CLASSIFICATION AND STATISTICS OF CUT-AND-PROJECT SETS

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ABSTRACT. We define *Ratner-Marklof-Strömbergsson measures* (following [MS14]). These are probability measures supported on cut-and-project sets in $\mathbb{R}^d$ ($d \geqslant 2$) which are invariant and ergodic for the action of the groups $\mathrm{ASL}_d(\mathbb{R})$ or $\mathrm{SL}_d(\mathbb{R})$. We classify the measures that can arise in terms of algebraic groups and homogeneous dynamics. Using the classification, we prove analogues of results of Siegel, Weil and Rogers about a Siegel summation formula and identities and bounds involving higher moments. We deduce results about asymptotics, with error estimates, of point-counting and patch-counting for typical cut-and-project sets.

1. INTRODUCTION

A *cut-and-project set* is a discrete subset of $\mathbb{R}^d$ obtained by the following construction. Fix a direct sum decomposition $\mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^m$, where the two summands in this decomposition are denoted respectively $V_{\mathrm{phys}}, V_{\mathrm{int}}$, so that

$$\mathbb{R}^n = V_{\mathrm{phys}} \oplus V_{\mathrm{int}},$$

and the corresponding projections are

$$\pi_{\mathrm{phys}} : \mathbb{R}^n \to V_{\mathrm{phys}}, \qquad \pi_{\mathrm{int}} : \mathbb{R}^n \to V_{\mathrm{int}}.$$

Also fix a lattice $\mathcal{L} \subset \mathbb{R}^n$ and a *window* $W \subset V_{\mathrm{int}}$; then the corresponding cut-and-project set $\Lambda = \Lambda(\mathcal{L}, W)$ is given by

$$\Lambda(\mathcal{L}, W) \stackrel{\mathrm{def}}{=} \pi_{\mathrm{phys}} \left(\mathcal{L} \cap \pi_{\mathrm{int}}^{-1}(W) \right). \tag{1.1}$$

We sometimes allow $\mathcal{L}$ to be a *grid*, i.e., the image of a lattice under a translation in $\mathbb{R}^n$, and sometimes require Λ to be *irreducible*, a notion we define in §2. Cut-and-project sets are prototypical aperiodic sets exhibiting long-term-order, and are sometimes referred to as *model sets* or *quasicrystals*. Beginning with work of Meyer [Mey70] in connection to Pisot numbers, they have been intensively studied from various points of view. See [BG13] and the references therein.

Date: September 1, 2022.

Given a cut-and-project set, a natural operation is to take the closure (with respect to a natural topology) of its orbit under translations. This yields a dynamical system for the translation group and has been studied by many authors under different names. In recent years several investigators have become interested in the orbit-closures under the group $\mathrm{SL}_d(\mathbb{R})$ (respectively $\mathrm{ASL}_d(\mathbb{R})$), which is the group of orientation- and volume-preserving linear (resp., affine) transformations of $\mathbb{R}^d$. In particular, in the important paper [MS14], motivated by problems in mathematical physics, Marklof and Strömbergsson introduced a class of natural probability measures on these orbit-closures. The goal of this paper is to classify and analyze such measures, and derive consequences for the statistics and large scale geometry of cut-and-project sets.

1.1. **Classification of Ratner-Marklof-Strömbergsson measures.** We say that a cut-and-project set is *irreducible* if it arises from the above construction, where the data satisfies the assumptions **(D), (I)** and **(Reg)** given in §2.1. Informally speaking, **(D)** and **(I)** imply that the set cannot be presented as a finite union of sets whose construction involves smaller groups in the cut-and-project construction, and **(Reg)** is a regularity assumption on the window set W. We denote by $\mathscr{C}(\mathbb{R}^d)$ the space of closed subsets of $\mathbb{R}^d$, equipped with the Chabauty-Fell topology. This is a compact metric topology whose definition is recalled in §2.2, and which is also referred to in the quasicrystals literature as the *local rubber topology* or the *natural topology*. Since the groups $\mathrm{ASL}_d(\mathbb{R})$ and $\mathrm{SL}_d(\mathbb{R})$ act on $\mathbb{R}^d$, they also act on $\mathscr{C}(\mathbb{R}^d)$. We say that a Borel probability measure μ on $\mathscr{C}(\mathbb{R}^d)$ is a *Ratner-Marklof-Strömbergsson measure*, or *RMS measure* for short, if it is invariant and ergodic under $\mathrm{SL}_d(\mathbb{R})$ and gives full measure to the set of irreducible cut-and-project sets. We call it *affine* if it is also invariant under $\mathrm{ASL}_d(\mathbb{R})$, and *linear* otherwise (i.e., if it is invariant under $\mathrm{SL}_d(\mathbb{R})$ but not under $\mathrm{ASL}_d(\mathbb{R})$).

A construction of RMS measures was given in [MS14], as follows. Let $\mathscr{Y}_n$ denote the space of grids of covolume one in $\mathbb{R}^n$, equipped with the Chabauty-Fell topology, or equivalently with the topology it inherits from its identification with the homogeneous space $\mathrm{ASL}_n(\mathbb{R})/\mathrm{ASL}_n(\mathbb{Z})$. Similarly, let $\mathscr{X}_n$ denote the space of lattices of covolume one in $\mathbb{R}^n$, which is identified with the homogeneous space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$. Fix the data $d, m, V_{\mathrm{phys}} \simeq \mathbb{R}^d, V_{\mathrm{int}} \simeq \mathbb{R}^m, \pi_{\mathrm{phys}}, \pi_{\mathrm{int}}$, as well as a set $W \subset V_{\mathrm{int}}$, and choose $\mathcal{L}$ randomly according to a probability measure $\bar{\mu}$ on $\mathscr{Y}_n$. This data determines a cut-and-project set Λ, which is random since $\mathcal{L}$ is. The resulting probability measure μ on cut-and-project

sets can thus be written as the pushforward of $\bar{\mu}$ under the map $\mathcal{L} \mapsto \Lambda(W, \mathcal{L})$, and is easily seen to be invariant and ergodic under $\mathrm{SL}_d(\mathbb{R})$ or $\mathrm{ASL}_d(\mathbb{R})$ if the same is true for $\bar{\mu}$. One natural choice for $\bar{\mu}$ is the so-called *Haar-Siegel measure,* which is the unique Borel probability measure invariant under the group $\mathrm{ASL}_n(\mathbb{R})$. Another is the Haar-Siegel measure on $\mathscr{X}_n$ (i.e., the unique $\mathrm{SL}_n(\mathbb{R})$-invariant measure). It is also possible to consider other measures on $\mathscr{Y}_n$ which are $\mathrm{ASL}_d(\mathbb{R})$- or $\mathrm{SL}_d(\mathbb{R})$-invariant. As observed in [MS14], a fundamental result of Ratner [Rat91] makes it possible to give a precise description of such measures on $\mathscr{Y}_n$. They correspond to certain algebraic groups which are subgroups of $\mathrm{ASL}_n(\mathbb{R})$ and contain $\mathrm{ASL}_d(\mathbb{R})$ (or $\mathrm{SL}_d(\mathbb{R})$).

Our first result is a classification of such measures. We refer to §2 and §3 for more precise statements, and for definitions of the terminology.

Theorem 1.1. *Let μ be an RMS measure on $\mathscr{C}(\mathbb{R}^d)$. Then, up to rescaling, there are fixed m and $W \subset \mathbb{R}^m$ such that μ is the pushforward via the map*

$$\mathscr{Y}_n \to \mathscr{C}(\mathbb{R}^d), \quad \mathcal{L} \mapsto \Lambda(\mathcal{L}, W)$$

of a measure $\bar{\mu}$ on $\mathscr{Y}_n$, where $n = d+m$, W satisfies **(Reg)**, *the measure $\bar{\mu}$ is supported on a closed orbit $H\mathcal{L}_1 \subset \mathscr{Y}_n$ for a connected real algebraic group $H \subset \mathrm{ASL}_n(\mathbb{R})$ and $\mathcal{L}_1 \in \mathscr{Y}_n$. There is an integer $k \geqslant d$, a real number field $\mathbb{K}$ and a $\mathbb{K}$-algebraic group $\mathbf{G}$, such that the Levi subgroup of H arises via restriction of scalars from $\mathbf{G}$ and $\mathbb{K}$, and one of the following holds for $\mathbf{G}$:*

- $\mathbf{G} = \mathrm{SL}_k$ *(as a $\mathbb{K}$-group) and* $n = k \cdot \deg(\mathbb{K}/\mathbb{Q})$.
- $\mathbf{G} = \mathrm{Sp}_{2k}$ *(as a $\mathbb{K}$-group), and* $d = 2$, $n = 2k \cdot \deg(\mathbb{K}/\mathbb{Q})$.

Furthermore, in the linear (resp. affine) case μ is invariant under none of (resp., all of) the translations by nonzero elements of V_{phys}.

Here the group Sp_{2k} is the group preserving the standard symplectic form in $2k$ variables; we caution the reader that this group is sometimes denoted by Sp_k in the literature. As we will see in Proposition 3.3, any choice of $\mathbb{K}$ and $\mathbf{G}$ satisfying the description in Theorem 1.1 gives rise to an affine and a linear RMS measure. We note that the vertex sets of the famous Ammann-Beenker and Penrose tililngs, which are well-known to have representations as cut-and-project constructions, are associated with the real quadratic fields $\mathbb{K} = \mathbb{Q}(\sqrt{2})$ and $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, resepctively, with $d = 2$ and $\mathbf{G} = \mathrm{SL}_2$, see also §5.

Theorem 1.1 is actually a combination of two separate results. The first extends work of Marklof and Strömbergsson [MS14]. They introduced the pushforward $\bar{\mu} \mapsto \mu$ described above, where $\bar{\mu}$ is a homogeneous measure on $\mathscr{Y}_n$, and noted that the measures $\bar{\mu}$ could be classified

using Ratner's work. Our contribution in this regard (see Theorem 3.1) is to give a full list of the measures $\bar{\mu}$ which can arise. The second result, contained in our Theorem 4.1, is that this construction is the only way to obtain RMS measures according to our definition (which is given in terms of V_{phys} rather than $\mathscr{Y}_n$).

1.2. **Formulae of Siegel-Weil and Rogers.** In geometry of numbers, computations with the Haar-Siegel probability measure on $\mathscr{X}_n$ are greatly simplified by the Siegel summation formula [Sie45], according to which for $f \in C_c(\mathbb{R}^n)$,

$$\int_{\mathscr{X}_n} \hat{f}(\mathcal{L})\, dm(\mathcal{L}) = \int_{\mathbb{R}^n} f(x)\, d\mathrm{vol}(x), \quad \text{where } \hat{f}(\mathcal{L}) = \sum_{v \in \mathcal{L} \smallsetminus \{0\}} f(v). \tag{1.2}$$

Here m is the Haar-Siegel probability measure on $\mathscr{X}_n$, and vol is the Lebesgue measure on $\mathbb{R}^n$. The analogous formula for RMS measures was proved in [MS14]. Namely[1], suppose μ is an RMS measure, and for each $\Lambda \in \operatorname{supp} \mu$, and for $f \in C_c(\mathbb{R}^d)$, set

$$\hat{f}(\Lambda) \stackrel{\text{def}}{=} \begin{cases} \displaystyle\sum_{v \in \Lambda \smallsetminus \{0\}} f(v) & \mu \text{ is linear} \\ \displaystyle\sum_{v \in \Lambda} f(v) & \mu \text{ is affine.} \end{cases} \tag{1.3}$$

We will refer to $\hat{f}$ as the *Siegel-Veech transform* of f. Then it is shown in [MS14, MS20], that for an explicitly computable constant $c > 0$, for any $f \in C_c(\mathbb{R}^d)$ one has

$$\int \hat{f}(\Lambda)\, d\mu(\Lambda) = c \int_{\mathbb{R}^d} f(x)\, d\mathrm{vol}(x). \tag{1.4}$$

A first step in the proof of (1.4) is to show that $\hat{f}$ is integrable, i.e., belongs to $L^1(\mu)$. As a corollary of Theorem 1.1, and using reduction theory for lattices in algebraic groups, we strengthen this and obtain the precise integrability exponent of the Siegel-Veech transform, as follows:

Theorem 1.2. *Let μ be an RMS measure, let $\mathbf{G}$ and $\mathbb{K}$ be as in Theorem 1.1, let $r \stackrel{\text{def}}{=} \operatorname{rank}_{\mathbb{K}}(\mathbf{G})$ denote the $\mathbb{K}$-rank of $\mathbf{G}$, and define*

$$q_\mu \stackrel{\text{def}}{=} \begin{cases} r+1 & \mu \textit{ is linear} \\ r+2 & \mu \textit{ is affine.} \end{cases} \tag{1.5}$$

[1]Our notations differ slightly from those of [MS14], but the result as stated here can be easily shown to be equivalent to the one in [MS14].

Then for any $f \in C_c(\mathbb{R}^d)$ and any $p < q_\mu$ we have $\hat{f} \in L^p(\mu)$. Moreover, if the window W contains a neighborhood of the origin in V_{int}, there are $f \in C_c(\mathbb{R}^d)$ for which $\hat{f} \notin L^{q_\mu}(\mu)$.

The proof involves integrating some characters over a Siegel set for a homogeneous subspace of $\mathscr{X}_n$. The special case for which $\mathbb{K} = \mathbb{Q}$, $\mathbf{G} = \mathrm{SL}_k$ and the measure μ is linear was carried out in [EMM98, Lemma 3.10]. Note that

$$\mathrm{rank}_{\mathbb{K}}(\mathbf{G}) = \begin{cases} k-1 & \text{if } \mathbf{G} = \mathrm{SL}_k \\ k & \text{if } \mathbf{G} = \mathrm{Sp}_{2k}. \end{cases} \tag{1.6}$$

We will say that the RMS measure μ is *of higher rank* when $q_\mu \geqslant 3$; in light of the above this happens unless $d = 2$, $\mathbf{G} = \mathrm{SL}_2$, and μ is linear. It follows immediately from Theorem 1.2 that $\hat{f} \in L^1(\mu)$, and in the higher-rank case, that $\hat{f} \in L^2(\mu)$.

The proof of (1.4) given in [MS14] follows a strategy of Veech [Vee98], and relies on a difficult result of Shah [Sha96]. Following Weil [Wei82], we will reprove the result with a more elementary argument. Combined with Theorem 1.2, the argument gives a strengthening of (1.4).

Given $p \in \mathbb{N}$, write $\bigoplus_1^p \mathbb{R}^d = \mathbb{R}^{dp}$, and for a compactly supported function f on $\mathbb{R}^{dp}$, define

$${}^p\widehat{f}(\Lambda) \stackrel{\text{def}}{=} \begin{cases} \displaystyle\sum_{v_1,\dots,v_p \in \Lambda \smallsetminus \{0\}} f(v_1,\dots,v_p) & \mu \text{ is linear} \\ \displaystyle\sum_{v_1,\dots,v_p \in \Lambda} f(v_1,\dots,v_p) & \mu \text{ is affine.} \end{cases} \tag{1.7}$$

Theorem 1.3. *Let μ be an RMS measure, and suppose $p < q_\mu$ where q_μ is as in (1.5). Then there is a countable collection $\{\tau_{\mathfrak{e}} : \mathfrak{e} \in \mathfrak{E}\}$ of Borel measures on $\mathbb{R}^{dp}$ such that $\tau \stackrel{\text{def}}{=} \sum \tau_{\mathfrak{e}}$ is locally finite, and for every $f \in L^1(\tau)$ we have*

$$\int {}^p\widehat{f}\, d\mu = \int_{\mathbb{R}^{dp}} f\, d\tau < \infty.$$

The measures $\tau_{\mathfrak{e}}$ are H-c&p-algebraic, for the group H appearing in Theorem 3.1 (see Definition 7.3).

This result is inspired by several results of Rogers for lattices, see e.g. [Rog55, Thm. 4]. Loosely speaking, c&p-algebraic measures are images of algebraically defined measures on $\mathbb{R}^{np}$ under a natural map associated with the cut-and-project construction.

Theorems 1.2 and 1.3 will be deduced from their more general counterparts Theorems 6.2 and 7.1, which deal with the homogenous subspace $H\mathcal{L}_1 \subset \mathscr{Y}_n$ arising in Theorem 1.1.

1.3. **Rogers-type bound on the second moment.** A fundamental problem in geometry of numbers is to control the higher moments of random variables associated with the Haar-Siegel measure on the space $\mathscr{X}_n$. In particular, regarding the second moment, the following important estimate was proved in [Rog55, Rog56, Sch60]: for the Haar-Siegel measure m on $\mathscr{X}_n$, $n \geqslant 3$ there is a constant $C > 0$ such that for any function $f \in C_c(\mathbb{R}^n)$ taking values in $[0,1]$ we have

$$\int_{\mathscr{X}_n} \left| \hat{f}(x) - \int_{\mathscr{X}_n} \hat{f}\, dm \right|^2 dm(x) \leqslant C \int_{\mathbb{R}^n} f \, d\mathrm{vol},$$

where $\hat{f}$ is as in (1.2). We will prove an analogous result for RMS measures of higher rank.

Theorem 1.4. *Let μ be an RMS measure of higher rank. For $p = 2$ let τ be the measure as in Theorem 1.3. In the notation of Theorem 1.1, assume that*

$$\mathbf{G} = \mathrm{SL}_k, \quad \text{or } \mu \text{ is affine.} \tag{1.8}$$

Then there is $C > 0$ such that for any Borel function $f : \mathbb{R}^d \to [0,1]$ belonging to $L^1(\tau)$ we have

$$\int_{\mathscr{C}(\mathbb{R}^d)} \left| \hat{f}(x) - \int_{\mathscr{C}(\mathbb{R}^d)} \hat{f}\, d\mu \right|^2 d\mu(x) \leqslant C \int_{\mathbb{R}^d} f \, d\mathrm{vol}. \tag{1.9}$$

The case in which (1.8) fails, that is, μ is linear and $\mathbf{G} = \mathrm{Sp}_{2k}$, and in which in addition $\mathbb{K} = \mathbb{Q}$, is treated in [KY18], where a similar bound is obtained. The symplectic case with $\mathbb{K}$ a proper field extension of $\mathbb{Q}$ is more involved, and we hope to investigate it further in future work.

