PARKING GARAGES WITH OPTIMAL DYNAMICS

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Abstract. We construct generalized polygons ('parking garages') in which the billiard flow satisfies the Veech dichotomy, although the associated translation surface obtained from the Zemlyakov-Katok unfolding is not a lattice surface. We also explain the difficulties in constructing a genuine polygon with these properties.

1. Introduction and Statement of results

A *parking garage* is an immersion $h : N \to \mathbb{R}^2$, where N is a two dimensional compact connected manifold with boundary, and $h(\partial N)$ is a finite union of linear segments. A parking garage is called *rational* if the group generated by the linear parts of the reflections in the boundary segments is finite. If h is actually an embedding, the parking garage is a polygon; thus polygons form a subset of parking garages, and rationals polygons (i.e. polygons all of whose angles are rational multiples of π) form a subset of rational parking garages.

The dynamics of the billiard flow in a rational polygon has been intensively studied for over a century; see [FK] for an early example, and [DeM, MT, Vo, Zo] for recent surveys. The definition of the billiard flow on a polygon readily extends to a parking garage: on the interior of N the billiard flow is the geodesic flow on the unit tangent bundle of N (with respect to the pullback of the Euclidean metric) and at the boundary, the flow is defined by elastic reflection (angle of incidence equals the angle of return). The flow is undefined at the finitely many points of N which map to 'corners', i.e. endpoints of boundary segments, and hence at the countable union of codimension 1 submanifolds corresponding to points in the unit tangent bundle for which the corresponding geodesics eventually arrive at corners in positive or negative time. Since the direction of motion of a trajectory changes at a boundary segment via a reflection in its side, for rational parking garages, only finitely many directions of motion are assumed. In other words, the phase space of the billiard flow decomposes into invariant two-dimensional subsets corresponding to fixing the directions of motion.

Veech [Ve] discovered that the billiard flow in some special polygons exhibits a striking dichotomy. Namely he found polygons for which, in any initial direction, the flow is either *completely periodic* (all orbits are periodic), or *uniquely ergodic* (all orbits are equidistributed). Following McMullen we will say that a polygon with these properties has *optimal dynamics*. We briefly summarize Veech's strategy of proof. A standard unfolding construction usually attributed to Zemlyakov and Katok [ZK][1], associates to any rational polygon $\mathcal{P}$ a translation surface $M_{\mathcal{P}}$, such that the billiard flow on $\mathcal{P}$ is essentially equivalent to the straightline flow on $M_{\mathcal{P}}$. Associated with any translation surface M is a Fuchsian group Γ_M, now known

[1]but dating back at least to Fox and Kershner [FK].

as the *Veech group* of M, which is typically trivial. Veech found M and $\mathcal{P}$ for which this group is a non-arithmetic lattice in $\mathrm{SL}_2(\mathbb{R})$. We will call these *lattice surfaces* and *lattice polygons* respectively. Veech investigated the $\mathrm{SL}_2(\mathbb{R})$-action on the moduli space of translation surfaces, and building on earlier work of Masur, showed that lattice surfaces have optimal dynamics. From this it follows that *lattice polygons have optimal dynamics.*

This chain of reasoning remains valid if one starts with a parking garage instead of a polygon; namely, the unfolding construction associates a translation surface to a parking garage, and one may define a lattice parking garage in an analogous way. The arguments of Veech then show that the billiard flow in a lattice parking garage has optimal dynamics. This generalization is not vacuous: lattice parking garages, which are not polygons, were recently discovered by Bouw and Möller [BM]. The term 'parking garage' was coined by Möller.

A natural question is whether Veech's result admits a converse, i.e. whether non-lattice polygons or parking garages may also have optimal dynamics. In [SW], Smillie and the second-named author showed that there are non-lattice translation surfaces which have optimal dynamics. However translation surfaces arising from billiards form a set of measure zero in the moduli space of translation surfaces, and it was not clear whether the examples of [SW] arise from polygons or parking garages. In this paper we show:

Theorem 1.1. *There are non-lattice parking garages with optimal dynamics.*

An example of such a parking garage is shown in Figure 1.

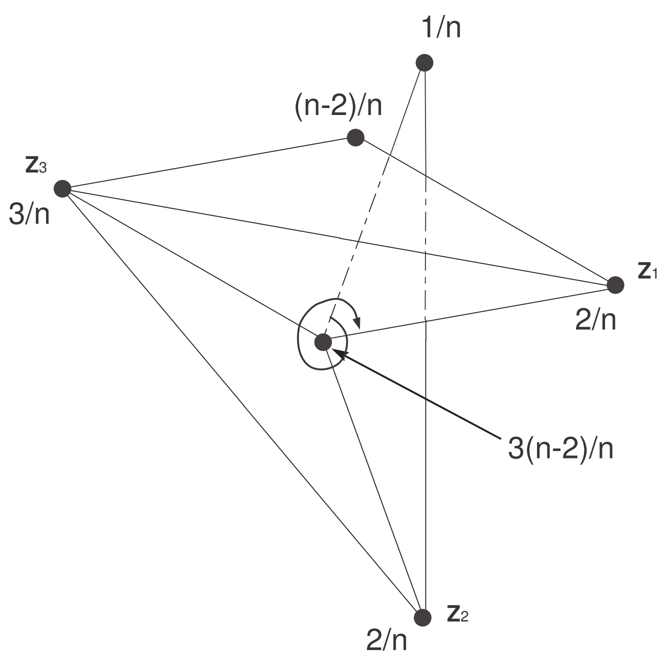

FIGURE 1. A non-lattice parking garage with optimal dynamics. (Here $2/n$ represents angle $2\pi/n$.)

