linear isomorphisms on the torus $\mathbb{T} = \mathbb{R}^d/\mathbb{Z}^d$. It is well known that an orbit Γx for this action is either finite (if $x \in \mathbb{T}(\mathbb{Q})$) or dense. Theorem 1.2 implies a quantitative strengthening of this fact.

Corollary 1.6. *Fix a norm $\|\cdot\|$ on $\mathrm{Mat}_d(\mathbb{R})$. Let $\varphi \in C_c(\mathbb{T})$. Then for any $x \in \mathbb{T} \smallsetminus \mathbb{T}(\mathbb{Q})$,*

$$\frac{\sum_{\|\gamma\|<T} \varphi(\gamma^{-1}x_0)}{\#\{\gamma \in \Gamma : \|\gamma\| < T\}} \underset{T\to\infty}{\longrightarrow} \int_{\mathbb{T}} \varphi \, d\mu$$

where μ is the Lebesgue probability measure on $\mathbb{T}$.

Corollary 1.6 was independently proved by N. Shah (private communication).

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2 Notation and Statement of Results

2.1 The general setup. Let G be a second countable locally compact noncompact topological group, let Γ be a lattice in G, and let $\pi : G \to G/\Gamma$ be the natural quotient map.

There is a natural left-action of G and any of its subgroups on G/Γ defined by $g_1\,\pi(g_2) = \pi(g_1 g_2)$.

Since it admits a lattice, G is unimodular, i.e. the Haar measure m is invariant under both left and right multiplication. Let m' denote the G-invariant measure on G/Γ induced by m, that is,

$$m'(X) \stackrel{\text{def}}{=} m\left(\Omega \cap \pi^{-1}(X)\right),$$

for some Borel fundamental domain Ω for the right-action of Γ on G. It follows from the invariance of m that m' is independent of the choice of Ω. Normalize m so that $m'(G/\Gamma) = 1$.

By a *distance function* on G we mean a function $G \to [0, \infty)$ which is continuous and proper. Let D be a distance function on G. The words *the general setup holds* mean that G, Γ, D, m, m' are as above.

Fixing D, for a subset $L \subset G$ we write

$$L_T \stackrel{\text{def}}{=} \left\{g \in L : D(g) < T\right\}.$$

We list some hypotheses about this setup.

UC Right uniform continuity of $\log D$. For any $\varepsilon > 0$ there is a neighborhood $\mathcal{U}$ of identity in G such that for all $g \in G$, $u \in \mathcal{U}$,

$$D(gu) < (1+\varepsilon) D(g). \tag{4}$$

I1 Moderate volume growth for balls in G. For any $\varepsilon > 0$ there are $\delta > 0$ and T_0 such that for all $T > T_0$:

$$m(G_{(1+\delta)T}) \leq (1+\varepsilon) m(G_T). \tag{5}$$

I2 Γ-points equidistributed in G w.r.t. D.

$$\#\Gamma_T \sim m(G_T).$$

2.2 Inducing the action. Suppose X is a space on which Γ acts on the right, preserving a finite invariant measure. There is a standard construction (see §3) of a left G-space Y with a G-map $\pi_{G/\Gamma} : Y \to G/\Gamma$, such that the fiber over $\pi(e) = [\Gamma] \in G/\Gamma$ is isomorphic to X. In case $X = H\backslash G$, it is simply the action of G on the product $H\backslash G \times G/\Gamma$, given by

$$g \cdot \left(\tau(g_2), \pi(g_1)\right) = \left(\tau(g_2 g^{-1}), \pi(g g_1)\right) \tag{6}$$

where $\tau : G \to H\backslash G$ denotes the natural factor map.

The following reduces the study of asymptotic behavior of Γ-orbits on X to that of G-orbits on Y.

PROPOSITION 2.1. *Suppose hypotheses I1, I2 and UC hold. Let $x_0 \in X$, and let $y_0 \in Y$ be the corresponding point in the fiber $\pi_{G/\Gamma}^{-1}([\Gamma])$. Suppose that there is a measure μ on X such that:*

(∗) *For any $F \in C_c(Y)$,*

$$\frac{1}{m(G_T)} \int_{G_T} F(g^{-1} \cdot y_0)\, dm(g) \underset{T\to\infty}{\longrightarrow} \int_Y F\, d\nu\,,$$

where $d\nu = d\mu\, dm'$.

Then for any $\varphi \in C_c(X)$,

$$\frac{1}{\#\Gamma_T} \sum_{\gamma\in\Gamma_T} \varphi(x_0\gamma) \underset{T\to\infty}{\longrightarrow} \int_X \varphi\, d\mu\,.$$

2.3 The general setup, continued. Suppose the general setup holds, let H be a closed subgroup of G, and let $\tau : G \to H\backslash G$ be the natural quotient map. Let λ be a left Haar measure on H. There is a natural right-action of G on $H\backslash G$ defined by $\tau(g_1)\, g_2 = \tau(g_1 g_2)$. We have the following diagram:

$$\begin{tikzcd} & G \arrow[dl, "\tau"'] \arrow[dr, "\pi"] & \\ H\backslash G & & G/\Gamma \end{tikzcd}$$

If H has been defined, the words *the general setup holds* mean in addition that H, λ and τ are as above.

If $g_1, g_2, g \in G$ and $L \subset G$, we write

$$L_T[g_1, g_2] \overset{\text{def}}{=} \left\{\ell \in L :\ D(g_1 \ell g_2) < T\right\}, \quad L_T[g] \overset{\text{def}}{=} L_T[e, g]\,. \tag{7}$$

We sometimes call these ‘skew balls’.

We list some additional hypotheses.

S Locally continuous section. For any $x \in H\backslash G$ there is a Borel map $\sigma : H\backslash G \to G$ which is continuous in a neighborhood of x and satisfies $\tau\circ\sigma = \mathrm{Id}_{H\backslash G}$.

D1 Uniform volume growth for skew-balls in H. For any bounded $B \subset G$ and any $\varepsilon > 0$ there are T_0 and $\delta > 0$ such that for all $T > T_0$ and all $g_1, g_2 \in B$ we have

$$\lambda\big(H_{(1+\delta)T}[g_1, g_2]\big) \le (1+\varepsilon)\lambda\big(H_T[g_1, g_2]\big)\,. \tag{8}$$

D2 Limit volume ratios. For any $g_1, g_2 \in G$, the limit

$$\alpha(g_1, g_2) \overset{\text{def}}{=} \lim_{T\to\infty} \frac{\lambda(H_T[g_1^{-1}, g_2])}{\lambda(H_T)} \tag{9}$$

exists and is positive and finite.

2.4 Duality. The following result establishes a link between the asymptotic behavior of a Γ-orbit on $H\backslash G$ and the behavior of a corresponding H-orbit on G/Γ.

Theorem 2.2. *Suppose the general setup holds, and conditions S, D1, UC are satisfied. Let $g_0 \in G$ and assume the following:*

(∗∗) *For any $b \in G$ and any $F \in C_c(G/\Gamma)$,*

$$\frac{1}{\lambda\left(H_T[g_0,b]\right)}\int_{H_T[g_0,b]} F(h^{-1}\pi(g_0))d\lambda(h) \underset{T\to\infty}{\longrightarrow} \int_{G/\Gamma} F(x)dm'(x)\,.$$

Then for every non-negative compactly supported φ on $H\backslash G$,

$$\sum_{\gamma\in\Gamma_T}\varphi\big(\tau(g_0)\gamma\big) \sim \int_{G_T}\varphi\big(\tau(g_0)g\big)\,dm(g)\,. \tag{10}$$

2.5 The limiting density. The conclusion of Theorem 2.2 describes the asymptotics of a Γ-orbit in terms of the asymptotics of a G-orbit. In order to calculate the latter we need some more terminology, and an additional assumption. Note that we are calculating the asymptotics of a G-orbit in a space on which G acts transitively. Nevertheless the computation is not trivial.

We have not assumed that H is unimodular, and therefore a G-invariant measure on $H\backslash G$ need not exist. To remedy this, we need to discuss measures on $H\backslash G$. We suppose that condition D2 is satisfied, and define $\alpha(g_1,g_2)$ by (9).

Let Y be a lift of $H\backslash G$ to G. That is, Y is a Borel subset of G such that the product map

$$H\times Y \to G : (h,y)\mapsto hy$$

is a Borel isomorphism. Since the measures m and λ are left H-invariant,

$$dm(hy) = d\lambda(h)\,d\nu_Y(y) \tag{11}$$

for some Borel measure ν_Y on Y. Moreover (11) determines ν_Y uniquely. When H is unimodular, ν_Y is identified with a measure on $H\backslash G$ which we denote by $\nu_{H\backslash G}$. It is independent of Y and is the unique (up to scaling) G-invariant measure on $H\backslash G$.

For $x_0 = \tau(g_0) \in H\backslash G$, define a measure ν_{x_0} on Y by

$$d\nu_{x_0}(y) \stackrel{\text{def}}{=} \alpha(g_0,y)d\nu_Y(y)\,. \tag{12}$$

This is easily seen to be well defined (independent of the choice of g_0). Pushing forward the measures ν_{x_0} and ν_Y via the map $\tau|_Y$ defines measures on $H\backslash G$, which we denote by the same letters. Although ν_Y (as a measure on $H\backslash G$) depends on the choice of Y, the measure ν_{x_0} depends only on x_0. See Proposition 5.1 below for more details.

Theorem 2.3. *Assume the general setup holds, and conditions UC, D1, D2 are satisfied. For $x_0 \in H\backslash G$ and for every compactly supported continuous function φ on $H\backslash G$,*

$$\frac{1}{\lambda(H_T)} \int_{G_T} \varphi(x_0 g) dm(g) \underset{T\to\infty}{\longrightarrow} \int_{H\backslash G} \varphi \, d\nu_{x_0} . \tag{13}$$

Note that the conclusion of Theorem 2.3 may fail if condition D2 does not hold, see §12.2.

As an immediate corollary we obtain an equidistribution result for certain Γ-orbits. To state it, for $A \subset H\backslash G$ and $x_0 \in H\backslash G$, let

$$N_T(A, x_0) \stackrel{\text{def}}{=} \#\{\gamma \in \Gamma_T : x_0 \, \gamma \in A\} .$$

Corollary 2.4. *Assume the general setup holds, and conditions S, UC, D1, D2 are satisfied. Let $g_0 \in G$, $x_0 = \tau(g_0)$, and assume $(**)$ holds. Then for any $\varphi \in C_c(H\backslash G)$ we have*

$$\frac{1}{\lambda(H_T)} \sum_{\gamma\in\Gamma_T} \varphi(x_0\gamma) \underset{T\to\infty}{\longrightarrow} \int_{H\backslash G} \varphi \, d\nu_{x_0} , \tag{14}$$

and, for any bounded $A \subset H\backslash G$ with $\nu_{x_0}(\partial A) = 0$, we have

$$\frac{N_T(A, x_0)}{\lambda(H_T)} \underset{T\to\infty}{\longrightarrow} \nu_{x_0}(A) . \tag{15}$$

2.6 Distance functions. In order to apply our general results we must show that the conditions listed above hold in some cases of interest. Showing this can be quite complicated, and requires specific methods for specific cases. We verify the conditions under several hypotheses on D, G and H. In all cases we consider in this paper, G will be a Lie group and H a Lie subgroup of G. In this setup, condition S holds by a standard application of the implicit function theorem, see e.g. [W, Th. 3.58]. Note that if G is not assumed to be a Lie group, condition S might not hold (see §12.1). In this paper we will always assume furthermore that at least one of H and G is semisimple and connected.

2.6.1 Linear norms. Suppose G is realized, via a linear representation $\Psi : G \to \mathrm{GL}(d, \mathbb{R})$, as a closed subgroup of $\mathrm{Mat}_d(\mathbb{R})$, and $\|\cdot\|$ is a norm on $\mathrm{Mat}_d(\mathbb{R})$ (considered as a vector space). Assume that $\ker \Psi$ is compact. In this setting we call

$$D(g) \stackrel{\text{def}}{=} \max\{1, \|\Psi(g)\|\}$$

a *matrix norm distance function.* Although D depends on both Ψ and $\|\cdot\|$, to simplify notation we omit this dependence from the notation.

2.6.2 Symmetric space distance functions. Suppose G is a connected, semisimple Lie group, K a maximal compact subgroup of G, and H a semisimple subgroup. Thus $K\backslash G$ (resp. G/K) is the right (resp. left) symmetric space of G. It is equipped with a natural G-invariant Riemannian metric induced by the Cartan–Killing form on the Lie algebra $\mathfrak{g}$ of G (see [H1, Ch. IV] or [E, Ch. 2]). Let $P : G \to K\backslash G$ (resp. $P' : G \to G/K$) be the natural projection. Let d (resp. d') be the corresponding metric on $K\backslash G$ (resp. G/K). In this setting we call

$$D(g) \stackrel{\text{def}}{=} \exp\big(d(P(e), P(g))\big)$$

(respectively $\exp(d'(P'(e), P'(g)))$) a *symmetric space distance function.* Note that D is proper since K is compact and $x \mapsto d(x_0, x)$ is proper on $K\backslash G$.

Definition 2.5. *Suppose the general setup holds. We say that G, D are* standard *if G is semisimple and connected, and D is either a matrix norm distance function or a symmetric space distance function. We say that G, H, D are* standard *in either of the following two cases:*

- *H is semisimple and connected and D is a matrix norm distance function.*
- *Both G and H are semisimple and connected and D is a symmetric space distance function.*

Proposition 2.6. *If G, D are standard and G is balanced (see section 2.7 below) then UC, I1 and I2 are satisfied. If G, H, D are standard then conditions UC, D1, and D2 are satisfied.*

We deduce the proposition from the following two results, which are of independent interest, regarding the asymptotics of volumes of 'balls' with respect to a standard distance function.

Theorem 2.7. *Let H be connected noncompact semisimple Lie group and let $\Psi : H \to \mathrm{GL}(V)$ be a representation with compact kernel. Then for any matrix norm distance function we have*

$$\lambda(H_T) \sim C(\log T)^{\ell} T^m,$$

where $\ell \in \mathbb{Z}_+$ and $m > 0$ are explicitly computable constants depending only on Ψ, and $C > 0$ is an explicitly computable constant depending continuously on the choice of norm on $\mathrm{Mat}_d(\mathbb{R})$.

In §7 we introduce a technical condition, called condition $\mathcal{G}$, about Ψ (see Definition 7.1). The condition holds for 'most' choices of Ψ (cf. Remark 7.2). We prove Theorem 7.4, which is a precise version of Theorem 2.7, under the assumption that condition $\mathcal{G}$ holds, and assuming that Ψ

is irreducible. The proof of Theorem 2.7 without making these assumptions is a rather lengthy computation which will appear in a separate paper.

In §9 we treat the case of a symmetric space distance function, and prove

Theorem 2.8. *Let G be a connected semisimple Lie group, H a connected semisimple subgroup, and let $D : G \to [0, \infty)$ be a symmetric space distance function. There are non-negative constants ℓ and m, and for any $g_1, g_2 \in G$, a positive constant $C = C(g_1, g_2)$ such that*

$$\lambda\big(H_T[g_1, g_2]\big) \sim C(\log T)^{\ell} T^m. \tag{16}$$

The constants $\ell \geq 0$, $m > 0$, $C(g_1, g_2) > 0$ are given explicitly, $C(g_1, g_2)$ is continuous in g_1, g_2, and the convergence in (16) is uniform for g_1, g_2 in compact subsets of G.

2.7 Balanced semisimple groups. We will see below that *when H is simple* and G, H, D are standard, condition $(**)$ holds whenever $H\pi(g_0)$ is dense in G/Γ. However, in the case of a general semisimple group, we need to make an additional assumption.

Suppose that H is a semisimple non-simple Lie group with finite center, and thus a nontrivial almost direct product of its simple factors. It may happen that some of these factors do not contribute to the volume growth of balls in H, and therefore should be ignored when computing the asymptotics of H-orbits. To make this precise, we make the following definition.

Let $H = H_1 \cdots H_t$ be the decomposition of H into an almost direct product of its simple factors. Fix measurable sections $\sigma_i : H \to H_i$, $i = 1, \dots, t$; that is, $\sigma_i(h) \in H_i$ and $h = \sigma_1(h) \cdots \sigma_t(h)$ for each $h \in H$.

Definition 2.9. *We say that H is* balanced *if for every $j \in \{1, \dots, t\}$, every $g_1, g_2 \in G$ and every compact $L \subset H_j$,*

$$\lim_{T \to \infty} \frac{\lambda\left(\{h \in H_T[g_1, g_2] : \sigma_j(h) \in L\}\right)}{\lambda(H_T[g_1, g_2])} = 0\,. \tag{17}$$

Note that this definition does not depend on the choice of the sections because for any two sections σ_i, σ_i' we have $\sigma_i'(h) \in Z\sigma_i(h)$ for every $h \in H$, where Z is the (finite) center of H. Note also that if H has compact factors then H is not balanced by our definition.

We remark that if D is a matrix norm distance function, corresponding to a representation $\Psi : G \to \mathrm{GL}_d(\mathbb{R}) \subset \mathrm{Mat}_d(\mathbb{R})$ with compact kernel, and a norm $\|\cdot\|$ on $\mathrm{Mat}_d(\mathbb{R})$, then the condition that H is balanced depends on Ψ but not on $\|\cdot\|$ (see Proposition 8.1).

The importance of the assumption that H is balanced lies in the following:

Theorem 2.10. *Suppose the general setup holds, G, H, D are standard and H is balanced. Suppose $g_0 \in G$ satisfies $\overline{H\pi(g_0)} = G/\Gamma$. Then $(**)$ holds.*

Note that the conclusion of Theorem 2.10 may fail if H is not balanced, see §12.3.

2.8 Equidistribution results. Collecting the results stated previously, we obtain the following results concerning the distribution of lattice orbits:

COROLLARY 2.11. *Suppose the general setup holds, G, D are standard, and G is balanced. Suppose there is a Γ-invariant probability measure μ on $H\backslash G$, and $x_0 \in H\backslash G$ satisfies $H\backslash G = \overline{x_0\Gamma}$.*

Then for any $\varphi \in C_c(H\backslash G)$,

$$\frac{1}{\#\Gamma_T} \sum_{\gamma \in \Gamma_T} \varphi(x_0\gamma) \underset{T\to\infty}{\longrightarrow} \int_{H\backslash G} \varphi \, d\mu \, .$$

COROLLARY 2.12. *Suppose the general setup holds, G, H, D are standard, and H is balanced. Suppose $x \in H\backslash G$ with $H\backslash G = \overline{x\Gamma}$. Then for any $\varphi \in C_c(V)$,*

$$S_{\varphi,x}(T) \sim \widetilde{S}_{\varphi,x}(T) \tag{18}$$

and

$$\frac{S_{\varphi,x}(T)}{\lambda(H_T)} \underset{T\to\infty}{\longrightarrow} \int_{H\backslash G} \varphi \, d\nu_x \, . \tag{19}$$

Also, for any bounded $A \subset H\backslash G$ with $\nu_{H\backslash G}(\partial A) = 0$,

$$\lim_{T\to\infty} \frac{N_T(A,x)}{\lambda(H_T)} = \nu_x(A) \, .$$

2.9 Organization of the paper. In section 3, we develop a framework for analyzing action preserving finite measure and prove a general version of Proposition 2.1. In section 4, we prove Theorem 2.2, which relates distributions of Γ orbits in $H\backslash G$ and H-orbits in G/Γ. In section 5, we discuss the limiting density and prove Theorem 2.3. In section 6, we verify some of the basic properties of distance functions and prove Theorem 2.6. In section 7, we compute volume asymptotics for sets defined by matrix norms proving Theorem 2.7, and in section 9, we compute asymptotics for the Riemannian metric case proving Theorem 2.8. Section 8 contains discussion of 'balancedness'. In section 10, we prove Theorem 2.10. Finally, the results stated in section 1 are proved in section 11. In section 12, we present examples to demonstrate necessity of conditions S, D2, and 'balanced', and discuss the example from section 1.1 in more detail.