There have been several recent papers proving an estimate like (1.9) for homogeneous measures associated with various algebraic groups. See [KS19] and references therein. The alert reader will have noted that, even though the measure μ is the pushforward of a measure supported on a homogeneous space $H\mathcal{L}_1$, we prove the bound (1.9) for functions defined on $\mathscr{C}(\mathbb{R}^d)$ rather than on $H\mathcal{L}_1$. Indeed, while we expect such a stronger result to be true, it requires a more careful analysis than the one needed for our application.

1.4. **The Schmidt theorem for cut-and-project sets, and patch-counting.** It is well-known that every irreducible cut-and-project set Λ has a *density*

$$D(\Lambda) \stackrel{\mathrm{def}}{=} \lim_{T \to \infty} \frac{\#\left(\Lambda \cap B(0,T)\right)}{\mathrm{vol}(B(0,T))} = \frac{\mathrm{vol}(W)}{\mathrm{covol}(\mathcal{L})}, \tag{1.10}$$

where $\Lambda = \Lambda(\mathcal{L}, W)$, $\mathrm{vol}(W)$ is the volume of W, and $\mathrm{covol}(\mathcal{L})$ is the covolume of $\mathcal{L}$ (for two proofs, which are valid for a larger class of nice

sets in place of $B(0,T)$, see [Moo02] and [MS14, §3], and see references therein). In particular, the limit exists and is positive. Following Schmidt [Sch60], we would like to strengthen this result and allow counting in even more general shapes, and with a bound on the rate of convergence. We say that a collection of Borel subsets $\{\Omega_T : T \in \mathbb{R}_+\}$ of $\mathbb{R}^d$ is an *unbounded ordered family* if

- $0 \leqslant T_1 \leqslant T_2 \implies \Omega_{T_1} \subset \Omega_{T_2}$;
- For all $T > 0$, $\mathrm{vol}(\Omega_T) < \infty$;
- $\mathrm{vol}(\Omega_T) \to_{T\to\infty} \infty$; and
- For all large enough $V > 0$ there is T such that $\mathrm{vol}(\Omega_T) = V$.

Theorem 1.5. *Let μ be an RMS measure of higher rank, such that* (1.8) *holds. Then for every $\varepsilon > 0$, for every unbounded ordered family $\{\Omega_T\}$, for μ-a.e. cut-and-project set Λ,*

$$\#\left(\Omega_T \cap \Lambda\right) = D(\Lambda) \cdot \mathrm{vol}(\Omega_T) + O\left(\mathrm{vol}(\Omega_T)^{\frac{1}{2}+\varepsilon}\right). \tag{1.11}$$

This result is a direct analogue of Schmidt's result for lattices, and its proof follows [Sch60]. In the special case $\Omega_T = B(0,T)$, we obtain an estimate for the rate of convergence in (1.10), valid for μ-a.e. cut-and-project set. For related work see [HKW14]. Note that for $B(0,T)$, and for lattices, Götze [Göt98] has conjectured that an error estimate $O\left(\mathrm{vol}(B(0,T))^{\frac{1}{2}-\frac{1}{2d}+\varepsilon}\right)$ should hold.

Even for $\Omega_T = B(0,T)$, one cannot expect (1.11) to hold for *all* cut-and-project sets; in fact, a Baire category argument as in [HKW14, §9] can be used to show that for any error function $E(T)$ with $E(T) = o(T^d)$ there are cut-and-project sets for which, along a subsequence $T_n \to \infty$,

$$\left|\#\left(B(0,T_n) \cap \Lambda\right) - D(\Lambda) \cdot \mathrm{vol}(B(0,T_n))\right| \geqslant E(T_n).$$

Thus, it is an interesting open problem to obtain error estimates like (1.11) for explicit cut-and-project sets. Note that for explicit cut-and-project sets which can also be described via substitution tilings, such as the vertex set of a Penrose tiling, there has been a lot of work in this direction, see [Sol14] and references therein.

We now discuss patch counting, which is a refinement which makes sense for cut-and-project sets but not for lattices. For any discrete set $\Lambda \subset \mathbb{R}^d$, any point $x \in \Lambda$ and any $R > 0$, we refer to the set

$$\mathcal{P}_{\Lambda,R}(x) \stackrel{\mathrm{def}}{=} B(0,R) \cap (\Lambda - x)$$

as the *R-patch of Λ at x*. Two points $x_1, x_2 \in \Lambda$ are said to be *R-patch equivalent* if $\mathcal{P}_{\Lambda,R}(x_1) = \mathcal{P}_{\Lambda,R}(x_2)$. It is well-known that any cut-and-project set Λ is of *finite local complexity*, which means that for any

$R > 0$,

$$\#\{\mathcal{P}_{\Lambda,R}(x) : x \in \Lambda\} < \infty.$$

Furthermore, it is known that whenever $\mathcal{P}_0 = \mathcal{P}_{\Lambda,R}(x_0)$ for some $x_0 \in \Lambda$ and some $R > 0$, the *density* or *absolute frequency*

$$D(\Lambda, \mathcal{P}_0) = \lim_{T\to\infty} \frac{\#\{x \in \Lambda \cap B(0,T) : \mathcal{P}_{\Lambda,R}(x) = \mathcal{P}_0\}}{\mathrm{vol}(B(0,T))} \tag{1.12}$$

exists; in fact, the set in the numerator of (1.12) is itself a cut-and-project set, see [BG13, Cor. 7.3]. Our analysis makes it possible to obtain an analogue of Theorem 1.5 for counting patches, namely:

Theorem 1.6. *Let μ be an RMS measure of higher rank, for which (1.8) holds. For any $\delta > 0$, set $\theta_0 \overset{\text{def}}{=} \frac{\delta}{m+2\delta}$, where $m = \dim V_{\text{int}}$. Suppose the window $W \subset V_{\text{int}}$ in the cut-and-project construction satisfies $\dim_B(\partial W) \leqslant m - \delta$, where $\dim_B$ denotes the upper box dimension (see §10). Then for every unbounded ordered family $\{\Omega_T\}$ in $\mathbb{R}^d$, for μ-a.e. Λ, for any patch $\mathcal{P}_0 = \mathcal{P}_{\Lambda,R}(x_0)$, and any $\theta \in (0, \theta_0)$, we have*

$$\#\{x \in \Omega_T \cap \Lambda : \mathcal{P}_{\Lambda,R}(x) = \mathcal{P}_0\} = D(\Lambda, \mathcal{P}_0)\,\mathrm{vol}(\Omega_T) + O\left(\mathrm{vol}\,(\Omega_T)^{1-\theta}\right). \tag{1.13}$$

For additional results on effective error terms for patch-counting in cut-and-project sets, see [HJKW19].

1.5. **Acknowledgements.** We are grateful to Mikhail Borovoi, Manfred Einsiedler, Dmitry Kleinbock, Henna Koivusalo, Jens Marklof, Dave Morris, Michel Skenderi, and Andreas Strömbergsson for useful discussions. Specifically, Morris supplied most of the arguments of Theorem 3.5, Borovoi supplied an argument used in Step 4 in the proof of Lemma 3.2, and Einsiedler supplied arguments for Lemmas 3.2 and 3.4. We gratefully acknowledge support of BSF grant 2016256, ISF grants 2919/19, 1570/17, 1149/18 and 264/22, Swiss National Science Foundation 168823 and European Research Council 754475, and the David and Rosa Orzen Endowment Fund. We thank the anonymous referees for a careful reading of the paper and for many helpful comments and suggestions.

2. Basics

2.1. **Cut-and-project sets.** In the literature, different authors impose slightly different assumptions on the data in the cut-and-project construction. For related discussions, see [BG13, Moo97, MS14]. Here are the assumptions which will be relevant in this paper:

(D) $\pi_{\text{int}}(\mathcal{L})$ is dense in V_{int}.

(I) $\pi_{\text{phys}}|_{\mathcal{L}}$ is injective.

(Reg) The window W is Borel measurable, bounded, has non-empty interior, and its boundary ∂W has zero measure with respect to Lebesgue measure on V_{int}.

We will say that the construction is *irreducible* if **(D)**, **(I)** and **(Reg)** hold.

In the literature, a more general cut-and-project scheme is discussed, in which the groups $V_{\text{phys}} \simeq \mathbb{R}^d, V_{\text{int}} \simeq \mathbb{R}^m$ may be replaced with general locally compact abelian groups. Note that if **(D)** fails, we can replace V_{int} with $\overline{\pi_{\text{int}}(\mathcal{L})}$, which is a proper subgroup of V_{int}, while if **(I)** fails, we can replace V_{int} with $V_{\text{int}}/(\mathcal{L} \cap \ker \pi_{\text{phys}})$. In both cases one can obtain the same set using smaller groups. Note that when **(D)** fails, the group $\overline{\pi_{\text{int}}(\mathcal{L})}$ might be disconnected, and in that case, using **(Reg)** we see that only finitely many of its connected components will intersect W, and $\Lambda(\mathcal{L}, W)$ will have a description as a finite union of cut-and-projects sets with an internal space of smaller dimension.

Regarding the regularity assumptions on W, note that if no regularity assumptions are imposed, one can let Λ be an arbitrary subset of $\pi_{\text{phys}}(\mathcal{L})$ by letting W be equal to $\pi_{\text{int}}\left(\mathcal{L} \cap \pi_{\text{phys}}^{-1}(\Lambda)\right)$. Also, the assumption that W is bounded (respectively, has nonempty interior) implies that Λ is uniformly discrete (respectively, relatively dense).

Finally, note that it is not W that plays a role in (1.1), but rather $\pi_{\text{int}}^{-1}(W)$. In particular, if convenient, one can replace the space V_{int} with any space V'_{int} which is complementary to V_{phys}, and with the obvious notations, replace W with $W' \stackrel{\text{def}}{=} \pi'_{\text{int}}(\pi_{\text{int}}^{-1}(W))$. Put otherwise, it would have been more natural to think of W as being a subset of the quotient space $\mathbb{R}^n/V_{\text{phys}}$. We refrain from doing so to avoid conflict with established conventions.

2.2. **Chabauty-Fell topology.** Let $\mathscr{C}(\mathbb{R}^d)$ denote the collection of all closed subsets of $\mathbb{R}^d$. Equip $\mathscr{C}(\mathbb{R}^d)$ with the topology induced by the following metric, which we will call the *Chabauty-Fell metric*: for $Y_0, Y_1 \in \mathscr{C}(\mathbb{R}^d)$, $d(Y_0, Y_1)$ is the infimum of all $\varepsilon \in (0, 1)$ for which, for both $i = 0, 1$,

$$Y_i \cap B\left(0, \varepsilon^{-1}\right) \text{ is contained in the } \varepsilon\text{-neighborhood of } Y_{1-i},$$

and $d(Y_0, Y_1) = 1$ if there is no such ε. It is known that with this metric, $\mathscr{C}(\mathbb{R}^d)$ is a compact metric space. In this paper, closures of collections in $\mathscr{C}(\mathbb{R}^d)$ and continuity of maps with image in $\mathscr{C}(\mathbb{R}^d)$ will always refer to this topology, and all measures will be regular measures on the Borel σ-algebra induced by this topology. We note that in the

quasicrystals literature this topology is often referred to as the *local rubber topology* or the *natural topology* .

We note that there are many topologies on the set of closed subsets $\mathscr{C}(X)$ of a topological space X. The Chabauty-Fell metric was introduced by Chabauty [Cha50] for $X = \mathbb{R}^d$ as well as for X a locally compact second countable group, and by Fell [Fel62] for general spaces X, particularly spaces arising in functional analysis. See also [LS03], where the connection to the Hausdorff metric is elucidated via stereographic projection. Many of the different topologies in the literature coincide on $\mathscr{C}\left(\mathbb{R}^d\right)$. Two notable exceptions are the Hausdorff topology, which is defined on the collection of *nonempty* closed subsets of X, and the weak-* topology of Borel measures on $\mathbb{R}^d$, studied in [Vee98, MS19], satisfying a certain growth condition and restricted to point processes. See [Bee93] for a comprehensive discussion of topologies on $\mathscr{C}(X)$.

We will need the following fact, which is well-known to experts, but for which we could not find a reference (see [MS19, §5.3] for a related discussion):

Proposition 2.1. *Suppose W is Borel measurable and bounded. Then the map*

$$\Psi : \mathscr{Y}_n \to \mathscr{C}(\mathbb{R}^d), \qquad \Psi(\mathcal{L}) \stackrel{\text{def}}{=} \Lambda(\mathcal{L}, W) \tag{2.1}$$

is a Borel map, and is continuous at any $\mathcal{L}$ for which $\pi_{\text{int}}(\mathcal{L}) \cap \partial W = \varnothing$.

Proof. We first prove the second assertion, that is, we assume that $\pi_{\text{int}}(\mathcal{L}) \cap \partial W = \varnothing$ and suppose by contradiction that $\mathcal{L}_j \to \mathcal{L}$ in $\mathscr{Y}_n$ but $\Psi(\mathcal{L}_j) \not\to \Lambda \stackrel{\text{def}}{=} \Psi(\mathcal{L})$. By passing to a subsequence and using the definition of the Chabauty-Fell metric on $\mathscr{C}(\mathbb{R}^d)$, we can assume that there is $\varepsilon > 0$ such that for all j, one of the following holds:

(a) There is $v \in \Lambda$, $\|v\| \leqslant \varepsilon^{-1}$ such that for all j, $\Psi(\mathcal{L}_j)$ does not contain a point within distance ε of v.
(b) There is $v_j \in \Psi(\mathcal{L}_j)$ such that $v_j \to v$, where $\|v\| \leqslant \varepsilon^{-1}$, and $v \notin \Lambda$.

In case (a), there is $u \in \mathcal{L}$ such that $v = \pi_{\text{phys}}(u)$ and $\pi_{\text{int}}(u) \in W$. By assumption $\pi_{\text{int}}(u)$ is in the interior of W. Since $\mathcal{L}_j \to \mathcal{L}$ there is $u_j \in \mathcal{L}_j$ such that $u_j \to u$ and for large enough j, $\pi_{\text{int}}(u_j) \in W$ and hence $v_j = \pi_{\text{phys}}(u_j) \in \Psi(\mathcal{L}_j)$. Clearly $v_j \to v$ and we have a contradiction.

In case (b), we let $u_j \in \mathcal{L}_j$ such that $v_j = \pi_{\text{phys}}(u_j)$. Then the images of v_j under both projections π_{phys}, π_{int} are bounded sequences, and hence the sequence (u_j) is also bounded. Passing to a subsequence and using that $\mathcal{L}_j \to \mathcal{L}$ we can assume $u_j \to u$ for some $u \in \mathcal{L}$. Since

$\pi_{\rm int}(u_j) \in W$ for each j, $\pi_{\rm int}(u) \in \overline{W}$ and hence, by our assumption, $\pi_{\rm int}(u)$ belongs to the interior of W, and in particular to W. This implies that $v = \pi_{\rm phys}(u) \in \Lambda$, a contradiction.

We now prove that Ψ is a Borel measurable map. For this it is enough to show that $\Psi^{-1}(B)$ is measurable in $\mathscr{Y}_n$, whenever $B = B(\Lambda, \varepsilon)$ is the ε-ball with respect to the Chabauty-Fell metric centered at $\Lambda = \Psi(\mathcal{L}) \in \mathscr{C}(\mathbb{R}^d)$. Let

$$F_1 \stackrel{\rm def}{=} \left\{x \in \mathcal{L} : \pi_{\rm phys}(x) \in B\left(0, \varepsilon^{-1}\right),\ \pi_{\rm int}(x) \in W\right\}$$

and

$$F_2 \stackrel{\rm def}{=} \Lambda \cap B\left(0, \varepsilon^{-1} + \varepsilon\right).$$

Then the definition of the Chabauty-Fell metric gives that $\mathcal{L}'$ belongs to $\Psi^{-1}(B)$ if and only if for any $u_1 \in F_1$, there is $u_1' \in \mathcal{L}'$ with $\pi_{\rm int}(u_1') \in W$ and $\|\pi_{\rm phys}(u_1) - \pi_{\rm phys}(u_1')\| < \varepsilon$, and additionally, for any $u_1' \in \mathcal{L}'$ with $\pi_{\rm int}(u_1') \in W$ and $\|\pi_{\rm phys}(u_1')\| < \varepsilon^{-1}$ there is $v \in F_2$ with $\|\pi_{\rm phys}(u_1') - v\| < \varepsilon$. Since lattices are countable, F_1, F_2 are finite, and $W \subset V_{\rm int}$ is Borel measurable, this shows that $\Psi^{-1}(B)$ is described by countably many measurable conditions. □

We use this to obtain a useful continuity property for measures. Given a topological space X, we denote by $\mathrm{Prob}(X)$ the space of regular Borel probability measures. We equip $\mathrm{Prob}(X)$ with the weak-* topology. Any Borel map $f : X \to Y$ induces a map $f_* : \mathrm{Prob}(X) \to \mathrm{Prob}(Y)$ defined by $f_*\mu = \mu \circ f^{-1}$.

Corollary 2.2. *Let Ψ be as in (2.1). Then any $\bar{\mu} \in \mathrm{Prob}(\mathscr{Y}_n)$ for which*

$$\bar{\mu}\left(\{\mathcal{L} \in \mathscr{Y}_n : \pi_{\rm int}(\mathcal{L}) \cap \partial W \neq \varnothing\}\right) = 0. \tag{2.2}$$

is a continuity point for Ψ_*. In particular, this holds if $\bar{\mu}$ is invariant under translations by elements of $V_{\rm int} \simeq \mathbb{R}^m$ and ∂W has zero Lebesgue measure.