Veech's work shows that for lattice polygons, the directions in which all orbits are periodic are precisely those containing a *saddle connection*, i.e. a billiard path connecting corners of the polygon which unfold to singularities of the corresponding surface. Following Cheung, Hubert and Masur [CHM], if a polygon $\mathcal{P}$ has optimal dynamics, and the periodic directions coincide with the directions of saddle connections, we will say that $\mathcal{P}$ satisfies *strict ergodicity and topological dichotomy.* It is not clear to us whether our example satisfies this stronger property. As we explain in Remark 3.2 below, this would follow if it were known that the center of the regular n-gon is a 'connection point' in the sense of Gutkin, Hubert and Schmidt [GHS] for some n which is an odd multiple of 3.

Veech also showed that for a lattice polygon $\mathcal{P}$, the number $N_{\mathcal{P}}(T)$ of periodic strips on $\mathcal{P}$ of length at most T satisfies a quadratic growth estimate of the form $N_{\mathcal{P}}(T) \sim cT^2$ for a positive constant c. As we explain in Remark 3.3, our examples also satisfy such a quadratic growth estimate.

It remains an open question whether there is a genuine polygon which has optimal dynamics and is not a lattice polygon. Although our results make it seem likely that such a polygon exists, in her M.Sc. thesis [C], the first-named author obtained severe restrictions on such a polygon. In particular she showed that there are no such polygons which may be constructed from any of the currently known lattice examples via the covering construction as in [Vo, SW]. We explain these results and prove a representative special case in §4.

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2. Preliminaries

In this section we cite some results which we will need, and deduce simple consequences. For the sake of brevity we will refer the reader to [MT, Zo, SW] for definitions of translation surfaces.

Suppose S_1, S_2 are compact orientable surfaces and $\pi : S_2 \to S_1$ is a branched cover. That is, π is continuous and surjective, and there is a finite $\Sigma_1 \subset S_1$, called the set of *branch points*, such that for $\Sigma_2 = \pi^{-1}(\Sigma_1)$, the restriction of π to $S_2 \smallsetminus \Sigma_2$ is a covering map of finite degree d, and for any $p \in \Sigma_1$, $\#\pi^{-1}(p) < d$. A *ramification point* is a point $q \in \Sigma_2$ for which there is a neighborhood $\mathcal{U}$ such that $\{q\} = \mathcal{U} \cap \pi^{-1}(\pi(q))$ and for all $u \in \mathcal{U} \smallsetminus \{q\}$, $\#\left(\mathcal{U} \cap \pi^{-1}(\pi(u))\right) \geq 2$.

If M_1, M_2 are translation surfaces, a *translation map* is a surjective map $M_2 \to M_1$ which is a translation in charts. It is a branched cover. In contrast to other authors (cf. [GHS, Vo]), we do not require that the set of branch points be distinct from the singularities of M_1, or that they be marked. It is clear that the ramification points of the cover are singularities on M_2.

If M is a lattice surface, a point $p \in M$ is called *periodic* if its orbit under the group of affine automorphisms of M is finite. A point $p \in M$ is called a *connection point* if any segment joining a singularity with p is contained in a saddle connection (i.e. a segment joining singularities) on M. The following proposition summarizes results discussed in [FK, MT, SW, HS]:

Proposition 2.1.

(a) *A non-minimal direction on a translation surface contains a saddle connection.*
(b) *If M_1 is a lattice surface, $M_2 \to M_1$ is translation map with a unique branch point, then any minimal direction on M_2 is uniquely ergodic.*
(c) *If $M_2 \to M_1$ is a translation map such that M_1 is a lattice surface, then all branch points are periodic if and only if M_2 is a lattice surface.*
(d) *If $M_2 \to M_1$ is a translation map with a unique branch point, such that M_1 is a lattice surface and the branch point is a connection point, then any saddle connection direction on M_2 is periodic.*

Corollary 2.2. *Let $M_2 \to M_1$ be a translation map such that M_1 is a lattice surface with a unique branch point p. Then:*

1. *M_2 has optimal dynamics.*
2. *If p is a connection point then M_2 satisfies topological dichotomy and strict ergodicity.*
3. *If p is not a periodic point then M_2 is not a lattice surface.*

Proof. To prove (1), by (b), the minimal directions are uniquely ergodic, and we need to prove that the remaining directions are either completely periodic or uniquely ergodic. By (a), in any non-minimal direction on M_2 there is a saddle connection δ, and there are three possibilities:

(i) δ projects to a saddle connection on M_1.
(ii) δ projects to a geodesic segment connecting the branch point p to itself.
(iii) δ projects to a geodesic segment connecting p to a singularity.