3 Induction

Let Γ act continuously on a locally compact Hausdorff space X from the right. In this section we recall the definition of the *induced action* of G, and relate the asymptotics of a Γ-orbit with that of a corresponding G-orbit.

On $\widetilde{Y} \stackrel{\text{def}}{=} X \times G$, there is a left action of G given by

$$g_1 \cdot (x, g) = (x, g_1 g)\,,$$

and a right action of Γ given by

$$(x, g) \cdot \gamma = (x \cdot \gamma, g\gamma)\,.$$

It is clear that these actions commute, so that G acts on the quotient $Y = \widetilde{Y}/\Gamma$, and the quotient map $\pi_{G/\Gamma} : Y \to G/\Gamma$ is a continuous G-map. Since Γ is discrete in G, the projection is a covering map, so that Y inherits (locally) the topological properties of $\widetilde{Y}$. As a G-space, Y is a fiber bundle with base G/Γ and fiber isomorphic to X. In particular, if μ is a Γ-invariant probability measure on X, then $d\mu\, dm'$ defines a finite G-invariant measure on Y.

Assume that $\sigma : G/\Gamma \to G$ is a Borel section. Then

$$\gamma(g) = g^{-1}\sigma(\pi(g)) \in \Gamma$$

and the map

$$\pi_X = \pi_X^\sigma : Y \to X\,, \quad \pi_X : y = [(x, g)] \mapsto x \cdot \gamma(g)$$

is well defined (does not depend on the representative $(x, g) \in \widetilde{Y}$ of the Γ-orbit $[(x, g)] \in Y$).

Note that when $X = H\backslash G$ is as in §2.2, the space Y is isomorphic to $H\backslash G \times G/\Gamma$, with the action given by (6) where the isomorphism is defined by the map

$$(x, g) \mapsto (xg^{-1}, g)\,, \quad x \in X\,,\ g \in G\,.$$

The following reduces the study of asymptotic behavior of Γ-orbits on X to that of G-orbits on Y. It immediately implies Proposition 2.1.

PROPOSITION 3.1. *Suppose hypotheses I1 and UC hold. Let $x_0 \in X$, let $\widetilde{y}_0 = (x_0, e) \in \widetilde{Y}$ and let $y_0 = [\widetilde{y}_0] \in Y$. Suppose that there is a measure μ on X such that for any $F \in C_c(Y)$,*

$$\frac{1}{m(G_T)} \int_{G_T} F(g^{-1}y_0)\,dm(g) \underset{T\to\infty}{\longrightarrow} \int_Y F\, d\nu\,, \tag{20}$$

where $\nu = d\mu\, dm'$.

Then for any $\varphi \in C_c(X)$,

$$\frac{1}{m(G_T)} \sum_{\gamma \in \Gamma_T} \varphi(x_0 \cdot \gamma) \underset{T\to\infty}{\longrightarrow} \int_X \varphi \, d\mu \, . \tag{21}$$

If in addition I2 holds then

$$\frac{1}{\#\Gamma_T} \sum_{\gamma \in \Gamma_T} \varphi(x_0 \cdot \gamma) \underset{T\to\infty}{\longrightarrow} \int_X \varphi \, d\mu \, . \tag{22}$$

Proof of Proposition 3.1. There is no loss of generality in assuming that $\varphi \geq 0$.

Let $\varepsilon > 0$. By condition I1, there are positive δ, T_0 such that, for all $T > T_0$,

$$\frac{m(G_{(1+\delta)T})}{m(G_T)} < 1 + \varepsilon \, . \tag{23}$$

By condition UC, there exists a symmetric neighborhood $\mathcal{O}$ of the identity in G such that for every $T > 0$,

$$G_T \mathcal{O} \subset G_{(1+\delta)T} \, . \tag{24}$$

Let $\chi \in C_c(G)$ be a non-negative function such that $\operatorname{supp} \chi \subset \mathcal{O}$ and $\int_G \chi \, dm = 1$. Define functions $\widetilde{F} : \widetilde{Y} \to \mathbb{R}$, $F : Y \to \mathbb{R}$ by

$$\widetilde{F}(\widetilde{y}) = \widetilde{F}(x, g) = \chi(g)\varphi(x) \, , \quad F(y) = \sum_{\gamma \in \Gamma} \widetilde{F}(\widetilde{y}\gamma) \, ,$$

where $\widetilde{y} \in \widetilde{Y}$, $y = [\widetilde{y}] \in Y$. Since $\widetilde{F}$ is compactly supported, the sum in the definition of F is actually finite, $F \in C_c(Y)$, and

$$\begin{aligned}
\int_Y F \, d\nu &= \int_{G/\Gamma} \int_X F(y) d\mu\big(\pi_X(y)\big) dm'\big(\pi_{G/\Gamma}(y)\big) \\
&= \sum_{\gamma \in \Gamma} \int_{\sigma(G/\Gamma)} \int_X \chi(g\gamma)\varphi(x \cdot \gamma) d\mu(x) dm(g) \\
&= \int_X \varphi \, d\mu \left(\sum_{\gamma \in \Gamma} \int_{\sigma(G/\Gamma)} \chi(g\gamma) dm(g) \right) \\
&= \int_X \varphi \, d\mu \int_G \chi \, dm = \int_X \varphi \, d\mu \, .
\end{aligned}$$

Applying (20) we obtain

$$\frac{1}{m(G_T)} \int_{G_T} F(g^{-1} y_0) dm(g) \underset{T\to\infty}{\longrightarrow} \int_X \varphi \, d\mu \, . \tag{25}$$

Now let

$$
\begin{aligned}
I_T(x_0) &\stackrel{\text{def}}{=} \sum_{\gamma\in\Gamma}\left(\int_{G_T}\chi(g^{-1}\gamma)dm(g)\right)\varphi(x_0\cdot\gamma)\\
&= \sum_{\gamma\in\Gamma}\int_{G_T}\widetilde{F}\big((x_0,g^{-1})\cdot\gamma\big)dm(g) \qquad (26)\\
&= \int_{G_T}F(g^{-1}y_0)dm(g)\,.
\end{aligned}
$$

We claim that

$$
I_{T/(1+\delta)}(x_0)\le\sum_{\gamma\in\Gamma_T}\varphi(x_0\cdot\gamma)\le I_{(1+\delta)T}(x_0)\,. \qquad (27)
$$

Let $g\in G\smallsetminus G_{(1+\delta)T}$. Then by (24), $g\notin G_T\mathcal{O}$, and hence $g^{-1}\gamma\notin\mathcal{O}$ for all $\gamma\in\Gamma_T$. This implies that $\chi(g^{-1}\gamma)=0$, that is

$$
\gamma\in\Gamma_T\Longrightarrow\int_{G_{(1+\delta)T}}\chi(g^{-1}\gamma)dm(g)=\int_G\chi(g^{-1}\gamma)dm(g)=1\,.
$$

The right-hand inequality in (27) follows.

Now if $\gamma\in\Gamma\smallsetminus\Gamma_T$ then for all $g\in G_{T/(1+\delta)}$ we have $g^{-1}\gamma\notin\mathcal{O}$, hence $\chi(g^{-1}\gamma)=0$ and so

$$
\gamma\notin\Gamma_T\Longrightarrow\int_{G_{T/(1+\delta)}}\chi(g^{-1}\gamma)dm(g)=0\,.
$$

This implies the second inequality in (27).

We obtain, for all large enough T,

$$
\begin{aligned}
\sum_{\gamma\in\Gamma_T}\varphi(x_0\cdot\gamma) &\stackrel{(27)}{\le} I_{T(1+\delta)}(x_0)\\
&\stackrel{(25)}{\le} (1+\varepsilon)m(G_{(1+\delta)T})\int_X\varphi\,d\nu\\
&\stackrel{(23)}{\le} (1+\varepsilon)^2m(G_T)\int_X\varphi\,d\nu\,.
\end{aligned}
$$

Since $\varepsilon>0$ was arbitrary,

$$
\limsup_{T\to\infty}\frac{1}{m(G_T)}\sum_{\gamma\in\Gamma_T}\varphi(x_0\cdot\gamma)\le\int_X\varphi\,d\nu\,.
$$

The opposite inequality for lim inf is similarly established. This proves (21), and (22) immediately follows in light of condition I2. $\square$

4 Duality

In this section we prove Theorem 2.2.

We first record the following useful consequence of condition UC.

PROPOSITION 4.1. *Suppose the general setup holds and condition UC is satisfied. Then for any bounded $B \subset G$ and any $\eta > 0$ there is a neighborhood $\mathcal{O}$ of identity in G such that, for all $g \in G$, $h \in \mathcal{O}$, $b \in B$, we have*

$$\left| \frac{D(ghb)}{D(gb)} - 1 \right| < \eta \,. \tag{28}$$

Proof. Let $\varepsilon = \eta$ and let $\mathcal{U}$ be as in the formulation of condition UC. Since B is bounded, there is a small enough neighborhood $\mathcal{O}$ of identity in G such that for all $h \in \mathcal{O}$, $b \in B$, we have $b^{-1}hb \in \mathcal{U} \cap \mathcal{U}^{-1}$. This implies that, for $g \in G$, $h \in \mathcal{O}$, $b \in B$,

$$D(ghb) = D(gb\,b^{-1}hb) < (1+\eta)D(gb)$$

and

$$D(gb) = D(ghb\,b^{-1}h^{-1}b) \leq (1+\eta)D(ghb)\,.$$

Putting these together we obtain

$$\frac{D(gb)}{1+\eta} < D(ghb) < (1+\eta)D(gb)\,,$$

and (28) follows. □

We now make some reductions:

CLAIM 1. *There is no loss of generality in assuming:*

(a) $g_0 = e$.

(b) *There is an open set $\mathcal{U} \subset H\backslash G$ and a Borel section $\sigma : H\backslash G \to G$ such that $\sigma|_{\mathcal{U}}$ is continuous and* $\operatorname{supp}\varphi \subset \mathcal{U}$.

In order to reduce to (a), let $H' \stackrel{\text{def}}{=} g_0^{-1}Hg_0$, λ' be a left-Haar measure on H', $\tau' : G \to H'\backslash G$ the natural quotient map and $\varphi'(\tau'(g)) \stackrel{\text{def}}{=} \varphi(\tau(g_0g))$, and note that the hypotheses and conclusion of the theorem for H',λ',τ',φ',e are equivalent to those for $H, \lambda, \tau, \varphi, g_0$.

Using condition S, for each $x \in \operatorname{supp}\varphi$ there is a section $\sigma_x : H\backslash G \to G$ which is continuous in a neighborhood $\mathcal{U}_x$ of x. Using the compactness of $\operatorname{supp}\varphi$ we may choose a finite subset of $\{\mathcal{U}_x : x \in \operatorname{supp}\varphi\}$ covering $\operatorname{supp}\varphi$, and using a standard partition of unity argument for this cover we may assume that, for some $x \in H\backslash G$, $\operatorname{supp}\varphi \subset \mathcal{U}_x$ and take $\sigma = \sigma_x$. This reduces our problem to the case (b). The claim is proved.

Let

$$B \stackrel{\text{def}}{=} \sigma(\operatorname{supp}\varphi)\,,$$

a compact subset of G, in view of Claim 1. Also let

$$h_g \overset{\text{def}}{=} \widetilde{\sigma}(g)g^{-1} \in H\,, \quad \text{where } \widetilde{\sigma} \overset{\text{def}}{=} \sigma \circ \tau : G \to G\,.$$

Given $c > 1$ we will find T_0 such that, for all $T > T_0$,

$$\sum_{\gamma \in \Gamma_T} \varphi(\tau(\gamma)) < c \int_{G_T} \varphi(g)dm(g)\,. \tag{29}$$

Let

$$0 < \varepsilon < \sqrt{c} - 1\,, \tag{30}$$

and let $\delta > 0$ be as in hypothesis D1. Let $\eta > 0$ be small enough so that

$$(1+\eta)^4 < 1 + \delta\,. \tag{31}$$

Using hypothesis UC and Proposition 4.1, let $\mathcal{O}$ be a small-enough neighborhood of the identity in H so that, for all $g \in G$, $h \in \mathcal{O}$, $b \in B$,

$$D(ghb) < (1+\eta)D(gb)\,.$$

Let $\psi : H \to \mathbb{R}$ be a non-negative continuous function such that

$$\operatorname{supp}\psi \subset \mathcal{O} \quad \text{and} \quad \int_H \psi(h)d\lambda(h) = 1\,.$$

Define a function $f : G \to \mathbb{R}$ by

$$f(g) = \psi(h_g)\varphi(\tau(g))\,.$$

Note that f is non-negative, continuous, of compact support, and

$$\tau(g) \notin \operatorname{supp}\varphi \Longrightarrow \forall h \in H\,,\ f(h^{-1}g) = 0\,. \tag{32}$$

We now show

Claim 2. *For any $T > 0$,*

$$\sum_{\gamma \in \Gamma_T} \varphi(\tau(\gamma)) \leq \sum_{\gamma \in \Gamma} \int_{H_{(1+\eta)T}[\widetilde{\sigma}(\gamma)]} f(h^{-1}\gamma)d\lambda(h) \tag{33}$$

and

$$\int_{G_T} \varphi(\tau(g))dm(g) \geq \int_G \int_{H_{T/(1+\eta)}[\widetilde{\sigma}(g)]} f(h^{-1}g)d\lambda(h)dm(g)\,. \tag{34}$$

Let $g \in G_T$, that is, $D(h_g^{-1}\widetilde{\sigma}(g)) = D(g) < T$. Suppose $\tau(g) \in \operatorname{supp}\varphi$. Then for $h \in \mathcal{O}$ one has

$$D\big(h_g^{-1}h\widetilde{\sigma}(g)\big) < (1+\eta)T\,,$$

that is, $h_g^{-1}h \in H_{(1+\eta)T}[\widetilde{\sigma}(g)]$. Since $\operatorname{supp}\psi \subset \mathcal{O}$, this shows that

$$\operatorname{supp}\psi \subset h_g H_{(1+\eta)T}[\widetilde{\sigma}(g)]\,.$$

So, using left-invariance of λ and the fact that $h_g h_0 = h_{h_0^{-1}g}$,

$$\begin{aligned}\varphi(\tau(g)) &= \varphi(\tau(g)) \int_{h_g H_{(1+\eta)T}[\widetilde{\sigma}(g)]} \psi(h) d\lambda(h) \\ &= \varphi(\tau(g)) \int_{H_{(1+\eta)T}[\widetilde{\sigma}(g)]} \psi(h_g h) d\lambda(h) \\ &= \int_{H_{(1+\eta)T}[\widetilde{\sigma}(g)]} \psi(h_{h^{-1}g}) \varphi\big(\tau(h^{-1}g)\big) d\lambda(h) \\ &= \int_{H_{(1+\eta)T}[\widetilde{\sigma}(g)]} f(h^{-1}g) d\lambda(h)\,.\end{aligned}$$

Specializing to $g = \gamma \in \Gamma_T$ one obtains (33).

Let $\widetilde{G} \stackrel{\text{def}}{=} \tau^{-1}(\operatorname{supp}\varphi)$. Similarly, it is easy to check that for $g \in \widetilde{G} \smallsetminus G_T$,

$$\operatorname{supp}\psi \cap h_g H_{T/(1+\eta)}[\widetilde{\sigma}(g)] = \varnothing\,.$$

Therefore, arguing as in the previous computation,

$$\begin{aligned}&\int_G \int_{H_{T/(1+\eta)}[\widetilde{\sigma}(g)]} f(h^{-1}g) d\lambda(h) dm(g) \\ &\quad\stackrel{(32)}{=} \int_{\widetilde{G}} \int_{H_{T/(1+\eta)}[\widetilde{\sigma}(g)]} f(h^{-1}g) d\lambda(h) dm(g) \\ &\quad\le \left[\int_{G_T} + \int_{\widetilde{G}\smallsetminus G_T}\right] \left(\varphi(\tau(g)) \int_{h_g H_{T/(1+\eta)}[\widetilde{\sigma}(g)]} \psi(h) d\lambda(h)\right) dm(g) \\ &\quad= \int_{G_T} \varphi(\tau(g)) \int_{h_g H_{T/(1+\eta)}[\widetilde{\sigma}(g)]} \psi(h) d\lambda(h) dm(g) \\ &\quad\le \int_{G_T} \varphi(\tau(g)) dm(g)\,.\end{aligned}$$

This proves (34).

We now claim

Claim 3. *There are $b_1, \dots, b_N \in B$ and a symmetric neighborhood $\mathcal{O}$ of the identity in G, such that, for $x \in \mathcal{O} b_i$, $i = 1, \dots, N$,*

$$H_{(1+\eta)T}[x] \subset H_{(1+\eta)^2 T}[b_i]$$

and such that

$$B \subset \bigcup_{i=1}^{N} \mathcal{O} b_i\,. \tag{35}$$

To see this, using hypothesis UC and Proposition 4.1 we let $\mathcal{O}$ be a small enough symmetric neighborhood of identity in G such that for all $g \in H$, $h \in \mathcal{O}$ and $b \in B$, we have $(1 - \eta/2) D(gb) < D(ghb)$. This implies that $H_{(1+\eta)T}[x] \subset H_{(1+\eta)^2T}[b]$ for all $x \in \mathcal{O}b$. Taking a finite subcover $\{\mathcal{O}b_i : i = 1, \ldots, N\}$ of the cover $\{\mathcal{O}b : b \in B\}$ proves the claim.