Proof. Suppose $\bar{\mu}_j \to \bar{\mu}$ in $\mathrm{Prob}(\mathscr{Y}_n)$, and let μ_j, μ denote respectively the pushforwards $\Psi_*\bar{\mu}_j$, $\Psi_*\bar{\mu}$. To establish continuity of Ψ_* we need to show $\mu_j \to \mu$. Since $\bar{\mu}_j \to \bar{\mu}$, we have $\int g\, d\bar{\mu}_j \to \int g\, d\bar{\mu}$ for any $g \in C_c(\mathscr{Y}_n)$. By the Portmanteau theorem this also holds for any g which is bounded, compactly supported, and for which the set of discontinuity points has $\bar{\mu}$-measure zero. Let f be a continuous function on $\mathscr{C}(\mathbb{R}^d)$ and let $\bar{f} = f \circ \Psi$. Then $\bar{f}$ is continuous at $\bar{\mu}$-a.e. point, by Proposition 2.1. The Portmanteu theorem then ensures that

$$\int_{\mathscr{C}(\mathbb{R}^d)} f d\mu_j = \int_{\mathscr{Y}_n} \bar{f} d\bar{\mu}_j \to \int_{\mathscr{Y}_n} \bar{f} d\bar{\mu} = \int_{\mathscr{C}(\mathbb{R}^d)} f d\mu.$$

That is, $\mu_j \to \mu$, as required.

For the last assertion, assuming that $\bar{\mu}$ is invariant under translations by elements of V_{int}, we need to show that (2.2) is satisfied. Letting $\mathbb{1}_{\partial W}$, $m_{V_{\text{int}}}$ denote respectively the indicator of ∂W and Lebesgue measure on V_{int}, and letting $B \subset V_{\text{int}}$ be a measurable set of finite and positive measure, we have by Fubini that

$$\begin{aligned}&\bar{\mu}\left(\{\mathcal{L} \in \mathscr{Y}_n : \pi_{\text{int}}(\mathcal{L}) \cap \partial W \neq \varnothing\}\right)\\ &= \int \left[\frac{1}{m_{V_{\text{int}}}(B)} \int_B \mathbb{1}_{\partial W} \circ \pi_{\text{int}}(\mathcal{L} + x)\, dm_{V_{\text{int}}}(x)\right] d\bar{\mu}(\mathcal{L}).\end{aligned}$$

It therefore suffices to show that for any $\mathcal{L}$,

$$m_{V_{\text{int}}}\left(\{x \in V_{\text{int}} : \pi_{\text{int}}(\mathcal{L} + x) \cap \partial W \neq \varnothing\}\right) = 0;$$

and indeed, this follows immediately from the countability of $\mathcal{L}$ and the assumption that $m_{V_{\text{int}}}(\partial W) = 0$. □

2.3. **Ratner's Theorems.** Ratner's measure classification and orbit-closure theorems [Rat91] are fundamental results in homogeneous dynamics. We recall them here, in the special cases which will be important for us. A Borel probability measure ν on $\mathscr{Y}_n$ (respectively, $\mathscr{X}_n$) is called *homogeneous* if there is x_0 in $\mathscr{Y}_n$ (respectively, $\mathscr{X}_n$) and a closed subgroup H of $\mathrm{ASL}_n(\mathbb{R})$ (respectively, $\mathrm{SL}_n(\mathbb{R})$) such that the H-action preserves ν, the orbit Hx_0 is closed and equal to $\operatorname{supp}\nu$, and $H_{x_0} \stackrel{\text{def}}{=} \{h \in H : hx_0 = x_0\}$ is a lattice in H. When we want to stress the role of H we will say that ν is *H-homogeneous*.

Recall that $\mathrm{ASL}_n(\mathbb{R})$ (respectively, $\mathrm{ASL}_n(\mathbb{Z})$) denotes the group of affine transformations of $\mathbb{R}^n$ whose derivative has determinant one (respectively, and which map the integer lattice $\mathbb{Z}^n$ to itself), and that $\mathscr{Y}_n$ is identified with $\mathrm{ASL}_n(\mathbb{R})/\mathrm{ASL}_n(\mathbb{Z})$, via the map which identifies the coset represented by the affine map φ with the grid $\varphi(\mathbb{Z}^n)$. Similarly, we have an identification of $\mathscr{X}_n$ with $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$. We view the elements of $\mathrm{ASL}_n(\mathbb{R})$ concretely as pairs (g, v), where $g \in \mathrm{SL}_n(\mathbb{R})$ and $x \in \mathbb{R}^n$ determine the map $x \mapsto gx + v$. In what follows two subgroups of $\mathrm{ASL}_n(\mathbb{R})$ play an important role, namely the groups $\mathrm{SL}_d(\mathbb{R})$ and $\mathrm{ASL}_d(\mathbb{R})$, which we will denote alternately by F, and embed concretely in $\mathrm{ASL}_n(\mathbb{R})$ in the upper left hand corner. That is, in the case $F = \mathrm{SL}_d(\mathbb{R})$, $g \in F$ is identified with

$$\left(\begin{pmatrix} g & \mathbf{0}_{d,m} \\ \mathbf{0}_{m,d} & \mathrm{Id}_m \end{pmatrix}, \mathbf{0}_n\right) \tag{2.3}$$

and in the case $F = \mathrm{ASL}_d(\mathbb{R})$, $(g, v) \in F$ is identified with

$$\left(\begin{pmatrix} g & \mathbf{0}_{d,m} \\ \mathbf{0}_{m,d} & \mathrm{Id}_m \end{pmatrix}, \begin{pmatrix} v \\ \mathbf{0}_m \end{pmatrix} \right). \tag{2.4}$$

Here Id_m, $\mathbf{0}_{k,\ell}$, $\mathbf{0}_k$ denote respectively an identity matrix of size $m \times m$, a zero matrix of size $k \times \ell$, and the zero vector in $\mathbb{R}^k$. We will refer to the embeddings of $\mathrm{SL}_d(\mathbb{R})$ and $\mathrm{ASL}_d(\mathbb{R})$ in $\mathrm{ASL}_n(\mathbb{R})$, given by (2.3) and (2.4), as the *top-left corner embeddings*.

The following is a special case of Ratner's result.

Theorem 2.3 (Ratner)**.** *Let $2 \leqslant d \leqslant n$, and let F be equal to either $\mathrm{ASL}_d(\mathbb{R})$ or $\mathrm{SL}_d(\mathbb{R})$ (with the top-left corner embedding in $\mathrm{ASL}_n(\mathbb{R})$). Then any F-invariant ergodic measure ν on $\mathscr{Y}_n$ is H-homogeneous, where H is a closed connected subgroup of $\mathrm{ASL}_n(\mathbb{R})$ containing F. Every orbit-closure $\overline{Fx}$ is equal to $\operatorname{supp}\nu$ for some homogeneous measure ν. The same conclusion holds for $\mathscr{X}_n$ and $F = \mathrm{SL}_d(\mathbb{R})$.*

The following additional results were obtained in [Sha91, Tom00]:

Theorem 2.4 (Shah, Tomanov)**.** *Let ν, H be as in Theorem 2.3, and let $x_0 = g_0\mathbb{Z}^n$ in $\mathscr{Y}_n$ or $\mathscr{X}_n$ such that $\operatorname{supp}\nu = Hx_0$. Then H is the connected component of the identity in the group of real points of an algebraic subgroup $\mathbf{H}$ of ASL_n, the solvable radical of H is equal to the unipotent radical of H, and the group $g_0^{-1}\mathbf{H}g_0$, as well as its unipotent radical, are defined over $\mathbb{Q}$. Furthermore, $\mathbf{H}$ is the smallest group with these properties, for which $F \subset H$.*

We will need a result of Shah which relies on Ratner's work (once more this is a special case of a more general result).

Theorem 2.5 ([Sha96])**.** *Let F be equal to either $\mathrm{ASL}_d(\mathbb{R})$ or $\mathrm{SL}_d(\mathbb{R})$ as above, let $\{g_t\}$ be a one-parameter diagonalizable subgroup of $\mathrm{SL}_d(\mathbb{R})$, and let $U = \{g \in F : \lim_{t\to\infty} g_{-t}gg_t \to e\}$ be the corresponding expanding horospherical subgroup. Let $\Omega \subset U$ be a relatively compact open subset of U and let m_U be the restriction of Haar measure to U, normalized so that $m_U(\Omega) = 1$. Then for every $x_0 \in \mathscr{Y}_n$, letting ν be the homogeneous measure such that $\operatorname{supp}\nu = \overline{Fx_0}$, we have*

$$\int_\Omega (g_t u)_* \delta_{x_0}\, dm_U(u) \to_{t\to\infty} \nu,$$

where δ_{x_0} is the Dirac measure at x_0 and the convergence is weak- convergence in $\mathrm{Prob}(\mathscr{Y}_n)$.*

2.4. **Number fields, geometric embeddings, and restriction of scalars.** For more details on the material in this subsection we refer the reader to [Wei82, PR94, Mor15, EW].

Let $\mathbb{K}$ be a number field of degree $D = \deg(\mathbb{K}/\mathbb{Q})$, and let $\mathcal{O} = \mathcal{O}_{\mathbb{K}}$ be its ring of integers. Let $\sigma_1, \dots, \sigma_r, \sigma_{r+1}, \overline{\sigma_{r+1}}, \dots, \sigma_{r+s}, \overline{\sigma_{r+s}}$ be the field embeddings of $\mathbb{K}$ in $\mathbb{C}$ where $r+2s = D$, $\sigma_1, \dots, \sigma_r$ are real embeddings and $\sigma_{r+1}, \dots, \sigma_{r+s}$ are complex (non-real) embeddings. An *order* in $\mathbb{K}$ is a subring of $\mathcal{O}$ which is of rank D as an additive group. The *geometric embedding* or *Minkowski embedding* of an order Δ is the set

$$\{(\sigma_1(x), \dots, \sigma_r(x), \sigma_{r+1}(x), \dots, \sigma_{r+s}(x)) : x \in \Delta\}.$$

It is a lattice in $\mathbb{R}^D \simeq \mathbb{R}^r \times \mathbb{C}^s$. Note that the geometric embedding depends on a choice of ordering of the field embeddings, and on representatives of each pair of complex conjugate embeddings. Thus, when we speak of 'the' geometric embeddings we will consider this data as fixed.

An algebraic group $\mathbf{G}$ defined over $\mathbb{K}$ (or $\mathbb{K}$-algebraic group) is a variety such that the multiplication and inversion maps $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$, $\mathbf{G} \to \mathbf{G}$ are $\mathbb{K}$-morphisms. A $\mathbb{K}$-homomorphism of algebraic groups is a group homomorphism which is a $\mathbb{K}$-morphism of algebraic varieties. We will work only with *linear algebraic groups* which means that they are affine varieties, i.e., for some N, they are the subset of affine space $\mathbb{A}^N$ satisfying a system of polynomial equations in N variables. We will omit the word 'linear' in the rest of the paper. A typical example of a $\mathbb{K}$-algebraic group is a Zariski closed matrix group, that is, a subgroup of the matrix group $\mathrm{SL}_m(\mathbb{C})$ for some m described by polynomial equations in the matrix entries, with coefficients in $\mathbb{K}$. If $\mathbf{G}_i$ are $\mathbb{K}$-algebraic groups realized as subgroups of $\mathrm{SL}_{m_i}(\mathbb{C})$ for $i = 1, 2$, and $\varphi : \mathbf{G}_1 \to \mathbf{G}_2$ is a $\mathbb{K}$-homomorphism, then there is a map $\hat{\varphi} : \mathrm{SL}_{m_1}(\mathbb{C}) \to \mathrm{SL}_{m_2}(\mathbb{C})$ which is polynomial in the matrix entries, with coefficients in $\mathbb{K}$, such that $\hat{\varphi}|_{\mathbf{G}_1} = \varphi$. For any field $L \subset \mathbb{C}$ containing $\mathbb{K}$, we will denote by $\mathbf{G}_L$ the collection of L-points of $\mathbf{G}$. It is a subgroup of $\mathrm{SL}_m(L)$, if $\mathbf{G}$ is realized as subgroup of $\mathrm{SL}_m(\mathbb{C})$.

We will do the same for rings $L = \mathbb{Z}$ or $L = \mathcal{O}$. In this case the group $\mathbf{G}_L$ depends on the concrete realization of $\mathbf{G}$ as a matrix group but the commensurability class of $\mathbf{G}_L$ is independent of choices (recall that two subgroups Γ_1, Γ_2 of some ambient group G are *commensurable* if $[\Gamma_i : \Gamma_1 \cap \Gamma_2] < \infty$ for $i = 1, 2$). By a *real algebraic group* we will mean a subgroup of finite index in $\mathbf{G}_{\mathbb{R}}$ for some $\mathbb{K}$-algebraic group $\mathbf{G}$, where $\mathbb{K} \subset \mathbb{R}$.

The *restriction of scalars* $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}$ is a functor from the category of $\mathbb{K}$-algebraic groups to $\mathbb{Q}$-algebraic groups. Given an algebraic group

$\mathbf{G}$ defined over $\mathbb{K}$, there is an algebraic group $\mathbf{H} = \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$ defined over $\mathbb{Q}$, such that $\mathbf{H}_{\mathbb{Q}}$ is naturally identified with $\mathbf{G}_{\mathbb{K}}$. For any $\mathbb{K}$-homomorphism of $\mathbb{K}$-algebraic groups $\varphi : \mathbf{G}_1 \to \mathbf{G}_2$ we have a $\mathbb{Q}$-homomorphism $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\varphi) : \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G}_1) \to \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G}_2)$. Given a matrix representation of $\mathbf{G}$ there is a corresponding matrix representation of $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$, defined as follows. We can realize $\mathbb{K}$ (as a ring) as a subalgebra of the $\mathbb{Q}$-algebra of $D \times D$ matrices with entries in $\mathbb{Q}$, and this leads to a corresponding identification of $\mathrm{SL}_m(\mathbb{K})$ with a subgroup of $\mathrm{SL}_{mD}(\mathbb{Q})$. A different choice of basis will produce a group that differs by a $\mathrm{SL}_{mD}(\mathbb{Q})$-conjugate. Now suppose $\mathbf{G} \subset \mathrm{SL}_m(\mathbb{C})$ is the solution set of polynomial equations $P_1, \ldots, P_\ell$ in the matrix entries, with coefficients in $\mathbb{K}$. Let $\hat{P}_1, \ldots, \hat{P}_\ell$ be the matrix valued polynomials where each $\mathbb{K}$-coefficient is replaced by its $\mathrm{Mat}_{D\times D}(\mathbb{Q})$ representative, and each variable (previously a matrix coefficient of $\mathrm{SL}_m(\mathbb{C})$) is an $\mathrm{Mat}_{D\times D}(\mathbb{C})$-block of $\mathrm{SL}_{mD}(\mathbb{C})$. These polynomials together with the (linear) polynomials that ensure that each $D \times D$ block is an element of the $\mathbb{Q}$-algebra $\mathbb{K}$, have coefficients in $\mathbb{Q}$, and $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$ is their solution set.

The $\mathbb{R}$-points of $\mathbf{H} = \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$ can be represented concretely as

$$\mathbf{G}_{\mathbb{R}}^{\sigma_1} \times \cdots \times \mathbf{G}_{\mathbb{R}}^{\sigma_r} \times \mathbf{G}_{\mathbb{C}}^{\sigma_{r+1}} \times \cdots \times \mathbf{G}_{\mathbb{C}}^{\sigma_{r+s}}, \tag{2.5}$$

where $\mathbf{G}^{\sigma_j}$ is the algebraic group defined by applying the field embedding σ_j to the polynomials in the matrix entries, with coefficients in $\mathbb{K}$, defining $\mathbf{G}$. Here, for a $\mathbb{C}$-algebraic group $\mathbf{M}$, $\mathbf{M}_{\mathbb{C}}$ is a shorthand notation for the $\mathbb{C}$-points of $\mathbf{M}$, thought of as an $\mathbb{R}$-group via the isomorphism $\mathbb{C} \cong \mathbb{R}^2$. More explicitly, a polynomial equation involving m^2 complex matrix entries $z_{ij} = a_{ij} + \mathbf{i}b_{ij}$, where $i, j \in \{1, \ldots, m\}$, is replaced with the same polynomial in the matrix algebra of 2×2 real matrices, with each appearance of z_{ij} replaced by $A^{(ij)} \stackrel{\text{def}}{=} \begin{pmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$, and with the $2m^2$ additional equations $(A^{(ij)})_{12} = -(A^{(ij)})_{21}, (A^{(ij)})_{11} = (A^{(ij)})_{22}$. Furthermore, denoting by $\bar{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$, there is a conjugation of $\mathrm{SL}_{mD}(\bar{\mathbb{Q}})$ by an element with coefficients in the Galois closure of $\mathbb{K}$, so that $\mathbf{H}(\bar{\mathbb{Q}})$ is embedded in $\mathrm{SL}_{mD}(\bar{\mathbb{Q}})$ in block form with $r + s$ blocks, where each block contains one of the factors in (2.5).