In case (i) and (ii) since M_1 is a lattice surface, the direction is periodic on M_1, hence on M_2 as well. In case (iii), there are two subcases: if δ projects to a part of a saddle connection on M_1, then it is also a periodic direction. Otherwise, in light of Proposition 2.1(a), the direction must be minimal in M_1, and hence, by Proposition 2.1(b), uniquely ergodic in M_2. This proves (1). Note also that if p is a connection point then the last subcase does not arise, so all directions which are non-minimal on M_2 are periodic. This proves (2). Statement (3) follows from (c). $\square$

We now describe the unfolding construction [FK, ZK], extended to parking garages. Let $\mathcal{P} = (h : N \to \mathbb{R}^2)$. An *edge* of $\mathcal{P}$ is a connected subset L of ∂N such that $h(L)$ is a straight segment and L is maximal with these properties (with respect to inclusion). A *vertex* of $\mathcal{P}$ is any point which is an endpoint of an edge. The *angle* at a vertex is the total interior angle, measured via the pullback of the Euclidean metric, at the vertex. By convention we always choose the positive angles. Note that for polygons, angles are less than 2π, but for parking garages there is no apriori upper bound on the angle at a vertex. Since our parking garages are rational, all angles are rational multiples of π, and we always write them as p/q, omitting π from the notation.

Let $G_{\mathcal{P}}$ be the dihedral group generated by the linear parts of reflections in $h(L)$, for all edges L. For the sake of brevity, if there is a reflection with linear part g fixing a line parallel to L, we will say that g fixes L. Let S be the topological space obtained from $N \times G_{\mathcal{P}}$ by identifying (x, g_1) with (x, g_2) whenever $g_1^{-1}g_2$ fixes an edge containing $h(x)$. Topologically S is a compact orientable surface, and the immersions $g \circ h$ on each $N \times \{g\}$ induce an atlas of charts to $\mathbb{R}^2$ which endows S with a translation surface structure. We denote this translation surface by $M_{\mathcal{P}}$, and write $\pi_{\mathcal{P}}$ for the map $N \times G_{\mathcal{P}} \to M_{\mathcal{P}}$.

We will be interested in a 'partial unfolding' which is a variant of this construction, in which we reflect a parking garage repeatedly around several of its edges to form a larger parking garage. Formally, suppose $\mathcal{P} = (h : N \to \mathbb{R}^2)$ and $\mathcal{Q} = (h' : N' \to \mathbb{R}^2)$ are parking garages. For $\ell \geq 1$, we say that $\mathcal{P}$ *tiles* $\mathcal{Q}$ *by reflections*, and that ℓ is *the number of tiles*, if the following holds. There are maps $h'_1, \dots h'_\ell : N \to N'$ and $g_1, \dots, g_\ell \in G_{\mathcal{P}}$ (not necessarily distinct) satisfying:

(A) The h'_i are homeomorphisms onto their images, and $N' = \bigcup h'_i(N)$.
(B) For each i, the linear part of $h' \circ h'_i \circ h^{-1}$ is everywhere equal to g_i.
(C) For each $1 \leq i < j \leq \ell$, let $L'_{ij} = h'_i(N) \cap h'_j(N)$ and $L = (h'_i)^{-1}(L'_{ij})$. Then $(h'_j)^{-1} \circ h'_i$ is the identity on L, and L is either empty, or a vertex, or an edge of $\mathcal{P}$. If L is an edge then $h'_i(N) \cup h'_j(N)$ is a neighborhood of L'_{ij}. If L_{ij} is a vertex then there is a finite set of $i = i_1, i_2, \dots, i_k = j$ such that $\bigcup h'_{i_s}(N)$ contains a neighborhood of L_{ij}, and each consecutive pair $h'_{i_t}(N), h'_{i_{t+1}}(N)$ intersect along an edge containing L_{ij}.

Vorobets [Vo] realized that a tiling of parking garages gives rise to a branched cover. More precisely:

Proposition 2.3. *Suppose $\mathcal{P}$ tiles $\mathcal{Q}$ by reflections with ℓ tiles, $M_{\mathcal{P}}, M_{\mathcal{Q}}$ are the corresponding translation surfaces obtained via the unfolding construction, and $G_{\mathcal{P}}, G_{\mathcal{Q}}$ are the corresponding reflection groups. Then there is a translation map $M_{\mathcal{Q}} \to M_{\mathcal{P}}$, such that the following hold:*

