VALUES OF BINARY QUADRATIC FORMS AT INTEGER POINTS AND SCHMIDT GAMES

DMITRY KLEINBOCK AND BARAK WEISS

ABSTRACT. We prove that for any countable set A of real numbers, the set of binary indefinite quadratic forms Q such that the closure of $Q(\mathbb{Z}^2)$ is disjoint from A has full Hausdorff dimension.

Dedicated to S.G. Dani on the occasion of his 65th birthday

1. INTRODUCTION

We start with the following statement, conjectured by Oppenheim and Davenport in the 1930s and proved by Margulis in the 1980s:

Theorem 1.1. *Let Q be a real nondegenerate indefinite quadratic form of $n > 2$ variables which is not proportional to a rational form. Then*

$$(1.1) \qquad 0 \text{ belongs to the closure of } Q(\mathbb{Z}^n \smallsetminus \{0\}).$$

Margulis used an approach which was suggested earlier by Raghunathan and implicitly used earlier by Cassels and Swinnerton-Dyer [CaSD]. Let Q_0 be a fixed quadratic form of the same signature as Q, then one can write $Q(\mathbf{v}) = aQ_0(g\mathbf{v})$ for some $g \in G = \mathrm{SL}_n(\mathbb{R})$ and $a \in \mathbb{R}$. Let F be the stabilizer of Q_0 and let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. Then (1.1) holds if and only if the orbit $Fg\mathbb{Z}^n$ in the space G/Γ of unimodular lattices in $\mathbb{R}^n$ is unbounded. The theorem proved by Margulis (in the case $n = 3$, to which the general case can be reduced) stated that any bounded orbit must be compact, from which it is not hard to derive Theorem 1.1.

Note that it was later proved by Dani and Margulis that for Q as above the set $Q(\mathbb{Z}^n)$, and even $Q(P(\mathbb{Z}^n))$, is dense in $\mathbb{R}$ (here $P(\mathbb{Z}^n)$ stands for the set of primitive integer points in $\mathbb{Z}^n$).

When $n = 2$ it is well known that conclusion of Theorem 1.1 fails. Namely, take

$$(1.2) \qquad Q(p,q) = p^2 - \lambda^2 q^2 = q^2 \left(\frac{p}{q} - \lambda\right)\left(\frac{p}{q} + \lambda\right)$$

such that λ is *badly approximable*, that is,

$$\inf_{p \in \mathbb{Z},\, q \in \mathbb{N}} q^2 \left|\frac{p}{q} - \lambda\right| > 0\,;$$

then the absolute value of $Q(p,q)$ is uniformly bounded away from 0 for any nonzero integer (p,q). It is known from the work of Jarník [J] that the set of such λ's, although null, is *thick*, that is, its intersection with any nonempty open set has full Hausdorff dimension. As in the reduction of the Oppenheim conjecture to a dynamical statement, choose $Q_0(x,y) \stackrel{\text{def}}{=} xy$, let

$$(1.3) \qquad G \stackrel{\text{def}}{=} \mathrm{SL}_2(\mathbb{R})\,,$$

and let F be the connected component of the identity in the stabilizer of Q_0, namely

$$(1.4) \qquad F = \{g_t : t \in \mathbb{R}\} \text{ where } g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

We also let

$$(1.5) \qquad X \stackrel{\text{def}}{=} G/\Gamma \text{ where } \Gamma \stackrel{\text{def}}{=} \mathrm{SL}_2(\mathbb{Z})\,, \text{ and } \mathcal{Q} \stackrel{\text{def}}{=} F\backslash G;$$

thus X can be identified with the space of unimodular lattices in $\mathbb{R}^2$, and $\mathcal{Q}$ can be identified with the space of binary indefinite quadratic forms, considered up to scaling. The set of integer values of a quadratic form is a (set-valued) function on $F\backslash G/\Gamma$ but since this double coset space has a complicated topological structure, it is more useful to consider it either as an F-invariant function on X, or dually, as a Γ-invariant function on $\mathcal{Q}$. This duality principle (already evident in the work of Cassels and Swinnerton-Dyer [CaSD] and developed explicitly by Dani [D1] in a related context) makes it possible to recast dynamical properties of the F-action on X as properties of quadratic forms. In particular (see e.g. [Ma2, Lemma 2.2.1]), it follows from Mahler's Compactness Criterion that for a quadratic form $Q(\mathbf{v}) = Q_0(g\mathbf{v}) \in \mathcal{Q}$, the set of values $Q\left(\mathbb{Z}^2\right)$ has a gap at zero if and only if the orbit $F(g\mathbb{Z}^2)$ is bounded in X. It is known [KM] that the set of points of X with bounded F-orbits is thick. Thus one has

Theorem 1.2. *Let X and Q_0 be as above. Then the set*
$$\left\{x \in X : 0 \notin \overline{Q_0(x \smallsetminus \{0\})}\right\}$$
is thick; dually, the set of binary indefinite quadratic forms whose set of values at integer points has a gap at 0 is thick in the space of all binary indefinite forms.

Note that the paper [KM] was motivated by the work of S.G. Dani [D1, D2], who used W. Schmidt's results and methods [S1, S2] to find thick sets of points in homogeneous spaces with bounded trajectories of one-parameter diagonal semigroups. See also [Ar, AL, KW2] for related results.

The goal of this paper is to strengthen Theorem 1.2 by considering even more complicated properties of the sets of values of quadratic forms at integer points, and, correspondingly, the behavior of F-orbits in X. Here is one of our main results:

Theorem 1.3. *For any countable subset $A \subset \mathbb{R}$, the set*
$$\left\{x \in X : \overline{Q_0(x \smallsetminus \{0\})} \cap A = \varnothing\right\}$$
is thick. Consequently, the set of binary indefinite quadratic forms whose set of values at integer points miss a given countable set is thick in the space of all binary indefinite forms.

Note that by a theorem of Lekkerkerker [L], for Q as in (1.2) the set of accumulation points of $Q\left(\mathbb{Z}^2\right)$ is discrete if and only if λ is quadratic irrational. See [TV, DN] for a precise description of the sets of accumulation points in these cases.

We will derive Theorem 1.3 from its more general dynamical counterpart. To state it, we need the following definition. Let H be a connected subgroup of G different from F, and let Z be a submanifold of X. Say that Z is (F, H)-*transversal* if the following two conditions hold:

  (F) for any $x \in Z$, $T_x(Fx)$ is not contained in T_xZ;

(H, F) for any $x \in Z$, $T_x(Hx)$ is not contained in $T_xZ \oplus T_x(Fx)$.