Similarly, for a $\mathbb{K}$-morphism $\varphi : \mathbf{G}_1 \to \mathbf{G}_2$, the restriction to the factor $\mathbf{G}_{\mathbb{R}}^{\sigma_j}$ in formula (2.5), of the $\mathbb{Q}$-morphism $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\varphi) : \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G}_1) \to \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G}_2)$, is the map φ_j obtained from φ by applying the field embedding σ_j to its coefficients. Thus, after writing both $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G}_1)$ and

$\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G}_2)$ in product form as in (2.5), we have

$$\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\varphi)(g_1,\dots,g_{r+s}) = (\varphi_1(g_1),\dots,\varphi_{r+s}(g_{r+s})). \tag{2.6}$$

We now note a connection between restriction of scalars, geometric embeddings of lattices, and the action on $\mathscr{X}_n$. Suppose that $\mathcal{O} = \mathcal{O}_{\mathbb{K}}$, Δ is an order in $\mathcal{O}$, and let $\mathcal{L}$ be the geometric embedding of Δ in $\mathbb{R}^D$. For $m \in \mathbb{N}$ set $n = Dm$ and let

$$\mathcal{L}' = c \cdot \underbrace{\mathcal{L} \oplus \cdots \oplus \mathcal{L}}_{m \text{ copies}},$$

where we choose the dilation factor c so that $\mathcal{L}' \in \mathscr{X}_n$, and we choose the ordering of the indices so that

$$\mathcal{L}' \stackrel{\text{def}}{=} c\left\{(\sigma_1(x),\dots,\sigma_{r+s}(x)) : x \in \Delta^m\right\}. \tag{2.7}$$

Now suppose $\mathbf{G}$ is an algebraic $\mathbb{K}$-group without $\mathbb{K}$-characters, $\varphi : \mathbf{G} \to \mathrm{SL}_m$ is a $\mathbb{K}$-morphism, and $\mathbf{H} \stackrel{\text{def}}{=} \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$. Since φ is a $\mathbb{K}$-morphism, there is a finite-index subgroup of $\mathbf{G}_{\mathcal{O}}$ whose image under φ is contained in $\mathrm{SL}_m(\mathcal{O})$, and hence preserves $\mathcal{O}^m$. This implies that a finite index subgroup of $\mathbf{H}_{\mathbb{Z}}$ preserves $\mathcal{L}'$. Since $\mathbf{H}_{\mathbb{Z}}$ is a lattice in $H \stackrel{\text{def}}{=} \mathbf{H}_{\mathbb{R}}$ (see [Bor19, §13]), we find that $H\mathcal{L}'$ is a closed orbit in $\mathscr{X}_n$ which is the support of an H-homogeneous measure.

3. Classification of invariant measures

Recall from the introduction that an affine (respectively, linear) RMS measure μ is a probability measure on $\mathscr{C}(\mathbb{R}^d)$ which gives full measure to the collection of all irreducible cut-and-project sets, and is invariant and ergodic under F, where

$$F \stackrel{\text{def}}{=} \begin{cases} \mathrm{SL}_d(\mathbb{R}) & \text{if } \mu \text{ is linear} \\ \mathrm{ASL}_d(\mathbb{R}) & \text{if } \mu \text{ is affine} \end{cases} \tag{3.1}$$

is the stabilizer group of μ. In this section we will give some more background on RMS measures, and two assertions (Theorem 3.1 and 4.1) which together imply Theorem 1.1. The careful reader will have noticed that we gave here a seemingly weaker definition of an affine RMS measure compared to the introduction, by requiring it to be ergodic under $\mathrm{ASL}_d(\mathbb{R})$ instead of $\mathrm{SL}_d(\mathbb{R})$. However, these two definitions are equivalent by the Howe-Moore ergodicity theorem (see [EW11]).

3.1. **RMS measures — background and basic strategy.** In order to motivate the definition of an RMS measure, we recall some crucial observations of [MS14]. Let F be as in (3.1). Let $\mathbb{R}^n = V_{\mathrm{phys}} \oplus V_{\mathrm{int}}, \pi_{\mathrm{phys}}, \pi_{\mathrm{int}}, \mathcal{L}, W$ be the data involved in a cut-and-project construction.

The observations of [MS14] consist of the following:

- From the fact that π_{phys} intertwines the action of F on $\mathbb{R}^n$ (via the top-left corner embedding in $\mathrm{ASL}_n(\mathbb{R})$) and on $\mathbb{R}^d$, for the map Ψ defined in (2.1), one obtains the equivariance property
$$\Psi \circ g = g \circ \Psi \tag{3.2}$$
for all $g \in F$; in other words, $g\Lambda(\mathcal{L}, W) = \Lambda(g\mathcal{L}, W)$.
- In particular, if we fix the data $\mathbb{R}^n = V_{\mathrm{phys}} \oplus V_{\mathrm{int}}$, W, then the map $\Psi_* : \mathrm{Prob}(\mathscr{Y}_n) \to \mathrm{Prob}(\mathscr{C}(\mathbb{R}^d))$ considered in Corollary 2.2 maps F-invariant measures to F-invariant measures.
- Due to Ratner's work described in §2.3, ergodic F-invariant measures on $\mathscr{Y}_n$ can be described in detail, in terms of certain real algebraic subgroups of $\mathrm{ASL}_n(\mathbb{R})$.
- Theorem 2.5 and other results from homogeneous dynamics can then be harnessed as a powerful tool for deriving information about cut-and-project sets.

In order to analyze measures on $\mathscr{Y}_n$, a basic strategy is to work first with the simpler space $\mathscr{X}_n$. Let
$$M \stackrel{\mathrm{def}}{=} \mathrm{ASL}_n(\mathbb{R}), \quad \Gamma \stackrel{\mathrm{def}}{=} \mathrm{ASL}_n(\mathbb{Z}).$$
Recall that $\mathscr{Y}_n$ is identified with M/Γ and under this identification, a closed orbit $H\mathcal{L}$ is identified with $Hg\Gamma = gH_1\Gamma$, where $g \in M$ is such that $\mathcal{L} = g\mathbb{Z}^n$, and $H_1 = g^{-1}Hg$. Also let
$$\underline{M} \stackrel{\mathrm{def}}{=} \mathrm{SL}_n(\mathbb{R}) \quad \text{and } \underline{\Gamma} \stackrel{\mathrm{def}}{=} \mathrm{SL}_n(\mathbb{Z}).$$
We think of $\underline{M}$ concretely as the stabilizer of the origin in the action of M on $\mathbb{R}^n$. Recall also that $\mathscr{X}_n$ is identified with $\underline{M}/\underline{\Gamma}$. Let
$$\pi : M \to \underline{M}, \qquad \underline{\pi} : \mathscr{Y}_n \to \mathscr{X}_n \tag{3.3}$$
denote respectively the natural quotient map, and the induced map on the quotients (which is well-defined since $\pi(\Gamma) = \underline{\Gamma}$). The map π is a $\mathbb{Q}$-morphism, and the map $\underline{\pi}$ is realized concretely by mapping a grid $\mathcal{L}$ to the underlying lattice $\mathcal{L} - \mathcal{L}$ obtained by translating $\mathcal{L}$ so that it has a point at the origin. It satisfies an equivariance property
$$\underline{\pi}(g\mathcal{L}) = \pi(g)\underline{\pi}(\mathcal{L}) \quad (\text{where } g \in M,\ \mathcal{L} \in \mathscr{Y}_n). \tag{3.4}$$
Every fiber of $\underline{\pi}$ is a torus and thus $\underline{\pi}$ is a proper map.

We summarize the spaces and maps we use in the following diagram.

$$\begin{array}{ccc} \mathscr{Y}_n = M/\Gamma & & \\ \Big\downarrow{\scriptstyle \underline{\pi}} & \searrow^{\Psi} & \\ \mathscr{X}_n = \underline{M}/\underline{\Gamma} & & \mathscr{C}(\mathbb{R}^d) \end{array}$$

Extending the terminology in the introduction, a homogeneous measure $\bar{\mu}$ on $\mathscr{Y}_n$ will be called *affine* if it is $\mathrm{ASL}_d(\mathbb{R})$-invariant, and *linear* if it is $\mathrm{SL}_d(\mathbb{R})$-invariant but not $\mathrm{ASL}_d(\mathbb{R})$-invariant. Here $\mathrm{ASL}_d(\mathbb{R})$ and $\mathrm{SL}_d(\mathbb{R})$ are embedded in M via the top-left corner embeddings (2.4) and (2.3).

3.2. **The homogeneous measures arising from the F-action on $\mathscr{Y}_n$.** In this section we state a more precise version of Theorem 1.1. Suppose k_0 is a subfield of $\mathbb{C}$. We say that a k_0-algebraic group $\mathbf{H}$ is k_0-*almost simple* if any normal k_0-subgroup $\mathbf{H}'$ satisfies $\dim \mathbf{H}' = \dim \mathbf{H}$ or $\dim \mathbf{H}' = 0$. In this case we will also say that a subgroup of finite index of $\mathbf{H}_{k_0}$ is k_0-almost simple.

Theorem 3.1. *Let $\bar{\mu}$ be an F-invariant ergodic measure on $\mathscr{Y}_n$, and let H and $\mathcal{L}_1$ denote respectively the subgroup of M and the point in $\mathscr{Y}_n$ involved in Theorem 2.3; i.e., $\bar{\mu}$ is H-invariant and supported on the closed orbit $H\mathcal{L}_1$. Let $g_1 \in M$ such that $\mathcal{L}_1 = g_1\mathbb{Z}^n$ and let $H_1 \stackrel{\mathrm{def}}{=} g_1^{-1}Hg_1$. Assume also that $\mathcal{L}_1$ satisfies conditions* **(D)** *and* **(I)**. *Then H, H_1 and $\mathcal{L}_1$ are described as follows:*

(i) *In the linear case, H_1 is semisimple and $\mathbb{Q}$-almost simple. In this case write $H' \stackrel{\mathrm{def}}{=} H_1$. In the affine case, we can write H_1 as a semidirect product $H' \ltimes \mathbb{R}^n$ where H' is semisimple and $\mathbb{Q}$-almost simple, and $\mathbb{R}^n$ denotes the full group of translations of $\mathbb{R}^n$.*

(ii) *The group H' in (i) is the connected component of the identity in the group of $\mathbb{R}$-points of $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$, where $\mathbb{K}$ is a real number field and $\mathbf{G}$ is a $\mathbb{K}$-group which is $\mathbb{K}$-isomorphic to either SL_k or Sp_{2k}, for some $k \geqslant d$. In the case $\mathbf{G} = \mathrm{SL}_k$ we have $n = k\deg(\mathbb{K}/\mathbb{Q})$, and there is a subspace V of $\mathbb{R}^n$ of dimension k containing V_{phys} which is H'-invariant and such that the action of H' on V gives the group $\mathrm{SL}(V)$. The case $\mathbf{G} = \mathrm{Sp}_{2k}$ only arises when $d = 2$, and in that case $n = 2k\deg(\mathbb{K}/\mathbb{Q})$, and there is a subspace V of $\mathbb{R}^n$ of dimension $2k$ equipped with a symplectic form ω' such that V is H'-invariant, the action of H' on V gives the symplectic group $\mathrm{Sp}(V, \omega')$, and V contains V_{phys} as a symplectic subspace.*

The proof will involve a reduction to the space $\mathscr{X}_n$ of lattices. We introduce some notation and give some preparatory statements.

As in §3.1, let $M = \mathrm{ASL}_n(\mathbb{R})$, $\Gamma = \mathrm{ASL}_n(\mathbb{Z})$, $\mathscr{Y}_n = M/\Gamma$, so that the closed orbit $H\mathcal{L}_1$ is identified with $Hg_1\Gamma = g_1H_1\Gamma$. By Theorem 2.3, $\Gamma_{H_1} \overset{\text{def}}{=} H_1 \cap \Gamma$ is a lattice in H_1 and $\bar{\mu}$ is the pushforward of the unique H_1-invariant probability measure on H_1/Γ_{H_1} under the map $h\Gamma_{H_1} \mapsto g_1h\Gamma$. By Theorem 2.4, H_1 is the group of real points of a $\mathbb{Q}$-algebraic group. In particular there are at most countably many possibilities for H_1.

Also let $\underline{M} = \mathrm{SL}_n(\mathbb{R})$, $\underline{\Gamma} = \mathrm{SL}_n(\mathbb{Z})$, $\mathscr{X}_n = \underline{M}/\underline{\Gamma}$ as above, and let $\pi, \underline{\pi}$ be the maps in (3.3). The orbit $\underline{\pi}(Hg_1\Gamma) = \underline{H}\underline{g}_1\underline{\Gamma} = \underline{g}_1\underline{H}_1\underline{\Gamma}$ is closed, where $\underline{H}_1 = \pi(H_1), \underline{g}_1 = \pi(g_1)$ and $\underline{H} = \pi(H)$ contains $\pi(F) \simeq \mathrm{SL}_d(\mathbb{R})$.

We say that property **(irred)** holds if there is no proper $\mathbb{Q}$-rational subspace of $\mathbb{R}^n$ that is $\underline{H}_1$-invariant (for the linear action by matrix multiplication). Note that by Theorem 2.4, $\underline{H}_1$ is the smallest $\mathbb{Q}$-subgroup of $\underline{M}$ containing $\underline{g}_1 \mathrm{SL}_d(\mathbb{R})\underline{g}_1^{-1}$, and thus **(irred)** is equivalent to requiring that there is no proper $\mathbb{Q}$-rational subspace of $\mathbb{R}^n$ that is $\underline{g}_1 \mathrm{SL}_d(\mathbb{R})\underline{g}_1^{-1}$-invariant.

We now state an analogue of Theorem 3.1 for the action on $\mathscr{X}_n$.

Lemma 3.2. *Assume* (**irred**) *holds. Then $\underline{H}_1$ is the connected component of the identity of the group of real points of a $\mathbb{Q}$-algebraic group, satisfying the properties listed in statement (ii) of Theorem 3.1 (for the group H').*

Lemma 3.2 is the main result of this section, and its proof will be given below in §3.3 and §3.4.

Proof of Theorem 3.1 assuming Lemma 3.2. Let U be the unipotent radical of H_1. Since $\underline{H}_1$ is semisimple, $U \subset \ker\pi$, and since $\ker\pi \cap H_1$ is a unipotent normal subgroup, $U = \ker\pi \cap H_1$. This means that in the affine map determined by $h \in H_1$ on $\mathbb{R}^n$, $\pi(h)$ is the linear part, and U acts on $\mathbb{R}^n$ by translations. This implies the equality

$$\mathrm{span}\{u(x) - x : x \in \mathbb{R}^n, u \in U\} = \mathrm{span}\{u(0) : u \in U\}, \tag{3.5}$$

and we denote the subspace of $\mathbb{R}^n$ appearing in (3.5) by V_0. Since H_1 is the group of $\mathbb{R}$-points of an algebraic group defined over $\mathbb{Q}$, the same holds for U. Since U is connected, it follows from [Bor91, Cor. 18.3] that $U_{\mathbb{Q}}$ is Zariski dense in U, and this implies that V_0 is a $\mathbb{Q}$-subspace of $\mathbb{R}^n$. Since H_1 normalizes U, V_0 is H_1-invariant, and since $\underline{H}_1 = \pi(H_1)$ is the group of linear parts of elements of H_1, $\underline{H}_1$ also preserves V_0. As we will see in Lemma 3.4, under the assumptions of Theorem 3.1, condition **(irred)** holds. By **(irred)** we must have $V_0 = \{0\}$ or $V_0 = \mathbb{R}^n$.

If $V_0 = \{0\}$ then $U = \{0\}$. If $V_0 = \mathbb{R}^n$ then U contains translations in n linearly independent directions and hence $U \simeq \mathbb{R}^n$ is the entire group of translations of $\mathbb{R}^n$. This gives the description of the translational part of H_1, in assertion (i). Assertion (ii) follows from Lemma 3.2. □

The next proposition shows that all the cases described in Theorem 3.1 do arise. Namely we have:

Proposition 3.3. *For any $k \geqslant d \geqslant 2$ and any real number field $\mathbb{K}$, there are $\mathbb{R}$-algebraic groups H and H' in M, and $\mathcal{L}_1 = g_1\mathbb{Z}^n \in \mathscr{Y}_n$, where $n = k\deg(\mathbb{K}/\mathbb{Q})$ and $g_1 \in M$, such that the following hold:*

- *H' is defined over $\mathbb{Q}$, and is $\mathbb{Q}$-isogenous to $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$, where $\mathbf{G}$ is $\mathbb{K}$-isomorphic to SL_k.*
- *H is either equal to H' (linear case) or to $H' \ltimes \mathbb{R}^n$ (affine case).*
- *The orbit $H\mathcal{L}_1$ is closed and supports an H-homogeneous probability measure ν. The pushforward $\Psi_*\nu$ is an RMS measure.*

The same statement is true with $d = 2$, $n = 2k\deg(\mathbb{K}/\mathbb{Q})$, and with $\mathbf{G}$ being $\mathbb{K}$-isomorphic to Sp_{2k} for some $k \geqslant 2$.

Proof. The proof amounts to reversing the steps in the preceding discussion. For concreteness, we give it for $\mathbf{G} = \mathrm{SL}_k$. Let $D \stackrel{\mathrm{def}}{=} \deg(\mathbb{K}/\mathbb{Q})$, $n \stackrel{\mathrm{def}}{=} Dk$ and $\underline{G} \stackrel{\mathrm{def}}{=} \mathrm{SL}_n(\mathbb{R})$. The standard action φ of $\mathbf{G}_{\mathbb{K}}$ on $\mathbb{K}^k$ gives rise to a $\mathbb{Q}$-embedding $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\varphi) : \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G}) \to \mathrm{SL}_n$. Let $\underline{H}_1$ denote the connected component of the identity in the group of $\mathbb{R}$-points in $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$. Similarly to (2.3) and (2.4), we refer to

$$g \mapsto \begin{pmatrix} g & \mathbf{0}_{d,n-d} \\ \mathbf{0}_{n-d,d} & \mathrm{Id}_{n-d} \end{pmatrix} \tag{3.6}$$

as the top-left corner embedding of $\mathrm{SL}_d(\mathbb{R})$ in $\underline{M}$. By the explicit description of restriction of scalars described in §2.4, there is $\underline{g}_1 \in \underline{M}$ such that $\underline{H} \stackrel{\mathrm{def}}{=} \underline{g}_1 H_1 \underline{g}_1^{-1}$ contains the top-left corner embedding of $\mathrm{SL}_d(\mathbb{R})$ in $\underline{M}$, and up to scaling, $\underline{g}_1\mathbb{Z}^n$ is the geometric embedding of $\mathcal{O}^k$ as in (2.7), where $\mathcal{O}$ is the ring of integers in $\mathbb{K}$. In particular, the orbit $\underline{H}\underline{g}_1\mathbb{Z}^n$ is a closed orbit supporting an $\underline{H}$-homogeneous measure in $\mathscr{X}_n$.