(1) *$G_{\mathcal{Q}} \subset G_{\mathcal{P}}$.*
(2) *The branch points are contained in the $G_{\mathcal{P}}$-orbit of the vertices of $\mathcal{P}$.*
(3) *The degree of the cover is $\frac{\ell}{[G_{\mathcal{P}} : G_{\mathcal{Q}}]}$.*
(4) *Let $z \in M_{\mathcal{P}}$ be a point which is represented (as an element of $N \times \{1, \dots, r\}$) by (x, k) with x a vertex in $\mathcal{P}$ with angle $\frac{m}{n}$ (where $\gcd(m, n) = 1$). Let $(y_i) \subset M_{\mathcal{Q}}$ be the pre-images of z, with angles $\frac{k_i m}{n}$ in $\mathcal{Q}$. Then z is a branch point of the cover if and only if $k_i \nmid n$ for some i.*

Proof. Assertion (1) follows from the fact that $\mathcal{Q}$ is tiled by $\mathcal{P}$. Since this will be important in the sequel, we will describe the covering map $M_{\mathcal{Q}} \to M_{\mathcal{P}}$ in detail. We will map $(x', g) \in N' \times G_{\mathcal{Q}}$ to $\pi_{\mathcal{P}}(x, gg_i) \in M_{\mathcal{P}}$, where $x' = h'_i(x)$. We now check that this map is independent of the choice of x, i, and descends to a well-defined map $M_{\mathcal{Q}} \to M_{\mathcal{P}}$, which is a translation in charts.

If $x' = h'_i(x_1) = h'_j(x_2)$ then $x_1 = x_2$ since $(h'_i)^{-1} \circ h'_j$ is the identity. If x' is in the relative interior of an edge L_{ij} then

$$\pi_{\mathcal{P}}(x', gg_i) = \pi_{\mathcal{P}}(x', gg_j) \tag{1}$$

since $(gg_i)^{-1} gg_j = g_i^{-1} g_j$ fixes an edge containing $h(x_1)$. If x_1 is a vertex of $\mathcal{P}$ then one proves (1) by an induction on k, where k is as in (C). This shows that the map is well-defined. We now show that it descends to a map $M_{\mathcal{Q}} \to M_{\mathcal{P}}$. Suppose $(x', g), (x', g')$ are two points in $N' \times G_{\mathcal{Q}}$ which are identified in $M_{\mathcal{Q}}$, i.e. $x' \in \partial N'$ is in the relative interior of an edge fixed by $g^{-1}g'$. By (C) there is a unique i such that x' is in the image of h'_i. Thus (x', g) maps to (x, gg_i) and (x', g') maps to $(x, g'g_i)$, and $g_i^{-1} g^{-1} g' g_i$ fixes the edge through $x = g_i^{-1}(x')$. It remains to show that the map we have defined is a translation in charts. This follows immediately from the chain rule and (B).

Assertion (2) is simple and left to the reader. For assertion (3) we note that $M_{\mathcal{P}}$ (resp. $M_{\mathcal{Q}}$) is made of $|G_{\mathcal{P}}|$ (resp. $\ell\,|G_{\mathcal{Q}}|$) copies of $\mathcal{P}$. The point z will be a branch point if and only if the total angle around $z \in M_{\mathcal{P}}$ differs from the total angle around one of the pre-images $y_i \in M_{\mathcal{Q}}$. The total angle at a singularity corresponding to a vertex with angle r/s (where $\gcd(r,s) = 1$) is $2r\pi$, thus the total angle at z is $2m\pi$ and the total angle at y_i is $\frac{2k_i m\pi}{\gcd(k_i,n)}$. Assertion (4) follows. □

3. Non-lattice dynamically optimal parking garages

In this section we prove the following result, which immediately implies Theorem 1.1:

Theorem 3.1. *Let $n \geq 9$ be an odd number divisible by 3, and let $\mathcal{P}$ be an isosceles triangle with equal angles $1/n$. Let $\mathcal{Q}$ be the parking garage made of four copies of $\mathcal{P}$ glued as in Figure 1, so that $\mathcal{Q}$ has vertices (in cyclic order) with angles $1/n, 2/n, 3/n, (n-2)/n, 2/n, 3(n-2)/n$. Then $M_{\mathcal{P}}$ is a lattice surface and $M_{\mathcal{Q}} \to M_{\mathcal{P}}$ is a translation map with one aperiodic branch point. In particular $\mathcal{Q}$ is a non-lattice parking garage with optimal dynamics.*

Proof. The translation surface $M_{\mathcal{P}}$ is the double n-gon, one of Veech's original examples of lattice surfaces [Ve]. The groups $G_{\mathcal{P}}$ and $G_{\mathcal{Q}}$ are both equal to the dihedral group D_n. Thus by Proposition 2.3, the degree of the cover $M_{\mathcal{Q}} \to M_{\mathcal{P}}$ is four. Again by Proposition 2.3, since n is odd and divisible by 3, the only vertices which correspond to branch points are the two vertices z_1, z_2 with angle $2/n$ (they correspond to the case $k_i = 2$ while the other vertices correspond to 1 or 3). In the surface $M_{\mathcal{P}}$ there are two points which correspond to vertices of equal angle in $\mathcal{P}$ (the centers of the two n-gons), and these points are known to be aperiodic [HS]. We need to check that z_1 and z_2 both map to the same point in $M_{\mathcal{P}}$. This follows from the fact that both are opposite the vertex z_3 with angle $3/n$, which also corresponds to the center of an n-gon, so in $M_{\mathcal{P}}$ project to a point which is distinct from z_3. □