For example, the above conditions are satisfied if Z consists of a single point. If $\dim(Z) = 1$, the (F, H)-transversality of Z is equivalent to saying that for each $x \in Z$ the lines $T_x(Hx)$, T_xZ and $T_x(Fx)$ are in general position, i.e. they generate the space T_xX.

We permit Z to be a manifold with boundary; in such a case, smoothness of maps and definitions of tangent spaces at the boundary points are defined by positing the existence of smooth extensions (see e.g. [GP, Chap. 2]). We also note that in the application to quadratic forms, Z is an analytic manifold, but in all our arguments, it suffices to assume that Z is C^1-smooth.

Now let us denote by H^+ and H^- the *expanding and contracting horospherical subgroups* with respect to g_1:

$$(1.6) \qquad H^+ \stackrel{\text{def}}{=} \left\{h_s \stackrel{\text{def}}{=} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R}\right\}, \quad H^- \stackrel{\text{def}}{=} \left\{\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} : s \in \mathbb{R}\right\}.$$

A dynamical statement which will imply Theorem 1.3 is as follows:

Theorem 1.4. *Let Z be a countable union of submanifolds of X which are both (F, H^+)- and (F, H^-)-transversal. Then the set*

$$(1.7) \qquad \left\{ x \in X : Fx \text{ is bounded and } \overline{Fx} \cap Z = \varnothing \right\}$$

is thick.

Note that for arbitrary homogeneous spaces $X = G/\Gamma$, non-quasiunipotent subgroups F and *finite* sets Z the thickness of the set (1.7) was conjectured by Margulis [Ma1]. Then in [KM] it was shown that, for mixing flows, the set of points with bounded orbits is thick, and in [K], for arbitrary homogeneous spaces and Z being a *finite* union of Z_i as in the above theorem[1] – that the set

$$\left\{ x \in X : \overline{Fx} \cap Z = \varnothing \right\}$$

is thick. Margulis' Conjecture was finally proved in [KW2] using the technique of Schmidt games, with the set Z upgraded from finite to countable. However the argument of [KW2] could not produce a result for Z being more than zero-dimensional, which, in particular, is needed for an application to quadratic forms.

Organization of the paper. In §2 we explain the reduction of Theorem 1.3 to Theorem 1.4. In §3 we describe a variant of Schmidt's (α, β)-game. This variant is very close to the *absolute game* and *hyperplane absolute game* introduced in [Mc] and [BFKRW] respectively, but adapted to a situation in which we want to play on a smooth manifold. In §4 we state our main technical result, Theorem 4.3 which implies Theorem 1.4, and is a result on the winning property of a certain set for the above game. We complete the proof of Theorem 4.3 in §5.

2. Dynamics and gaps between values of quadratic forms

In this section we explain how to reduce Theorem 1.3 to Theorem 1.4. As was mentioned in the introduction, the equivalence between the set of values of the form at integer points being bounded away from 0 and the F-orbit of the corresponding lattice being bounded in X is well-known. We need to understand how to dynamically describe the set of binary quadratic forms whose values at $\mathbb{Z}^2$ miss a fixed $a \neq 0$. So fix a nonzero a and choose $\mathbf{v} \in \mathbb{R}^2$ such that $Q_0(\mathbf{v}) = a$. Not much will depend on this choice, yet the most obvious one seems to be choosing

$$(2.1) \qquad \mathbf{v} = \begin{cases} (\sqrt{a}, \sqrt{a}) & \text{if } a > 0 \\ (-\sqrt{|a|}, \sqrt{|a|}) & \text{otherwise} \end{cases}$$

which is what we will do. Now define

$$\tilde{Z}_{\mathbf{v}} \stackrel{\text{def}}{=} \left\{ x \in X : \mathbf{v} \in x \right\},$$

that is, the set of unimodular lattices in $\mathbb{R}^2$ containing $\mathbf{v}$. The structure of this set is easy to describe: $\tilde{Z}_{\mathbf{v}} = \bigcup_{n \in \mathbb{N}} Z_{\frac{1}{n}\mathbf{v}}$ where

$$Z_{\mathbf{v}} \stackrel{\text{def}}{=} \left\{ x \in X : \mathbf{v} \in P(x) \right\},$$

the set of unimodular lattices in $\mathbb{R}^2$ containing $\mathbf{v}$ as a *primitive* vector. The latter is simply a closed horocycle, namely a periodic orbit under the action of the subgroup V of G stabilizing

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[1]Technically the statement in [K, Corollary 4.4.2] is weaker than quoted here, namely Z_i are assumed to satisfy condition (F) and have dimension smaller than dimensions of H^+ and H^- – but the argument relies precisely on the combination of (F, H^+)- and (F, H^-)-transversality.

$\mathbf{v}$, which is easily seen to be, for our choice of $\mathbf{v}$, equal to

$$(2.2) \qquad V \stackrel{\text{def}}{=} \begin{cases} \left\{ \begin{pmatrix} 1+s & -s \\ s & 1-s \end{pmatrix} : s \in \mathbb{R} \right\} & \text{if } a > 0 \\[2em] \left\{ \begin{pmatrix} -1-s & s \\ s & -1+s \end{pmatrix} : s \in \mathbb{R} \right\} & \text{otherwise} \end{cases}.$$

Now let us state a proposition which relates escaping $\tilde{Z}_{\mathbf{v}}$ with a gap in values of quadratic forms at a:

Proposition 2.1. *Suppose $x \in X$ is such that Fx is bounded, let $a \neq 0$, and define $\mathbf{v}$ by (2.1). Then $\overline{Fx} \cap \tilde{Z}_{\mathbf{v}} = \varnothing$ if and only if $a \notin \overline{Q_0(x \smallsetminus \{0\})}$.*

Proof. For the 'if' direction, suppose that $x_0 \in \overline{Fx}$ contains $\mathbf{v}$. Then there are t_n such that $g_{t_n} x \to x_0$ and so there are $\mathbf{w}_n \in x$ with $g_{t_n} \mathbf{w}_n \to \mathbf{v}$. In particular $\mathbf{w}_n \neq 0$. Hence $Q_0(\mathbf{w}_n) = Q_0(g_{t_n} \mathbf{w}_n) \to Q_0(\mathbf{v}) = a$, so $a \in \overline{Q_0(x \smallsetminus \{0\})}$, a contradiction.