Recall that there is an embedding of $\underline{M}$ in M and of $\mathscr{X}_n$ in $\mathscr{Y}_n$ (respectively as the stabilizer of the origin in the standard action on $\mathbb{R}^n$, and as the set of lattices in the space of grids). We let H' denote the image of $\underline{H}_1$ under this embedding, and in the linear case we set $H \stackrel{\mathrm{def}}{=} H'$ and let $H\mathcal{L}_0$ be the image of $\underline{H}\underline{g}_1\mathbb{Z}^n$ under this embedding, and let ν be the H-homogeneous measure on $H\mathcal{L}_1$. Because the action of $\mathrm{SL}_d(\mathbb{R})$ is ergodic with respect to ν, we can find g_1 so that for $\mathcal{L}_1 = g_1\mathbb{Z}^n$ we have $\overline{\mathrm{SL}_d(\mathbb{R})\mathcal{L}_1} = H\mathcal{L}_1 = H\mathcal{L}_0$. It is not hard to check that with

these choices, the desired conclusions hold. The proof in the affine case is similar, taking $H = H' \ltimes \mathbb{R}^n$ and $\underline{\pi}^{-1}(\underline{H}\underline{g}_1\mathbb{Z}^n)$. □

3.3. **Preparations for the proof of Lemma 3.2.** Recall that $\underline{\mathcal{L}}_1 = \underline{\pi}(\mathcal{L}_1)$. A vector space $V \subset \mathbb{R}^n$ is called $\mathcal{L}_1$-*rational* if $V \cap \underline{\mathcal{L}}_1$ is a lattice in V. In other words a subspace V is $\mathcal{L}_1$-rational if it is of the form $\underline{g}_1 W$ for some rational subspace $W \subset \mathbb{R}^n$, i.e., a subspace spanned by vectors with rational entries.

Lemma 3.4. *The following implications hold.*

(a) (**D**) $\Rightarrow$ V_{phys} *is not contained in a proper* $\mathcal{L}_1$*-rational subspace.*
(b) (**I**) $\Rightarrow$ V_{int} *contains no nontrivial* $\mathcal{L}_1$*-rational subspace.*
(c) (**I**) *and* (**D**) $\Rightarrow$ (**irred**).

Variants of statements (a) and (b) are given in [Ple03], but we give a complete proof for the convenience of the reader.

Proof. We will prove all three statements by contradiction. Suppose that (a) fails, so that there is a proper $\mathcal{L}_1$-rational subspace W containing V_{phys}. Let $W^\perp$ be an $\mathcal{L}_1$-rational complement of W. Since $W^\perp$ is $\mathcal{L}_1$-rational, $\underline{\mathcal{L}}_1$ is mapped to a lattice in $W^\perp$ under the projection $\mathbb{R}^n \to W^\perp$, and hence the projection of $\mathcal{L}_1$ to $W^\perp$ is discrete. On the other hand, $\mathbb{R}^n \to W^\perp$ factors through V_{int} since $V_{\text{phys}} \subset W$, and by **(D)** the image of $\mathcal{L}_1$ is dense in V_{int}. Thus, the projection of $\mathcal{L}_1$ is dense in $W^\perp$, a contradiction.

Now suppose that (b) fails, and V_{int} contains a nontrivial $\mathcal{L}_1$-rational subspace W. Then V_{int}, which is the kernel of the map $\mathbb{R}^n \to V_{\text{phys}}$, contains $W \cap \underline{\mathcal{L}}_1$, which by assumption is nontrivial. This contradicts **(I)**.

Now suppose **(D)** and **(I)** hold but **(irred)** fails, so that there is a proper $\underline{H}_1$-invariant $\mathbb{Q}$-rational subspace W. From (b) we know that $\underline{g}_1 W$ is not contained in V_{int}. Hence some $u \in \underline{g}_1 W$ can be written as

$$u = u_p + u_i,\ u_p \in V_{\text{phys}} \smallsetminus \{0\},\ u_i \in V_{\text{int}}.$$

Since $\mathrm{SL}_d(\mathbb{R}) \subset \underline{H} = \underline{g}_1 \underline{H}_1 \underline{g}_1^{-1}$, $\underline{g}_1 W$ is also $\mathrm{SL}_d(\mathbb{R})$-invariant. Since $\mathrm{SL}_d(\mathbb{R})$ acts trivially on V_{int}, for any $g \in \mathrm{SL}_d(\mathbb{R})$ we have

$$gu - u = gu_p - u_p \in V_{\text{phys}}.$$

We can find $g \in \mathrm{SL}_d(\mathbb{R})$ such that $gu_p \neq u_p$, and hence $\underline{g}_1 W \cap V_{\text{phys}}$ is nontrivial. Since $\mathrm{SL}_d(\mathbb{R})$ acts irreducibly on V_{phys}, $V_{\text{phys}} \subset \underline{g}_1 W$. This contradicts the conclusion of (a). □

Theorem 3.5 (Morris)**.** *Let $n \geqslant d \geqslant 2$, and let S be a connected real algebraic group which is $\mathbb{R}$-almost simple, and contains the image of*

$\mathrm{SL}_d(\mathbb{R})$ *under the top-left corner embedding (see* (3.6)*). Then there are* $k \geqslant d, \ell \geqslant d$ *and* $g \in \mathrm{SL}_n(\mathbb{R})$ *such that* gSg^{-1} *is the image of either* $\mathrm{SL}_k(\mathbb{R})$ *or* $\mathrm{Sp}_{2\ell}(\mathbb{R})$ *under the top-left corner embedding, and the latter can only occur when* $d = 2$*.*

In this statement, by the 'top-left corner embedding of $\mathrm{Sp}_{2k}(\mathbb{R})$', we mean the image under (3.6), that is, the elements of $\mathrm{SL}_{2k}(\mathbb{R})$ stabilizing a non-degenerate alternating bilinear form on $\mathbb{R}^{2k}$. As is well-known, such a form can be taken to be defined by

$$\omega(\vec{x}_i, \vec{y}_j) = -\omega(\vec{y}_j, \vec{x}_i) = \delta_{ij}, \quad \omega(\vec{x}_i, \vec{x}_j) = \omega(\vec{y}_i, \vec{y}_j) = 0$$

for some basis $\vec{x}_1, \dots, \vec{x}_k, \vec{y}_1, \dots, \vec{y}_k$ of $\mathbb{R}^{2k}$.

This result was proved by Dave Morris in 2014, in connection with prior work of one of the authors and Solomon. Namely, the result appeared in an initial ArXiV version [SW14] (in a slightly different form) but eventually did not appear in the published version [SW16].

We will refer to the image of $\mathrm{SL}_d(\mathbb{R})$ under the top-left corner embedding in (3.6) as the *top-left copy of* $\mathrm{SL}_d(\mathbb{R})$. Clearly, with respect to the decomposition

$$\mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^{n-d}, \tag{3.7}$$

the top-left copy of $\mathrm{SL}_d(\mathbb{R})$ acts via its standard action on the first summand, and the second summand is the set of vectors fixed by the action.

Let k be maximal, such that S contains a conjugate (over $\mathrm{SL}_n(\mathbb{R})$) of the top-left copy of $\mathrm{SL}_k(\mathbb{R})$. To make the ideas more transparent we separate the proof into cases according to whether $k \geqslant 3$ (the easier case) or $k = 2$. The proofs in these cases are not independent – readers interested in the case $k = 2$ are encouraged to first read the proof for $k \geqslant 3$.

Proof in case $k \geqslant 3$*.* We recall the following result of Mostow [Mos55]: If $G_1 \subset \cdots \subset G_r \subset \mathrm{SL}_n(\mathbb{R})$ are connected reductive real algebraic groups, then there is $x \in \mathrm{SL}_n(\mathbb{R})$ such that $x^{-1}G_i x$ is self-adjoint for every i. That is, if $g \in x^{-1}G_i x$, then the transpose of g is also in $x^{-1}G_i x$.

Replacing S by a conjugate, we may assume that S contains the top-left embedding of $\mathrm{SL}_k(\mathbb{R})$, which we denote by F. By Mostow's theorem, there is $x \in \mathrm{SL}_n(\mathbb{R})$, such that $x^{-1}Fx$ and $x^{-1}Sx$ are self-adjoint. Let V be the $(n-k)$-dimensional subspace of $\mathbb{R}^n$ which is pointwise fixed by F. Since $\mathrm{SO}_n(\mathbb{R})$ acts transitively on the set of subspaces of any given dimension, there is some $h \in \mathrm{SO}_n(\mathbb{R})$, such that $xh(V) = V$. After replacing x with xh, we may assume that $x^{-1}Fx$ fixes pointwise the second summand in the splitting (3.7), and $x^{-1}Fx$

and $x^{-1}Sx$ are self-adjoint (because this property is not affected by conjugation by an element of $\mathrm{SO}_n(\mathbb{R})$). We conclude that $x^{-1}Fx = F$. Thus, we may assume that S is self-adjoint and contains F. We will assume that $S \neq F$ and derive a contradiction to the maximality of k. Since $F \subsetneq S$ are connected, their Lie algebras $\mathfrak{f}, \mathfrak{s}$ satisfy $\dim \mathfrak{f} < \dim \mathfrak{s}$.

For $1 \leqslant i, j \leqslant n$, let $e_{i,j}$ be the elementary matrix with 1 in the (i,j) entry, and all other entries 0. Write

$$\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{f} \oplus \mathfrak{z} \oplus X_1 \oplus \cdots \oplus X_k \oplus Y_1 \oplus \cdots \oplus Y_k, \tag{3.8}$$

where

- $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{f}$ are the Lie algebras of $\mathrm{SL}_n(\mathbb{R})$ and F, respectively,
- $\mathfrak{z}$ is the subspace of $\mathfrak{sl}_n(\mathbb{R})$ fixed pointwise by $\mathrm{Ad}(F)$, where $\mathrm{Ad} : \mathrm{SL}_n(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{sl}_n(\mathbb{R}))$ is the adjoint representation,
- X_i is the linear span of $\{ e_{i,j} : k+1 \leqslant j \leqslant n \}$, and
- Y_j is the linear span of $\{ e_{i,j} : k+1 \leqslant i \leqslant n \}$.

Now we denote by A the group of diagonal matrices in F with positive entries. We write an element $a \in A$ as

$$a = \mathrm{diag}\big(a_1, a_2, \ldots, a_{k-1}, (a_1 a_2 \cdots a_{k-1})^{-1}, 1, \ldots, 1\big), \tag{3.9}$$

and denote by χ_i the characters $a \mapsto a_i$, where $a_k \stackrel{\text{def}}{=} (a_1 \cdots a_{k-1})^{-1}$. Since $k \geqslant 3$, the characters χ_i, χ_i^{-1} are distinct, for $i = 1, \ldots, k$, and the subspaces $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$ are the corresponding weight spaces, that is,

- $X_i = \{x \in \mathfrak{sl}_n(\mathbb{R}) : \mathrm{Ad}(a)(x) = \chi_i(a)x \text{ for all } a \in A\}$, and
- $Y_j = \{x \in \mathfrak{sl}_n(\mathbb{R}) : \mathrm{Ad}(a)(x) = \chi_j^{-1}(a)x \text{ for all } a \in A\}$.

We will use repeatedly the fact that if $\mathfrak{l}$ is an $\mathrm{Ad}(A)$-invariant subspace of $\mathfrak{sl}_n(\mathbb{R})$, and $v \in \mathfrak{l}$ has a nontrivial projection onto some weight space, then this projection is contained in $\mathfrak{l}$.

Since $A \subset S$, $\mathfrak{s}$ is invariant under $\mathrm{Ad}(A)$. Since S is $\mathbb{R}$-almost simple and $\dim \mathfrak{f} < \dim \mathfrak{s}$, $\mathfrak{s}$ cannot be contained in $\mathfrak{f} \oplus \mathfrak{z}$, and hence $\mathfrak{s}$ projects nontrivially to some X_i or Y_j. In fact, since S is self-adjoint, it must project nontrivially to both X_i and Y_i, for some i. Since X_i is a weight space of $\mathrm{Ad}(A)$, we find that $X_i \cap \mathfrak{s}$ is nontrivial. Conjugating by an element of $I_k \times \mathrm{SO}_{n-k}(\mathbb{R})$, we may assume that $\mathfrak{s}$ contains the matrix $e_{i,k+1}$. Applying an appropriate element of $\mathrm{Ad}(\mathrm{SO}_k(\mathbb{R}))$ shows that $e_{k,k+1} \in \mathfrak{s}$. Then, since S is self-adjoint, $\mathfrak{s}$ also contains $e_{k+1,k}$. Therefore, $\mathfrak{s}$ contains the Lie subalgebra generated by $\mathfrak{f}$, $e_{k,k+1}$, and $e_{k+1,k}$, which is the Lie subalgebra of F', the top-left copy of $\mathrm{SL}_{k+1}(\mathbb{R})$. Thus S contains F', contradicting the maximality of k, and completing the proof in case $k \geqslant 3$. □

Proof in case $k = 2$. In this case we also have $d = 2$. Arguing as in the case $k \geqslant 3$ we may assume that S properly contains F, the top-left copy of $\mathrm{SL}_2(\mathbb{R})$, and is self-adjoint. Let ℓ be the maximal number so that S contains a copy of $H \stackrel{\mathrm{def}}{=} F_1 \times \cdots \times F_\ell$, where each F_r is isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and there is an H-invariant direct sum decomposition $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_\ell \oplus V_0$, where the spaces $V_1, \ldots, V_\ell$ are two dimensional, and each F_r acts linearly on V_r and trivially on $\bigoplus_{s \neq r} V_s$. By assumption $\ell \geqslant 1$, and there is a conjugation taking H into a top-left copy of $\mathrm{SL}_t(\mathbb{R})$, where $t = 2\ell \geqslant 2$. We replace H and S by their images under this conjugacy (retaining the same names H and S). By Mostow's theorem we can assume that H and S are both self-adjoint.

Our first goal is to show that

$$S \text{ is also contained in the top-left copy of } \mathrm{SL}_t(\mathbb{R}). \tag{3.10}$$

Indeed, in analogy with (3.8), consider the decomposition

$$\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{l} \oplus \mathfrak{z} \oplus \mathfrak{m}, \text{ where } \mathfrak{m} = X_1 \oplus \cdots \oplus X_t \oplus Y_1 \oplus \cdots \oplus Y_t,$$

and

- $\mathfrak{l}$ is the Lie algebra of the top-left $\mathrm{SL}_t(\mathbb{R})$,
- $\mathfrak{z}$ is the Lie algebra of the centralizer of the top-left $\mathrm{SL}_t(\mathbb{R})$,
- X_i is the linear span of $\{ e_{i,j} : t+1 \leqslant j \leqslant n \}$, and
- Y_j is the linear span of $\{ e_{i,j} : t+1 \leqslant i \leqslant n \}$.

With this notation, our claim (3.10) is that $\mathfrak{s} \subset \mathfrak{l}$.

We note that

$$\mathfrak{s} \text{ does not contain a nonzero element in some } X_i \text{ or some } Y_i. \tag{3.11}$$

Indeed, if $v \in (\mathfrak{s} \cap X_i) \smallsetminus \{0\}$, we could re-index to assume $i = 1$, and conjugate by an element of $I_t \times \mathrm{SO}_{n-t}(\mathbb{R})$ and rescale to assume $v = e_{1,t+1}$. Since $\mathfrak{s}$ is self-adjoint, we also have $e_{t+1,1} \in \mathfrak{s}$. Since $\mathfrak{f}_1$, $e_{t+1,1}$ and $e_{1,t+1}$ generate a Lie algebra isomorphic to $\mathfrak{sl}_3(\mathbb{R})$, this gives a contradiction to the choice of k and proves (3.11).

If $\mathfrak{s} \not\subset \mathfrak{l}$, using that $\mathfrak{s}$ is simple and the Lie algebras $\mathfrak{l}$, $\mathfrak{z}$ commute, we see that the projection of $\mathfrak{s}$ onto $\mathfrak{m}$ is nontrivial; indeed, if $\mathfrak{s} \subset \mathfrak{l} \oplus \mathfrak{z}$ then the kernel of the projection of $\mathfrak{s}$ to $\mathfrak{z}$ contains $\mathfrak{f}$ and by simplicity is equal to $\mathfrak{s}$.