Remark 3.2. *As of this writing, it is not known whether the center of the regular n-gon is a connection point on the double n-gon surface. If this turns out to be the case for some n which is an odd multiple of 3, then by Corollary 2.2(2), our construction satisfies strict ergodicity and topological dichotomy. See* [AS] *for some recent related results.*

Remark 3.3. *Since our examples are obtained by taking branched covers over lattice surfaces, a theorem of Eskin, Marklof and Morris* [EMM, Thm. 8.12] *shows that our examples also satisfy a quadratic growth estimate of the form $N_{\mathcal{P}}(T) \sim cT^2$; moreover §9 of* [EMM] *explains how one may explicitly compute the constant c.*

4. Non-lattice optimal polygons are hard to find

In this section we present results indicating that the above considerations will not easily yield a non-lattice polygon with optimal dynamics. Isolating the properties necessary for our proof of Theorem 3.1, we say that a pair of polygons $(\mathcal{P}, \mathcal{Q})$ is *suitable* if the following hold:

- $\mathcal{P}$ is a lattice polygon.
- $\mathcal{P}$ tiles $\mathcal{Q}$ by reflections.
- The corresponding cover $M_{\mathcal{Q}} \to M_{\mathcal{P}}$ as in Proposition 2.3 has a unique branch point which is aperiodic.

In her M.Sc. thesis at Ben Gurion University, the first-named author conducted an extensive search for a suitable pair of polygons. By Corollary 2.2, such a pair will have yielded a non-lattice polygon with optimal dynamics. The search begins with a list of candidates for $\mathcal{P}$, i.e. a list of currently known lattice polygons. At present, due to work of many authors, there is a fairly large list of known lattice polygons but there is no classification of all lattice polygons. In [C], the full list of lattice polygons known as of this writing is given, and the following is proved:

Theorem 4.1 (M. Cohen)**.** *Among the list of lattice surfaces given in* [C]*, there is no $\mathcal{P}$ for which there is $\mathcal{Q}$ such that $(\mathcal{P}, \mathcal{Q})$ is a suitable pair.*

The proof of Theorem 4.1 contains a detailed case-by-case analysis for each of the different possible $\mathcal{P}$. These cases involve some common arguments which we will illustrate in this section, by proving the special case in which $\mathcal{P}$ is any of the obtuse triangles investigated by Ward [W]:

Theorem 4.2. *For $n \geq 4$, let $\mathcal{P} = \mathcal{P}_n$ be the (lattice) triangle with angles $\left(\frac{1}{n}, \frac{1}{2n}, \frac{2n-3}{2n}\right)$. Then there is no polygon $\mathcal{Q}$ for which $(\mathcal{P}, \mathcal{Q})$ is a suitable pair.*

Our proof relies on some auxilliary statements which are of independent interest. In all of them, $M_{\mathcal{Q}} \to M_{\mathcal{P}}$ is the branched cover with unique branch point corresponding to a suitable pair $(\mathcal{P}, \mathcal{Q})$. These statements are also valid in the more general case in which $\mathcal{P}, \mathcal{Q}$ are parking garages.

Recall that an affine automorphism of a translation surface is a homeomorphism which is linear in charts. We denote by $\mathrm{Aff}(M)$ the group of affine automorphisms of M and by $D : \mathrm{Aff}(M) \to \mathrm{GL}_2(\mathbb{R})$ the homomorphism mapping an affine automorphism to its linear part. Note that we allow orientation-reversing affine automorphisms, i.e. $\det \varphi$ may be 1 or -1.

We now explain how $G_{\mathcal{P}}$ acts on $M_{\mathcal{P}}$ by translation equivalence. Let $\pi_{\mathcal{P}} : N \times G_{\mathcal{P}} \to M_{\mathcal{P}}$ and S be as in the discussion preceding Proposition 2.3, and let $g \in G_{\mathcal{P}}$. Since the left action of g on G is a permutation and preserves the gluing rule $\pi_{\mathcal{P}}$, the map $N \times G_{\mathcal{P}} \to N \times G_{\mathcal{P}}$ sending (x, g') to $(x, g^{-1}g')$ induces a homeomorphism $\varphi : S \to S$ and $g \circ h \circ \varphi$ is a translation in charts. Thus $g \in G_{\mathcal{P}}$ gives a translation isomorphism of $M_{\mathcal{P}}$, and similarly $g \in G_{\mathcal{P}}$ gives a translation isomorphism of $M_{\mathcal{Q}}$.