Suppose the converse does not hold, that is, there exist vectors $\mathbf{v}_k \in x$ such that $Q_0(\mathbf{v}_k) \to a$ as $k \to \infty$. For each k, choose $t_k \in \mathbb{R}$ such that $g_{t_k} \mathbf{v}_k$ belongs to the line passing through $\mathbf{v}$; then it is easy to see that

$$(2.3) \qquad\qquad g_{t_k} \mathbf{v}_k \to \mathbf{v} \text{ as } k \to \infty.$$

But Fx is relatively compact, hence there exists a limit point x_0 of the sequence $g_{t_k} x$ of lattices, and from (2.3) it follows that x_0 contains $\mathbf{v}$, or, equivalently, $x_0 \in \tilde{Z}_{\mathbf{v}}$, contrary to assumption. $\qquad\square$

Remark 2.2. The boundedness of Fx was only used in proving the implication $\Longrightarrow$. If Fx is not bounded (or equivalently, $0 \in \overline{Q_0(x \smallsetminus \{0\})}$) then the situation is more interesting. A simple argument involving multiplication by integers, shows that when $Q_0(x \smallsetminus \{0\})$ contains sequences approaching 0 from both sides, then the set of values $Q_0(x \smallsetminus \{0\})$ is dense. However, as shown by Oppenheim [O], lattices x for which 0 is only a one-sided limit of $Q_0(x \smallsetminus \{0\})$ do exist.

We record the following:

Lemma 2.3. *For $\mathbf{v}$ as in (2.1) (in fact, the same is true for every $\mathbf{v}$ with $Q_0(\mathbf{v}) \neq 0$), the manifold $Z_{\mathbf{v}}$ is both (F, H^+)-transversal and (F, H^-)-transversal.*

Proof. It suffices to show that the Lie algebras of F, V and H^+, as well as those of F, V and H^-, span the Lie algebra of G. This is an easy computation using (1.4), (1.6) and (2.2). $\qquad\square$

We can now see that Theorem 1.3 follows from Theorem 1.4:

Proof of Theorem 1.3 assuming Theorem 1.4. We have already mentioned that the boundedness of the F-orbit of x implies that 0 is not in the closure of the set $Q_0(x \smallsetminus \{0\})$. Now for each $a \in A \smallsetminus 0$ take $\mathbf{v} = \mathbf{v}(a)$ as in (2.1), then, in view of Lemma 2.3, the set

$$Z = \bigcup_{a \in A \smallsetminus \{0\}} \tilde{Z}_{\mathbf{v}(a)}$$

satisfies the assumption of Theorem 1.4, hence the set (1.7) is thick. On the other hand, Proposition 2.1 implies that for every x from the set, $\overline{Q_0(x \smallsetminus \{0\})} \cap A = \varnothing$. $\qquad\square$

3. Schmidt games

We first recall Schmidt's (α, β)-game, introduced in [S1]. The game is played by two players Alice and Bob on a complete metric space[2] X equipped with a target set S and two

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[2]In this section, following tradition, we denote this metric space by X; elsewhere X continues to denote the space of two-dimensional unimodular lattices.

fixed parameters $\alpha, \beta \in (0,1)$. Bob begins the (α, β)-game by choosing $x_1 \in X, r_1 > 0$, thus specifying the closed ball $B_1 \stackrel{\text{def}}{=} \bar{B}(x_1, r_1)$, where

$$\bar{B}(z, \rho) \stackrel{\text{def}}{=} \{x \in X : \text{dist}(x, z) \le \rho\}.$$

Then Alice and Bob continue by alternately choosing x_i', x_{i+1} so that

$$\text{dist}(x_i, x_i') \le (1 - \alpha)r_i, \quad \text{dist}(x_{i+1}', x_i) \le (1 - \beta)r_i', \quad \text{where } r_i' \stackrel{\text{def}}{=} \alpha r_i, \quad r_{i+1} \stackrel{\text{def}}{=} \beta r_i'.$$

This implies that the closed balls

$$A_i \stackrel{\text{def}}{=} \bar{B}(x_i', r_i'), \quad B_{i+1} \stackrel{\text{def}}{=} \bar{B}(x_{i+1}, r_{i+1})$$

are nested, i.e.

$$B_1 \supset A_1 \supset B_2 \supset \cdots$$

The set S is said to be α-*winning* if for any $\beta > 0$ Alice has a strategy in the (α, β)-game guaranteeing that the unique point of intersection $\bigcap_{i=1}^{\infty} B_i = \bigcap_{i=1}^{\infty} A_i$ of all the balls belongs to S, regardless of the way Bob chooses to play. It is called *winning* if it is α-winning for some α.

In [BFKRW], inspired by ideas of McMullen [Mc], the *absolute hyperplane game* was introduced. This modification is played on $\mathbb{R}^d$. Let $S \subset \mathbb{R}^d$ be a target set and let $\beta \in \left(0, \frac{1}{3}\right)$. As before Bob begins by choosing a closed ball B_1 of radius r_1 and then Alice and Bob alternate moves. The sets B_i chosen by Bob are closed balls of radii r_i satisfying

$$r_{i+1} \ge \beta r_i.$$

The sets A_i chosen by Alice are r_i'-neighborhoods of affine hyperplanes, where the r_i' satisfy $r_i' \le \beta r_i$. Additionally Bob's choices must satisfy

$$B_{i+1} \subset B_i \setminus A_i.$$

Then $S \subset \mathbb{R}^d$ is said to be β-*HAW* (where HAW is an acronym for *hyperplane absolute winning*) if Alice has a strategy which leads to

$$(3.1) \qquad \bigcap_{i=1}^{\infty} B_i \cap S \ne \varnothing$$

regardless of how Bob chooses to play; S is said to be *HAW* if it is β-HAW for all $0 < \beta < \frac{1}{3}$. It is easy to see that β-HAW implies β'-HAW whenever $\beta \le \beta' < 1/3$; thus a set is HAW iff it is β-HAW for arbitrary small positive β. In the case $d = 1$ hyperplanes are points, and thus the HAW property coincides with the *absolute winning* property[3] introduced by McMullen in [Mc].

The following proposition summarizes properties of winning and HAW subsets of $\mathbb{R}^d$:

Proposition 3.1. (a) *Winning sets are thick.*
  (b) *HAW implies winning.*
  (c) *The countable intersection of α-winning (resp., HAW) sets is again α-winning (resp., HAW).*
  (d) *The image of a HAW set under a C^1 diffeomorphism $\mathbb{R}^d \to \mathbb{R}^d$ is HAW.*

For the proofs, see [S1, Mc, BFKRW].

Remark 3.2. Note that in the hyperplane absolute version the radii of balls do not have to tend to zero, therefore $\cap_i B_i$ does not have to be a single point. However the outcome with radii not tending to 0 is clearly winning for Alice as long as S is dense. Thus in all the proofs of the HAW property it will be safe to assume that Bob plays so that the radii of his balls tend to zero as $n \to \infty$. Indeed, if Alice has a strategy which is guaranteed to win the game whenever $r_n \to 0$, then the target set must be dense, and hence she will automatically win if $r_n \not\to 0$.