Let A' be the intersection of H with the diagonal subgroup and let $\mathfrak{a}'$ be its Lie algebra. For each odd index $i < t$, the spaces $X_i \oplus Y_{i+1}$ and $X_{i+1} \oplus Y_i$ are weight spaces for $\mathrm{Ad}(A')$, and hence there is some i such that $\mathfrak{s} \cap (X_i \oplus Y_{i+1} \cup X_{i+1} \oplus Y_i)$ contains a nonzero element u. Re-indexing, conjugating and rescaling as in the proof of (3.11), we can assume $u = e_{1,t+1} + \sum_{j \geqslant t+1} a_j e_{j,2}$, where the a_j are not all zero. By a

further conjugation by an element of $I_t \times \mathrm{SO}_{n-t}(\mathbb{R})$ that fixes $e_{1,t+1}$, we can also assume that $a_j = 0$ for $j > t+2$, that is, we can write

$$u = e_{1,t+1} + ae_{t+1,2} + be_{t+2,2}, \quad \text{with } (a,b) \neq (0,0).$$

Using brackets to denote the commutator $[x,y] = xy - yx$, we compute

$$\begin{aligned} w &\stackrel{\text{def}}{=} [u,[u,e_{2,1}]] = [e_{1,t+1} + ae_{t+1,2} + be_{t+2,2}, -e_{2,t+1} + ae_{t+1,1} + be_{t+2,1}] \\ &= a(e_{1,1} + e_{2,2} - 2e_{t+1,t+1}) - 2be_{t+2,t+1} \end{aligned}$$

and

$$[w,u] = 3ae_{1,t+1} - 3a^2 e_{t+1,2} - 3abe_{t+2,2},$$

so that

$$6ae_{t+1,1} = [w,u] + 3au \in \mathfrak{s}.$$

It follows from (3.11) that $a = 0$, and thus $\mathfrak{s}$ contains $\frac{-1}{2b}w = e_{t+2,t+1}$. Since $\mathfrak{s}$ is self-adjoint it also contains $e_{t+1,t+2}$, and since these two vectors generate a copy of $\mathfrak{sl}_2(\mathbb{R})$ which is contained in $\mathfrak{z}$, and acts on $\mathbb{R}^n$ by the standard two-dimensional representation, we have a contradiction to the definition of ℓ. This proves (3.10).

Since S properly contains F we have $\ell > 1$. We will now show that $\mathfrak{s}$ is the Lie algebra $\mathfrak{sp}(2\ell,\mathbb{R})$ of the top-left corner embedding of $\mathrm{Sp}_{2\ell}(\mathbb{R})$. We will begin with the case $\ell = 2$ as it will make the argument more transparent. That is, up to a conjugation in $\mathrm{SL}_n(\mathbb{R})$, we want to show that

$$\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{s}_{1,3} \oplus \mathfrak{s}_{1,4} \oplus \mathfrak{s}_{2,3} \oplus \mathfrak{s}_{2,4}, \tag{3.12}$$

where $\mathfrak{h} \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{s}$ is the Lie algebra of H, and

$$\begin{aligned} \mathfrak{s}_{1,3} &\stackrel{\text{def}}{=} \mathrm{span}(e_{1,3} - e_{4,2}) \qquad \mathfrak{s}_{1,4} \stackrel{\text{def}}{=} \mathrm{span}(e_{1,4} + e_{3,2}) \\ \mathfrak{s}_{2,3} &\stackrel{\text{def}}{=} \mathrm{span}(e_{2,3} + e_{4,1}) \qquad \mathfrak{s}_{2,4} \stackrel{\text{def}}{=} \mathrm{span}(e_{2,4} - e_{3,1}). \end{aligned} \tag{3.13}$$

To this end, let

$$\begin{aligned} \mathfrak{l}_{1,3} &\stackrel{\text{def}}{=} \mathrm{span}(e_{1,3}, e_{4,2}), \qquad \mathfrak{l}_{1,4} \stackrel{\text{def}}{=} \mathrm{span}(e_{1,4}, e_{3,2}), \\ \mathfrak{l}_{2,3} &\stackrel{\text{def}}{=} \mathrm{span}(e_{2,3}, e_{4,1}), \qquad \mathfrak{l}_{2,4} \stackrel{\text{def}}{=} \mathrm{span}(e_{2,4}, e_{3,1}) \end{aligned} \tag{3.14}$$

be the weight spaces for the action of $\mathrm{Ad}(A')$, which are not in $\mathfrak{h}$. Let

$$\mathfrak{s}'_{i,j} \stackrel{\text{def}}{=} \mathfrak{l}_{i,j} \cap \mathfrak{s},$$

where the indices (i,j) range over $\{1,2\} \times \{3,4\}$. Our goal is to show that

$$\text{for every } i,j, \quad \mathfrak{s}'_{i,j} = \mathfrak{s}_{i,j}. \tag{3.15}$$

We first show that

$$\text{for every } i,j, \quad \dim(\mathfrak{s}'_{i,j}) = 1. \tag{3.16}$$

To this end, note that the ad-action of the off-diagonal elements of $\mathfrak{h}$ permutes the spaces $\mathfrak{l}_{i,j}$ transitively. For example,

$$\mathfrak{l}_{1,3} = [e_{1,2}, \mathfrak{l}_{2,3}]\,,\ \mathfrak{l}_{1,3} = [e_{4,3}, \mathfrak{l}_{1,4}]\,,$$

and so on. Since $e_{1,2}, e_{2,1}, e_{3,4}, e_{4,3} \in \mathfrak{s}$, this ad-action also permutes the intersections $\mathfrak{s}'_{i,j}$, and thus they all have the same dimension. If this dimension is 0 then $\mathfrak{s} = \mathfrak{h}$, contradicting the fact that $\mathfrak{s}$ is simple, and if this dimension is 2, then $\mathfrak{s} = \mathfrak{sl}(4, \mathbb{R})$, contradicting the definition of k. We have shown (3.16).

We now claim that

$$\mathfrak{s}'_{1,3} \text{ is equal to either } \mathfrak{s}_{1,3} = \operatorname{span}(e_{1,3} - e_{4,2}) \text{ or } \operatorname{span}(e_{1,3} + e_{4,2}). \quad (3.17)$$

To see this, let $u = ae_{1,3} + be_{4,2} \in \mathfrak{s}'_{1,3} \smallsetminus \{0\}$. By (3.11), a, b are both nonzero. Since $\mathfrak{s}$ is self-adjoint $v \stackrel{\text{def}}{=} ae_{3,1} + be_{2,4} \in \mathfrak{s}$ and hence $v \in \mathfrak{s}'_{2,4}$. Also we have

$$w \stackrel{\text{def}}{=} [e_{2,1}, [e_{3,4}, u]] = [e_{2,1}, -ae_{1,4} + be_{3,2}] = -ae_{2,4} - be_{3,1} \in \mathfrak{s}.$$

Since w and v are both nonzero elements of $\mathfrak{s}'_{2,4}$, by (3.16) they are scalar multiples of each other and thus there is $c \neq 0$ so that $w = cv$. This forces $-a = cb$ and $-b = ca$ and so $c = \pm 1$, proving (3.17).

Using the ad-action as before we see that in order to obtain (3.15), it suffices to show that after a conjugation, we have $\mathfrak{s}'_{1,3} = \mathfrak{s}_{1,3}$. Suppose that $\mathfrak{s}'_{1,3} = \operatorname{span}(e_{1,3} + e_{4,2})$. Then

$$\mathfrak{s}'_{1,4} = \operatorname{span}([e_{3,4}, e_{1,3} + e_{4,2}]) = \operatorname{span}(e_{1,4} - e_{3,2}),$$

and we can apply a permutation matrix swapping the indices 3,4 to obtain

$$\mathfrak{s}'_{1,3} = \operatorname{span}(e_{1,3} - e_{4,2}) = \mathfrak{s}_{1,3}.$$

We have shown (3.15), completing the proof in case $\ell = 2$.

Note that for the case $\ell = 2$ we only applied one conjugation, namely the conjugation swapping the indices 3,4. Thus, by induction on ℓ, we see that after a conjugation, we have the following. For $i \in \{1, \ldots, \ell - 1\}$, let $\mathrm{SL}_4^{(i)}(\mathbb{R})$ be the copy of $\mathrm{SL}_4(\mathbb{R})$ embedded in $\mathrm{SL}_n(\mathbb{R})$ in a 4×4 block corresponding to indices $2i-1, 2i, 2i+1, 2i+2$. Let $H^{(i)} = F_i \times F_{i+1} \subset \mathrm{SL}_4^{(i)}(\mathbb{R})$ be the corresponding diagonal copies of $\mathrm{SL}_2(\mathbb{R})$, and let $\mathfrak{s}^{(i)}$ be the intersection of $\mathfrak{s}$ with the Lie algebra of $\mathrm{SL}_4^{(i)}(\mathbb{R})$. Then $\mathfrak{s}^{(i)}$ is the obvious embedding of $\mathfrak{sp}(4, \mathbb{R})$ (namely, the embedding given for $i = 1$ by (3.12) and (3.13)). The Lie algebras $\mathfrak{s}^{(i)}$ generate $\mathfrak{sp}(2\ell, \mathbb{R})$ (namely, the Lie algebra of the top-left $\mathrm{Sp}_{2\ell}(\mathbb{R})$). This implies that H contains $\mathrm{Sp}_{2\ell}(\mathbb{R})$. Since $\mathrm{Sp}_{2\ell}(\mathbb{R})$ is a maximal subgroup of $\mathrm{SL}_{2\ell}(\mathbb{R})$, we must have that $S = \mathrm{Sp}_{2\ell}(\mathbb{R})$. $\square$

3.4. **Proof of Lemma 3.2.** Since $\underline{\pi}$ is proper, we have

$$\underline{H}\underline{g}_1\underline{\Gamma} = \underline{\pi}(Hg_1\Gamma) = \underline{\pi}\left(\overline{Fg_1\Gamma}\right) = \overline{\mathrm{SL}_d(\mathbb{R})\underline{g}_1\underline{\Gamma}}.$$

Since $\underline{H}_1 = \underline{g}_1^{-1}\underline{H}\underline{g}_1$, by Theorem 2.4, $\underline{H}_1$ is the connected component of the identity in the group of real points of a $\mathbb{Q}$-algebraic group $\mathbf{H}$. From now on we replace F with its image under $\underline{\pi}$, i.e., denote $F = \mathrm{SL}_d(\mathbb{R})$. We also write

$$F' \stackrel{\mathrm{def}}{=} \underline{g}_1^{-1}F\underline{g}_1, \quad \text{so that } \overline{F'\underline{\Gamma}} = \underline{H}_1\underline{\Gamma}.$$

We need to show that $\mathbf{H}$ admits the description given in the statement. We divide the proof into steps.

Step 1: H is semisimple. Let $\mathbf{U}$ be the radical of $\mathbf{H}$. By Theorem 2.4, it is defined over $\mathbb{Q}$ and unipotent, $U = \mathbf{U}^\circ_{\mathbb{R}}$ is the unipotent radical of $\underline{H}_1$, and $\mathbf{U}$ is connected ([Bor91, 11.21]). Let V^U be the subspace of $\mathbb{R}^n$ fixed by U. Since $\mathbf{U}_{\mathbb{Q}} \subset U$ is Zariski dense in $\mathbf{U}$ (see [Bor91, Cor. 18.3]), we have

$$V^U = \{z \in \mathbb{R}^n : uz = z \text{ for all } u \in \mathbf{U}_{\mathbb{Q}}\}.$$

Thus V^U is defined over $\mathbb{Q}$.

Furthermore, since every unipotent subgroup can be put in an upper triangular form, $V^U \neq \{0\}$, and is a proper subspace of $\mathbb{R}^n$ unless U is trivial. Since U is normal in $\underline{H}_1$, the space V^U is $\underline{H}_1$-invariant, and thus by assumption **(irred)**, V^U is not a proper subspace of $\mathbb{R}^n$. It follows that U is trivial, and hence $\underline{H}_1$ is semisimple. Therefore so is $\mathbf{H}$.

For a group M and normal subgroups $M_1, \dots, M_k$, the *product* is the subgroup

$$\prod M_i \stackrel{\mathrm{def}}{=} \{m_1 \cdots m_k : m_i \in M_i,\ i = 1, \dots, k\}.$$

Note that $\prod M_i$ is also normal and does not depend on the ordering of the M_i. Let k_0 be one of the fields $\mathbb{Q}$ or $\mathbb{R}$. Recall that an *almost direct product* is the image of a direct product under a homomorphism with finite kernel (that is, isogenous to a direct product). A semisimple k_0-group is an almost direct product of its k_0-almost simple normal subgroups, and such a decomposition is unique up to permuting the k_0-almost simple factors.

We write $\mathbf{H}$ in two ways: as an almost direct product of its $\mathbb{R}$-almost simple factors $\mathbf{S}_i$, and as an almost direct product of its $\mathbb{Q}$-almost simple factors $\mathbf{T}_j$, and let S_i and T_j denote respectively the connected component of the identity in the group of $\mathbb{R}$-points of $\mathbf{S}_i$ and $\mathbf{T}_j$. Since every $\mathbf{T}_j$ can be further decomposed into $\mathbb{R}$-almost simple factors, and

since these decompositions are unique, the decomposition of $\mathbf{H}$ into the $\mathbf{S}_i$ refines the decomposition of $\mathbf{H}$ into the $\mathbf{T}_j$. In other words, there is a partition of the $\mathbf{S}_i$ into subsets such that each $\mathbf{T}_j$ is a product of the $\mathbf{S}_i$ in one subset of the partition. Then $\underline{H}_1$ is the product of the S_i. For $h \in \underline{H}_1$, we can write $h = h_1 \cdots h_t$, where $h_i \in S_i$, and if $h = h'_1 \cdots h'_t$ is another such presentation, then for each i, $h'_i h_i^{-1}$ belongs to the finite center of $\underline{H}_1$.

Step 2: F' is contained in one of the S_i, and $\mathbf{H}$ is $\mathbb{Q}$-almost simple. The second assertion follows from the first one. Indeed, by reindexing, let S_1 and T_1 denote respectively the connected component of the identity in the real points of the $\mathbb{R}$- and $\mathbb{Q}$-simple factors containing F'. Then $S_1 \subset T_1$ and T_1 does not properly contain the real points of any $\mathbb{Q}$-subgroup containing S_1, and by the last assertion of Theorem 2.4 we have that $\underline{H}_1 = T_1$.

Turning to the first assertion, let $\mathrm{Z}(S)$ denote the center of S, for each i let S'_i be the quotient group $S/\left(\mathrm{Z}(S) \cdot \prod_{j \neq i} S_j\right)$, and let F'_i denote the image of the projection of F' to S'_i. Let

$$\underline{H}_2 \stackrel{\text{def}}{=} \prod_{i \in \mathcal{I}} S_i, \quad \text{where } \mathcal{I} \stackrel{\text{def}}{=} \{i : F'_i \text{ is nontrivial}\}.$$

Note that $i_0 \in \mathcal{I}$ if and only if for any subset $\mathcal{F}' \subset F'$ which generates a dense subgroup, there is $f' \in \mathcal{F}'$ which can be written as a product of elements f'_i in S_i, where f'_{i_0} is not central in S_{i_0}. Clearly $F' \subset \underline{H}_2$, and our goal is to show that $\underline{H}_2$ is equal to one of the S_i, or in other words that $\#\mathcal{I} = 1$. Also, for $i \in \mathcal{I}$, F'_i is isogenous to $\mathrm{SL}_d(\mathbb{R})$.

Recall that a representation of a group H on a vector space V is *isotypic* if V is the direct sum of $k \in \mathbb{N}$ isomorphic irreducible representations for H, where k is referred to as the *multiplicity*. We will also use the term *H-isotypic*, if we want to make the dependence on H explicit. A linear representation of a semisimple group has a unique presentation as a direct sum of isotypic representations (up to permuting factors). Let $V'_{\text{phys}} \stackrel{\text{def}}{=} \underline{g}_1^{-1}(V_{\text{phys}})$ and $V'_{\text{int}} \stackrel{\text{def}}{=} \underline{g}_1^{-1}(V_{\text{int}})$. Then the decomposition $\mathbb{R}^n = V'_{\text{phys}} \oplus V'_{\text{int}}$, is the decomposition of $\mathbb{R}^n$ into F'-isotypic representations, and the action of F' on V'_{phys} is irreducible. In particular, the multiplicity of the representation on V'_{phys} is equal to one.

Let $V_1 \oplus \cdots \oplus V_t$ be a decomposition of $\mathbb{R}^n$ into $\underline{H}_2$-isotypic representations. Since $F' \subset \underline{H}_2$, each V_ℓ is F'-invariant, and decomposes further into isotypic representations for F'. Since V'_{phys} is an isotypical component of F' of multiplicity one, V'_{phys} is contained in one of the V_ℓ. By renumbering we can assume $V'_{\text{phys}} \subset V_1$. Since F' acts on V'_{phys}

irreducibly, the action of $\underline{H}_2$ on V_1 is irreducible, and the $\underline{H}_2$-isotypic component associated to V_1 has multiplicity one. Since F' acts trivially on V'_{int}, which is a complementary subspace to V'_{phys}, the action of F' on each V_ℓ is trivial for $\ell = 2, \dots, t$, that is,

$$F' \subset \bigcap_{\ell=2}^{t} \ker\left(\underline{H}_2|_{V_\ell}\right). \tag{3.18}$$

The right-hand side of (3.18) is a normal subgroup of $\underline{H}_2$, and thus a product $\prod_{i\in\mathcal{J}} S_i$ for some $\mathcal{J} \subset \mathcal{I}$. By the assumption that F'_i is nontrivial for each $i \in \mathcal{I}$, we must have that $\mathcal{J} = \mathcal{I}$, that is, the group on the right-hand side of (3.18) must coincide with $\underline{H}_2$. This means that for $\ell \geqslant 2$, the V_ℓ are trivial representations for $\underline{H}_2$, and hence of S_i for each $i \in \mathcal{I}$.

Let $\mathcal{F}'$ denote the elements of F' whose eigenvalues on V'_{phys} are all real, distinct from each other, and not equal to 1. Since these conditions are invariant under conjugation and F' is simple, $\mathcal{F}'$ generates a dense subgroup of F'. Write f' as a product of elements f'_i, where $f'_i \in S_i$. Then the elements f'_i commute with each other and with f'. Thus each f'_i fixes the eigenspaces for f' and hence each f'_i preserves the eigenspace decomposition of the action of f' on $\mathbb{R}^n$. In particular, f'_i preserves V'_{phys} for each $i \in \mathcal{I}$.