Lemma 4.3. *The branch point of the cover $p : M_{\mathcal{Q}} \to M_{\mathcal{P}}$ is fixed by $G_{\mathcal{Q}}$.*

Proof. Since $G_{\mathcal{Q}} \subset G_{\mathcal{P}}$, any $g \in G_{\mathcal{Q}}$ induces translation isomorphisms of both $M_{\mathcal{P}}$ and $M_{\mathcal{Q}}$. We denote both by g. The definition of p given in the first paragraph of the proof of Proposition 2.3 shows that $p \circ g = g \circ p$; namely both maps are induced by sending $(x', g') \in N' \times G_{\mathcal{Q}}$ to $\pi_{\mathcal{P}}(x, gg'g_i)$, where $x' = h'_i(x)$. Since the cover p has a unique branch point, any $g \in G_{\mathcal{Q}}$ must fix it. □

Lemma 4.4. *If an affine automorphism φ of a translation surface has infinitely many fixed points then $D\varphi$ fixes a nonzero vector, in its linear action on $\mathbb{R}^2$.*

Proof. Suppose by contradiction that the linear action of $D\varphi$ on the plane has zero as a unique fixed point, and let $\mathcal{F}_\varphi$ be the set of fixed points for φ. For any $x \in \mathcal{F}_\varphi$ which is not a singularity, there is a chart from a neighborhood U_x of x to $\mathbb{R}^2$ with $x \mapsto 0$, and a smaller neighborhood $V_x \subset U_x$, such that $\varphi(V_x) \subset U_x$ and when expressed in this chart, $\varphi|_{V_x}$ is given by the linear action of $D\varphi$ on the plane. In particular x is the only fixed point in V_x. Similarly, if $x \in \mathcal{F}_\varphi$ is a singularity, then there is a neighborhood U_x of x which maps to $\mathbb{R}^2$ via a finite branched cover ramified at $x \mapsto 0$, such that the action of φ in $V_x \subset U_x$ covers the linear action of $D\varphi$. Again we see that x is the only fixed point in V_x. By compactness we find that $\mathcal{F}_\varphi$ is finite, contrary to hypothesis. □

Lemma 4.5. *Suppose M is a lattice surface and $\varphi \in \mathrm{Aff}(M)$ has $D\varphi = -\mathrm{Id}$. Then a fixed point for φ is periodic.*

Proof. Let
$$F_1 = \{\sigma \in \mathrm{Aff}(M) : D\sigma = -\mathrm{Id}\}.$$
Then $\varphi \in F_1$ and F_1 is finite, since it is a coset for the group $\ker D$ which is known to be finite. Let $\mathcal{A} \subset M$ be the set of points which are fixed by some $\sigma \in F_1$. By Lemma 4.4 this is a finite set, which contains the fixed points for φ. Thus in order to prove the Lemma, it suffices to show that $\mathcal{A}$ is $\mathrm{Aff}(M)$-invariant.

Let $\psi \in \mathrm{Aff}(M)$, and let $x \in \mathcal{A}$, so that $x = \sigma(x)$ with $D\sigma = -\mathrm{Id}$. Since -Id is central in $\mathrm{GL}_2(\mathbb{R})$, $D(\sigma\,\psi) = D(\psi\,\sigma)$, so there is $f \in \ker D$ such that $\psi\,\sigma = f\,\sigma\,\psi$. Therefore
$$\psi(x) = \psi\,\sigma(x) = f\sigma\,\psi(x), \quad \text{and } f\sigma \in F_1.$$
This proves that $\psi(x) \in \mathcal{A}$. □

Remark 4.6. *This improves Theorem 10 of* [GHS]*, where a similar conclusion is obtained under the additional assumptions that M is hyperelliptic and* $\mathrm{Aff}(M)$ *is generated by elliptic elements.*

The following are immediate consequences:

Corollary 4.7. *Suppose $(\mathcal{P}, \mathcal{Q})$ is a suitable pair. Then*
- $-\mathrm{Id} \notin D(G_{\mathcal{Q}})$.
- *None of the angles between two edges of $\mathcal{Q}$ are of the form p/q with* $\gcd(p,q) = 1$ *and q even.*

Proof of Theorem 4.2. We will suppose that $\mathcal{Q}$ is such that $(\mathcal{P}, \mathcal{Q})$ are a suitable pair and reach a contradiction. If n is even, then $\mathrm{Aff}(M_{\mathcal{P}})$ contains a rotation by π which fixes the points in $M_{\mathcal{P}}$ coming from vertices of $\mathcal{P}$. Thus by Lemma 4.5 all vertices of $\mathcal{P}$ give rise to periodic points, contradicting Proposition 2.1(c). So n must be odd.

Let x_1, x_2, x_3 be the vertices of $\mathcal{P}$ with corresponding angles $1/n, 1/2n, (2n-3)/2n$. Then x_3 gives rise to a singularity, hence a periodic point. Also using Lemma 4.5 and the rotation by π, one sees that x_2 also gives rise to a periodic point. So the unique branch point must correspond to the vertex x_1. The images of the vertex x_1 in $\mathcal{P}$ give rise to two regular points in $M_{\mathcal{P}}$, marked c_1, c_2 in Figure 2. Any element of $G_{\mathcal{P}}$ acts on $\{c_1, c_2\}$ by a permutation, so by Lemma 4.3, $G_{\mathcal{Q}}$ must be contained in the subgroup of index two fixing both of the c_i. Let e_1 be the edge of $\mathcal{P}$ opposite x_1. Since the reflection in e_1, or any edge which is an image of e_1 under $G_{\mathcal{P}}$, swaps the c_i, we have:

$$e_1 \text{ is not a boundary edge of } \mathcal{Q}. \tag{2}$$

We now claim that in $\mathcal{Q}$, any vertex which corresponds to the vertex x_3 from $\mathcal{P}$ is always doubled, i.e. consists of an angle of $(2n-3)/n$. Indeed, for any polygon $\mathcal{P}_0$, the group $G_{\mathcal{P}_0}$ is the dihedral group D_N where N is the least common multiple of