Using a partition of unity subordinate to the cover (35), there is no loss of generality in assuming that for some $b \in B$,

$$\operatorname{supp}\varphi \subset \tau(\mathcal{O}b)\,. \tag{36}$$

Thus for some $b \in B$, and for all $\gamma \in \Gamma$,

$$\int_{H_{(1+\eta)T}[\widetilde{\sigma}(\gamma)]} f(h^{-1}\gamma)d\lambda(h) \le \int_{H_{(1+\eta)^2T}[b]} f(h^{-1}\gamma)d\lambda(h)\,. \tag{37}$$

Now defining

$$F(\pi(g)) \stackrel{\text{def}}{=} \sum_{\gamma\in\Gamma} f(g\gamma)$$

(actually a finite sum for each g) one obtains, via the monotone convergence theorem,

$$\begin{aligned}
&\sum_{\gamma\in\Gamma}\int_{H_{(1+\eta)T}[\widetilde{\sigma}(\gamma)]} f(h^{-1}\gamma)d\lambda(h)\\
&\quad\stackrel{(37)}{\le} \sum_{\gamma\in\Gamma}\int_{H_{(1+\eta)^2T}[b]} f(h^{-1}\gamma)d\lambda(h) \qquad (38)\\
&\quad= \int_{H_{(1+\eta)^2T}[b]} F\big(h^{-1}\pi(e)\big)d\lambda(h)\,.
\end{aligned}$$

Using $(**)$, there is T_1 such that for all $T > T_1$,

$$\left|\frac{\int_{H_{(1+\eta)^2T}[b]} F(h^{-1}\pi(e))d\lambda(h)}{\lambda(H_{(1+\eta)^2T}[b])} - \int_{G/\Gamma} F(x)dm'(x)\right| < \varepsilon\,,$$

hence

$$\int_{H_{(1+\eta)^2T}[b]} F\big(h^{-1}\pi(e)\big)d\lambda(h) \le (1+\varepsilon)\lambda\big(H_{(1+\eta)^2T}[b]\big)\int_{G/\Gamma} F(x)dm'(x)\,. \tag{39}$$

Reversing the argument with G in place of Γ yields

$$\begin{aligned}
&\lambda\big(H_{(1+\eta)^2T}[b]\big)\int_{G/\Gamma} F(x)dm'(x)\\
&\quad\stackrel{(8),(31)}{\le} (1+\varepsilon)\lambda\big(H_{T/(1+\eta)^2}[b]\big)\int_G f(g)dm(g)\\
&\quad= (1+\varepsilon)\int_{H_{T/(1+\eta)^2}[b]}\int_G f(h^{-1}g)dm(g)d\lambda(h)\\
&\quad= (1+\varepsilon)\int_G\int_{H_{T/(1+\eta)^2}[b]} f(h^{-1}g)d\lambda(h)dm(g)
\end{aligned}$$

$$\overset{(32),(36)}{\leq} (1+\varepsilon)\int_G \int_{H_{T/(1+\eta)}[\widetilde{\sigma}(g)]} f(h^{-1}g)d\lambda(h)dm(g)$$

$$\overset{(34)}{\leq} (1+\varepsilon)\int_{G_T} \varphi(\tau(g))dm(g)\,. \tag{40}$$

Now putting together (33), (38), (39), (40) and (30) proves (29).

The proof of the second inequality

$$\sum_{\gamma\in\Gamma_T} \varphi(\tau(\gamma)) > \frac{1}{c}\int_{G_T} \varphi(\tau(g))dm(g) \tag{41}$$

is similar and is left to the reader. The two inequalities (29) and (41), along with the assumption that $g_0 = e$, imply (10). □

5 The Limiting Density

Let the notation be as in §2.5. In particular assume condition D2. In this section we discuss the measures $\{\nu_x : x \in H\backslash G\}$ and prove Theorem 2.3. We first introduce some additional notation.

Let $N_G(H)$ denote the normalizer of H in G. There is a homomorphism $\Delta : N_G(H) \to \mathbb{R}_+$ such that for any Borel subset A of H and any $n \in N_G(H)$,

$$\lambda(nAn^{-1}) = \Delta(n)\lambda(A)\,. \tag{42}$$

Note that $\Delta|_H$ is the modular function of H.

PROPOSITION 5.1. *Let $g_0 \in G$, $x_0 = \tau(g_0)$. Suppose that condition D2 is satisfied. Then $\alpha(g_1, g_2)$ and the measure ν_{x_0} have the following properties:*

(i) *For $h \in H$, $n \in N_G(H)$ and $g_1, g_2 \in G$,*

$$\alpha(nhg_1, ng_2) = \Delta(n)\alpha(g_1, g_2)\,.$$

In particular, for $h_1, h_2 \in H$, $g_1, g_2 \in G$,

$$\alpha(h_1g_1, h_2g_2) = \Delta(h_2)\alpha(g_1, g_2)\,.$$

(ii) *ν_{x_0} is well defined (does not depend on g_0) and does not depend on the section Y.*

(iii) *If G, H are Lie groups then ν_{x_0} is absolutely continuous with respect to the smooth measure class on $H\backslash G$.*

(iv) *If H is unimodular then*

$$\alpha_{x_0} : H\backslash G \to \mathbb{R}_+\,, \quad \alpha_{x_0}(\tau(g)) = \alpha(g_0, g)\,,$$

is well defined and $d\nu_{x_0}(x) = \alpha_{x_0}(x)d\nu_{H\backslash G}(x)$, where $\nu_{H\backslash G}$ is the G-invariant measure on $H\backslash G$.

Proof. For $h \in H$, $n \in N_G(H)$, $g_1, g_2 \in G$ we have

$$\begin{aligned} H_T\big[(nhg_1)^{-1}, ng_2\big] &= \big\{h' \in H : D(g_1^{-1}(nh)^{-1}h'ng_2) < T\big\} \\ &= \big\{h' \in H : D(g_1^{-1}(nh)^{-1}nn^{-1}h'ng_2) < T\big\} \\ &= n\big\{h'' \in H : D(g_1^{-1}h^{-1}h''g_2) < T\big\}n^{-1} \\ &= nh\big\{h''' \in H : D(g_1^{-1}h'''g_2) < T\big\}n^{-1} \\ &= nhH_T[g_1^{-1}, g_2]n^{-1}, \end{aligned}$$

hence, using (42) and the left-invariance of λ,

$$\lambda\big(H_T[(nhg_1)^{-1}, ng_2]\big) = \Delta(n)\lambda\big(H_T[g_1, g_2]\big)$$

and (i) follows.

It follows from (i) that $\alpha(hg_0, y) = \alpha(g_0, y)$ for all $h \in H$, so ν_{x_0} is well defined. Suppose Y, Z are two different sections for τ. There are maps $\bar{h} : Z \to H$, $\bar{y} : Z \to Y$ defined by the formula $z = \bar{h}(z)\bar{y}(z)$ for all $z \in Z$. Let $\bar{y}_*\nu_Z$ denote the measure on Y obtained by pushing forward ν_Z via $\bar{y}$. Using (11) we have for each $h \in H$,

$$\begin{aligned} dm(hz) = dm\big(h\bar{h}(z)\bar{y}(z)\big) &= d\lambda(h\bar{h}(z))d\nu_Y(\bar{y}(z)) \\ &= \Delta(\bar{h}(z))d\lambda(h)d\bar{y}_*\nu_Z(y)\,. \end{aligned}$$

By the uniqueness of the decomposition (11) we have $\Delta(\bar{h}(z))d\bar{y}_*\nu_Z(y) = d\nu_Y(y)$. In particular, setting $\bar{y}(z) = y$, $\bar{h}(z) = h = zy^{-1}$ we have

$$\begin{aligned} \alpha(g_0, y)d\nu_Y(y) &= \alpha(g_0, yz^{-1}z)\Delta(zy^{-1})d\bar{y}_*\nu_Z(y) \\ &= \Delta(yz^{-1})\alpha(g_0, z)\Delta(zy^{-1})d\nu_Z(z) \\ &= \alpha(g_0, z)d\nu_Z(z)\,, \end{aligned}$$

proving (ii).

If H is a Lie subgroup of a Lie group G then there is a manifold structure on $H\backslash G$ which is obtained locally by finding, for each $x_0 \in H\backslash G$, a submanifold $\mathcal{U} \subset G$ such that $x_0 \in \tau(\mathcal{U})$ and $\tau|_{\mathcal{U}}$ is one to one. Now, using (ii), we can assume that the lift Y contains $\mathcal{U}$. It is clear from (11) and G-invariance of m that $\nu_Y|_{\mathcal{U}}$ is smooth and (iii) follows.

The last assertion follows immediately from the preceding ones. □

Proof of Theorem 2.3. By separating φ into its negative and positive parts, we may assume without loss of generality that $\varphi \geq 0$. Assume also that $\varphi \neq 0$ (otherwise there is nothing to prove). Let $Y = \sigma(H\backslash G)$ where $\sigma : H\backslash G \to G$ is a Borel section, and let $B = \sigma(\operatorname{supp}\varphi)$. We may assume that B is bounded, e.g. by applying condition S and a partition of unity argument. Let ν_Y be the measure on Y satisfying (11). In view of

Proposition 5.1, there is no loss of generality in defining ν_{x_0} by (12). We have

$$\begin{aligned}\int_{G_T} \varphi\big(\tau(g_0)g\big)dm(g) &= \int_{\{g:D(g_0^{-1}g)<T\}} \varphi(\tau(g))dm(g)\\ &= \int_Y \int_{\{h:D(g_0^{-1}hy)<T\}} \varphi(\tau(y))d\lambda(h)d\nu_Y(y) \qquad (43)\\ &= \int_Y \varphi(\tau(y))\lambda\big(H_T[g_0^{-1},y]\big)d\nu_Y(y)\,.\end{aligned}$$

Let T_0 be as in condition D1. We claim that there is C such that for all $T > T_0$ and all $y \in B$,

$$\frac{\lambda(H_T[g_0^{-1},y])}{\lambda(H_T)} < C\,.$$

Let $C_1 > 1$. By Proposition 4.1, there exists a neighborhood $\mathcal{O}$ of identity in G such that for all $h \in \mathcal{O}$ and $b \in B$, $D(g_0^{-1}hb) < C_1 D(g_0^{-1}b)$. Choose a finite cover $B \subset \cup_{i=1}^N \mathcal{O}b_i$ for some $b_i \in B$. Then for every $y \in B$,

$$H_T[g_0^{-1},y] \subset \bigcup_{i=1}^N H_{C_1T}[g_0^{-1},b_i]$$

and

$$\lambda\big(H_T[g_0^{-1},y]\big) \le N \max_i \lambda\big(H_{C_1T}[g_0^{-1},b_i]\big)\,.$$

Using condition D1 a finite number of times (depending on C_1) we find a constant C_2 such that $\lambda(H_{C_1T}[g_0^{-1},b_i]) \le C_2\lambda(H_T[g_0^{-1},b_i])$. Since the limit defining $\alpha(g_0,b_i)$ exists, there is a constant $C_3 > 0$ such that

$$\frac{\lambda(H_T[g_0^{-1},b_i])}{\lambda(H_T)} < C_3\,, \quad \text{for all } T > T_0 \text{ and } i = 1,\dots,N\,.$$

Therefore

$$\frac{\lambda(H_T[g_0^{-1},y])}{\lambda(H_T)} < N\,C_2\,C_3 \stackrel{\text{def}}{=} C\,.$$

Hence, by (9), (43), and Lebesgue's dominated convergence theorem,

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{\lambda(H_T)}\int_{G_T}\varphi\big(\tau(g_0)g\big)dm(g) &= \int_B \varphi(\tau(y))\lim_{T\to\infty}\frac{\lambda(H_T[g_0^{-1},y])}{\lambda(H_T)}\,d\nu_Y(y)\\ &= \int_B \varphi(\tau(y))\alpha(g_0,y)d\nu_Y(y)\\ &= \int_Y \varphi\,d\nu_{x_0}\,. \qquad \square\end{aligned}$$

Proof of Corollary 2.4. Statement (14) is immediate from (10) and (13), and (15) follows by a standard argument for approximating 1_A from above and below by continuous functions, which we omit. □

6 Properties of Distance Functions

Our next goal will be to verify the conditions we have listed. Most of the verification is reduced to the computation of asymptotics of volumes of certain sets. We make the reduction in this section, and also list some simple relations between our conditions.

We first show that I2 follows from I1, UC, and a very special case of $(**)$. Namely, we have

PROPOSITION 6.1. *Suppose that I1 and UC hold, and for any* $F \in C_c(G/\Gamma)$,

$$\frac{1}{m(G_T)} \int_{G_T} F\big(g^{-1}\pi(e)\big)dm(g) \underset{T\to\infty}{\longrightarrow} \int_{G/\Gamma} F\,dm'. \tag{44}$$

Then I2 holds, that is, $\#\Gamma_T \sim m(G_T)$ *as* $T \to \infty$.

Proof. Let $\varepsilon > 0$. By UC, there exists a symmetric neighborhood $\mathcal{O}$ of e in G such that

$$G_T\mathcal{O} \subset G_{(1+\varepsilon)T} \tag{45}$$

for every $T > 0$. Let $f \in C_c(G)$ be such that

$$f \geq 0\,, \quad \operatorname{supp}(f) \subset \mathcal{O}\,, \quad \int_G f\,dm = 1\,, \tag{46}$$

and let $F \in C_c(G/\Gamma)$ be defined by $F(\pi(g)) = \sum_{\gamma\in\Gamma} f(g\gamma)$. We have

$$\begin{aligned}
\int_{G_T} F\big(\pi(g^{-1})\big)dm(g) &= \sum_{\gamma\in\Gamma}\int_{G_T} f(g^{-1}\gamma)dm(g)\\
&\overset{(45),(46)}{=} \sum_{\gamma\in\Gamma_{(1+\varepsilon)T}} \int_{G_T} f(g^{-1}\gamma)dm(g)\\
&= \sum_{\gamma\in\Gamma_{(1+\varepsilon)T}} \int_{G_T^{-1}\gamma} f\,dm \overset{(46)}{\leq} \#\Gamma_{(1+\varepsilon)T}\,,
\end{aligned}$$

and

$$\begin{aligned}
\int_{G_T} F\big(\pi(g^{-1})\big)dm(g) &= \sum_{\gamma\in\Gamma}\int_{G_T} f(g^{-1}\gamma)dm(g)\\
&\overset{(45),(46)}{\geq} \sum_{\gamma\in\Gamma_{T/(1+\varepsilon)}} \int_{G_T} f\,dm = \#\Gamma_{T/(1+\varepsilon)}\,.
\end{aligned}$$

This implies that

$$\begin{aligned}
\limsup_{T\to\infty}\frac{\#\Gamma_T}{m(G_T)} &\leq \limsup_{T\to\infty}\frac{1}{m(G_T)}\int_{G_{(1+\varepsilon)T}} F\big(\pi(g^{-1})\big)dm(g)\\
&\overset{(44)}{\leq} \limsup_{T\to\infty}\frac{m(G_{(1+\varepsilon)T})}{m(G_T)}\int_{G/\Gamma} F\,dm' = \limsup_{T\to\infty}\frac{m(G_{(1+\varepsilon)T})}{m(G_T)}
\end{aligned}$$

for every $\varepsilon > 0$. Thus, by I1,

$$\limsup_{T\to\infty} \frac{\#\Gamma_T}{m(G_T)} \leq 1 .$$

The lower estimate for lim inf is proved similarly. □

PROPOSITION 6.2. *If D is a matrix norm distance function then D satisfies condition UC.*

Proof. The set

$$\mathrm{Mat}^1_d(\mathbb{R}) \stackrel{\text{def}}{=} \left\{ z \in \mathrm{Mat}_d(\mathbb{R}) : \|z\| = 1 \right\}$$

is compact. Hence, given $\varepsilon > 0$, there is a small enough neighborhood $\mathcal{V}$ of identity in $\mathrm{Mat}_d(\mathbb{R})$ such that for all $x \in \mathcal{V}$ and all $z \in \mathrm{Mat}^1_d(\mathbb{R})$,

$$\|zx\| < (1+\varepsilon) .$$

Let $\mathcal{U} = \Psi^{-1}(\mathcal{V})$. There is a bounded set $B \subset G$ such that for all $g \in G \smallsetminus B$ and all $g' \in \mathcal{U}$ we have $\|\Psi(g)\| > 1$ and $\|\Psi(gg')\| > 1$ so that

$$D(g) = \|\Psi(g)\| \quad \text{and} \quad D(gg') = \left\|\Psi(gg')\right\| .$$

For any $g \in G \smallsetminus B$, $z = \frac{\Psi(g)}{\|\Psi(g)\|} \in \mathrm{Mat}^1_d(\mathbb{R})$, hence for any $g' \in \mathcal{U}$ we have

$$\frac{D(gg')}{D(g)} = \frac{\|\Psi(gg')\|}{\|\Psi(g)\|} = \left\|z\Psi(g')\right\| < 1+\varepsilon .$$

By making $\mathcal{U}$ smaller if necessary we can also ensure that

$$\frac{D(gg')}{D(g)} < 1+\varepsilon$$

for all $g \in B$ and $g' \in \mathcal{U}$. This implies UC. □

PROPOSITION 6.3. *Suppose (X,d) is a metric space equipped with a right (resp. left) action of G which is continuous and isometric. Then for any $x_0 \in X$, the function*

$$D(g) = \exp\big(d(x_0, x_0 g)\big) \qquad (\text{resp. } \exp(d(x_0, gx_0)))$$

satisfies UC.

Proof. Suppose the action is on the right. Given $\varepsilon > 0$, let

$$\mathcal{U} \stackrel{\text{def}}{=} \left\{ g \in G : d(x_0, x_0 g) < \log(1+\varepsilon) \right\} .$$

Then for every $g \in G$ and $u \in \mathcal{U}$,

$$\begin{aligned} d(x_0, x_0 gu) - d(x_0, x_0 g) &= d(x_0 u^{-1}, x_0 g) - d(x_0, x_0 g) \\ &\leq d(x_0 u^{-1}, x_0) < \log(1+\varepsilon) . \end{aligned}$$

This implies UC. The proof for a left action is similar. □

PROPOSITION 6.4. *Suppose $F : [0,\infty) \to [0,\infty)$ and $C : G \times G \to (0,\infty)$ are continuous functions such that for all $\delta > 0$,*

$$\frac{F((1+\delta)T)}{F(T)} \underset{T\to\infty}{\longrightarrow} 1\,.$$

If the general setup holds and $m(G_T) \sim F(T)$ then I1 holds.

If the general setup holds and

$$\lambda\big(H_T[g_1, g_2]\big) \sim C(g_1, g_2)F(T)$$

as $T \to \infty$, uniformly for g_1, g_2 in compact subsets of G, then conditions D1 and D2 are satisfied.

Proof. Given a bounded $B \subset G$ and $\varepsilon > 0$, there is T_1 such that for all $T > T_1$ and all $g_1, g_2 \in B$,

$$\frac{C(g_1, g_2)F(T)}{(1+\varepsilon)^{1/3}} < \lambda\big(H_T[g_1, g_2]\big) < (1+\varepsilon)^{1/3} C(g_1, g_2)F(T)\,.$$

Also there is $T_2 > 0$ such that for all $T > T_2$,

$$\frac{F((1+\delta)T)}{F(T)} < (1+\varepsilon)^{1/3}.$$

Then setting $T_0 = \max\{T_1, T_2\}$ we obtain for $T > T_0$,

$$\begin{aligned}\lambda\big(H_{(1+\delta)T}[g_1, g_2]\big) &< (1+\varepsilon)^{1/3} C(g_1, g_2) F\big((1+\delta)T\big)\\ &< (1+\varepsilon)^{2/3} C(g_1, g_2)F(T)\\ &< (1+\varepsilon)\lambda\big(H_T[g_1, g_2]\big)\,.\end{aligned}$$

This proves D1.

Now for any $g_1, g_2 \in G$ we have

$$\lim_{T\to\infty} \frac{\lambda(H_T[g_1^{-1}, g_2])}{\lambda(H_T)} = \lim_{T\to\infty} \frac{\lambda(H_T[g_1^{-1}, g_2])}{F(T)} \frac{F(T)}{\lambda(H_T)} = \frac{C(g_1^{-1}, g_2)}{C(e,e)}\,,$$

and we obtain D2. The proof of I1 is similar and is omitted. □

Proof of Proposition 2.6. Condition UC follows from Propositions 6.2, 6.3. If G, D are standard and G is balanced, then condition I1 follows from Theorems 2.7 and 2.8 and Proposition 6.4, and condition I2 follows from Proposition 6.1 and Theorem 2.10.