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[3]More generally, the paper [BFKRW] introduced k-dimensional absolute winning for every $0 \le k < d$; the case $k = 0$ was considered earlier by McMullen.

We will be interested in playing variants of the two games described above on a differentiable manifold. Note that a manifold is not equipped with an intrinsic metric, nor with an intrinsic notion of affine submanifolds, and thus the definitions given above cannot be applied directly. Our approach will be to work in a given coordinate system and argue using Proposition 3.1 that the class of winning sets does not depend on the choice of a coordinate system. It will be technically simpler to work with the hyperplane absolute game and this is all that we require for the present paper. We proceed to the details.

We first define the absolute hyperplane game on an open subset $W \subset \mathbb{R}^d$. This is defined just as the absolute hyperplane game on $\mathbb{R}^d$, except for requiring that Bob's first move B_1 be contained in W. If Alice has a winning strategy, we say that S is HAW on W. Now suppose X is a C^1 d-dimensional manifold equipped with an atlas of charts $\mathcal{A} = (U_\alpha, \varphi_\alpha)$; that is, X is a separable topological space, the sets U_α are open subsets of X with $X = \bigcup_\alpha U_\alpha$, each $\varphi_\alpha : U_\alpha \to \mathbb{R}^d$ is a homeomorphism onto its image, and the transition functions $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^d$ are C^1. To define the absolute hyperplane game on $(X, \mathcal{A})$, we specify a target set $S \subset X$. Bob begins play by choosing one coordinate chart $(U, \varphi) = (U_\alpha, \varphi_\alpha)$ in $\mathcal{A}$ and a closed ball $B_1 \subset \mathbb{R}^d$ contained in $\varphi(U)$. From this point on the game continues as before, where Alice and Bob alternate moves in $\varphi(U) \subset \mathbb{R}^d$. To decide the winner they check whether the point of intersection $\bigcap_i B_i$ belongs to $\varphi(S)$. If Alice has a winning strategy, we say that S is HAW on $(X, \mathcal{A})$.

Note that when W is an open subset of $\mathbb{R}^d$, the definition of the game on W, given at the beginning of the previous paragraph, coincides with the definition of the game on $(X, \mathcal{A})$ if we take $X = W$ and take $\mathcal{A}$ to be the atlas consisting of one chart (W, Id). Also note that Bob has been given the additional prerogative of choosing a coordinate chart within which to work, at the start of play, and this appears to make the winning property very restrictive. However we have:

Proposition 3.3. *Suppose X is a C^1 manifold with an atlas $\mathcal{A}$ and (U_i, φ_i) is a system of coordinate charts in $\mathcal{A}$, such that $X = \bigcup U_i$, and $S \subset X$. Then S is HAW on $(X, \mathcal{A})$ if and only if for each i, $\varphi_i(S)$ is HAW on $\varphi_i(U_i)$.*

Proof. The direction $\implies$ is immediate from the definitions, since Bob may select to work with each of the charts φ_i on his first move. For the direction $\impliedby$, note first that by Proposition 3.1(a,b), $\varphi_i(S)$ is dense in each $\varphi_i(U_i)$ and hence S is dense in X. Now suppose Bob chose to work in some chart φ_α distinct from the φ_i. If the diameters of the balls B_n chosen by Bob do not decrease to zero, then $\bigcap B_n$ has interior. Since S is dense in X, each $\varphi_\alpha(S)$ is dense in $\varphi_\alpha(U_\alpha)$, so Alice wins. Otherwise, since B_1 is compact and is covered by the open sets $\varphi_\alpha(U_i)$, there is a Lebesgue number for this cover, that is $\delta > 0$ such that each subset of B_1 of diameter at most δ is contained in one of the $\varphi_\alpha(U_i)$. Thus there are n, i such that

$$(3.2) \qquad\qquad \varphi_\alpha^{-1}(B_n) \subset U_i.$$

In light of Proposition 3.1(d), $\varphi_\alpha(S) = \varphi_\alpha \circ \varphi_i^{-1}(\varphi_i(S))$ is HAW on $\varphi_\alpha(U_\alpha \cap U_i)$. In light of (3.2), the latter set contains Bob's choice B_n and so (applying her strategy for the case that B_n is the first ball chosen by Bob), Alice wins in this case as well. $\square$

Note that our definition above depended on the choice of atlas $\mathcal{A}$. We now deduce from Proposition 3.3 that the HAW property in fact depends only on the manifold structure on X, and not on a specific atlas. Namely, recall that two atlases of charts $\mathcal{A}_1 \overset{\text{def}}{=} (U_\alpha, \varphi_\alpha)$, $\mathcal{A}_2 \overset{\text{def}}{=} (V_\beta, \psi_\beta)$ on the same manifold X are said to be C^1-*compatible* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a C^1-atlas of in the above sense.

Corollary 3.4. *Suppose $\mathcal{A}_1, \mathcal{A}_2$ are two compatible atlases of charts, and $S \subset X$. Then S is HAW on $(X, \mathcal{A}_1)$ if and only if it is HAW on $(X, \mathcal{A}_2)$.*

Proof. The atlases $\mathcal{A}_1, \mathcal{A}_2$ are compatible if and only if the maximal atlas (with respect to inclusion) $\mathcal{A}_{\max}$ containing $(U_\alpha, \varphi_\alpha)$ coincides with the maximal atlas containing (V_β, ψ_β).

So it is enough to show that S is HAW on $(X, \mathcal{A}_1)$ if and only if it is HAW on $(X, \mathcal{A}_{\max})$. For this, apply Proposition 3.3 with $\mathcal{A} = \mathcal{A}_{\max}$ and $\{(U_i, \varphi_i)\} = \mathcal{A}_1$. $\qquad\square$

In light of Corollary 3.4, we will be justified below in omitting the atlas from the terminology and saying that $S \subset X$ is *HAW* if it is HAW on $(X, \mathcal{A})$ for some atlas of charts $\mathcal{A}$ defining the manifold structure on X. It is clear that Proposition 3.1 immediately implies

Proposition 3.5. *Let X be a C^1 manifold. Then:*

(a) *HAW subsets of X are thick.*
(b) *The countable intersection of HAW subsets of X is again HAW.*
(c) *The image of a HAW subset of X under a C^1 diffeomorphism $X \to X$ is HAW.*

Now, and for the rest of the paper, let X be as in (1.5) and F as in (1.4). Also define $F^+ \stackrel{\text{def}}{=} \{g_t : t \geq 0\}$. It turns out that Theorem 1.4 can be reduced to the following statement about one-sided orbits:

Theorem 3.6. (a) *The set*

$$(3.3) \qquad \{x \in X : F^+x \text{ is bounded}\}$$

   is HAW.
(b) *Let Z be a compact (F, H^+)-transversal submanifold of X; then the set*

$$(3.4) \qquad \{x \in X : \overline{F^+x} \cap Z = \varnothing\}$$

   is HAW.