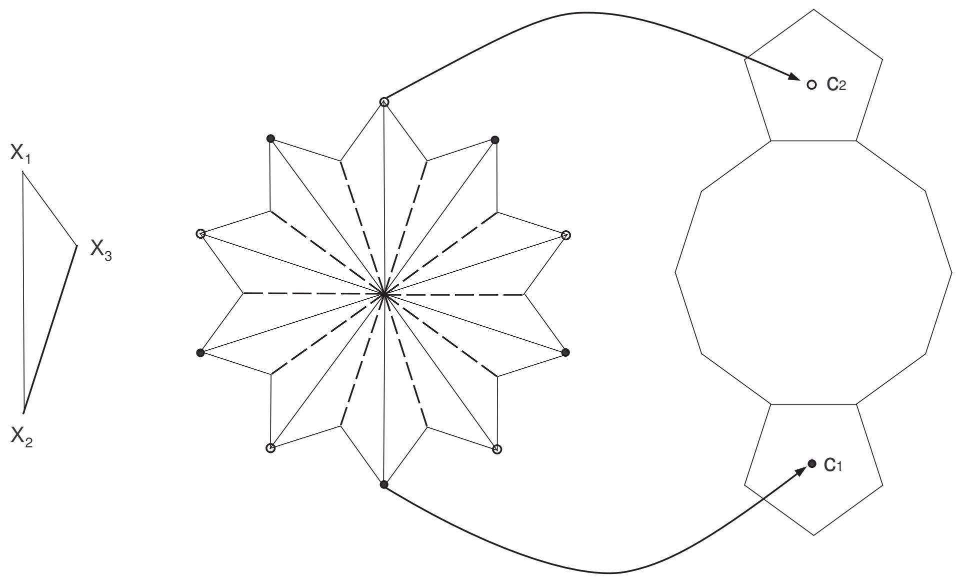

FIGURE 2. Ward's surface, $n = 5$

the denominators of the angles at vertices of $\mathcal{P}_0$. In particular it contains -Id when N is even. Writing $(2n-3)/2n$ in reduced form we have an even denominator, and since, by Corollary 4.7, $-\mathrm{Id} \notin G_{\mathcal{Q}}$, in $\mathcal{Q}$ the angle at vertex x_3 must be multiplied by an even integer $2k$. Since $2k(2n-3)/2n$ is bigger than 2 if $k > 1$, and since the total angle at a vertex of a polygon is less than 2π, we must have $k = 1$, i.e. any vertex in $\mathcal{Q}$ corresponding to the vertex x_3 is always doubled. This establishes the claim. *It is here that we have used the assumption that $\mathcal{Q}$ is a polygon and not a parking garage.*

There are two possible configurations in which a vertex x_3 is doubled, as shown in Figure 3. The bold lines indicate lines which are external, i.e. boundary edges of $\mathcal{Q}$. By (2), the configuration on the right cannot occur.

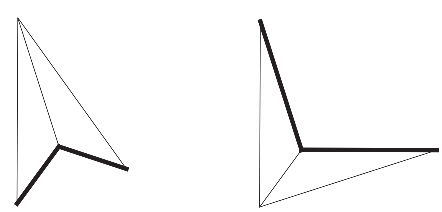

FIGURE 3. Two options to start the construction of Q

Let us denote the polygon on the left hand side of Figure 3 by $\mathcal{Q}_0$. It cannot be equal to $\mathcal{Q}$, since it is a lattice polygon. We now enlarge $\mathcal{Q}_0$ by adding copies of $\mathcal{P}$ step by step, as described in Figure 4. Without loss of generality we first add triangle number 1. By (2), the broken line indicates a side which must be internal in $\mathcal{Q}$. Therefore, we add triangle number 2. We denote the resulting polygon by $\mathcal{Q}_1$. One can check by computing angles, using the fact that n is odd, and using Proposition 2.3(4) that the cover $M_{\mathcal{Q}_1} \to M_{\mathcal{P}}$ will branch over the points a corresponding to vertex x_2. Since the allowed branching is only over the points corresponding to x_1, we must have $\mathcal{Q}_1 \subsetneq \mathcal{Q}$, so we continue the construction. Without loss of generality we add triangle number 3. Again, by (2), the broken line indicates a side which must be internal in $\mathcal{Q}$. Therefore, we add triangle number 4, obtaining $\mathcal{Q}_2$. Now, using Proposition 2.3(4) again, in the cover $M_{\mathcal{Q}_2} \to M_{\mathcal{P}}$ we have branching over two vertices u and v which are both of type x_1 and correspond to distinct points c_1 and c_2 in $M_{\mathcal{P}}$. This implies $\mathcal{Q}_2 \subsetneq \mathcal{Q}$.