If G, H, D are standard, then conditions D1, D2 follow from Theorems 2.7 and 2.8 and Proposition 6.4. □

7 Matrix Norms and Volume Computations

In this section we consider the asymptotics of volumes of ‘balls’ H_T with respect to a matrix norm distance function. We first recall some standard details on Cartan (or polar) decomposition and Haar measure.

Let H be a connected semisimple Lie group. Let K be a maximal compact subgroup of H and let A be the associated *split Cartan* subgroup of H; by this we mean that A is a maximal connected group which is invariant under a Cartan involution associated to K, such that $\mathrm{Ad}(A)$ is diagonalizable over $\mathbb{R}$. Denote by $\mathfrak{h}$ and $\mathfrak{a}$ the Lie algebras of H and A respectively. One can write

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{h}_\alpha$$

for $\Phi \subset \mathfrak{a}^*$, the dual space of $\mathfrak{a}$, where

$$\mathfrak{h}_\alpha = \left\{ X \in \mathfrak{h} : \forall Y \in \mathfrak{a},\ \mathrm{Ad}(\exp(Y))X = e^{\alpha(Y)} X \right\},$$

and Φ is the restricted root system of H relative to A. Let Φ^+ be the set of positive roots with respect to some ordering and

$$\mathfrak{a}^+ = \left\{ Y \in \mathfrak{a} : \forall \alpha \in \Phi^+,\ \alpha(Y) \geq 0 \right\}$$

the corresponding (closed) Weyl chamber. Let $\rho \in \mathfrak{a}^*$ be defined by

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Phi} m_\alpha \alpha, \quad \text{where } \ m_\alpha = \dim \mathfrak{h}_\alpha . \tag{47}$$

Let $\Delta = \{\alpha_1, \dots, \alpha_r\} \subset \Phi^+$ be the set of simple roots that corresponds to $\mathfrak{a}^+$ and $\{\widetilde{\beta}_1, \dots, \widetilde{\beta}_r\}$ the dual basis of $\mathfrak{a}$, that is,

$$\alpha_i(\widetilde{\beta}_j) = \delta_{ij} .$$

It will be convenient to rescale each $\widetilde{\beta}_i$ according to ρ. Namely, let

$$\beta_i = \frac{\widetilde{\beta}_i}{2\rho(\widetilde{\beta}_i)},$$

so that

$$2\rho\left(\sum_{i=1}^r t_i \beta_i \right) = \sum_{i=1}^r t_i \quad \text{ and } \quad i \neq j \Longrightarrow \alpha_i(\beta_j) = 0 . \tag{48}$$

Every $h \in H$ can be written as $h = k_1 a k_2$, where $k_1, k_2 \in K$ and $a \in A^+ \stackrel{\text{def}}{=} \exp(\mathfrak{a}^+)$. Note that this decomposition is not unique. Nevertheless, λ can be expressed in terms of this decomposition of G (see [H2, p. 186]). Namely, letting dk denote the probability Haar measure on K we have, for $f \in C_c(H)$,

$$\int_H f(h) d\lambda(h) = \int_K \int_{\mathfrak{a}^+} \int_K f\big(k_1 \exp(Y) k_2\big) \xi(Y) dk_1\, dY\, dk_2 , \tag{49}$$

where dY is a scalar multiple of the Lebesgue measure on $\mathfrak{a}$, and

$$\xi(Y) = \prod_{\alpha \in \Phi^+} \sinh^{m_\alpha}(\alpha(Y)) . \tag{50}$$

Now suppose $\Psi : H \to \mathrm{GL}(V)$ is an irreducible representation. Let $d = \dim V$ and $V = V_1 \bigoplus \cdots \bigoplus V_s$ be a direct sum decomposition into the weight-spaces of $\mathfrak{a}$ with weights $\lambda_1,\dots,\lambda_s \in \mathfrak{a}^*$, that is, for all $Y \in \mathfrak{a}$, $v \in V_i$,

$$\Psi(\exp(Y))v = e^{\lambda_i(Y)}v\,. \tag{51}$$

One of the λ_i's is a highest weight – indeed, this is well known over $\mathbb{C}$ and holds also over $\mathbb{R}$, see e.g. [GuJT, Ch. IV]. By re-ordering the basis of V assume with no loss of generality that the highest weight is λ_1, i.e.

$$\forall Y \in \mathfrak{a}^+\,,\ \forall j \in \{2,\dots,s\}\,,\quad \lambda_1(Y) \geq \lambda_j(Y)\,. \tag{52}$$

Now let

$$m_1 = \min_{i=1,\dots,r} \lambda_1(\beta_i)\,. \tag{53}$$

By re-ordering $\beta_1,\dots,\beta_r$ assume with no loss of generality that

$$m_1 = \lambda_1(\beta_1)\,.$$

We now make the following assumption.

Definition 7.1. *Say that* Ψ satisfies condition $\mathcal{G}$ *if*

$$m_2 = \min_{j=2,\dots,r} \lambda_1(\beta_j) > m_1 = \lambda_1(\beta_1)\,. \tag{54}$$

Remark 7.2. Irreducible representations of H are usually described in terms of the corresponding dominant weights. Note that if condition $\mathcal{G}$ fails, then for some $i \neq j$, the dominant weight λ of Ψ satisfies

$$\rho(\widetilde{\beta}_j)\lambda(\widetilde{\beta}_i) = \rho(\widetilde{\beta}_i)\lambda(\widetilde{\beta}_j)\,.$$

Thus the condition fails only for a subset of Ψ whose dominant weight is contained in a finite union of proper linear subspaces of $\mathfrak{a}^*$.

Let

$$\mathcal{J} = \big\{j \in \{1,\dots,s\} : \lambda_j(\beta_1) = m_1\big\}\,.$$

Note that $1 \in \mathcal{J}$, and by (52), we have for $j \in \mathcal{J}$,

$$\lambda_j(\beta_1) = \max_{t=1,\dots,s} \lambda_t(\beta_1)\,.$$

We fix a basis V such that $\Psi(A)$ is a diagonal subgroup. For $j = 1,\dots,d$ let $E_j \in \mathrm{Mat}_d(\mathbb{R})$ denote the matrix whose j,j-th entry is 1 and all other entries are zero, and for $\tau = (t_2,\dots,t_r) \in \mathbb{R}^{r-1}$, define

$$\bar{\tau} = \sum_{i=2}^{r} t_i\beta_i\,,\quad E_\tau = \sum_{(j,k):k\in\mathcal{J},E_j\in V_k} e^{\lambda_k(\bar{\tau})}E_j\,. \tag{55}$$

Also, let

$$\hat{\Phi} = \big\{\alpha \in \Phi : \alpha(\beta_1) = 0\big\}\,,$$

that is, $\hat{\Phi}$ contains those roots whose expression as a linear combination of simple roots does not involve α_1, and let

$$\hat{\xi}(\tau) = \left(\frac{1}{2}\right)^{\sum_{\alpha\notin\hat{\Phi}} m_\alpha} \prod_{\alpha\in\hat{\Phi}} \left(\frac{1}{2} - \frac{1}{2e^{2\alpha(\bar{\tau})}}\right)^{m_\alpha} e^{\sum_{i=2}^{r} t_i}.$$

We say that a collection $\mathcal{N}$ of norms on $\mathrm{Mat}_d(\mathbb{R})$ is *bounded* if there is $c > 1$ such that for any two norms $\|\cdot\|_1$, $\|\cdot\|_2$ in $\mathcal{N}$, and any nonzero $A \in \mathrm{Mat}_d(\mathbb{R})$, $1/c < \|A\|_1/\|A\|_2 < c$.

Proposition 7.3. *For any norm $\|\cdot\|$ on $\mathrm{Mat}_d(\mathbb{R})$, the indefinite integral*

$$D = \int_{\tau\in[0,\infty)^{r-1}} \frac{\hat{\xi}(\tau)}{\|E_\tau\|^{1/m_1}}\, d\tau \tag{56}$$

converges, and the convergence is uniform for $\|\cdot\|$ in a bounded collection of norms.

Theorem 7.4. *Let H be connected and semisimple, and let $\Psi : H \to \mathrm{GL}(V)$ be an irreducible representation satisfying condition $\mathcal{G}$. Choosing a basis of V, identify $\Psi(H)$ with a subset of $\mathrm{Mat}_d(\mathbb{R})$, where $d = \dim V$. Then for any linear norm $\|\cdot\|$ on $\mathrm{Mat}_d(\mathbb{R})$ we have*

$$\lambda(H_T) \sim CT^m,$$

where $m = 1/m_1$, m_1 is given by (53), and

$$C = \int_K \int_K \int_{\tau\in[0,\infty)^{r-1}} \frac{\hat{\xi}(\tau)}{\|\Psi(k_1)E_\tau\Psi(k_2)\|^m}\, d\tau\, dk_1\, dk_2\,.$$

Remark 7.5. 1. In the above expression, m depends only on Ψ, and C depends continuously on $\|\cdot\|$; this means that, for any $\varepsilon > 0$ there is $\delta > 0$ such that for a norm $|\cdot|$,

$$\forall v \neq 0\,, \quad \left|1 - \tfrac{\|v\|}{|v|}\right| < \delta \quad \Longrightarrow \quad \big|C(\|\cdot\|) - C(|\cdot|)\big| < \varepsilon\,.$$

2. Since for any $k_1, k_2 \in G$, the map $A \mapsto \|\Psi(g_1)A\Psi(g_2)\|$ is a linear norm on $\mathrm{Mat}_d(\mathbb{R})$, the integral for C converges in light of Proposition 7.3.

3. In the general case, that is when Ψ is reducible or does not satisfy condition $\mathcal{G}$, there exist $k \in \mathbb{Z}_+$, $m > 0$ and $C > 0$, where k and m depend only on Ψ, and C depends continuously on $\|\cdot\|$, such that

$$\lambda(H_T) \sim C(\log T)^k T^m.$$

Details will appear elsewhere.

Proposition 7.6. *Suppose $\mathcal{N}$ is a bounded collection of norms on $\mathrm{Mat}_d(\mathbb{R})$. Then for any $\delta > 0$ there is T_0 such that for all $T \geq T_0$ and all $\|\cdot\| \in \mathcal{N}$ we have*

$$\left| \int_{\mathfrak{a}^+(T,\|\cdot\|)} \xi(Y)dY - DT^m \right| < \delta\, T^m, \tag{57}$$

where

$$\mathfrak{a}^+\big(T, \|\cdot\|\big) = \big\{Y \in \mathfrak{a}^+ : \|\Psi(\exp(Y))\| < T\big\}\,,$$

D is as in (56), and $m = 1/m_1$.

Proof of Theorem 7.4 assuming Proposition 7.6. Note that the collection of norms

$$\left\{x \mapsto \|\Psi(k_1)x\Psi(k_2)\| : k_1, k_2 \in K\right\}$$

is bounded. Thus the result follows using (49). □

Proof of Proposition 7.3. Let $\mathcal{N}$ be a bounded collection of norms on $\mathrm{Mat}_d(\mathbb{R})$. For the purpose of this proof, the notation $X \ll Y$ will mean that X and Y are quantities depending on various parameters, and there is C, depending only on $\mathcal{N}$ and independent of the other parameters, such that $X \leq CY$.

First note that for a fixed basis $\mathcal{B}$ of $\mathrm{Mat}_d(\mathbb{R})$, for every $\|\cdot\| \in \mathcal{N}$, and every $A = \sum_{E \in \mathcal{B}} a_E E \in \mathrm{Mat}_d(\mathbb{R})$,

$$\|A\| \ll \max_{E \in \mathcal{B}} |a_E| \ll \|A\|.$$

Since $E_1, \ldots, E_d$ can be completed to a basis of $\mathrm{Mat}_d(\mathbb{R})$, this implies that $\|E_\tau\|$ is bounded below by a positive constant independent of $\tau \in [0,\infty)^{r-1}$ and of $\|\cdot\| \in \mathcal{N}$. Thus we need only consider the behavior of the integrand as $\tau \to \infty$. For $E_j \in V_1$, we have

$$e^{m_2 \sum_{i=2}^r t_i} \overset{(54)}{\ll} e^{\sum_{i=2}^r t_i \lambda_1(\beta_i)} \|E_j\| = e^{\lambda_1(\bar{\tau})} \|E_j\| \overset{(55)}{\ll} \|E_\tau\|. \tag{58}$$

On the other hand it is clear that

$$\hat{\xi}(\tau) \ll e^{\sum_{i=2}^r t_i}. \tag{59}$$

Putting together (58) and (59), and using condition $\mathcal{G}$ we obtain

$$\begin{aligned} D &= \int_{\tau \in [0,\infty)^{r-1}} \frac{\hat{\xi}(\tau)}{\|E_\tau\|^{1/m_1}} \, d\tau \\ &\overset{(58)}{\ll} \int_{\tau \in [0,\infty)^{r-1}} \frac{e^{\sum_{i=2}^r t_i}}{\left(e^{m_2 \sum_{i=2}^r t_i}\right)^{1/m_1}} d\tau \\ &= \int_{\tau \in [0,\infty)^{r-1}} e^{(1-m_2/m_1)\sum_{i=2}^r t_i} d\tau \overset{(54)}{\ll} \infty. \end{aligned}$$

□

Proof of Proposition 7.6. We first show that for all sufficiently large M, for all $T > 0$, we have

$$\int_{\mathfrak{b}^+(M,T,\|\cdot\|)} \xi(Y) dY < \left(\frac{\delta}{3}\right) T^m, \tag{60}$$

where

$$\mathfrak{b}^+\big(M,T,\|\cdot\|\big) = \left\{ \sum_{i=1}^r t_i \beta_i \in \mathfrak{a}^+\big(T,\|\cdot\|\big) : \exists j \in \{2,\ldots,r\},\, t_j \geq M \right\}. \tag{61}$$

It suffices to show that for any $j \in \{2, \dots, r\}$ there is M_0 such that for $M > M_0$ and any $T > 0$

$$\int_{\mathfrak{b}} \xi(Y)dY < \left(\frac{\delta}{3r}\right) T^m, \tag{62}$$

where

$$\mathfrak{b} = \left\{ \sum_{i=1}^{r} t_i\beta_i \in \mathfrak{a}^+(T, \|\cdot\|) : t_j \geq M \right\}.$$

To prove (62) we will need the following easy lemma, which is proved by induction on k:

Lemma 7.7. *Given $k \in \mathbb{N}$ and $0 < m_1 < \cdots < m_k$, there is $c > 0$ such that for all $S > 0$,*

$$\int_{\Delta} e^{\sum_{i=1}^{k} t_i} dt_1 \cdots dt_k \leq c e^{S/m_1},$$

where

$$\Delta = \left\{ (t_1, \dots, t_k) : \forall i\,,\ t_i \geq 0\,,\ \sum_{1}^{k} m_i t_i \leq S \right\}.$$

By (52), and by comparing with the max-norm, we find that there is $\widetilde{C}$ such that for all $\|\cdot\| \in \mathcal{N}$ and all $T > 0$,

$$\mathfrak{a}^+\big(T, \|\cdot\|\big) \subset \left\{ \sum_{i=1}^{r} s_i\beta_i : \forall i\,,\ s_i \geq 0\,,\ \sum_{i=1}^{r} s_i\lambda_1(\beta_i) \leq \log T + \widetilde{C} \right\}. \tag{63}$$

Suppose $j \in \{2, \dots, r\}$ and $t_j \geq M$. To simplify notation suppose $j = r$. Then we obtain that

$$\mathfrak{b} \subset \left\{ \sum_{i=1}^{r} s_i\beta_i : \forall i\,,\ s_i \geq 0\,,\ s_r \geq M\,,\ \sum_{i=1}^{r} s_i\lambda_1(\beta_i) < \log T + \widetilde{C} \right\}.$$

Now using condition $\mathcal{G}$, choose $\widetilde{m}_1 < \widetilde{m}_2 < \cdots < \widetilde{m}_r$ such that

$$\widetilde{m}_1 = m_1 \quad \text{and} \quad \widetilde{m}_j \leq \lambda_1(\beta_j)\,, \quad j \in \{2, \dots, r\}\,.$$

Then it follows that $\mathfrak{b} \subset M\beta_r + \mathfrak{c}$ (that is, the translation of $\mathfrak{c}$ by the vector $M\beta_r$) where

$$\mathfrak{c} = \left\{ \sum_{i=1}^{r} s_i\beta_i : \forall i\,,\ s_i \geq 0\,,\ \sum_{i=1}^{r} s_i\widetilde{m}_i < \log T + \widetilde{C} - m_2 M \right\}.$$

Applying Lemma 7.7 we find that

$$\begin{aligned} \int_{\mathfrak{b}} \xi(Y)dY &\leq \int_{\mathfrak{b}} e^{2\rho(\sum_1^r t_i\beta_i)} dt_1 \cdots dt_r \leq \int_{M\beta_r + \mathfrak{c}} e^{\sum_1^r t_i} dt_1 \cdots dt_r \\ &= e^M \int_{\mathfrak{c}} e^{\sum_1^r t_i} dt_1 \cdots dt_r \leq c e^{M + (\log T + \widetilde{C} - m_2 M)/m_1} \\ &\leq c e^{\widetilde{C}} e^{(1 - m_2/m_1)M}\, T^{1/m_1}. \end{aligned}$$

Since $m_2 > m_1$, this implies (62), and we have proved (60).