Proof of Theorem 1.4 assuming Theorem 3.6. It is clear that a statement analogous to Theorem 3.6 holds for the semigroup $F^- \stackrel{\text{def}}{=} \{g_t : t \leq 0\}$ in place of F^+ and with the roles of H^+ and H^- exchanged: namely, the sets

$$(3.5) \qquad \{x \in X : F^-x \text{ is bounded}\}$$

and

$$(3.6) \qquad \{x \in X : \overline{F^-x} \cap Z = \varnothing\},$$

where Z is a compact (F, H^-)-transversal submanifold of X, are HAW. The set

$$(3.7) \qquad \{x \in X : Fx \text{ is bounded and } \overline{Fx} \cap Z = \varnothing\}$$

is the intersection of sets (3.3)–(3.6); hence, in view of Proposition 3.5(b), it is HAW whenever $Z \subset X$ is compact and both (F, H^+)- and (F, H^-)-transversal. In Theorem 1.4 our set Z is a countable union of manifolds, and hence (replacing if necessary a manifold with a countable union of compact manifolds with boundary) a countable union of compact manifolds with boundary. Thus the set (1.7) is the countable intersection of sets of the form (3.7), and Theorem 1.4 follows by another application of Proposition 3.5(b). $\qquad\square$

We will finish this section by proving part (a) of the above theorem, which is in fact a rather straightforward variation on some well-known results. In most of the earlier work concerning winning properties of sets of bounded trajectories, the games were actually played on expanding leaves for the F^+-action on X, which in our case can be parametrized as orbits of the expanding horospherical group H^+. An example is McMullen's strengthening [Mc, Theorem 1.3] of a theorem of Dani [D2] on the winning property of the set of directions in hyperbolic manifolds of finite volume with bounded geodesic rays, a special case of which can be restated as follows:

Theorem 3.7. *For any $y \in X$, the set*

$$(3.8) \qquad \{s \in \mathbb{R} : F^+h_sy \text{ is bounded}\}$$

is absolutely winning (which, for games played on $\mathbb{R}$, is the same as HAW).

To reduce Theorem 3.6(a) to Theorem 3.7, let us fix an atlas of coordinate charts for X as follows. Let $\mathfrak{g} \overset{\text{def}}{=} \mathrm{Lie}(G)$, and for any $y \in X$ denote by $\exp_y$ the map

$$(3.9) \qquad \exp_y : \mathfrak{g} \to X, \quad \mathbf{x} \mapsto \exp(\mathbf{x})y \,.$$

For any $y \in X$ one can choose a neighborhood W_y of $0 \in \mathfrak{g}$ such that $\exp_y |_{W_y}$ is one-to-one. Denote

$$(3.10) \qquad U_y \overset{\text{def}}{=} \exp_y(W_y) \text{ and } \varphi_y \overset{\text{def}}{=} \exp_y^{-1}|_{U_y} \,.$$

We will identify $\mathfrak{g}$ with $\mathbb{R}^3$ and use the collection $\{(U_y, \varphi_y) : y \in X\}$ as an atlas of coordinate charts. Since for an arbitrary y we have $y \in U_y$, X is indeed covered by the union of the charts.

Proof of Theorem 3.6(a). In view of Proposition 3.3 it suffices to show that for any $y \in X$, the set

$$(3.11) \qquad \varphi_y\big(\{x \in U_y : F^+ x \text{ is bounded}\}\big) = \{\mathbf{x} \in W_y : F^+ \exp(\mathbf{x})y \text{ is bounded}\}$$

is HAW on W_y. We know from Theorem 3.7 that the set (3.8) is HAW. Note that conjugation by g_t, $t \geq 0$, does not expand elements of $H^- F$. Therefore, for any $x \in X$,

$$(3.12) \qquad F^+ x \text{ is bounded} \iff F^+ g x \text{ is bounded } \forall g \in H^- F \,.$$

The set U_y is foliated by connected components of orbits for the action of $H^- F$. By composing φ_y with a suitable diffeomorphism of W_y, which we are allowed to do by Proposition 3.1(d), we can assume that this foliation is mapped into the foliation of $\mathfrak{g}$ by translates of $\mathfrak{p} \overset{\text{def}}{=} \mathrm{Lie}(H^-) \oplus \mathrm{Lie}(F)$. Let us denote by $\mathfrak{h}$ the Lie algebra of H^+, by $\pi : \mathfrak{g} \to \mathfrak{h}$ the projection associated with the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, and let $W_y^+ \overset{\text{def}}{=} \pi(W_y)$.

Fix $0 < \beta < 1/3$. Bob begins with a ball $B_1 \subset W_y$. Alice consults the strategy she, in view of Theorem 3.7, is assumed to have for playing on $\mathbb{R}$ for the chosen value of β with the target set (3.8) and Bob's first move being $B_1' \overset{\text{def}}{=} \pi(B_1)$ (here we identify $\mathfrak{h}$ with $\mathbb{R}$ via $s \leftrightarrow h_s$). The strategy specifies an interval (neighborhood of a point) $A_1' \subset \mathfrak{h}$, and in the game on W_y Alice chooses $A_1 \overset{\text{def}}{=} \pi^{-1}(A_1')$, which is a hyperplane neighborhood. Continuing iteratively, suppose Bob has chosen the ball $B_i \subset W_y$. The ball $B_i' \overset{\text{def}}{=} \pi(B_i)$ is a legal move for Bob playing on W_y^+, since the projection π does not affect the radii of balls and since

$$B_i \subset B_{i-1} \smallsetminus A_{i-1} \implies B_i' \subset B_{i-1}' \smallsetminus A_{i-1}' \,.$$

Thus Alice's strategy for playing on W_y^+ specifies a move $A_i' \subset \mathfrak{h}$, and in the new game Alice chooses $A_i \overset{\text{def}}{=} \pi^{-1}(A_i')$. This defines her strategy for playing on W_y and guarantees that $\bigcap B_i'$ belongs to the set (3.8). By (3.12), the point $\bigcap B_i$ belongs to the set (3.11). $\square$

4. Transversality and reduction to discrete time actions

It would seem natural to attempt to prove an analogue of Theorem 3.7 for orbits escaping Z, that is, show that for Z as in Theorem 3.6(b) and $y \in X$, the set $\{s \in \mathbb{R} : \overline{F^+ h_s y} \cap Z = \varnothing\}$ is HAW. And indeed the above statement is true; however it would not be enough for proving Theorem 3.6(b). A reason for that is that one can state a version of the equivalence (3.12) for this situation, namely, that

$$\overline{F^+ x} \cap Z = \varnothing \iff \overline{F^+ \exp(\mathbf{p})x} \cap Z = \varnothing \ \forall \mathbf{p} \in \mathfrak{p} \text{ with } \|\mathbf{p}\| \leq \varepsilon \,,$$

where $\varepsilon > 0$ depends on x. However to derive a winning property of the set (3.4) one would need a uniform lower bound on ε, which is not available here. Thus we will need to play on X itself.