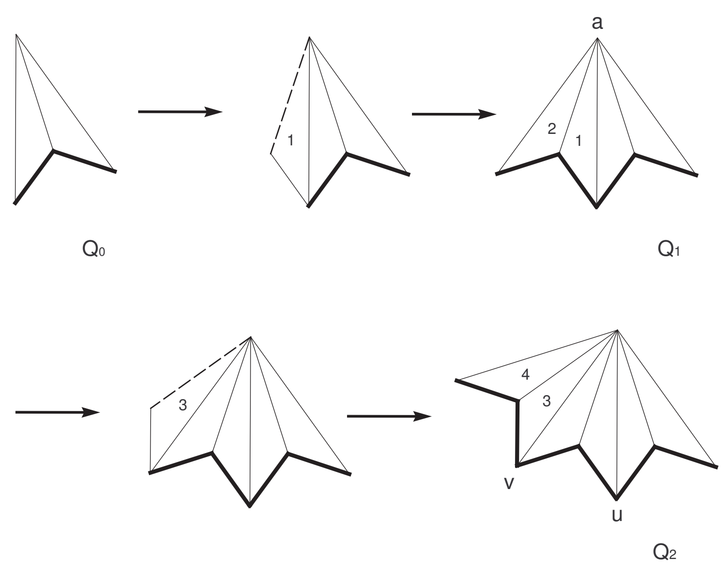

Figure 4. Steps of the construction of Q

Since both vertices u and v are delimited by 2 external sides, we cannot change the angle to prevent the branching over one of these points. This means that no matter how we continue to construct $\mathcal{Q}$, the branching in the cover $M_{\mathcal{Q}} \to M_{\mathcal{P}}$ will occur over at least two points – a contradiction. $\square$

References

[AS] P. Arnoux and T. Schmidt, *Veech surfaces with non-periodic directions in the trace field*, J. Mod. Dyn. **3** (2009), no. 4, 611–629.

[BM] I. Bouw and M. Möller, *Teichmüller curves, triangle groups, and Lyapunov exponents*, Ann. Math. **172** (2010) 139–185.

[CHM] Y. Cheung, P. Hubert and H. Masur, *Topological dichotomy and strict ergodicity for translation surfaces*, Ergodic Theory Dynam. Systens **28** (2008), 1729-1748.

[C] M. Cohen, *Looking for a Billiard Table which is not a Lattice Polygon but satisfies the Veech dichotomy*, M. Sc. thesis, Ben-Gurion University (2010), `arxiv.org/pdf/1011.3217`

[DeM] L. DeMarco, *The conformal geometry of billiards*, Bull. AMS, **48** No. 1, 2011, 33–52.

[EMM] A. Eskin, J. Marklof and D. Morris, *Unipotent flows on the space of branched covers of Veech surfaces*, Erg. Th. Dyn. Sys. **26** (2006), no. 1, 129-162.

[FK] R. H. Fox and R. B. Kershner, *Concerning the transitive properties of geodesics on a rational polyhedron,* Duke Math. J. **2(1)** (1936) 147–150.

[GHS] E. Gutkin, P. Hubert, and T. Schmidt, *Affine diffeomorphisms of translation surfaces: Periodic points, Fuchsian groups, and arithmeticity*, Ann. Sci. École Norm. Sup. (4) **36** (2003), 847-866.

[HS] P. Hubert and T. Schmidt, *Infinitely generated Veech groups*, Duke Math. J. **123** (2004), no. 1, 49–69.

[MT] H. Masur and S. Tabachnikov, *Rational billiards and flat structures*, in **Handbook of dynamical systems**, Vol. 1A, North-Holland, Amsterdam (2002), 1015-1089.

[SW] J. Smillie and B. Weiss, *Veech dichotomy and the lattice property*, Ergod. Th. and Dynam. Sys. **28** (2008), 1959-1972.

[Ve] W. A. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. **97** (1989), 553-583.

[Vo] Y. Vorobets, *Planar structures and billiards in rational polygons: the Veech alternative* (Russian); translation in Russian Math. Surveys **51** (1996), no. 5, 779-817.

[W] C. C. Ward, *Calculation of Fuchsian groups associated to billiards in a rational triangle*, Ergodic Theory Dynam. Systems **18** (1998), 1019-1042.

[ZK] A. Zemlyakov and A. Katok, *Topological transitivity of billiards in polygons*, Math. Notes of the USSR Acad. Sci. **18:2** (1975), 291-300. (English translation in Math. Notes **18:2** (1976) 760-764.)

[Zo] A. Zorich, *Flat surfaces*, in **Frontiers in number theory, physics and geometry,** P. Cartier, B. Julia, P. Moussa and P. Vanhove (eds), Springer (2006).

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