Now set

$$C_\tau = \frac{\hat{\xi}(\tau)}{\|E_\tau\|^m}\,, \tag{64}$$

so that

$$D = \int_{[0,\infty)^{r-1}} C_\tau d\tau\,. \tag{65}$$

Using Proposition 7.3, we may assume by enlarging M that

$$\left|\int_{\mathbb{R}^{r-1}\smallsetminus[0,M]^{r-1}} C_\tau d\tau\right| < \delta/3\,. \tag{66}$$

We will show below that

CLAIM 4. *There is T_0 such that for all $\|\cdot\| \in \mathcal{N}$, all $T > T_0$ and all $\tau = (t_2,\dots,t_r) \in [0,M]^{r-1}$,*

$$\left|\frac{\int_{\mathfrak{d}(\tau,T)} \xi\big(\sum_{i=1}^r t_i\beta_i\big)dt_1}{C_\tau T^m} - 1\right| < \frac{\delta}{3M^{r-1}\max_{\tau\in[0,M]^{r-1}} C_\tau}\,, \tag{67}$$

where

$$\mathfrak{d}(\tau,T) = \big\{t_1 : (t_1,t_2,\dots,t_r) \in \mathfrak{a}^+(T,\|\cdot\|)\big\}\,.$$

Assuming the validity of Claim 4, and writing

$$\mathfrak{a} = \mathfrak{a}^+\big(T,\|\cdot\|\big)\,,\quad \mathfrak{b} = \mathfrak{b}^+\big(M,T,\|\cdot\|\big)\,,$$

we have, for all $T > T_0$,

$$\begin{aligned}
&\left|\int_{\mathfrak{a}} \xi(Y)dY - DT^m\right| \\
&\overset{(65)}{<} \left|\int_{\mathfrak{a}\smallsetminus\mathfrak{b}} \xi(Y)dY - T^m\int_{[0,M]^{r-1}} C_\tau d\tau\right| \\
&\qquad + \left|\int_{\mathfrak{b}} \xi(Y)dY\right| + T^m\left|\int_{\mathbb{R}^{r-1}\smallsetminus[0,M]^{r-1}} C_\tau d\tau\right| \\
&\overset{(60),(66)}{<} \left|\int_{[0,M]^{r-1}}\left[\int_{\mathfrak{d}(\tau,T)} \xi(t_1\beta_1+\bar{\tau})dt_1 - C_\tau T^m\right]d\tau\right| + \frac{2\delta}{3}T^m \\
&\overset{(67)}{<} \delta T^m.
\end{aligned}$$

In order to prove the claim, we will first show that, for any $\eta > 0$, there is T_0 such that, for all $\|\cdot\| \in \mathcal{N}$ and all $T > T_0$,

$$\left[0, \frac{\log T - \log\|E_\tau\| - \eta}{m_1}\right] \subset \mathfrak{d}(\tau,T) \subset \left[0, \frac{\log T - \log\|E_\tau\| + \eta}{m_1}\right]. \tag{68}$$

Indeed, suppose that $t_1 \in \mathfrak{d}(\tau,T)$, that is,

$$\left\|\Psi\left(\exp\left(\sum_{i=1}^r t_i\beta_i\right)\right)\right\| < T\,.$$

Write $u = \max_{s \notin \mathcal{J}} \lambda_s(\beta_1) < m_1$. Then we have

$$\begin{aligned}\left\|\Psi\left(\exp\left(\sum_{i=1}^{r} t_i\beta_i\right)\right)\right\| &= \Big\| \sum_{(j,k):\, E_j \in V_k} e^{\lambda_k\left(\sum_{i=1}^{r} t_i\beta_i\right)} E_j \Big\| \\ &= \Big\| \sum_{(j,k):\, E_j \in V_k} e^{t_1\lambda_k(\beta_1)}\, e^{\sum_{i=2}^{r} t_i\lambda_k(\beta_i)} E_j \Big\| \\ &= \Big\| e^{m_1 t_1} E_\tau + \sum_{(j,k): k\notin\mathcal{J}, E_j\in V_k} e^{t_1\lambda_k(\beta_1)}\, e^{\sum_{i=2}^{r} t_i\lambda_k(\beta_i)} E_j \Big\| \\ &\geq e^{m_1 t_1}\|E_\tau\| - Ce^{ut_1}\end{aligned}$$

(here C is a constant depending only on $\mathcal{N}$ and M). Adjusting C if necessary, we obtain

$$e^{m_1 t_1}\|E_\tau\|\big(1 - Ce^{(u-m_1)t_1}\big) < T\,. \tag{69}$$

Taking t_1 and T large enough we can ensure that

$$-\log\big(1 - Ce^{(u-m_1)t_1}\big) < \eta\,,$$

and by plugging this in (69) and taking logs, we find

$$m_1 t_1 < \log T - \log\|E_\tau\| + \eta\,.$$

This proves the right-hand inclusion in (68). The proof of the left-hand inclusion is similar.

For any $\alpha \in \Phi \smallsetminus \hat{\Phi}$, $\alpha\big(\sum_{i=1}^{r} t_i\beta_i\big) \to_{t_1\to\infty} \infty$ and hence

$$\frac{\sinh\big(\alpha\big(\sum_{i=1}^{r} t_i\beta_i\big)\big)}{e^{\alpha(\sum_{i=1}^{r} t_i\beta_i)}} \underset{t_1\to\infty}{\longrightarrow} \frac{1}{2}\,.$$

Hence for fixed τ,

$$\begin{aligned}\frac{\xi\big(\sum_{i=1}^{r} t_i\beta_i\big)}{e^{t_1}} &= \frac{\xi\big(\sum_{i=1}^{r} t_i\beta_i\big)}{e^{\sum_{i=1}^{r} t_i}} e^{\sum_{i=2}^{r} t_i} \\ &= \frac{\prod_{\alpha\in\Phi} \sinh\big(\alpha\big(\sum_{i=1}^{r} t_i\beta_i\big)\big)^{m_\alpha}}{e^{2\rho\left(\sum_{i=1}^{r} t_i\beta_i\right)}}\, e^{\sum_{i=2}^{r} t_i} \\ &= \prod_{\alpha\in\Phi}\left(\frac{\sinh\big(\alpha\big(\sum_{i=1}^{r} t_i\beta_i\big)\big)}{e^{\alpha(\sum_{i=1}^{r} t_i\beta_i)}}\right)^{m_\alpha} e^{\sum_{i=2}^{r} t_i} \\ &\underset{t_1\to\infty}{\longrightarrow} \left(\frac{1}{2}\right)^{\sum_{\alpha\notin\hat{\Phi}} m_\alpha} \prod_{\alpha\in\hat{\Phi}}\left(\frac{\sinh\big(\alpha\big(\sum_{i=1}^{r} t_i\beta_i\big)\big)}{e^{\alpha\left(\sum_{i=1}^{r} t_i\beta_i\right)}}\right)^{m_\alpha} e^{\sum_{i=2}^{r} t_i} \\ &= \hat{\xi}\,(\tau)\,.\end{aligned}$$

Note also that the convergence is uniform for τ in compact sets. It follows that

$$\frac{\hat{\xi}(\tau)\int_{\mathfrak{d}(\tau,T)} e^{t_1}dt_1}{\int_{\mathfrak{d}(\tau,T)} \xi\big(\sum_{i=1}^{r} t_i\beta_i\big)dt_1} \underset{T\to\infty}{\longrightarrow} 1\,, \tag{70}$$

and the convergence is uniform for $\tau \in [0, M]^{r-1}$.

We now have

$$\begin{aligned}
\lim_{T\to\infty} \frac{\int_{\mathfrak{d}(\tau,T)} \xi\big(\sum_{i=1}^r t_1\beta_i\big)\,dt_1}{C_\tau T^m} &\overset{(70)}{=} \lim_{T\to\infty} \frac{\hat{\xi}(\tau)\int_{\mathfrak{d}(\tau,T)} e^{t_1}\,dt_1}{C_\tau T^m} \\
&\overset{(68)}{=} \lim_{T\to\infty} \frac{\hat{\xi}(\tau)\int_0^{(\log T - \log\|E_\tau\|)/m_1} e^{t_1}\,dt_1}{C_\tau T^m} \\
&= \lim_{T\to\infty} \frac{\hat{\xi}(\tau)}{C_\tau T^m}\left(\frac{T}{\|E_\tau\|}\right)^{1/m_1} \overset{(64)}{=} 1\,.
\end{aligned}$$

The convergence in the above expression is uniform in τ, therefore Claim 4 is valid. This completes the proof of the proposition. □

8 Balanced Semisimple Groups

As a corollary of the computations in the previous sections, we obtain some information about balanced semisimple groups (cf. Definition 2.9). First we have

PROPOSITION 8.1. *If D is a matrix norm distance function, corresponding to a representation $\Psi : G \to \mathrm{GL}_d(\mathbb{R}) \subset \mathrm{Mat}_d(\mathbb{R})$ with compact kernel and a norm $\|\cdot\|$ on $\mathrm{Mat}_d(\mathbb{R})$, then the condition that H is balanced depends on Ψ but not on $\|\cdot\|$.*

Proof. Let $\|\cdot\|$, $\|\cdot\|'$ be two norms on $\mathrm{Mat}_d(\mathbb{R})$, and suppose that H is balanced with respect to $\|\cdot\|$. Then for some $C > 1$ and every $x \in \mathrm{Mat}_d(\mathbb{R})$,

$$\frac{\|x\|}{C} \le \|x\|' \le C\|x\|\,. \tag{71}$$

Let H_i, $i = 1, \dots, t$, be the simple factors of H and let $\sigma_i : H \to H_i$ be measurable sections. Given $j \in \{1, \dots, t\}$, $g_1, g_2 \in G$ and a compact $L \subset H_j$, let

$$\begin{aligned}
S_T &= S_T[g_1, g_2] = \big\{h \in H_T[g_1, g_2] : \sigma_j(h) \in L\big\}\,, \\
H'_T &= H'_T[g_1, g_2] = \big\{h \in H : \|g_1 h g_2\|' < T\big\}\,, \\
S'_T &= S'_T[g_1, g_2] = \big\{h \in H'_T[g_1, g_2] : \sigma_j(h) \in L\big\}\,.
\end{aligned}$$

It follows from (71) that

$$S'_T \subset S_{CT} \quad \text{and} \quad H_{T/C} \subset H'_T\,.$$

It is a consequence of Theorem 2.7 that

$$\limsup_{T\to\infty} \frac{\lambda(H_{CT})}{\lambda(H_{T/C})} < \infty\,.$$

Therefore

$$\frac{\lambda(S'_T)}{\lambda(H'_T)} \le \frac{\lambda(S_{CT})}{\lambda(H_{T/C})} = \frac{\lambda(S_{CT})}{\lambda(H_{CT})}\,\frac{\lambda(H_{CT})}{\lambda(H_{T/C})} \underset{T\to\infty}{\longrightarrow} 0\,.$$

This shows that H is balanced with respect to the norm $\|\cdot\|'$. □

We also have the following result, showing that balanced representations of semisimple nonsimple groups are rather atypical.

PROPOSITION 8.2. *Let H be a semisimple nonsimple Lie group, realized as a matrix group via an irreducible representation $\Psi : H \to \mathrm{GL}(V)$. Suppose condition $\mathcal{G}$ is satisfied. Then H is not balanced.*

Proof. Let $H = H_1 \cdots H_t$, $t \ge 2$, be a representation of H as an almost direct product, let K be a maximal compact subgroups of H, let A be an associated split Cartan subgroup of H, and for each $i \in \{1,\dots,t\}$ write $A_i = H_i \cap A$, so that A_i is a split Cartan subgroup of H_i. We write $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$ where Φ_i is the root system corresponding to (H_i, A_i). Also fix measurable sections $\sigma_i : H \to H_i$.

Let the notation be as in §7. There is a partition

$$\Delta = \Delta_1 \cup \cdots \cup \Delta_t$$

of the simple roots of $\mathfrak{a}$ such that, for each $j \in \{1,\dots,t\}$, Δ_j is a set of simple roots for $\mathfrak{a}_j$. Assume by reordering that $\alpha_1 \in \Delta_1$, and let $j \in \{2,\dots,t\}$. Since Δ is indexed by $\{1,\dots,r\}$, we think of Δ_j as a subset of $\{1,\dots,r\}$. For each $M > 0$ and $T > 0$, and each norm $\|\cdot\|$ on $\mathrm{Mat}_d(\mathbb{R})$, define

$$\mathfrak{a}^+\big(M,T,\|\cdot\|\big) = \left\{ Y = \sum_{i=1}^{r} s_i\beta_i \in \mathfrak{a}^+\big(T,\|\cdot\|\big) : \forall i \in \Delta_j\,,\ s_i \le M \right\}.$$

Then $\mathfrak{a}_j \cap \mathfrak{a}^+(M,T,\|\cdot\|)$ is compact for each $M > 0$ and hence

$$L = L(M) = \bigcup_T \sigma_j\big(K \exp(\mathfrak{a}^+(M,T,\|\cdot\|))K\big) \subset H_j$$

is precompact.

Now let $g_1 = g_2 = e$, let C be as in Theorem 7.4, let $0 < \delta < C$, and let $\mathfrak{b}(M,T,\|\cdot\|)$ be as in (61). Since the complement of $\mathfrak{b}(M,T,\|\cdot\|)$ is contained in $\mathfrak{a}^+(M,T,\|\cdot\|)$, we see that for large M (60) contradicts (17). □

9 Riemannian Skew Balls and Volume Computations

Suppose G is a connected semisimple Lie group and H is its connected semisimple Lie subgroup. The main result of this section is a computation of the asymptotics of the volume growth for certain 'skew balls' in H, with respect to a symmetric space distance function.

Before formulating the precise result we introduce some notation. We are given a semisimple Lie group G, a semisimple Lie subgroup H and a maximal compact subgroup K. It follows from a theorem of Mostow [Mo] that for a conjugate H' of H, we can choose a maximal compact subgroup L of H' respectively so that $L \subset K$ and split Cartan subgroups D and A of G and H' (associated to K and L) respectively so that $A \subset D$. Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{d}, \mathfrak{a}$ denote the corresponding Lie algebras.

Let $X = K\backslash G$ be the right-symmetric space of G, and let $P : G \to X$, $d(\cdot,\cdot)$ be as in §2.6.2. For the case of the left-symmetric space G/K, see Remark 9.5 below. We will write $\bar{g} = P(g)$ to simplify notation. Our goal will be to determine the asymptotics of the volumes of all skew balls $H_T[g_1, g_2]$, as in (7). Since applying a conjugation in G permutes the set of skew balls, with no loss of generality we will replace H with H'. The Riemannian metric on X induces a scalar product $(\cdot,\cdot)$ on $\mathfrak{d}$. We denote the corresponding norm by $\|\cdot\|$, and write

$$\mathfrak{d}^0 = \left\{Y \in \mathfrak{d} : \|Y\| = 1\right\}, \quad \mathfrak{a}^0 = \mathfrak{a} \cap \mathfrak{d}^0.$$

We recall the following facts about the geometry of X, see e.g. [B], [E] for more details.

Proposition 9.1. (1) *X is a complete Riemannian manifold of nonpositive curvature.*

(2) *The map $\mathfrak{d} \to X$, $Y \mapsto P(\exp(Y))$ is an isometry. In particular, the submanifold $P(D)$ is a totally geodesic subset and for any $Y \in \mathfrak{d}^0$, the path*

$$t \mapsto \gamma_Y(t) = \bar{e}\exp(tY)$$

is a unit speed geodesic. There is a continuous Busemann function *$\beta : X \times \mathfrak{d}^0 \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} d\big(\gamma_Y(t), x\big) - t = \beta(x, Y)$$

and the convergence is uniform over compact subsets of $X \times \mathfrak{d}^0$.

(3) *For any $g \in G$, $x \in X$, and $Y \in \mathfrak{d}^0$,*

$$\lim_{t\to\infty} d\big(\bar{g}\exp(tY), x\big) - d\big(\bar{g}\exp(tY), \bar{e}\big) = \beta(x, Y)$$

and the convergence is uniform over compact subsets of $G \times X \times \mathfrak{d}^0$.

Proof. Parts (1) and (2) are well known. To prove (3), note that

$$d\big(\bar{e}\exp(tY), \bar{g}\exp(tY)\big) = d(\bar{e}, \bar{g}),$$

that is, $\{\bar{g}\exp(tY) : t \geq 0\}$ stays within bounded distance of the geodesic $\{\bar{e}\exp(tY) : t \geq 0\}$. Now the result can be proved using [B, Prop. 2.5]. ◻

Let $\mathfrak{a}^+$ be a positive Weyl chamber in $\mathfrak{a}$ that corresponds to a system of positive roots $\Phi^+ \subset \Phi$. For $T > 0$, put $\mathfrak{a}_T^+ = \{a \in \mathfrak{a}^+ : \|a\| \leq T\}$. We will require the following standard fact concerning the functional ρ defined in (47).

LEMMA 9.2. *There is a unique* $Y_{\max} \in \mathfrak{a}_1^+ \cap \operatorname{int} \mathfrak{a}^+$ *such that* $\rho(Y_{\max}) = \max_{Y \in \mathfrak{a}_1^+} \rho(Y)$.

With this notation we have

Theorem 9.3. *Let G be a semisimple Lie group and H a connected semisimple subgroup. Then for any g_1, $g_2 \in G$,*

$$\lambda\big(H_T[g_1, g_2]\big) \sim C(g_1, g_2) T^{(d-1)/2} e^{\delta T}, \tag{72}$$

where $d = \dim A$, $\delta = 2\rho(Y_{\max})$, and

$$C(g_1, g_2) = \left(\int_L \exp\big(-\delta\beta(\bar{g_1}\ell, -Y_{\max})\big)\, d\ell \right) \times \left(\int_L \exp\left(-\delta\beta\left(\overline{g_2^{-1}}\ell^{-1}, Y_{\max}\right)\right) d\ell \right).$$

The convergence in (72) *is uniform for g_1, g_2 in compact subsets of G.*

We will need another standard result about integration of exponential functions on balls. Lebesgue measure on $\mathbb{R}^r$ is denoted by dY. We call $S \subset \mathbb{R}^r$ a *convex cone* S is convex and for any $\mathbf{s} \in S$, the ray $\{t\mathbf{s} : t > 0\}$ is contained in S.

LEMMA 9.4. *Let λ be a linear functional on $\mathbb{R}^r$, and let $\delta = \max_{Y \in \mathfrak{a}_1^+} \lambda(Y)$. Assume that $S \subset \mathfrak{a}^+$ is an open convex cone such that $\delta = \max_{Y \in \mathfrak{a}_1^+ \cap S} \lambda(Y)$. Then there is a constant C such that*

$$\int_{\mathfrak{a}_T^+ \cap S} e^{\lambda(Y)}\, dY \sim \int_{\mathfrak{a}_T^+} e^{\lambda(Y)}\, dY \sim C T^{(d-1)/2} e^{\delta T}.$$

Proof of Theorem 9.3 assuming Lemmas 9.2 and 9.4. Define ξ by (50). Let

$$\mathfrak{a}_T^+[g_1, g_2] = \big\{ Y \in \mathfrak{a}^+ : d(\bar{g_1} \exp(Y) g_2, \bar{e}) \leq T \big\}$$
$$\mathfrak{a}_T^+ = \mathfrak{a}_T^+[e, e] = \big\{ Y \in \mathfrak{a}^+ : \|Y\| \leq T \big\}.$$

First, we determine the asymptotics of

$$\psi_{g_1, g_2}(T) \overset{\text{def}}{=} \int_{\mathfrak{a}_T^+[g_1, g_2]} \xi(Y) dY$$

as $T \to \infty$ with uniform convergence for g_1 and g_2 in a fixed compact set $E \subset G$. To do this we will replace ξ by an exponential function.