Our first step will be a reduction to discrete time actions. The argument here loosely follows [K, §4]. For a parameter $\tau > 0$ to be defined later, denote by

$$F_\tau^+ \overset{\text{def}}{=} \{g_{n\tau} : n \in \mathbb{Z}_+\}$$

the cyclic subsemigroup of F generated by g_τ. We will make a reduction showing that we may replace the continuous semigroup F^+ with F_τ^+. For $Z \subset X$ we define

$$Z_{[t_1, t_2]} \stackrel{\text{def}}{=} \bigcup_{t_1 \le t \le t_2} g_t Z.$$

We have:

Lemma 4.1. *Suppose Z is a compact C^1 submanifold of X and H a connected subgroup of G. Then:*

    (a) *If condition (F) holds, then there exists $\sigma = \sigma(Z) > 0$ such that $Z_{[-\sigma, \sigma]}$ is a C^1 manifold.*

    (b) *If, in addition, condition (H, F) holds, then there exists positive $\tau = \tau(Z) \le \sigma(Z)$ such that for any x in $Z_{[0,\tau]}$, $T_x(Hx)$ is not contained in $T_x\left(Z_{[0,\tau]}\right)$.*

Note that the conclusion of part (b) of the above lemma coincides with condition (F) with F replaced by H and Z replaced by $Z_{[0,\tau]}$. It will be convenient to introduce more notation and, for a subgroup H of G and a smooth submanifold Z of X, say that Z is H-*transversal at* $x \in Z$ if $T_x(Hx)$ is not contained in $T_x Z$, and that Z is H-*transversal* if it is H-transversal at every point of Z. In other words, if condition (F) holds with F replaced by H. This way, condition (F) says that Z is F-transversal, and the conclusion of Lemma 4.1(b) states that $Z_{[0,\tau]}$ is H-transversal.

Proof. Let $\dim(Z) = k$. Using a finite covering of Z by appropriate coordinate charts of X, one can without loss of generality assume that Z is of the form $\psi\left(\overline{U}\right)$ for some bounded open $U \subset \mathbb{R}^k$, with ψ being a C^1, nonsingular embedding defined on an open $U' \subset \mathbb{R}^k$ strictly containing $\overline{U}$. Define $\tilde{\psi} : U' \times \mathbb{R} \to X$ by putting $\tilde{\psi}(u, t) = g_t\left(\psi(u)\right)$. From the F-transversality of Z it follows that $\tilde{\psi}$ is nonsingular at $t = 0$ and $u \in \overline{U}$. Hence $\tilde{\psi}$ is a nonsingular embedding of $U'' \times [-\sigma, \sigma]$ into X for some $\sigma > 0$ and an open set U'' strictly containing $\overline{U}$, and (a) is proved.

Clearly the tangent space to $Z_{[-\sigma, \sigma]}$ at $x \in Z$ is equal to $T_x Z \oplus T_x(Fx)$. Therefore condition (H, F) implies that $Z_{[-\sigma, \sigma]}$ is H-transversal at any point of Z. But the H-transversality is clearly an open condition, hence it holds at any point of $Z_{[-\sigma, \sigma]}$ which is close enough to Z. By compactness one can choose a positive $\tau \le \sigma$ such that $Z_{[0,\tau]}$ is H-transversal. $\qquad\square$

Here is another way to express H-transversality: fix a Riemannian metric 'dist' on the tangent bundle of X, and for a C^1 submanifold Z of X consider the function $\theta_H : Z \to \mathbb{R}$,

$$\theta_H(x) \stackrel{\text{def}}{=} \sup_{v \in T_x(Hx), \|v\| = 1} \text{dist}(v, T_x Z).$$

It is clear that $\theta_H(x) \ne 0$ iff Z is H-transversal at x, and that θ_H is continuous in $x \in Z$. Therefore the following holds:

Lemma 4.2. *A compact C^1 submanifold Z of X is H-transversal iff there exist $c = c(Z) > 0$ such that $\theta_H(x) \ge c$ for all $x \in Z$.*

A right-invariant metric on G induces a well-defined Riemannian metric on X and we now change notation slightly, writing 'dist' for the resulting path metric on X. Now let Z be as in Theorem 3.6, that is, compact and (H^+, F)-transversal, and take positive $\tau \le \tau(Z)$ as in Lemma 4.1 satisfying in addition:

(4.1) $$x, y \in X, 0 \le t \le \tau \implies \text{dist}(g_t x, g_t y) \le 2\,\text{dist}(x, y).$$

This can be done because $g_t, |t| \le \tau$ is bounded and hence there is a uniform bound on the amount by which it distorts the Riemannian metric. Suppose that for some $x \in X$ and $\varepsilon > 0$ there exists $t \ge 0$ and $z \in Z$ such that the distance between $g_t x$ and z is less than ε. Choose $0 \le t_1 < \tau$ such that $t + t_1 = n\tau$ for some $n \in \mathbb{N}$. It follows from (4.1) that $\text{dist}(g_{n\tau} x, g_{t_1} z) < 2\varepsilon$. This rather elementary argument proves that

$$\overline{F_\tau^+ x} \cap Z_{[0,\tau]} = \varnothing \implies \overline{F^+ x} \cap Z = \varnothing.$$

Thus to establish Theorem 3.6(b) it suffices to prove the following:

Theorem 4.3. *For any compact H^+-transversal submanifold Z of X and any $\tau > 0$ the set*

$$(4.2) \qquad\qquad\qquad \left\{ x \in X : F_\tau^+ x \cap Z = \varnothing \right\}$$

is HAW.

Theorem 3.6(b) can then be obtained from Theorem 4.3 with Z replaced by $Z_{[0,\tau]}$.

5. Completion of the proof: the percentage game

In this final section we prove Theorem 4.3. The argument below originates with an idea of Moshchevitin [Mo], which was developed further in the Ph.D. Thesis of Broderick, and in the papers [B, BFK, BFS]. We follow the streamlined presentation of [BFS], which consists of defining yet another game. In what follows, for a subset Y in a metric space and $\varepsilon > 0$ we will denote by $Y^{(\varepsilon)}$ the ε-neighborhood of Y.