Re-indexing if necessary we can assume that $1 \in \mathcal{I}$, and suppose by contradiction that there is $i_0 \in \mathcal{I} \smallsetminus \{1\}$. There is $f' \in \mathcal{F}'$ such that, when writing f' as a product of elements $f'_i \in S_i$, f'_1 acts on V'_{phys} with infinite order (this property does not depend on the presentation of f as a product of the f'_i). Then the action of f'_1 on V'_{phys} preserves an eigenspace V', with $d' \stackrel{\mathrm{def}}{=} \dim V' < d = \dim V'_{\mathrm{phys}}$. Since the action of S_{i_0} commutes with the action of f'_1, the space V' is preserved by S_{i_0}, and hence by f'_{i_0}. The group generated by all such elements f'_{i_0} is isogenous to F'_{i_0} and hence to $\mathrm{SL}_d(\mathbb{R})$. Thus, it has no nontrivial representations on any d'-dimensional real vector space, for $d' < d$. This implies that the action of S_{i_0} on V' has an infinite kernel, but since S_{i_0} is simple, the action of S_{i_0} on V' must also be trivial.

So the space

$$V'' \stackrel{\mathrm{def}}{=} \operatorname{span} S_1(V') \subset \operatorname{span} S_1(V'_{\mathrm{phys}}) \subset V_1$$

is acted on trivially by S_{i_0} for any $i_0 \in \mathcal{I} \smallsetminus \{1\}$. In particular, V'' is $\underline{H}_2$-invariant. By the irreducibility of the $\underline{H}_2$-action on V_1, this means that $V_1 = V''$, and therefore S_{i_0} acts trivially on V_1. It follows that F'_{i_0} acts trivially on V_1 for each $i_0 \in \mathcal{I} \smallsetminus \{1\}$. Since S_{i_0} acts trivially on V_ℓ

for all $i_0 \in \mathcal{I}$ and all $\ell \geqslant 2$, we get that in any decomposition of $f' \in F'$, all the elements f'_i for $i \geqslant 2$ act trivially on $\mathbb{R}^n$. That is, $\mathcal{I} = \{1\}$.

Step 3: Restriction of scalars, in explicit form. Since $\mathbf{H}$ is $\mathbb{Q}$-almost simple, it is obtained by restriction of scalars from an absolutely almost simple Lie algebraic group defined over a number field $\mathbb{K}$ – see [BT65, 6.21] for a proof. We will reprove this result in our setup, obtaining more information about the embedding of H in $\mathrm{SL}_n(\mathbb{R})$.

Using Step 2 and re-indexing, let $S_1 = (\mathbf{S}_1)^\circ_{\mathbb{R}}$ be the connected component of the identity in the $\mathbb{R}$-almost simple group containing F', and set $\mathbf{G} \stackrel{\text{def}}{=} \mathbf{S}_1$, $G \stackrel{\text{def}}{=} S_1$. From Theorem 3.5, we only have two possibilities for $\mathbf{G}_{\mathbb{R}}$, and in particular we have that $\mathbf{G}$ is actually $\mathbb{C}$-almost simple. Since $\mathbf{H}$ is defined over $\mathbb{Q}$, the $\mathbb{C}$-almost simple factors of $\mathbf{H}$ are defined over a finite extension of $\mathbb{Q}$; this is well-known (see e.g. [BT65, §2.15b]) but we were unable to find a suitable reference, so we sketch the argument. The group $\mathbf{H}$ has a maximal torus which is defined over $\mathbb{Q}$ and split over a finite extension $\mathbb{L}$ of $\mathbb{Q}$ by [Bor91, §8, §18]. For each root α, the group G_α, which is the centralizer of the connected component of the identity in $\ker \alpha$, is defined over $\mathbb{L}$ (see [Bor91, Proof of Thm. 18.7]). The groups G_α generate $\mathbf{H}$ [Bor91, §14] and each $\mathbb{C}$-almost simple factor either contains G_α, or intersects it trivially. Thus, any $\mathbb{C}$-almost simple factor $\mathbf{S}$ can be described as the elements commuting with all the G_α not contained in $\mathbf{S}$. In particular, the $\mathbb{C}$-almost simple factors are defined over $\mathbb{L}$.

Replacing $\mathbb{L}$ if necessary with its Galois extension, suppose that $\mathbb{L}$ is the smallest Galois extension of $\mathbb{Q}$ such that all $\mathbb{C}$-simple factors of $\mathbf{H}$ are defined over $\mathbb{L}$. Let $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ denote the Galois group of $\mathbb{L}$, which we can think of explicitly as the group of field automorphisms of $\mathbb{L}$. If $\mathbf{V} \subset \mathbb{A}^n$ is an affine variety defined over $\mathbb{L}$ then for any $\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ there is a new affine variety, which we will denote by $\mathbf{V}^\sigma$, obtained by acting on the coefficients of the defining polynomial equations, and σ acts on the points of $\mathbb{L}^n$ by acting separately on each component. The assignments $\mathbf{V} \mapsto \mathbf{V}^\sigma$ and $\sigma : \mathbb{L} \to \mathbb{L}$ are compatible in the sense that for $x \in \mathbb{L}^n$, $x \in \mathbf{V}_{\mathbb{L}}$ if and only if $\sigma(x) \in \mathbf{V}^\sigma_{\mathbb{L}}$. Moreover, if $\mathbf{V}$ is defined over $\mathbb{L}$, then it is defined over $\mathbb{Q}$ if and only if $\mathbf{V}^\sigma = \mathbf{V}$ for every $\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{Q})$; this follows from the more general fact (see [Bor91, §AG12-§AG14]), that if $\mathbb{L}'$ is a number field then $\mathbf{V}$ is defined over $\mathbb{L}'$ if and only if for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma|_{\mathbb{L}'} = \mathrm{Id}$ we have $\mathbf{V}^\sigma = \mathbf{V}$, where $\bar{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$.

Let D denote the number of $\mathbb{C}$-almost simple factors of $\mathbf{H}$, or equivalently, the number of $\mathbb{L}$-almost simple factors of $\mathbf{H}$. The action of

$\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ permutes these factors, and this permutation action is transitive since $\mathbf{H}$ is $\mathbb{Q}$-almost simple. Thus, the subgroup

$$\Delta \stackrel{\mathrm{def}}{=} \{\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{Q}) : \mathbf{G}^\sigma = \mathbf{G}\}$$

is of index D in $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$, and the $\mathbb{C}$-almost simple factors are the (distinct) images of $\mathbf{G}$ by elements $\sigma_1, \ldots, \sigma_D \in \mathrm{Gal}(\mathbb{L}/\mathbb{Q})$, where the σ_i are coset representatives of $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})/\Delta$.

Let

$$\mathbb{K} \stackrel{\mathrm{def}}{=} \{x \in \mathbb{L} : \forall \sigma \in \Delta,\, \sigma(x) = x\}.$$

Complex conjugation $z \mapsto \bar{z}$ induces an automorphism of $\mathbb{L}$ belonging to Δ, and since $\mathbf{G}$ is defined over $\mathbb{R}$, we see that $\mathbb{K} \subset \mathbb{R}$. By the Galois correspondence, $\deg(\mathbb{K}/\mathbb{Q}) = D$ and

$$\Delta = \{\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{Q}) :\ \text{for all } x \in \mathbb{K},\ \sigma(x) = x\}.$$

We claim that $\mathbf{G}$ is defined over $\mathbb{K}$, and $\mathbf{G}$ is not defined over any proper subfield of $\mathbb{K}$. Indeed, if $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ satisfies $\sigma|_{\mathbb{K}} = \mathrm{Id}$, then $\sigma|_{\mathbb{L}} \in \Delta$ and hence $\mathbf{G}^\sigma = \mathbf{G}$. Furthermore, if $\mathbf{G}$ were defined over a proper subfield $\mathbb{K}' \subsetneq \mathbb{K}$, then its stability group Δ' would be of index $D' < D$ and therefore the collection $\{\mathbf{G}^\sigma : \sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{Q})\}$ would have cardinality D'.

We will show that $\mathbf{H}$ is isomorphic (as a $\mathbb{Q}$-algebraic group) to $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$. Moreover, we will show that the given inclusion $\mathbf{H} \hookrightarrow \mathrm{SL}_n$ is, up to a conjugation over $\mathrm{SL}_n(\mathbb{R} \cap \bar{\mathbb{Q}})$, the matrix presentation described in §2.4. By Theorem 3.5 G is, up to a conjugation in $\mathrm{SL}_n(\mathbb{R})$, either the top-left copy of $\mathrm{SL}_k(\mathbb{R})$ or the top-left copy of $\mathrm{Sp}_{2k}(\mathbb{R})$ for some $k \geqslant 2$ (and the latter can only arise when $d = 2$). In the remainder of the proof we will refer to these two cases as the SL_k *case* and the Sp_{2k} *case.*

We know that G is conjugate over $\mathrm{SL}_n(\mathbb{R})$ to the top-left copy of $\mathrm{SL}_k(\mathbb{R})$ (in the SL_k case) or $\mathrm{Sp}_{2k}(\mathbb{R})$ (in the Sp_{2k} case). Therefore there is a G-invariant subspace $V \subset \mathbb{R}^n$, of dimension k (in the SL_k case) and $2k$ (in the Sp_{2k} case) and a complementary subspace V_0 such that $\mathbb{R}^n = V \oplus V_0$, the action of G on V is irreducible, and V_0 is the subspace of G-fixed vectors in $\mathbb{R}^n$. We claim that we can recover V explicitly as

$$V = \mathrm{span}\,\{gx - x : g \in G,\, x \in \mathbb{R}^n\}\,. \tag{3.19}$$

Indeed, denote the RHS of (3.19) by W. We clearly have $W \subset V$, and for the reverse inclusion, it is enough to show that W is G-invariant. To see this, let $g_0, g \in G$ and $x \in \mathbb{R}^n$. Then

$$g_0(gx - x) = g_0 g g_0^{-1} g_0 x - g_0 x = g'x' - x',$$

where $g' \stackrel{\text{def}}{=} g_0 g g_0^{-1}$ and $x' \stackrel{\text{def}}{=} g_0 x$. This shows that the generators of W are mapped to W by any $g_0 \in G$.

From (3.19) and since $\mathbf{G}$ is defined over $\mathbb{K} \subset \mathbb{R}$, we deduce that $V = \mathbf{V}_{\mathbb{R}}$ for a subspace $\mathbf{V} \subset \mathbb{A}^n$ defined over $\mathbb{K}$. Clearly V_0 is also defined over $\mathbb{K}$. Arguing as in (3.19), but using F' in place of G and V'_{phys} in place of V, we have $V'_{\text{phys}} = \text{span}\{f'x - x : f' \in F',\ x \in \mathbb{R}^n\}$, and therefore $V'_{\text{phys}} \subset V$.

We can think of $\mathbf{V}_{\bar{\mathbb{Q}}}$ as a $\bar{\mathbb{Q}}$-linear subspace of $\bar{\mathbb{Q}}^n$, and can discuss the action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ as before. We have that $(\mathbf{G}_i)_{\bar{\mathbb{Q}}}$ preserves the decomposition $\bar{\mathbb{Q}}^n = \mathbf{V}^{\sigma_i}_{\bar{\mathbb{Q}}} \oplus (\mathbf{V}^{\sigma_i}_0)_{\bar{\mathbb{Q}}}$. We claim that

$$\bar{\mathbb{Q}}^n = \bigoplus_{i=1}^{D} \mathbf{V}^{\sigma_i}_{\bar{\mathbb{Q}}}. \tag{3.20}$$

To see this, let $\mathbf{W}$ denote the vector subspace of $\mathbb{A}^n$ spanned by $\bigcup_i \mathbf{V}^{\sigma_i}$. Since it is $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-invariant, it is defined over $\mathbb{Q}$. Since $V'_{\text{phys}} = \underline{g}_1^{-1} V_{\text{phys}}$ and $\mathbb{Z}^n = \underline{g}_1^{-1}\mathcal{L}_1$, Lemma 3.4 implies that V'_{phys} is not contained in any proper rational subspace of $\mathbb{R}^n$. This implies that $\mathbf{W}_{\mathbb{R}} = \mathbb{R}^n$ and thus $\mathbf{W} = \mathbb{A}^n$. The groups $\mathbf{G}_i$ commute, and $\mathbf{V}^{\sigma_i}$ is a $\mathbf{G}_i$-isotypic component of multiplicity one. For each pair of distinct i, j, each $g \in \mathbf{G}_i$ defines an intertwining operator for the action of $\mathbf{G}_j$, and thus by Schur's lemma (see e.g. [Kna02, Cor. 4.9]), the action of $\mathbf{G}_i$ on $\mathbf{V}^{\sigma_j}$ factors through an abelian group. Since $\mathbf{G}_i$ is simple, this means that each $\mathbf{G}_i$ acts trivially on $\mathbf{V}^{\sigma_j}$ for $j \neq i$. In particular, $\mathbf{V}^{\sigma_i} \cap \mathbf{V}^{\sigma_j} = \{0\}$ for $i \neq j$, and we have shown (3.20).

It follows from (3.20) that $\mathbb{R}^n$ is the space of $\mathbb{R}$-points of $\text{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{V})$. Write $D = r + 2s$ as in §2.4. Since $\dim \mathbf{V}^{\sigma_i} = \dim \mathbf{V}^{\sigma_j}$ for every $i \neq j$, we have that $\underline{H}_1$ is realized explicitly in $r + s$ blocks. For real embeddings σ_i, $i = 1, \ldots, r$ we have that the dimension (over $\mathbb{R}$) of $\mathbf{V}^{\sigma_i}_{\mathbb{R}}$ is k (in the SL_k case) and $2k$ (in the Sp_{2k} case), and for σ_{r+j}, $j = 1, \ldots, s$ which are non-conjugate complex embeddings of $\mathbb{K}$ we have that the dimension (over $\mathbb{R}$) of $\mathbf{V}^{\sigma_{r+j}}_{\mathbb{R}}$ is $2k$ (in the SL_k case) and $4k$ (in the Sp_{2k} case). Putting this together we get that $n = Dk$ (in the SL_k case) and $n = 2Dk$ (in the Sp_{2k} case), and the embedding of $\underline{H}_1$ in $\text{SL}_n(\mathbb{R})$ is the one given in (2.6), where $\varphi : \text{SL}_k \to \text{SL}_k$ is the identity map (in the SL_k case), and $\varphi : \text{Sp}_{2k} \to \text{SL}_{2k}$ is the natural embedding (in the Sp_{2k} case). In particular, we have proved that $\mathbf{H} = \text{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$, with the explicit form of restriction of scalars given in §2.4.

Step 4: G as a $\mathbb{K}$-group. It remains to identify the $\mathbb{K}$-isomorphism type of $\mathbf{G}$. We proved in Step 3 that $\mathbb{K} \subset \mathbb{R}$, the decomposition $\mathbb{R}^n = V \oplus V_0$ into G-invariant subspaces is defined over $\mathbb{K}$, and there is

a conjugacy over $\mathrm{SL}_n(\mathbb{K})$ sending G to the top-left corner embedding of $\mathrm{SL}_k(\mathbb{R})$ or of $\mathrm{Sp}_{2k}(\mathbb{R})$ (as defined after the statement of Theorem 3.5). We now show that as a $\mathbb{K}$-group, $\mathbf{G}$ is $\mathbb{K}$-isomorphic to either SL_k or Sp_{2k}.

Consider first the SL_k-case. Let $\mathbf{G}'$ denote the conjugate of $\mathbf{G}$ contained in $\mathrm{SL}(\mathbf{V})$. Since $\mathbf{V}$ is a $\mathbb{K}$-subspace, we can take the conjugating element in $\mathrm{SL}_n(\mathbb{K})$, that is, $\mathbf{G}' = g\mathbf{G}g^{-1}$ for some $g \in \mathrm{SL}_n(\mathbb{K})$. In particular, the groups $\mathbf{G}$ and $\mathbf{G}'$ are $\mathbb{K}$-isomorphic, and $G' = \mathbf{G}'_{\mathbb{R}}$ is $\mathbb{R}$-isomorphic to the top-left $\mathrm{SL}_k(\mathbb{R})$. Let $\mathbf{G}'' = \mathrm{SL}(V)$ considered as a $\mathbb{K}$-group. Then $\mathbf{G}''_{\mathbb{R}}$ is also $\mathbb{R}$-isomorphic to $\mathrm{SL}_k(\mathbb{R})$, and thus $\mathbf{G}'$ and $\mathbf{G}''$ have the same dimension (as algebraic varieties). Since $\mathbf{G}'_{\mathbb{K}} = g\mathbf{G}_{\mathbb{K}}g^{-1} \subset \mathbf{G}''$, there is a $\mathbb{K}$-embedding $\mathbf{G} \hookrightarrow \mathbf{G}''$, and since these groups have the same dimension and are Zariski connected, $\mathbf{G}$ and $\mathbf{G}''$ are $\mathbb{K}$-isomorphic.

Now consider the Sp_{2k} case. Again let $g \in \mathrm{SL}_n(\mathbb{K})$ be the conjugating element sending $\mathbf{G}$ to $\mathbf{G}' = g\mathbf{G}g^{-1} \subset \mathrm{SL}(V)$. We have shown that $\dim V = 2k$ is even, and $G'_{\mathbb{R}}$ is $\mathbb{R}$-isomorphic to $\mathrm{Sp}_{2k}(\mathbb{R})$, that is, there is a nondegenerate alternating bilinear form ω on $V_{\mathbb{R}}$ such that $G'_{\mathbb{R}}$ is the group of all $\mathbb{R}$-linear transformations of V preserving ω. Note that ω is $\mathbb{R}$-bilinear and takes values in $\mathbb{R}$. We claim that there is a form ω' which is defined over $\mathbb{K}$ on V (that is, takes values in $\mathbb{K}$ when evaluated on elements of $V_{\mathbb{K}}$), so that $G'_{\mathbb{R}}$ is contained in the group of $\mathbb{R}$-linear transformations of V preserving ω'. Once the claim is proved, we will have that there is a $\mathbb{K}$-embedding $\mathbf{G} \hookrightarrow \mathrm{Sp}(V, \omega')$ (the group of linear transformations of V preserving ω') which will be an isomorphism by dimension considerations as in the preceding case, thus proving that $\mathbf{G}$ is $\mathbb{K}$-isomorphic to $\mathrm{Sp}(V, \omega') \cong \mathrm{Sp}_{2k}$.