By the triangle inequality, there exists $C = C(E) > 0$ such that for $g_1, g_2 \in E$,

$$\mathfrak{a}_{T-C}^+[e, e] \subset \mathfrak{a}_T^+[g_1, g_2] \subset \mathfrak{a}_{T+C}^+[e, e]. \tag{73}$$

Expanding (50) we obtain that there are $\lambda_1, \ldots, \lambda_k \in \mathfrak{a}^*$, $a_1, \ldots, a_k \in \mathbb{R}$ such that

$$\xi(Y) = \frac{e^{2\rho(Y)}}{2^m} + \sum_{i=1}^{k} a_i e^{\lambda_i(Y)},$$

where $m = \sum_{\alpha \in \Phi} m_\alpha$, and for all $Y \in \operatorname{int} \mathfrak{a}^+$,

$$2\rho(Y) > \max_i \lambda_i(Y).$$

Therefore

$$\psi_{g_1,g_2}(T) = \frac{1}{2^m} \int_{\mathfrak{a}_T^+[g_1,g_2]} e^{2\rho(Y)} dY + \sum_{i=1}^{k} a_i \int_{\mathfrak{a}_T^+[g_1,g_2]} e^{\lambda_i(Y)} dY.$$

Let C' be a constant such that for all $T > 0$ and all $g_1, g_2 \in E$, the Lebesgue measure of $\mathfrak{a}_T^+[g_1, g_2]$ is at most $C' T^r$. Let $Y_{\max}$ be as in Lemma 9.2. Using the compactness of $\mathfrak{a}_1^+$, we find that there is $\eta > 0$ such that for each $i \in \{1, \ldots, k\}$,

$$\max_{Y \in \mathfrak{a}_1^+} \lambda_i(Y) \le \delta - \eta.$$

This implies that

$$\begin{aligned}
\frac{\psi_{g_1,g_2}(T) - 2^{-m} \int_{\mathfrak{a}_T^+[g_1,g_2]} e^{2\rho(Y)} dY}{e^{\delta T}} &= \frac{\sum_{i=1}^{k} a_i \int_{\mathfrak{a}_T^+[g_1,g_2]} e^{\lambda_i(Y)} dY}{e^{\delta T}} \\
&\le \frac{\sum_{i=1}^{k} C' T^r \, |a_i| \max_{Y \in \mathfrak{a}_{T+C}^+} e^{\lambda_i(Y)}}{e^{\delta T}} \\
&\le \left(e^{\max_i \lambda_i(C)} C' T^r \sum_{i=1}^{k} |a_i| \right) e^{-\eta T} \\
&\xrightarrow[T \to \infty]{} 0. \qquad (74)
\end{aligned}$$

Hence, in order to derive the asymptotics of $\psi_{g_1,g_2}(T)$, it suffices to find $D_{g_1,g_2} > 0$ such that

$$\int_{\mathfrak{a}_T^+[g_1,g_2]} e^{2\rho(Y)} \, dY \sim D_{g_1,g_2} \, T^{(d-1)/2} e^{\delta T} \qquad (75)$$

as $T \to \infty$ uniformly on $g_1, g_2 \in E$.

We will use polar coordinates on $\mathfrak{a}^+$. Thus we will represent each $Y \in \mathfrak{a}^+$ as $t\omega$, where $t = \|Y\|$ and $\omega \in \mathfrak{a}^1 = \mathfrak{a}^0 \cap \mathfrak{a}^+$.

Let

$$\begin{aligned}
r_T = r_T(\omega, g_1, g_2) &= \inf \left\{ r > 0 : r\omega \notin \mathfrak{a}_T^+[g_1, g_2] \right\} \\
&= \inf \left\{ r > 0 : d\left(\overline{g_1} \exp(r\omega), \overline{g_2^{-1}} \right) > T \right\}, \\
R_T = R_T(\omega, g_1, g_2) &= \sup \left\{ r > 0 : r\omega \in \mathfrak{a}_T^+[g_1, g_2] \right\} \\
&= \sup \left\{ r > 0 : d\left(\overline{g_1} \exp(r\omega), \overline{g_2^{-1}} \right) \le T \right\}.
\end{aligned}$$

Then

$$\{r\omega : \omega \in \mathfrak{a}^1,\ 0 \le r < r_T\} \subset \mathfrak{a}_T^+[g_1, g_2] \subset \{r\omega : \omega \in \mathfrak{a}^1,\ 0 \le r \le R_T\}\,. \quad (76)$$

By (73), $r_T(\omega, g_1, g_2) \to \infty$ as $T \to \infty$ uniformly for $g_1, g_2 \in E$ and $\omega \in \mathfrak{a}^1$.

Let $s_T > 0$ be such that $d\big(\overline{g_1}\exp(s_T\omega), \overline{g_2^{-1}}\big) = T$ and $s_T \to \infty$. It follows from continuity that this condition holds for $s_T = r_T$ as well as for $s_T = R_T$. By Proposition 9.1(3),

$$\begin{aligned}
&\lim_{T\to\infty}(T - s_T)\\
&= \lim_{T\to\infty}\Big[d\left(\overline{g_1}\exp(s_T\omega), \overline{g_2^{-1}}\right) - d\left(\overline{g_1}\exp(s_T\omega), \overline{e}\right) + d\left(\overline{g_1}\exp(s_T\omega), \overline{e}\right) - s_T\Big]\\
&= \beta\left(\overline{g_2^{-1}}, \omega\right) + \lim_{T\to\infty}\big(d\left(\bar{e}\exp(-s_T\omega), \overline{g_1}\right) - s_T\big) \qquad (77)\\
&= \beta\left(\overline{g_2^{-1}}, \omega\right) + \beta\left(\overline{g_1}, -\omega\right).
\end{aligned}$$

In particular, this shows that

$$R_T(\omega, g_1, g_2) - r_T(\omega, g_1, g_2) \to 0 \quad \text{as} \quad T \to \infty \qquad (78)$$

with uniform convergence for $\omega \in \mathfrak{a}^1$ and $g_1, g_2 \in E$. By (76),

$$\begin{aligned}
\int_{\mathfrak{a}^1} \Lambda\big(r_T(\omega, g_1, g_2)\big)d\omega &\le \int_{\mathfrak{a}_T^+[g_1,g_2]} e^{2\rho(Y)}dY\\
&\le \int_{\mathfrak{a}^1} \Lambda\big(R_T(\omega, g_1, g_2)\big)d\omega\,,
\end{aligned}$$

where

$$\Lambda(r) = \int_0^r e^{2s\rho(\omega)}s^{d-1}ds\,,$$

and $d\omega$ is a volume form on $\mathfrak{a}^1$ such that $dY = t^{d-1}dt\,d\omega$. One can check that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\Lambda(r+\delta) \le (1+\varepsilon)\Lambda(r)$ for all sufficiently large r. Thus, it follows from (78) that

$$\int_{\mathfrak{a}^1} \Lambda\big(R_T(\omega, g_1, g_2)\big)d\omega \sim \int_{\mathfrak{a}^1} \Lambda\big(r_T(\omega, g_1, g_2)\big)d\omega$$

as $T \to \infty$, uniformly on $g_1, g_2 \in E$, and to prove (75), it suffices to derive asymptotics of one of these integrals.

Let

$$c_{g_1,g_2} = \beta\left(\overline{g_2^{-1}}, Y_{\max}\right) + \beta\left(\overline{g_1}, -Y_{\max}\right).$$

Let $\varepsilon > 0$. Since the Busemann function is continuous, we may choose an open cone $S \subset \mathfrak{a}^+$ sufficiently close to the ray through $Y_{\max}$, so that for $\omega \in S \cap \mathfrak{a}^1$ and $g_1, g_2 \in E$,

$$\left|\beta\left(\overline{g_2^{-1}}, \omega\right) + \beta\left(\overline{g_1}, -\omega\right) - c_{g_1,g_2}\right| < \varepsilon\,.$$

Then by (77) for all sufficiently large T, $g_1, g_2 \in E$, and $\omega \in S \cap \mathfrak{a}^1$,

$$\left|T - c_{g_1,g_2} - r_T(\omega, g_1, g_2)\right| < 2\varepsilon\,.$$

This implies that

$$S \cap \mathfrak{a}^+_{T-c_{g_1,g_2}-2\varepsilon} \subset \{r\omega : \omega \in \mathfrak{a}^0 \cap S\,,\ 0 \le r < r_T\} \subset S \cap \mathfrak{a}^+_{T-c_{g_1,g_2}+2\varepsilon}\,,$$

hence, using Lemma 9.4,

$$\limsup_{T\to\infty} \frac{\int_{\mathfrak{a}^1} \Lambda(r_T(\omega, g_1, g_2))\,d\omega}{T^{(d-1)/2}\,e^{\delta T}} = \limsup_{T\to\infty} \frac{\int_{\mathfrak{a}^+_{T-c_{g_1,g_2}+2\varepsilon}} e^{2\rho(Y)}\,dY}{T^{(d-1)/2}\,e^{\delta T}}$$
$$\le \limsup_{T\to\infty} \frac{(T - c_{g_1,g_2} + 2\varepsilon)^{(d-1)/2} e^{\delta(T-c_{g_1,g_2}+2\varepsilon)}}{T^{(d-1)/2} e^{\delta T}} = e^{\delta(-c_{g_1,g_2}+2\varepsilon)}.$$

Since c_{g_1,g_2} is bounded, convergence is uniform for $g_1, g_2 \in E$.

Similarly, one has

$$\liminf_{T\to\infty} \frac{\int_{\mathfrak{a}^1} \Lambda(r_T(\omega, g_1, g_2))\,d\omega}{T^{(d-1)/2} e^{\delta T}} \ge e^{\delta(-c_{g_1,g_2}-2\varepsilon)},$$

and since ε was arbitrary, we have

$$\int_{\mathfrak{a}^1} \Lambda\big(r_T(\omega, g_1, g_2)\big)d\omega \sim e^{-\delta c_{g_1,g_2}} \cdot T^{(d-1)/2} e^{\delta T}$$

as $T \to \infty$. This proves (75) with

$$D_{g_1,g_2} = e^{-\delta c_{g_1,g_2}} = \exp\left(-\delta\left(\beta\left(\overline{g_2^{-1}}, Y_{\max}\right) + \beta\left(\overline{g_1}, -Y_{\max}\right)\right)\right).$$

Thus we have proved that

$$\psi_{g_1,g_2}(T) \sim D_{g_1,g_2} T^{(d-1)/2} e^{\delta T},$$

as $T \to \infty$, with uniform convergence for $g_1, g_2 \in E$.

We now obtain (72) via (49). □

Remark 9.5. We use notation from section 2.6.2. To treat the case of the left-symmetric space G/K, we observe that the transformation $g \mapsto g^{-1}$ maps Riemannian balls $\{g : d(P(e), P(g)) < T\}$ to the balls $\{g : d'(P'(e), P'(g)) < T\}$. Hence, denoting

$$H'_T[g_1, g_2] = \left\{h \in H : d'(P'(e), P'(g_1 h g_2)) < T\right\},$$

we have

$$H'_T[g_1, g_2] = H_T[g_2^{-1}, g_1^{-1}]^{-1}.$$

This implies Theorem 9.3 for the balls $H'_T[g_1, g_2]$.

Proof of Lemma 9.2. For any $\chi \in \mathfrak{a}^*$, let $v_\chi \in \mathfrak{a}$ so that for all $Y \in \mathfrak{a}$,

$$(v_\chi, Y) = \chi(Y)\,.$$

Since $\mathfrak{a}_1$ is strictly convex, $\max_{Y\in\mathfrak{a}_1} \rho(Y)$ is attained at a unique point which we denote by $Y_{\max}$. It is a standard application of Lagrange multipliers that $Y_{\max} = v_\rho/\|v_\rho\|$, so it remains to show that $v_\rho \in \operatorname{int}\mathfrak{a}^+$, that is, for all $\alpha \in \Phi$,

$$(v_\alpha, v_\rho) > 0\,. \tag{79}$$

The inner product $(\cdot\,,\cdot)$ is invariant under the action of the Weyl group, that is, for any $\alpha, \beta, \gamma \in \Phi$,

$$\big(v_{r_\alpha(\beta)}, v_{r_\alpha(\gamma)}\big) = (v_\beta, v_\gamma)\,,$$

where $r_\alpha : \mathfrak{a}^* \to \mathfrak{a}^*$ denotes the reflection in the root α. Let

$$\Psi = \Phi^+ \smallsetminus \operatorname{span}(\alpha)\,, \quad \rho' = \tfrac{1}{2}\sum_{\beta\in\Psi} m_\beta \beta\,,$$

so that

$$\rho = \rho' + \tfrac{1}{2}\sum_{\beta\in\Phi^+\smallsetminus\Psi} m_\beta\beta\,.$$

It is a standard fact about root systems (see e.g. [Wa, Prop. 1.1.2.5]) that $r_\alpha(\rho') = \rho'$. This implies that

$$(v_\alpha, v_{\rho'}) = \big(v_{r_\alpha(\alpha)}, v_{r_\alpha(\rho')}\big) = (v_{-\alpha}, v_{\rho'}) = -(v_\alpha, v_{\rho'})\,,$$

so that $(v_\alpha, v_{\rho'}) = 0$. Moreover any $\beta \in \Phi^+ \smallsetminus \Psi$ is a positive multiple of α and hence satisfies $(v_\alpha, v_\beta) > 0$. Therefore

$$(v_\alpha, v_\rho) = (v_\alpha, v_{\rho'}) + \tfrac{1}{2}\sum_{\beta\in\Phi^+\smallsetminus\Psi} m_\beta\,(v_\alpha, v_\beta) > 0\,,$$

and we have (79). □

Proof of Lemma 9.4. Let

$$g_T(z) = \begin{cases} e^{-z} z^{(r-1)/2}\left(2 - \frac{z}{T}\right)^{(r-1)/2} & \text{when } 0 \le z \le 2T\,, \\ 0 & \text{when } z > 2T\,, \end{cases}$$

and let

$$g(z) = 2^{(r-1)/2}\, e^{-z} z^{(r-1)/2}.$$

By Lebesgue's dominated convergence theorem,

$$\int_0^\infty g_T(z)dz \underset{T\to\infty}{\longrightarrow} \int_0^\infty g(z)dz = C\,. \tag{80}$$

By rescaling, we may assume that $\delta = 1$. Let Vol_{r-1} denote Lebesgue measure on the subspace $\ker\lambda$. By translation it induces a measure on each affine subspace parallel to $\ker\lambda$. For each $x \in [-T, T]$,

$$B_{T,x} = \big\{Y \in B(0,T) : \lambda(Y) = x\big\}$$

is a ball of radius $\sqrt{T^2 - x^2}$ in a translate of $\ker\lambda$ and hence

$$\mathrm{Vol}_{r-1}(B_{T,x}) = C'\left(T^2 - x^2\right)^{(r-1)/2}\,,$$

where C' is a constant depending only on r.

Decomposing the integration into slices parallel to $\ker\lambda$, we have

$$\begin{aligned}\frac{\int_{B(0,T)} e^{\lambda(Y)}dY}{e^T T^{(r-1)/2}} &= \int_{-T}^{T} \frac{e^x \mathrm{Vol}_{r-1}(B_{T,x})}{e^T T^{(r-1)/2}}dx \\ &= C' \int_{-T}^{T} \frac{e^x \left(T^2 - x^2\right)^{(r-1)/2}}{e^T T^{(r-1)/2}}dx \\ &= C' \int_0^{2T} e^{-z} \left(\frac{z(2T-z)}{T}\right)^{(r-1)/2} dz \\ &= C' \int_0^{\infty} g_T(z)dz\,.\end{aligned}$$

Applying (80) we obtain

$$\int_{B(0,T)} e^{\lambda(Y)}\, dY \sim C\, T^{(r-1)/2}\, e^T. \tag{81}$$

Since $B(0,1)$ is strictly convex, the maximum $\max_{Y\in B(0,1)} \lambda(Y)$ is attained at a unique point Y_0. By the hypothesis $Y_0 \in S \subset \operatorname{int} \mathfrak{a}^+$. There is $\varepsilon > 0$ such that for all $Y \in B(0,1) \smallsetminus S$, $\lambda(Y) < \delta - \varepsilon$. Furthermore, there is a cube $\mathcal{C} \subset \mathbb{R}^r$ of side length 2, such that

$$B(0,1) \smallsetminus S \subset \mathcal{C}$$

and

$$\max_{Y\in\mathcal{C}} \lambda(Y) \le \delta - \varepsilon\,.$$

For $T > 0$, let

$$T\mathcal{C} = \left\{tc : c \in \mathcal{C}, t \in [0,T]\right\}.$$

We have

$$\begin{aligned}\int_{B(0,T)\smallsetminus S} e^{\lambda(Y)}dY &\le \int_{T\mathcal{C}} e^{\lambda(Y)}dY \\ &\le (2T)^r \max_{Y\in T\mathcal{C}} e^{\lambda(Y)} \\ &\le C'' T^r e^{(\delta-\varepsilon)T}.\end{aligned}$$

Comparing with (81) we obtain

$$\frac{\int_{B(0,T)\smallsetminus S} e^{\lambda(Y)}dY}{\int_{B(0,T)} e^{\lambda(Y)}dY} \longrightarrow_{T\to\infty} 0\,,$$

and so $\int_{B(0,T)\cap S} e^{\lambda(Y)}\, dY \sim \int_{B(0,T)} e^{\lambda(Y)}\, dY$. The assertion follows. □

10 Ratner Theory and Linearization

Our goal in this section is to prove Theorem 2.10. We will derive it from the following result:

Theorem 10.1. *Suppose the general setup holds, where H is connected, semisimple, and balanced, and assume also that UC and D1 hold. Suppose $g_0 \in G$ satisfies $\overline{H\pi(g_0)} = G/\Gamma$. Then for every $F \in C_c(G/\Gamma)$ and any $g_1, g_2 \in G$ we have*

$$\frac{1}{\lambda\left(H_T[g_1,g_2]\right)} \int_{H_T[g_1,g_2]} F(h\pi(g_0))\, d\lambda(h) \underset{T\to\infty}{\longrightarrow} \int_{G/\Gamma} F\, dm'. \tag{82}$$

The proof of Theorem 10.1 relies on the fundamental results of Ratner on the dynamics of unipotent flows on homogeneous spaces, and subsequent work of Dani, Margulis, Mozes, Shah, and others. Specifically, we use a result of Nimish Shah.

We say that a sequence $\{h_n\} \subset H$ is *strongly divergent* if its projection on every simple factor is divergent. In the case of a matrix group, and using the notation of §2.7, $\{h_n\}$ is strongly divergent if and only if $\{\sigma_i(h_n)\} \subset H_i$ has no convergent subsequence for every simple factor H_i of H.

Theorem 10.2 (Shah)**.** *Let H be a connected semisimple Lie subgroup of a Lie group G. Let A be a split Cartan subgroup of H, A^+ a closed Weyl chamber and K a maximal compact subgroup of H. Let Γ be a lattice in G. Let μ be a finite Borel measure on K that is absolutely continuous with respect to Haar measure on K. Suppose that for $x \in G/\Gamma$, Hx is dense in G/Γ. Then for every $f \in C_c(G/\Gamma)$ and every strongly divergent sequence $\{a_n\} \subset A^+$,*

$$\int_K f(a_n kx)d\mu(k) \to \mu(K) \int_{G/\Gamma} f\, dm' \quad \text{as } n \to \infty\,.$$

REMARK 10.3. Theorem 10.2 is proved, but not formulated explicitly, in [S] (see the proof of Corollary 1.2).

Proof of Theorem 10.1. Let $F \in C_c(G/\Gamma)$, and assume without loss of generality that $F \geq 0$. Let g_0, g_1, g_2 be as in the statement of the theorem. In proving (82), to conserve ink and computer memory, we will omit g_1 and g_2 from the notation. Thus H_T stands for $H_T[g_1, g_2]$, $D(x)$ stands for $D(g_1 x g_2)$ and so on.