Fix $\beta > 0$ and a target set $S \subset \mathbb{R}^d$. The *hyperplane percentage game* is defined as follows: Bob begins as usual by choosing a closed ball $B_1 \subset \mathbb{R}^d$. Then, for each $i \geq 1$, once B_i (of radius r_i) is chosen, Alice chooses finitely many affine hyperplanes $L_{i,j}$ and numbers $\varepsilon_{i,j}$, where $j = 1, \ldots, N_i$, satisfying $0 < \varepsilon_{i,j} \leq \beta r_i$. Here N_i can be any positive integer that Alice chooses. Bob then chooses a ball $B_{i+1} \subset B_i$ with radius $r_{i+1} \geq \beta r_i$ such that

$$B_{i+1} \cap L_{i,j}^{(\varepsilon_{i,j})} = \varnothing \text{ for at least } \frac{N_i}{2} \text{ values of } j \,.$$

Thus we obtain as before a nested sequence of closed balls $B_1 \supset B_2 \supset \cdots$ and declare Alice the winner if and only if (3.1) holds. If Alice has a strategy to win regardless of Bob's play, we say that S is β-*hyperplane percentage winning*, or β-*HPW*. Note that for large values of β it is possible for Alice to leave Bob with no available moves after finitely many turns. However, an elementary argument (see [Mo, Lemma 2] or [BFK, §2]) shows that Bob always has a legal move if β is smaller than some constant $\beta_0(d)$. In particular $\beta_0(1) = 1/5$. If S is β-HPW for each $0 < \beta < \beta_0(d)$, we say that S is *hyperplane percentage winning (HPW)*. It is clear that β-HPW implies β'-HPW if $\beta \leq \beta'$; thus HPW is equivalent to β-HPW for arbitrary small values of β.

We remark that the game defined above is actually a special case of the (β, p)-game defined in [BFS], corresponding to the choice $p = 1/2$. Here p represents the percentage of hyperplanes that Bob is obliged to stay away from.

One sees that in the hyperplane percentage game the rules are more favorable to Alice than in the hyperplane absolute game, and so any HAW set is automatically HPW. Surprisingly, the converse is true, see [BFS, Lemma 2.1]:

Lemma 5.1. *For any $\beta \in (0, 1/3)$ there exists $\beta' \in \big(0, \beta_0(d)\big)$ such that any set which is β'-HPW is β-HAW. In particular the HPW and HAW properties are equivalent.*

Proof of Theorem 4.3. Recall that we are given $\tau, \beta > 0$ and a compact H^+-transversal submanifold Z of X. Our goal (after using Lemma 5.1 and rewriting β for β') is to show that the set (4.2) is β-HPW. We will assume, without loss of generality, that

$$(5.1) \qquad\qquad\qquad\qquad \beta < e^{-2\tau} \,.$$

Let us say that a map between two metric spaces is *C-bi-Lipschitz* if the ratio of $\mathrm{dist}\big(f(x), f(y)\big)$ and $\mathrm{dist}(x, y)$ is uniformly bounded between $1/C$ and C.

The first step is to fix an atlas of coordinate charts for X. We will do it as in the proof of Theorem 3.6(a), that is, using charts (3.10) where $\exp_y$ is defined by (3.9) and $\exp_y |_{W_y}$ is one-to-one for all $y \in X$. As before, it suffices to show that for any $y \in X$, the set

$$\varphi_y\big(\{x \in U_y : F_\tau^+ x \cap Z = \varnothing\}\big) = \{\mathbf{x} \in W_y : F_\tau^+ \exp(\mathbf{x}) y \cap Z = \varnothing\}$$

is HAW on W_y.

The next step is to collect some information about Z. Since it is bounded, one can choose $\sigma_1 < 1$ such that the restriction of $\exp_y$ to $B_{\mathfrak{g}}(0, 4\sigma_1)$ is 2-bi-Lipschitz (in particular, injective) whenever $B_X(y, 2\sigma_1) \cap Z \neq \varnothing$ (here, as before, we work with a path-metric on X

coming from a right-invariant Riemannian metric on G obtained from some inner product on $\mathfrak{g}$). In light of Corollary 3.4, we can (by replacing the U_y with smaller sets, depending on Z) also assume that $U_y \subset B_X(y, 2\sigma_1)$ for any $y \in Z^{(4\sigma_1)}$. Then, because of compactness and C^1-smoothness of Z, for every $b > 0$ there exists $0 < \sigma_2(b) \le \sigma_1$ such that

(5.2)
$$\sigma \le \sigma_2(b) \text{ and } B_X(y, \sigma) \cap Z \ne \varnothing \quad \Rightarrow \quad \text{there exists}$$
$$\text{a } \dim(Z)\text{-dimensional subspace } L \text{ of } \mathfrak{g} \text{ such that } \varphi_y\big(B_X(y, \sigma) \cap Z\big) \subset L^{(b\sigma)}.$$

(More precisely, L is the tangent space to $\varphi_y(Z)$ at $\varphi_y(z)$, where z is some point in the intersection of $B_X(y, \sigma)$ and Z.) Also, recall that Z is H^+-transversal, and let $c = c(Z)$ be as in Lemma 4.2.

Now choose m large enough so that

(5.3)
$$\beta^{-n} < e^{2m\tau}, \quad \text{where } n = \lfloor \log_2 m \rfloor + 1.$$

This is possible since the left (resp., right) side of the inequality in (5.3) depends polynomially (resp., exponentially) on m.

Pick an arbitrary $y \in X$ and let us suppose that Bob chooses a ball $B_0 \subset W_y$ of radius r_0. Set

(5.4)
$$b \stackrel{\text{def}}{=} \frac{c\beta^{n+2}e^{-2m\tau}}{16},$$

(5.5)
$$\sigma \stackrel{\text{def}}{=} \min\left(\frac{1}{4}\sigma_2(b), e^{2m\tau}r_0\right),$$

and let

(5.6)
$$\delta \stackrel{\text{def}}{=} e^{-2m\tau}\sigma \quad \text{and} \quad \varepsilon \stackrel{\text{def}}{=} b\sigma = \frac{1}{16}c\beta^{n+2}\delta.$$

We will show that Alice can play the β-hyperplane percentage game in such a way that the intersection of all the balls belongs to the set

$$\{\mathbf{x} \in W_y : F_\tau^+ \exp(\mathbf{x})y \cap Z^{(\varepsilon)} = \varnothing\}.$$

Note that it follows from (5.5) and (5.6) that $\delta \le r_0$. The game will start with Alice making moves in an arbitrary way until the first time Bob's ball has radius $r_1 \le \delta$. (Recall that, in view of Remark 3.2, it suffices to assume that Alice has a winning strategy whenever the radii of Bob's balls tend to zero.) Re-indexing if necessary, let us call this ball B_1 and its radius r_1; note that we have $r_1 \ge \beta\delta$.