Fix $\varepsilon > 0$, let $\varepsilon_0 > 0$ such that $(1+\varepsilon_0)^2 < 1+\varepsilon$, and by D1, let $\delta > 0$ such that for all large enough T,

$$\lambda\left(H_{(1+\delta)^2 T}\right) \leq (1+\varepsilon_0)\lambda\left(H_T\right).$$

By UC, there exists a symmetric neighborhood $\mathcal{O}$ of identity in K such that for all $g \in G$ and $k \in \mathcal{O}$,

$$D(gk) \leq (1+\delta) D(g)\,. \tag{83}$$

Since K is compact, there exists a finite cover

$$K = \bigcup_{i=1}^{N} \ell_i \mathcal{O}, \qquad \ell_1, \dots, \ell_N \in K\,.$$

Denote by κ the probability Haar measure on K. Using a partition of unity, there are measures μ_i, $i = 1, \dots, N$ on K, absolutely continuous with respect to κ, such that

$$\operatorname{supp} \mu_i \subset \ell_i \mathcal{O}\,, \quad i = 1, \dots, N \quad \text{and} \quad \sum_{i=1}^{N} \mu_i = \kappa\,. \tag{84}$$

Let ν denote the measure on $\mathfrak{a}^+$ which is equal to $\xi(Y)\,dY$, where dY is Lebesgue measure. We decompose λ as in (49), so that

$$d\lambda(h) = d\kappa(k_1)\,d\nu(Y)\,d\kappa(k_2)$$

for $k_i \in K$, $Y \in \mathfrak{a}^+$, $h = k_1 \exp(Y) k_2$.

Let $H = H_1 \cdots H_t$ be the decomposition of H into an almost direct product of simple factors and let $\sigma_j : H \to H_j$ be measurable sections. Taking $x = \pi(g_0)$ in Theorem 10.2, we find that there are compact subsets $C_j \subset H_j$, $j = 1, \dots, t$ such that for any $h \in \widetilde{H}$

$$\left| \int_K F(hk\pi(g_0)) d\mu_i(k) - \mu_i(K) \int_{G/\Gamma} F\,dm' \right| < \varepsilon_0\,, \tag{85}$$

where

$$\widetilde{H} = \left\{ h \in H : \sigma_j(h) \notin C_j\,,\ j = 1, \dots, t \right\}.$$

For each $k_1, k_2 \in K$ and $T > 0$ write

$$\mathfrak{b}(k_1, k_2, T) = \left\{ Y \in \mathfrak{a}^+ : \forall j\,,\ \sigma_j(\exp(Y)) \notin C_j\,,\ D(k_1 \exp(Y) k_2) < T \right\}.$$

By enlarging each C_j we may assume that (85) holds for any $h = k_1 \exp(Y)$, with $k_1 \in K$, $Y \in \mathfrak{a}$ such that $\sigma_j(\exp(Y)) \notin C_j$ for $j = 1, \dots, t$. Further, we may assume that

$$\widetilde{H}_T = \bigcup_{k_1, k_2 \in K} k_1 \exp\left(\mathfrak{b}(k_1, k_2, T)\right) k_2\,.$$

It follows from (83) and (84) that if $k_2 \in \operatorname{supp} \mu_i$ then for all $T > 0$,

$$\mathfrak{b}(k_1, k_2, T) \subset \mathfrak{b}(k_1, \ell_i, (1+\delta)T) \subset \mathfrak{b}\big(k_1, k_2, (1+\delta)^2 T\big)\,. \tag{86}$$

Writing $x = \pi(g_0)$, we have

$$
\begin{aligned}
&\int_{\widetilde{H}_T} F(hx)d\lambda(h) \\
&\quad = \int_K d\kappa(k_1) \int_K d\kappa(k_2) \int_{\mathfrak{b}(k_1,k_2,T)} F\big(k_1 \exp(Y) k_2 x\big) d\nu(Y) \\
&\quad \overset{(84)}{=} \sum_{i=1}^{N} \int_K d\kappa(k_1) \int_{\ell_i \mathcal{O}} d\mu_i(k_2) \int_{\mathfrak{b}(k_1,k_2,T)} F\big(k_1 \exp(Y) k_2 x\big) d\nu(Y) \\
&\quad \overset{(86)}{\le} \sum_{i=1}^{N} \int_K d\kappa(k_1) \int_{\mathfrak{b}(k_1,\ell_i,(1+\delta)T)} d\nu(Y) \int_K F\big(k_1 \exp(Y) k_2 x\big) d\mu_i(k_2) \\
&\quad \overset{(85)}{\le} (1+\varepsilon_0) \int_{G/\Gamma} F\, dm' \sum_{i=1}^{N} \int_K d\kappa(k_1) \int_{\mathfrak{b}(k_1,\ell_i,(1+\delta)T)} \mu_i(K) d\nu(Y) \\
&\quad \overset{(86)}{\le} (1+\varepsilon_0) \int_{G/\Gamma} F\, dm' \sum_{i=1}^{N} \int_K d\kappa(k_1) \int_K d\mu_i(k_2) \int_{\mathfrak{b}(k_1,k_2,(1+\delta)^2 T)} d\nu(Y) \\
&\quad \le (1+\varepsilon_0) \lambda\big(\widetilde{H}_{(1+\delta)^2 T}\big) \int_{G/\Gamma} F\, dm' \\
&\quad \le (1+\varepsilon_0)^2 \lambda\left(H_T\right) \int_{G/\Gamma} F\, dm'.
\end{aligned}
$$

Thus

$$\frac{1}{\lambda\left(H_T\right)} \int_{\widetilde{H}_T} F(hx)\, d\lambda(h) < (1+\varepsilon) \int_{G/\Gamma} F\, dm'.$$

Since H is balanced and ε was arbitrary we have

$$\limsup_{T\to\infty} \frac{1}{\lambda\left(H_T\right)} \int_{H_T} F(hx)\, d\lambda(h) \le \int_{G/\Gamma} F dm'.$$

The proof of the opposite inequality for lim inf is similar, and we have proved (82). □

Proof of Theorem 2.10. Clearly (∗∗) follows from (82) if we replace h by h^{-1}. To justify this, let

$$\widetilde{D}(g) = D(g^{-1})$$

and apply Theorem 10.1 to $G, H, \widetilde{D}$ instead of G, H, D. Note that D1 for $\widetilde{D}$ holds since H is unimodular, and UC for $\widetilde{D}$ can be proved as in Propositions 6.2 and 6.3. □

11 Applications

In this section we prove the results stated in section 1.5.1. Note that Theorem 1.1 follows from Corollary 2.12.

Proof of Theorem 1.2. Suppose that the general setup holds for a balanced semisimple Lie group G and a standard distance function D. Let Γ be a lattice in G.

First, we suppose that G is a closed subgroup of a Lie group L, Λ is a lattice in L, and Γ acts on $X = L/\Lambda$ by right multiplication. Let $H = G \times L$ and $\tilde{G}$ be the diagonal embedding of G in H, and let $D((g,g)) = D(g)$, $g \in H$. Our choices of $\tilde{G}, D$ show that the general setup holds, and that $\tilde{G}, D$ are standard. Since $\Gamma\pi(g_0)$ is dense in L/Λ, $\tilde{G}\pi'(y_0)$ is dense in $H/(\Gamma \times \Lambda)$, where $y_0 = (g_0, e)$ and $\pi' : H \to H/(\Gamma \times \Lambda)$ is the quotient map. By Theorem 2.10, we find that $(**)$ holds for $\tilde{G}$. Since the $\tilde{G}$-action on $H/(\Gamma \times \Lambda)$ is isomorphic with the $\tilde{G}$-action on $Y = \Gamma\backslash G \times L/\Lambda$, it follows from Theorem 2.10 that $(*)$ holds for the $\tilde{G}$-action on Y. Using Proposition 2.6 we find that I1, I2 and UC hold, and using Proposition 3.1 we obtain the desired result.

Now we suppose that G acts smoothly on a Lie group L by automorphisms and $\Gamma\cdot\Lambda \subset \Lambda$. Thus, Γ acts on $X = L/\Lambda$. By Proposition 2.6, I1, I2 and UC are satisfied. Define $\tilde{Y} = G \times X$ and $Y = Y/\Gamma$ as in §3. The map $G \ltimes L \to G \times X$ induces a G-equivariant homeomorphism of $(G \ltimes L)/(\Gamma \ltimes \Lambda)$ and Y. Let $\pi' : \tilde{Y} \to Y$ be the quotient map. Since $\overline{G\pi'(e,x_0)} = \tilde{Y}$, the statement follows from Theorem 2.10 and Proposition 3.1. □

Remark 11.1. Arguing as in the proof of Theorem 2.10, it is possible in Theorem 1.2 to replace γ^{-1} with γ.

Proof of Corollary 1.3. In terms of the identification (3) of $\mathcal{F}_d$ with G, the map

$$\mathcal{F}_d \to \mathrm{Mat}_d(\mathbb{R}), \quad f \mapsto \bar{Q}(f)$$

is given by

$$G \to \mathrm{Mat}_d(\mathbb{R}), \quad g \mapsto {}^t g A_Q g, \tag{87}$$

where A_Q is the matrix of the quadratic form Q with respect to the standard basis $\mathbf{e}$. The set

$$\bar{Q}(\mathcal{F}_d) = \left\{{}^t g A_Q g : g \in G\right\}$$

consists of symmetric matrices that have the same determinant and the same signature as A_Q, and we have an identification

$$\rho : H\backslash G \to \bar{Q}(\mathcal{F}_d), \quad \rho(\tau(g)) = {}^t g A_Q g = \bar{Q}(g\,\mathbf{e}),$$

where H is the orthogonal group of Q. It can be shown that $\bar{Q}(\mathcal{F}_d)$ is an algebraic variety and that ρ is an isomorphism of algebraic varieties.

Given a norm $\|\cdot\|$ on $\mathbb{R}^d$, we define a norm on $\mathrm{Mat}_d(\mathbb{R})$, which we also denote by $\|\cdot\|$, by

$$\|g\| = \max_{j=1,\dots,d} \|g_{*j}\|\,, \quad g = (g_{ij}) \in \mathrm{Mat}_d(\mathbb{R})\,.$$

Then for $g \in G$, $\|g\,\mathbf{e}\| < T$ if and only if $\|g\| < T$. Let $\Gamma = G(\mathbb{Z})$ and $A \subset \bar{Q}(\mathcal{F}_d)$ as in the statement of the corollary. We have

$$\begin{aligned}
&\#\big\{f \in \mathcal{F}_d(\mathbb{Z}) : \|f\| < T\,,\ \bar{Q}(f) \in A\big\}\\
&\quad = \#\big\{\gamma \in \Gamma : \|\gamma\| < T\,,\ \bar{Q}(\gamma\,\mathbf{e}) \in A\big\}\\
&\quad = \#\big\{\gamma \in \Gamma : \|\gamma\| < T\,,\ \tau(\gamma) \in \rho^{-1}(A)\big\}\,.
\end{aligned}$$

Let H^o be the connected component of the identity in H. Since $d \ge 3$, the group H^o is a connected semisimple Lie group, and it is simple unless Q is of signature $(2,2)$. If H is simple, then it is balanced. If H is of signature $(2,2)$, then it can be shown by a direct computation that it is balanced.

Let $P : H^o\backslash G \to H\backslash G$ be the projection map. Since P has finite fibers, the set $\widetilde{A} = P^{-1}(\rho^{-1}(A))$ is a relatively compact subset with boundary of measure zero with respect to the smooth measure class on $H^o\backslash G$. It is a well-known consequence of Ratner's orbit-closure theorem (see e.g. dense in G/Γ. Thus we may apply Corollary 2.12, and obtain that

$$\begin{aligned}
&\#\big\{\gamma \in \Gamma : \|\gamma\| < T\,,\ \gamma H^o \in \widetilde{A}\big\}\\
&\quad \sim m\big(\{g \in G : \|g\| < T\,,\ gH^o \in \widetilde{A}\}\big)\\
&\quad = m\big(\{g \in G : \|g\| < T\,,\ \bar{Q}(g\,\mathbf{e}) \in A\}\big)\\
&\quad = m\big(\{f \in \mathcal{F}_d : \|f\| < T\,,\ \bar{Q}(f) \in A\}\big)\,.
\end{aligned}$$

This proves the first assertion, and shows that these quantities are asymptotic to $\nu(A)\lambda(H_T)$, where ν is the measure ν_{x_0} for $x_0 = \tau(e)$ defined by (12).

For the second assertion, we need to compute the asymptotics of $\lambda(H_T)$. We use the notation of §7. For the volume computation, we may assume that Q is given by

$$Q(x_1,\dots,x_{p+q}) = x_1x_{p+q} + \dots + x_px_{q+1} + x_{p+1}^2 + \ldots + x_q^2\,.$$

Then the split Cartan subgroup of $\mathrm{SO}(Q)$ can be chosen to be

$$A = \mathrm{diag}(e^{s_1},\dots,e^{s_p},1,\dots,1,e^{-s_p},\dots,e^{-s_1})\,,$$

and the weights of the representation map $Y = (s_1,\dots,s_p,0,-s_p,\dots,-s_1)$ to its coordinates $s_1,\dots,s_p,0,-s_1,\dots,-s_p$. Choosing an order on roots so that the positive root spaces are upper triangular, we find that the positive roots are

$$\begin{array}{lll} s_i - s_j\,, & 1 \le i < j \le p\,, & \text{multiplicity} = 1\,, \\ s_i\,, & 1 \le i \le p\,, & \text{multiplicity} = q - p\,, \\ s_i + s_j\,, & 1 \le i < j \le p\,, & \text{multiplicity} = 1\,. \end{array}$$

The dominant weight is $\lambda_1(s) = s_1$ and

$$2\rho(s) = \sum_{i=1}^{p} (p + q - 2i)s_i\,.$$

First, we consider the case when $p < q$. Then the system of simple roots is

$$\{s_1 - s_2, \dots, s_{p-1} - s_p, s_p\}\,.$$

The dual basis of $\mathfrak{a}$ is $\tilde{\beta}_k$, $k = 1, \dots, p$, where

$$\tilde{\beta}_k = (1, \dots, 1, 0, \dots, 0) \quad (k \text{ ones})\,.$$

and

$$\beta_k = \frac{\tilde{\beta}_k}{2\rho(\tilde{\beta}_k)} = \left(\frac{1}{k(p+q-k-1)}, \dots, \frac{1}{k(p+q-k-1)}, 0, \dots, 0\right).$$

We have $\lambda_1(\beta_k) = \frac{1}{k(p+q-k-1)}$. Note that $\lambda_1(\beta_k)$ is strictly decreasing for $0 \le k \le p \le \frac{p+q-1}{2}$. Thus, $m_1 = \frac{1}{p(q-1)}$ and condition $\mathcal{G}$ is satisfied, so by Theorem 7.4

$$\lambda(H_T) \sim CT^{p(q-1)}.$$

Now if $p = q$, it may be shown that condition $\mathcal{G}$ is not satisfied. Recall that in Remark 7.5(3) we mentioned a generalization of Theorem 7.4 in the case when condition **G** does not hold. Using this generalization we are able to show that in this case the asymptotics is $C(\log T)T^{p(p-1)}$. We omit the details. □

Proof of Corollary 1.4. For both assertions we apply Corollary 2.12 to the matrix norm distance function corresponding to Ψ and $\|\cdot\|$, with $H\backslash G = S$ and $x = P(s_0)$. Note that all conditions are satisfied, and ν_x in our case is equal to $\alpha(s_0, s)\, d\nu(s)$. So it remains to calculate the density α.

We begin with the first case. Since $V = V_H \oplus V_S$, for any $s \in S$, $h \in H$, $\Psi(sh) = \Psi(s) + \Psi(h) - \mathrm{Id}$, where Id is the identity matrix on V. Therefore, given a bounded $S_0 \subset S$ there is $M > 0$ such that for $h \in H$ and $s \in S_0$,

$$D(h) - M \le D(sh) \le D(h) + M\,.$$

This implies that for $s_1, s_2 \in S$,

$$\lambda\big(H_{T-M}[e,e]\big) \le \lambda\big(H_T[s_1, s_2]\big) \le \lambda\big(H_{T+M}[e,e]\big)\,.$$

Via Theorem 2.7 we obtain that $\alpha \equiv 1$, and the first assertion follows.

For the second assertion, we proceed as before to calculate α. Note that for $s \in S$ and $h \in H$ and using the p-norms on V_H and V_S,

$$\left\|\Psi(sh)\right\| = (\dim V_S \cdot \dim V_H)^{-1/p} \cdot \left\|\Psi_S(s)\right\| \cdot \left\|\Psi_H(h)\right\|.$$

Therefore

$$\begin{aligned} H_T[s_0, s] &= \left\{h \in H : \|\Psi(s_0 h s)\| < T\right\} \\ &= \left\{h \in H : \|\Psi(h s_0 s)\| < T\right\} \\ &= \left\{h \in H : \|\Psi_H(h)\| < c_1 T\right\}, \end{aligned}$$

where

$$c_1 = \frac{(\dim V_S \cdot \dim V_H)^{1/p}}{\|\Psi_S(s_0 s)\|}.$$

It follows using Theorem 2.7 that

$$\begin{aligned} \alpha(s_0, s) &= \lim_{T\to\infty} \frac{\lambda\left(H_T[s_0, s]\right)}{\lambda\left(H_T\right)} \\ &= \lim_{T\to\infty} \frac{C\left(\log c_1 T\right)^{\ell}(c_1 T)^m}{C\left(\log T\right)^{\ell} T^m} \\ &= c\|\Psi_S(s_0 s)\|^{-m} \end{aligned}$$

with

$$c = (\dim V_S \cdot \dim V_H)^{m/p}.$$

This proves the second statement. □

12 Examples

In this section we collect some examples showing that our hypotheses are not automatically satisfied, and that they are important for the validity of our results.

12.1 Condition S. Let

$$S^1 \stackrel{\text{def}}{=} \left\{z \in \mathbb{C} : |z| = 1\right\}, \quad G \stackrel{\text{def}}{=} \prod_1^\infty S^1, \quad H \stackrel{\text{def}}{=} \prod_1^\infty \{\pm 1\},$$

equipped with the Tychonov topology, and let $\tau : G \to H\backslash G$ be the natural map. Then G is compact and H is a compact subgroup. Any section $\sigma : H\backslash G \to G$ induces a section $\{\pm 1\}\backslash S^1 \to S^1$ in each factor. Let $\mathcal{U}$ be an open subset of $H\backslash G$. By the definition of the product topology on G, $\tau^{-1}(\mathcal{U})$ contains a subset of the form $V_1 \times \cdots \times V_r \times \prod_{r+1}^\infty S^1$, where $V_1, \ldots, V_r$ are open, and in particular contains a copy of S^1 in one factor. Since there is no continuous section $\{\pm 1\}\backslash S^1 \to S^1$, there is no continuous section defined on $\mathcal{U}$.

12.2 Condition D2. Let $G \stackrel{\text{def}}{=} \mathrm{SL}(3,\mathbb{R})$ and let

$$H = \left\{ \begin{pmatrix} e^t \cos t & -e^t \sin t & x \\ e^t \sin t & e^t \cos t & y \\ 0 & 0 & e^{-2t} \end{pmatrix} : x,y,t \in \mathbb{R} \right\}.$$

In coordinates t,x,y, the left-Haar measure on H is given by $d\lambda{=}e^{2t}dx\,dy\,dt$.