In order to specify Alice's strategy we will partition her moves into windows. For each $j, k \in \mathbb{N}$, we will say that k *lies in the jth window* if

(5.7)
$$\beta^{-n(j-1)} \le e^{2k\tau} < \beta^{-jn}.$$

By (5.3), for every $j \in \mathbb{N}$, there are at most m indices k lying in the jth window, and (5.1) guarantees that every $k \in \mathbb{N}$ lies in some window. We will call the indices i for which

(5.8)
$$\beta^{nj}r_1 < r_i \le \beta^{n(j-1)}r_1$$

the *jth stage of the game*. The first stage begins with Bob's initial ball B_1, and the rules of the game imply that each stage contains at least n indices. Loosely speaking, Alice will use her moves indexed by numbers in the jth stage, to ensure that the points $\mathbf{x}$ contained in the balls chosen by Bob satisfy $g_\tau^k \exp(\mathbf{x})y \notin Z^{(\varepsilon)}$ for any k in the jth window. We now specify Alice's strategy in more detail.

Fix j and suppose that $i = i(j)$ is the first index of stage j. For any k belonging to the jth window, denote

$$A_{j,k} \stackrel{\text{def}}{=} \{g_\tau^k \exp(\mathbf{x})y : \mathbf{x} \in B_i\}.$$

In view of (5.8), the diameter of $A_{j,k}$ is at most

$$2e^{2k\tau}\beta^{n(j-1)}r_1 \stackrel{(5.7)}{\le} 2\beta^{-n}r_1 \stackrel{(5.3)}{\le} 2e^{2m\tau}r_1 \stackrel{(5.6)}{\le} 2\sigma \stackrel{(5.5)}{\le} \frac{1}{2}\sigma_2(b).$$

If $A_{j,k}$ does not intersect $Z^{(\varepsilon)}$ for any k in the jth window, Alice makes all her moves in the jth stage of the game in an arbitrary way (e.g. she could put $N_i = 0$, that is, decide not to specify any hyperplanes on her ith move). Suppose that one of them does; let $x \overset{\text{def}}{=} g_\tau^k y$. Then $A_{j,k} \cap B_X(x, \sigma_2(b)) \neq \varnothing$. Now let us use φ_x to map everything to $\mathfrak{g}$. In view of (5.2), there exists $z \in Z$ and a $\dim(Z)$-dimensional subspace $L = T_{\varphi_x(z)}(\varphi_x(Z))$ of $\mathfrak{g}$ such that

$$\varphi_x\left(A_{j,k} \cap Z^{(\varepsilon)}\right) \subset \varphi_x(A_{j,k}) \cap (\varphi_x Z)^{(2\varepsilon)} \overset{(5.2)}{\subset} \varphi_x(A_{j,k}) \cap L^{(2\varepsilon + 2b\sigma)}$$

$$\overset{(5.4),\,(5.6)}{\subset} \varphi_x(A_{j,k}) \cap L^{\left(c\beta^{n+2}e^{-2m\tau}\sigma/4\right)}.$$

In view of Lemma 4.2 and the 2-bi-Lipschitz property of maps φ_x and φ_z, the intersection of $L^{\left(c\beta^{n+2}e^{-2m\tau}\sigma/8\right)}$ with any translate of $\mathfrak{h}$ has length at most $2\beta^{n+2}e^{-2m\tau}\sigma$. Consequently, the intersection of

$$(5.9) \quad \left\{\mathbf{x} \in B_i : g_\tau^k \exp(\mathbf{x})y \in Z^{(\varepsilon)}\right\} = \varphi_y\left(g_\tau^{-k}(A_{j,k} \cap Z^{(\varepsilon)})\right) = \mathrm{Ad}_{g_\tau^{-k}}\left(\varphi_x(A_{j,k} \cap Z^{(\varepsilon)})\right)$$

with any translate of $\mathfrak{h}$ has length at most

$$2e^{-2k\tau}\beta^{n+2}e^{-2m\tau}\sigma \overset{(5.7),\,(5.6)}{\leq} 2\beta^{n(j-1)}\beta^{n+2}\delta \leq 2\beta^{n(j-1)}\beta^{n+1}r_1 \leq 2\beta^n r_i.$$

This implies that $\left\{\mathbf{x} \in B_i : g_\tau^k \exp(\mathbf{x})y \in Z^{(\varepsilon)}\right\}$ is contained in a $\beta^n r_i$-neighborhood of some hyperplane, and hence the union of sets (5.9) over all indices k contained in the jth window is covered by at most m $\beta^n r_i$-neighborhoods of hyperplanes. Let us denote these hyperplanes by $L_{i,j}$ and let $\varepsilon_{i,j} = \beta^n r_i$. These will be Alice's choices in the ith move; the above discussion ensures that they constitute a valid move for Alice. In each of the remaining steps belonging to the jth stage, Alice will choose those neighborhoods which still intersect the ball chosen by Bob. That is, if we write the indices belonging to the jth stage as $i(j), i(j) + 1, \ldots, i(j+1) - 1$, Alice's choices in stage j will be those hyperplanes $L_{i,j}$ for which $L_{i,j}^{(\varepsilon_{\ell,j})} \cap B_i \neq \varnothing$, equipped with $\varepsilon_{i,j} = \beta^\ell r_{i(j)}$. This choice and (5.8) ensure that all of these moves are valid moves for Alice.

Since the jth stage contains at least n indices, and in every one of his moves Bob must choose a ball intersecting at most $1/2$ of the intervals chosen by Alice, we have that out of those neighborhoods, at most $2^{-n}m$ can intersect the first ball $B_{i(j+1)}$ in the first move of the $(j+1)$-th stage of the game. But $2^{-n}m < 1$ in view of (5.3). Consequently, for k in the jth window and $\mathbf{x} \in B_{i(j+1)}$, we have that $g_\tau^k \exp(\mathbf{x})y$ is not in $Z^{(\varepsilon)}$, and therefore, if $\mathbf{x}_\infty$ is the intersection point of all Bob's balls, it follow that $\exp(\mathbf{x}_\infty)y$ belongs to the set (4.2). The theorem is proved. $\qquad \square$

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Department of Mathematics, Brandeis University, Waltham MA 02454-9110, USA
E-mail address: kleinboc@brandeis.edu

Department of Mathematics, Tel Aviv University, Tel Aviv, Israel
E-mail address: barakw@post.tau.ac.il