For a constant $c > 1$ to be specified below, let D be a matrix norm distance function given by $D(g) = \|g\|$, where

$$\|(a_{ij})\| \stackrel{\text{def}}{=} \max\left\{ \sqrt{ca_{11}^2 + a_{12}^2}, \sqrt{ca_{22}^2 + a_{21}^2}, \sqrt{a_{13}^2 + a_{23}^2}, |a_{31}|, |a_{32}|, |a_{33}| \right\}.$$

We first compute the volume $\lambda(H_T)$ along two subsequences. In (t,x,y) coordinates, H_T is given by the inequalities

$$\sqrt{x^2+y^2} < T\,, \quad e^{-2t} < T\,,$$

and

$$f_c(t) < T\,, \quad \text{where } f_c(t) = e^t\sqrt{c^2\cos^2 t + \sin^2 t}\,. \tag{88}$$

For $c > 1$ small enough, $f_c(t)$ is a monotonically increasing function of t. This means that (88) is satisfied along a ray of the form $(-\infty,\tau)$, where $T = f_c(\tau)$.

$$\begin{aligned} \lambda(H_T) &= \int_{H_T} e^{2t}\,dt\,dx\,dy \\ &= \int_{-\log T/2}^{\tau} \int_{x^2+y^2<T^2} e^{2t}\,dt\,dx\,dy \\ &= \pi T^2 \int_{-\log T/2}^{\tau} e^{2t}dt \\ &= \tfrac{\pi}{2}T^2\left(e^{2\tau} - (2/T)^2\right). \end{aligned} \tag{89}$$

Now let $\tau = \tau_n = 2n\pi$, for $n \in \mathbb{N}$. We have for $T_n = f_c(\tau_n) = ce^{2n\pi}$,

$$\lambda(H_{T_n}) = \frac{\pi}{2}T_n^2\left((T_n/c)^2 - (2/T_n)^2\right) \sim \frac{\pi}{2c^2}T_n^4 \quad \text{as } n \to \infty\,. \tag{90}$$

Similarly, for $\tau_n \stackrel{\text{def}}{=} (2n+1/2)\pi$, $S_n = f_c(\tau_n)$ we have

$$\lambda(H_{S_n}) = \frac{\pi}{2}S_n^2\left(S_n^2 - (2/S_n)^2\right) \sim \frac{\pi}{2}S_n^4 \quad \text{as } n \to \infty\,. \tag{91}$$

In particular the quantity $\lambda(H_T)$ is not asymptotic to a function of the form kT^4 as $T \to \infty$ for any constant k.

Now taking

$$g_1 = g_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and computing the shape of $H_T[g_1^{-1}, g_2]$ in (t, x, y) coordinates we find the role of sin and cos switched in (88). This implies that the roles of T_n and S_n are reversed, that is,

$$\lambda\big(H_{T_n}[g_1^{-1}, g_2]\big) \sim \frac{\pi}{2} T_n^4 , \quad \lambda\big(H_{S_n}[g_1^{-1}, g_2]\big) \sim \frac{\pi}{2c^2} S_n^4 .$$

In particular,

$$\frac{\lambda(H_{T_n}[g_1^{-1}, g_2])}{\lambda(H_{T_n})} \to_{n\to\infty} c^2, \quad \frac{\lambda(H_{S_n}[g_1^{-1}, g_2])}{\lambda(H_{S_n})} \to_{n\to\infty} \frac{1}{c^2} ,$$

and D2 is not satisfied.

To show that condition D2 is indeed necessary for the validity of our results, we have the following:

Theorem 12.1. *Let H, G, $\|\cdot\|$, $\{S_n\}$, $\{T_n\}$ be as in the above example. Then there are two equivalent but different measures ν_1, ν_2 on $H\backslash G$ such that for any lattice Γ in G, any $x_0 \in H\backslash G$ such that $H\backslash G = \overline{x_0\,\Gamma}$, and any bounded $A \subset H\backslash G$ with $\nu_i(\partial A) = 0$ we have*

$$\frac{N_{S_n}(A, x_0)}{\lambda(H_{S_n})} \underset{n\to\infty}{\longrightarrow} \nu_1(A) \tag{92}$$

and

$$\frac{N_{T_n}(A, x_0)}{\lambda(H_{T_n})} \underset{n\to\infty}{\longrightarrow} \nu_2(A) . \tag{93}$$

Sketch of proof. Suppose Γ is given. The subgroup of H given by the requirement $t = 0$ is the unipotent radical of a parabolic subgroup of G. Using the methods of [S], one can show that condition $(**)$ holds for any $g_0 \in G$ for which $\overline{x_0\,\Gamma} = H\backslash G$ (where $x_0 = \tau(g_0)$) .

We now claim that D1 is satisfied. Let $\mathcal{N}$ be any bounded collection of norms on $\mathrm{Mat}_3(\mathbb{R})$. For $|\cdot| \in \mathcal{N}$, let

$$H_T^{|\cdot|} = \big\{h \in H : |h| < T\big\} .$$

We will show that there is a function $f(|\cdot|, T)$, which depends continuously on both its arguments, such that

$$\lambda\big(H_T^{|\cdot|}\big) \sim f\big(|\cdot|, T\big) T^4 \tag{94}$$

and such that, for all $|\cdot| \in \mathcal{N}$ and all $T > 0$,

$$f\big(|\cdot|, aT\big) = f\big(|\cdot|, T\big) , \qquad \text{where } a = e^{2\pi}. \tag{95}$$

It is easily seen that this implies D1.

For

$$\omega = (x_0, y_0, s_0) \in S^2 = \big\{(x_0, y_0, s_0) : x_0^2 + y_0^2 + s_0^2 = 1\big\}$$

and for T, $r > 0$, write

$$h(\omega, r) = \begin{pmatrix} rs_0 \cos\log(rs_0) & -rs_0 \sin\log(rs_0) & rx_0 \\ rs_0 \sin\log(rs_0) & rs_0 \cos\log(rs_0) & ry_0 \\ 0 & 0 & (rs_0)^{-2} \end{pmatrix}$$

and
$$\tau\big(\omega,T,|\cdot|\big)=\big\{r\geq 0:|h(\omega,r)|<T\big\}\,.$$
Then, letting $d\omega$ denote the standard volume form on S^2, we have
$$\lambda\big(H_T^{|\cdot|}\big)=\int_{S^2}\int_{\tau(\omega,T,|\cdot|)}s_0r^3dr\,d\omega\,.$$
Also write
$$E_\theta=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix},\qquad E_1=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},$$
$$E_2=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\qquad E_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.$$
Then we have, for $\omega=(x_0,y_0,s_0)\in S^2$,
$$\tau\big(\omega,T,|\cdot|\big)=\left\{r\geq 0:\left|rs_0E_{\log(rs_0)}+rx_0E_1+ry_0E_2+\frac{1}{(rs_0)^2}E_3\right|<T\right\}.$$
Define
$$\begin{aligned}\widetilde{\tau}\big(\omega,T,|\cdot|\big)&=\big\{r\geq 0:|rs_0E_{\log(rs_0)}+rx_0E_1+ry_0E_2|<T\big\}\\&=\big\{r\geq 0:r<T/|E(r,\omega)|\big\}\,,\end{aligned}$$
where
$$E(r,\omega)=s_0E_{\log(rs_0)}+x_0E_1+y_0E_2\,.$$
By the computation (89), and since the collection $\mathcal{N}$ is bounded, there are constants c_1,c_2 such that for all $|\cdot|\in\mathcal{N}$:
$$0<c_1\leq\liminf_{T\to\infty}\frac{\lambda(H_T^{|\cdot|})}{T^4}\leq\limsup_{T\to\infty}\frac{\lambda(H_T^{|\cdot|})}{T^4}\leq c_2<\infty\,.\tag{96}$$
Given $\varepsilon>0$, there is a constant C such that for each ω with $s_0\geq\varepsilon$, and each $|\cdot|\in\mathcal{N}$, the symmetric difference of the sets $\widetilde{\tau}(\omega,T,|\cdot|)$ and $\tau(\omega,T,|\cdot|)$ is contained in an interval of length C/T. Using this and (96) it is not hard to show that
$$\lambda\big(H_T^{|\cdot|}\big)\sim\int_{S^2}\int_{\widetilde{\tau}(\omega,T,|\cdot|)}r^3dr\,d\omega\,.\tag{97}$$
Since $E(r,\omega)=E(ar,\omega)$ we have, for all $T>0$ and $\omega\in S^2$,
$$\widetilde{\tau}(\omega,aT)=a\widetilde{\tau}(\omega,T)\,,$$
therefore by a change of variables, for any $\omega\in S^2$,
$$\int_{\widetilde{\tau}(\omega,aT)}r^3dr\,d\omega=a^4\int_{\widetilde{\tau}(\omega,T)}r^3dr\,d\omega\,,\tag{98}$$
hence
$$f\big(|\cdot|,T\big)=\frac{\int_{S^2}\int_{\widetilde{\tau}(\omega,T,|\cdot|)}r^3dr\,d\omega}{T^4}$$

depends continuously on both its parameters and satisfies (95). Now (94) follows from (97) and (98).

Now we may apply Theorem 2.2 to obtain, for any $\varphi \in C_c(H\backslash G)$,

$$S_{\varphi,x_0}(T) \sim \widetilde{S}_{\varphi,x_0}(T)\,.$$

Fix a sequence $t_n = e^{2\pi n}t_0$, for any $t_0 > 0$. The ratio

$$\alpha_{t_0}(g_1,g_2) = \lim_{n\to\infty} \frac{\lambda(H_{t_n}[g_1^{-1},g_2])}{\lambda\,(H_{t_n})}$$

exists, is positive, and depends continuously on g_1, g_2 by (94). Let ν_{t_0} be defined by (12), but using α_{t_0} in place of α. Now repeating the arguments of §5, but taking limits as $n \to \infty$, we obtain the conclusion of Corollary 15. In particular, taking in turn $t_n = S_n$, $t_n = T_n$ we obtain (92) and (93). Note that the two limiting measures in this case are different by (90) and (91). ◻

12.3 Non-balanced semisimple groups. We now construct a non-balanced semisimple group and show that Theorem 2.10 fails for this group, i.e. find an action of this group on a homogeneous space which does not satisfy (∗∗).

Let

$$H_1 = H_2 = \mathrm{SL}(2\,,\mathbb{R}), \quad H = H_1 \times H_2\,.$$

Abusing notation, we consider H_1 and H_2 as subgroups of H.

Denote by A_i a Cartan subgroup of H_i, $\mathfrak{a}_i$ its (one-dimensional) Lie algebra, Y_i a generator of $\mathfrak{a}_i$. Then $A = A_1A_2$ is a Cartan subgroup of H. We will write $(s_1,s_2) = s_1Y_1 + s_2Y_2 \in \mathfrak{a}$. With respect to these coordinates, a root system of (H,A) is $\{\pm\alpha_1, \pm\alpha_2\}$, where

$$\alpha_i(s_1,s_2) = 2s_i$$

and

$$\mathfrak{a}^+ = \big\{(s_1,s_2) : s_1 \ge 0, s_2 \ge 0\big\}\,.$$

In the notation of §7 we have

$$\rho = \tfrac{1}{2}\,(\alpha_1 + \alpha_2)\ , \quad \beta_i = \tfrac{Y_i}{2}\,, \quad i = 1,2\,.$$

We consider H as a subgroup of $G = \mathrm{SL}(2\ell,\mathbb{R})$ where H is embedded in G via the tensor product of irreducible representations of H_1 and H_2 of dimensions 2 and $\ell > 2$ respectively. Note that the set of weights of $\mathfrak{a}$ is

$$\big\{(s_1,s_2) \mapsto is_1 + js_2 : i \in \{\pm 1\}\,,\ j \in \{1-\ell, 3-\ell, \dots, \ell-1\}\big\}\,,$$

and a highest weight corresponding to the choice of $\mathfrak{a}^+$ is

$$\lambda_1(s_1,s_2) = s_1 + (\ell-1)s_2\,.$$

We have

$$\lambda_1(\beta_1) = \frac{1}{2} < \frac{\ell-1}{2} = \lambda_1(\beta_2)\,,$$

so condition $\mathcal{G}$ is satisfied, and by Proposition 8.2, H is not balanced, that is, there is a bounded open $L \subset H_2$ such that

$$c = \limsup_{T\to\infty} \frac{\lambda(H_T^L)}{\lambda(H_T)} > 0\,, \quad \text{where } H_T^L = \big\{h = (h_1, h_2) : \|h\| < T\,,\ h_2 \in L\big\}$$

for some (any, see Proposition 8.1) norm $\|\cdot\|$ on $\mathrm{Mat}_{2\ell}(\mathbb{R})$. We have

CLAIM 5. *There is a $\mathbb{Q}$-subgroup M of G containing H_1 and an element $m \in M$ such that $m^{-1}Hm$ is not contained in any proper $\mathbb{Q}$-subgroup of G. In particular, setting $x = \pi(m)$ and $\Gamma = \mathrm{SL}(2\ell, \mathbb{Z})$ we have that $H_1 x \subset M\pi(e) = \overline{M\pi(e)}$ and $\overline{Hx} = G/\Gamma$.*

Assuming the claim is true, let $X_0 = LM\pi(e) \subset G/\Gamma$. This is the closure of a locally closed submanifold of G/Γ, being the image of $M/\Delta \times L \subset H$ under the proper map $(g\Delta, \ell) \mapsto \ell g x$, where $\Delta = M \cap m\Gamma m^{-1}$. Since $\dim M < \dim G$, we have $m'(X_0) = 0$. Therefore, for any compact $K_0 \subset G/\Gamma$ we can find $\varphi \in C_c(G/\Gamma)$ such that $\int_{G/\Gamma} \varphi\, dm' < c/2$ and $\varphi|_{X_0 \cap K_0} \equiv 1$. It follows from the non-divergence results used in [S] that we can make K_0 large enough so that

$$\limsup_{T\to\infty} \frac{\lambda\left\{h \in H_T : hx \notin K_0\right\}}{\lambda(H_T)} < \frac{c}{2}\,.$$

This yields

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{\lambda(H_T)} \int_{H_T} \varphi(hx)\, d\lambda(h) \\
&\quad \geq \liminf_{T\to\infty} \frac{\lambda\left\{h \in H_T : hx \in K_0 \cap X_0\right\}}{\lambda(H_T)} \\
&\quad \geq \liminf_{T\to\infty} \frac{\lambda(H_T^L)}{\lambda(H_T)} - \limsup_{T\to\infty} \frac{\lambda\left\{h \in H_T : hx \notin K_0\right\}}{\lambda(H_T)} \\
&\quad \geq c - c/2 > \int_{G/\Gamma} \varphi\, dm',
\end{aligned}$$

and $(**)$ fails.

It remains to prove Claim 5. Let $\{u_1, u_2\}$ and $\{v_1, \dots, v_\ell\}$ be the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^\ell$ respectively, and let $\mathcal{B} = \{u_i \otimes v_j\}$, a basis of $\mathbb{R}^2 \otimes \mathbb{R}^\ell = \mathbb{R}^{2\ell}$. The $\mathbb{Q}$-structure on G is defined via $\mathcal{B}$, and the H_1-action (respectively, the H_2) action on $\mathbb{R}^{2\ell}$ is induced by its action on the u_i's (respectively, v_j's). Now let M be the subgroup of G leaving invariant each of the subspaces $V_j = \mathrm{span}\{u_1 \otimes v_j, u_2 \otimes v_j\}$. Clearly each V_j is H_1-invariant and hence $H_1 \subset M$. It is also clear that M is defined over $\mathbb{Q}$ and hence $M\pi(e)$ is closed. It remains to show that there exists $m \in M$ so that $m^{-1}Hm$ is not contained in any proper $\mathbb{Q}$-subgroup of G. Suppose otherwise; since the number of $\mathbb{Q}$-subgroups of G is countable, this would imply that there is a fixed proper $\mathbb{Q}$-subgroup $T \subset G$ such that for all $m \in M$, $m^{-1}Hm \subset T$. However it is not difficult to show (we omit the computation) that the set $\{m^{-1}hm : m \in M, h \in H\}$ generates G, and this is a contradiction.

12.4 A simple case revisited. Our results also enable us to generalize Ledrappier's result, discussed in §1.1, to general norms. Namely we have

Theorem 12.2. *Let Γ be a lattice in $\mathrm{SL}(2,\mathbb{R})$ and let $\|\cdot\|$ be a norm on $\mathrm{Mat}_2(\mathbb{R})$. Suppose that $v \in V = \mathbb{R}^2$ satisfies $\overline{v \cdot \Gamma} = V$, and let dw denote Lebesgue measure on $\mathbb{R}^2$. Then for every $\varphi \in C_c(V)$,*

$$S_{\varphi,v}(T) \sim \left(c_\Gamma \int_V \varphi(w)\alpha_v(w)dw \right) T\,, \tag{99}$$

where $c_\Gamma > 0$ is a constant depending on Γ and

$$\alpha_v(w) = \left\| \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} -v_2w_1 & -v_2w_2 \\ v_1w_1 & v_1w_2 \end{pmatrix} \right\|^{-1}.$$

Proof. Via the map

$$\tau : G \to V\,, \quad \tau : g \mapsto (1,0)\cdot g\,,$$

the space $V \smallsetminus \{0\}$ is identified with $H\backslash G$, where $G = \mathrm{SL}(2,\mathbb{R})$ and

$$H = \left\{ u_t = \begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Note that Haar measures on H and G are only defined up to a constant multiple; we equip H with the Haar measure dt, and choose Haar measure μ on G so that $\nu_{H\backslash G}$ is Lebesgue measure. This induces a choice μ' of G-invariant measure on G/Γ. Since our results were formulated for the choice making μ' a probability measure, we set $c_\Gamma = 1/\mu'(G/\Gamma)$, and $m' = c_\Gamma \mu'$.

We check that the hypotheses of Theorems 2.2 and 2.3 hold in this case. For $g_1, g_2 \in G$,

$$H_T[g_1,g_2] = \{u_t : \|a + tb\| < T\}$$

for $a = g_1g_2$ and $b = g_1 \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) g_2$. This implies

$$\left\{ u_t : |t| < \frac{T - \|a\|}{\|b\|} \right\} \subset H_T[g_1,g_2] \subset \left\{ u_t : |t| < \frac{T + \|a\|}{\|b\|} \right\}.$$

Hence

$$\lambda(H_T[g_1,g_2]) \sim \frac{2T}{\|b\|}\,, \tag{100}$$

with uniform convergence for g_1, g_2 in a compact subset of G. In particular, hypotheses D1 and D2 are satisfied. Hypothesis (∗∗) follows from the equidistribution of the horocycle flow [DS]. Thus, Corollary 2.4 applies, and (99) holds for the function $\alpha_v(w)$ as in Proposition 5.1(iv).

To calculate $\alpha_v(w)$, define $V_0 = \{(x_1, x_2) \in V : x_1 \neq 0\}$ and assume that $v \in V_0$ and supp $\varphi \subset V_0$. Consider a measurable section $\sigma : V \to G$ whose restriction to V_0 is continuous and defined by

$$\sigma(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \\ 1 & \frac{x_2+1}{x_1} \end{pmatrix}.$$

By (9) and (100),

$$\begin{aligned} \alpha_v(w) &= \lim_{T\to\infty} \frac{\lambda(H_T[\sigma(v)^{-1}, \sigma(w)])}{\lambda(H_T)} \\ &= \left\| \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\| \cdot \left\| \sigma(v)^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \sigma(w) \right\|^{-1} \end{aligned}$$

for $v, w \in V_0$. This implies the corollary. If $v \notin V_0$ or supp $\varphi \not\subset V_0$ we complete the proof by taking a different section in the obvious way. $\square$

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