To prove the claim, consider the collection $\bigwedge^2(V^*)$ of alternating bilinear forms on V. This collection is a linear space, and the nondegenerate forms form a Zariski open subset (since nondegeneracy is equivalent to the non-vanishing of the determinant of the Gram matrix of the form). Since $\mathbf{G}'$ is a $\mathbb{K}$-group, the subspace $\bigwedge^2(V^*)^{\mathbf{G}'}$ of $\mathbf{G}'$-invariant forms is a $\mathbb{K}$-subspace, which is nonempty since its collection of $\mathbb{R}$-points contains ω. Since $\mathbb{K}$-points are Zariski dense in $\mathbb{K}$-subspaces, we find that there are nondegenerate symplectic $\mathbb{K}$-forms which are $\mathbf{G}'$-invariant.

Finally, the proof of Theorem 3.5 shows that in the symplectic case, the space $V_{\mathrm{phys}} \cong \mathbb{R}^2$ is spanned by two vectors $\vec{x}, \vec{y}$ satisfying $\omega(\vec{x}, \vec{y}) = 1$; that is, V_{phys} is a symplectic subspace for ω. Write ω as a linear combination of forms ω' which are defined over $\mathbb{K}$ and $\mathbf{G}'$-invariant.

Since $\omega(\vec{x},\vec{y}) \neq 0$, there has to be some $\omega' \in \left(\bigwedge^2 (V^*)^{\mathbf{G}'}\right)_{\mathbb{K}}$ for which $\omega'(\vec{x},\vec{y}) \neq 0$. □

Remark 3.6. *In the symplectic case, Step 4 also shows that there is a symplectic form on the entire space $\mathbb{R}^n$ that is preserved by the entire group $\underline{H}_1$. Indeed, the form ω', which is symplectic and defined over $\mathbb{K}$, can be 'pushed' using the field embeddings σ_i to induce symplectic forms on the spaces $\mathbf{V}^{\sigma_i}$. We will not be using this fact and we leave the details to the reader.*

4. An intrinsic description of the measures arising via Ψ_*

The following result shows that all RMS measures arise via the map Ψ_*. For a given constant $c > 0$, we denote by $\rho_c : \mathscr{C}(\mathbb{R}^d) \to \mathscr{C}(\mathbb{R}^d)$ the map induced by the dilation by c, that is, $\rho_c(F) = \{cx : x \in F\}$.

Theorem 4.1. *Let F be as in* (3.1) *and embedded in G via the top-left corner embedding. For any ergodic F-invariant Borel probability measure μ on $\mathscr{C}(\mathbb{R}^d)$ which assigns full measure to irreducible cut-and-project sets, there is an irreducible cut-and-project construction with $\mathbb{R}^n = V_{\text{phys}} \oplus V_{\text{int}}, \pi_{\text{phys}}, \pi_{\text{int}}, W$ and with Ψ as in* (2.1)*, a constant $c > 0$, and an F-invariant ergodic homogeneous measure $\bar{\mu}$ on $\mathscr{Y}_n$, such that $\mu = \rho_{c*}\Psi_*\bar{\mu}$. For μ-a.e. Λ we have*

$$c = \left(\frac{\operatorname{vol}(W)}{D(\Lambda)}\right)^{\frac{1}{n}}, \tag{4.1}$$

where $D(\Lambda)$ is the density of Λ as defined in (1.10)*.*

We will split the proof into the linear and affine case.

Proof of Theorem 4.1, affine case. Suppose μ is $\mathrm{ASL}_d(\mathbb{R})$-invariant and $F = \mathrm{ASL}_d(\mathbb{R})$, and let $\{g_t\}$ be a one-parameter diagonalizable subgroup of $\mathrm{SL}_d(\mathbb{R}) \subset F$. By the Mautner phenomenon (see [EW11]), the action of $\{g_t\}$ on $\left(\mathscr{C}(\mathbb{R}^d), \mu\right)$ is ergodic. Thus, by the Birkhoff pointwise ergodic theorem, there is a subset $X_0 \subset \mathscr{C}(\mathbb{R}^d)$ of full μ-measure such that for all $\Lambda \in X_0$ we have

$$\frac{1}{T}\int_0^T (g_t)_*\delta_\Lambda \, dt \to_{T\to\infty} \mu.$$

Since the function $\Lambda \mapsto D(\Lambda)$ is measurable and invariant, we can further assume that the value of $D(\Lambda)$ is the same for each $\Lambda \in X_0$.

Let U, Ω, m_U be as in Theorem 2.5. Then by Fubini's theorem, and since μ is U-invariant, we have

$$\begin{aligned} 1 = \mu(X_0) &= \frac{1}{m_U(\Omega)} \int_\Omega \mu(u^{-1} X_0)\, dm_U(u) \\ &= \int \left[\frac{1}{m_U(\Omega)} \int_\Omega \mathbf{1}_{X_0}(u\Lambda)\, dm_U(u) \right] d\mu(\Lambda), \end{aligned}$$

where $\mathbf{1}_{X_0}$ is the indicator function of X_0. Thus the inner integral on the RHS is equal to one on a subset of full measure; i.e., there is $X_1 \subset \mathscr{C}(\mathbb{R}^d)$ of full measure such that for every $\Lambda \in X_1$ we have $u\Lambda \in X_0$ for m_U-a.e. $u \in \Omega$. This implies that for $\Lambda \in X_1$ we have

$$\frac{1}{T} \int_0^T \int_\Omega (g_t u)_* \delta_\Lambda \, dm_U(u)\, dt \to_{T\to\infty} \mu. \tag{4.2}$$

Let $\Lambda \in X_1$ be an irreducible cut-and-project set, that is, $\Lambda = \Psi(\mathcal{L})$, where $\mathcal{L}$ is a grid and Ψ is defined using data $d, m, n, V_{\text{phys}}, V_{\text{int}}, W$ satisfying **(D), (I), (Reg)**. We can simultaneously rescale $\mathcal{L}$, the window W, and the metric on V_{phys} by the same positive scalar, in order to assume that $\mathcal{L} \in \mathscr{Y}_n$. Namely, set $c_1 \stackrel{\text{def}}{=} \operatorname{covol}(\mathcal{L})^{-\frac{1}{n}}$, so that $\mathcal{L}_1 \stackrel{\text{def}}{=} c_1 \mathcal{L} \in \mathscr{Y}_n$ satisfies

$$\Lambda = \Lambda(\mathcal{L}, W) = \frac{1}{c_1} \Lambda(\mathcal{L}_1, c_1 W).$$

Now solving for $c = \frac{1}{c_1}$ in (1.10) gives (4.1).

Define a sequence of measures η_T on $\mathscr{Y}_n$ by

$$\eta_T \stackrel{\text{def}}{=} \frac{1}{T} \int_0^T \int_\Omega (g_t u)_* \delta_{\mathcal{L}} \, dm_U(u)\, dt.$$

That is, the measures η_T are defined by the same averaging as in (4.2), but for the action on $\mathscr{Y}_n$ rather than on $\mathscr{C}(\mathbb{R}^d)$. By (3.2), their push-forward under Ψ are the measures appearing on the LHS of (4.2). By Theorem 2.5 we have $\eta_T \to_{T\to\infty} \bar{\mu}$ for some homogeneous measure $\bar{\mu}$ on $\mathscr{Y}_n$. By assertion (i) of Theorem 3.1, $\bar{\mu}$ is invariant under translation by any element of $\mathbb{R}^n$, and in particular any element of V_{int}. Hence, by Corollary 2.2, $\bar{\mu}$ is a continuity point of the map Ψ_*. By (4.2), $\Psi_* \eta_T \to \mu$ and by continuity, $\mu = \Psi_* \bar{\mu}$. □

For the case in which μ is $\mathrm{SL}_d(\mathbb{R})$-invariant but not $\mathrm{ASL}_d(\mathbb{R})$-invariant, we will need the following result:

Lemma 4.2. *With the notation of Theorem 3.1, let*

$$H_1' \stackrel{\text{def}}{=} g_1 H' g_1^{-1}$$

(so that $H_1' = H$ in the linear case and H_1' is a Levi subgroup of H in the affine case), and let v be a nonzero vector in $\underline{\mathcal{L}}_1$. Then the orbit of v under the linear action of H_1' is an open dense subset of $\mathbb{R}^n$.

Proof. Write $v = g_1 u$ for $u \in \mathbb{Z}^n \smallsetminus \{0\}$. It suffices to show that the orbit $H'u$ is open and dense in $\mathbb{R}^n$. The linear action of H' on $\mathbb{R}^n$ factors through the group $\underline{H}_1$ so we may replace H' with $\underline{H}_1$.

The action of $\mathrm{SL}_k(\mathbb{R})$ on $\mathbb{R}^k$ has the property that the orbit of every nonzero vector is dense. The same is true for the action of $\mathrm{Sp}_{2k}(\mathbb{R})$ on $\mathbb{R}^{2k}$ (since any vector can be completed to a symplectic basis), for the action of $\mathrm{SL}_k(\mathbb{C})$ on $\mathbb{C}^k \simeq \mathbb{R}^{2k}$, and for the action of $\mathrm{Sp}_{2k}(\mathbb{C})$ on $\mathbb{C}^{2k} \simeq \mathbb{R}^{4k}$. By Step 3 of the proof of Lemma 3.2, $\underline{H}_1$ is the product of groups G_i, and we have a direct product $\mathbb{R}^n = \oplus_{i=1}^{r+s} V_i$, with the following properties:

- For $i = 1, \dots, r$ we have a real field embedding σ_i, and $V_i = \sigma(\mathbf{V})_{\mathbb{R}}$; for $i = r+1, \dots, r+s$ we have representatives σ_i of pairs of complex embeddings, and $V_i = \sigma(\mathbf{V})_{\mathbb{C}}$.
- For $i = 1, \dots, r$ we have $G_i = \sigma_i(\mathbf{G})_{\mathbb{R}}$ and for $i = r+1, \dots, s$ we have $G_i = \sigma_i(\mathbf{G})_{\mathbb{C}}$.
- In the SL_k-case (resp., the Sp_{2k} case), V_1 is isomorphic to $\mathbb{R}^k$ (resp., $\mathbb{R}^{2k}$), with the standard action.
- The action of G_i on V_i is the obtained from the action of G_1 on V_1 by applying σ_i. In particular, for real embeddings it is isomorphic to the standard action of $\mathrm{SL}_k(\mathbb{R})$ or $\mathrm{Sp}_{2k}(\mathbb{R})$, and for complex embeddings it is isomorphic to the standard action of $\mathrm{SL}_k(\mathbb{C})$ or $\mathrm{Sp}_{2k}(\mathbb{C})$.

Thus, it is enough to show that for any $u \in \mathbb{Z}^n \smallsetminus \{0\}$, and for any field embedding σ_j of $\mathbb{K}$, the projection u_j of u to the factor corresponding to σ_j is nonzero.

Suppose to the contrary that $u_j = 0$ for some j, and let $a \in \mathrm{SL}_n(\mathbb{R})$ be a diagonalizable matrix, such that a acts on the ℓ-th factor of $\mathbb{R}^n$ corresponding to the field embedding σ_ℓ as a scalar matrix $\lambda_\ell \cdot \mathrm{Id}$, where the λ_ℓ are positive real scalars satisfying

$$\lambda_j > 1, \quad \lambda_i < 1 \text{ for } i \neq j, \quad \text{and} \quad \prod_\ell \lambda_\ell = 1.$$

That is, a belongs to the centralizer of H' in $\mathrm{SL}_n(\mathbb{R})$, and $a^i u \to_{i\to\infty} 0$. This implies by Mahler's compactness criterion that the sequence $a^i \mathbb{Z}^n$ is divergent (eventually escapes every compact subset of $\mathscr{X}_n$). In particular, the orbit of the identity coset $\mathrm{SL}_n(\mathbb{Z})$ under the centralizer of H' is not compact. From this, via the implication $3 \implies 2$ in [EMS97,

Lemma 5.1], we see that H' is contained in a proper $\mathbb{Q}$-parabolic subgroup of $\mathrm{SL}_n(\mathbb{R})$, and hence (see e.g. [Bor19, §11.14]) leaves invariant a proper $\mathbb{Q}$-subspace of $\mathbb{R}^n$. This is a contradiction to **(irred)**. □

Proof of Theorem 4.1, linear case. We repeat the argument given for the affine case. The only complication is in establishing $\eta_T \to \bar{\mu}$ implies $\Psi_*\eta_T \to \Psi_*\bar{\mu}$, as in the last paragraph of the proof. In the proof for the affine case, this was obtained from Corollary 2.2, which shows that $\bar{\mu}$ is a continuity point for the map Ψ_*, using the fact that $\bar{\mu}$ is invariant under translations by elements of V_{int}. In the linear situation $\bar{\mu}$ no longer has this continuity property.

To overcome this difficulty we argue as follows. We note that if

$$\bar{\mu}\left(\{\mathcal{L} \in \mathscr{Y}_n : \pi_{\mathrm{int}}(\mathcal{L}) \cap \partial W \neq \varnothing\}\right) = 0 \tag{4.3}$$

then Corollary 2.2 can still be applied to show that $\bar{\mu}$ is a continuity point for Ψ_*. Thus, we can assume from now on that (4.3) fails. Since $\operatorname{supp}\bar{\mu} = H\mathcal{L}_1$, this implies that the Haar measure m_H of H satisfies

$$m_H\left(\{h \in H : \pi_{\mathrm{int}}(h\mathcal{L}_1) \cap \partial W \neq \varnothing\}\right) > 0. \tag{4.4}$$

Since $\mathcal{L}_1$ is countable, there must be some $v \in \mathcal{L}_1$ such that

$$m_H\left(\{h \in H : \pi_{\mathrm{int}}(hv) \in \partial W\}\right) > 0. \tag{4.5}$$

By Lemma 4.2, there is a unique element $v_1 \in \mathbb{R}^n$ which is fixed by H (namely $v_1 = g_1(0)$), and for any $v \neq v_1$, the orbit of v under the action of H is an open dense subset of $\mathbb{R}^n$. In particular, if $v \neq v_1$ then the map $h \mapsto hv$ sends m_H to an absolutely continuous measure on $\mathbb{R}^n$, and for such v (4.5) cannot hold by **(Reg)**.

Thus, we must have $v = v_1$. In this case $hv = v$ and $\pi_{\mathrm{int}}(hv) \in \partial W$ for all $h \in H$. By examining the proof of Proposition 2.1, we see that the map

$$H \to \mathscr{C}(\mathbb{R}^d), \qquad h \mapsto \Psi(h\mathcal{L}_1)$$

is still continuous at any point outside a set of zero measure; namely, the set of h for which there is $v \neq v_1$ such that $\pi_{\mathrm{int}}(hv) \in \partial W$. Furthermore, the measure $\bar{\mu}$ and the measures η_T are all supported on the orbit $H\mathcal{L}_1$. Thus, we can apply the argument proving Corollary 2.2, to see that the restriction of Ψ_* to measures supported on the orbit $H\mathcal{L}_1$ is continuous. This is sufficient to conclude that $\Psi_*\eta_T \to \Psi_*\bar{\mu}$ as $T \to \infty$. □

Remark 4.3. *Theorem 4.1 remains valid when one considers other topologies (and potentially, Borel structures) on $\mathscr{C}(\mathbb{R}^d)$, as is done for example in* [Vee98, MS19]. *Thus, in the terminology of* [Vee98]*, the*

theorem is valid if $\bar{\mu}$ is a Siegel measure giving full measure to cut-and-project sets. The proof is essentially identical as Proposition 2.1 and Corollary 2.2 are still valid in this setting.

5. Some consequences of the classification

With Theorem 3.1 in hand it is easy to obtain explicit descriptions of RMS measures in low dimensions. Recall that we refer to the unique $\mathrm{ASL}_n(\mathbb{R})$-invariant probability measure on $\mathscr{Y}_n$ and the unique $\mathrm{SL}_n(\mathbb{R})$-invariant probability measure on $\mathscr{X}_n$ as the *Haar-Siegel measures*.

Corollary 5.1. *With the notation above, suppose that* $\dim V_{\mathrm{phys}} > \dim V_{\mathrm{int}}$. *Then the only affine RMS measure is the one for which $\bar{\mu}$ is the Haar-Siegel measure on $\mathscr{Y}_n$, and the only linear RMS measure is the one for which $\bar{\mu}$ is the Haar-Siegel measure on $\mathscr{X}_n$.*

This reproves a result stated without proof in [MS14, Prop. 2.1].

Proof. In our classification result, there is $k \in \{d, \dots, n\}$ and $D = \deg(\mathbb{K}/\mathbb{Q})$ such that $n = Dk$ in the SL_k-case and $n = 2Dk$ in the Sp_{2k}-case. Since

$$k \geqslant d = \dim V_{\mathrm{phys}} > \dim V_{\mathrm{int}} = n - d \geqslant n - k, \tag{5.1}$$

we obtain $k > (D-1)k$ in the SL_k-case and $k > (2D-1)k$ in the Sp_{2k}-case. This is only possible if $D = 1$ and we are in the SL_k-case. That is, the only possible case is $H' = \mathrm{SL}_n(\mathbb{R})$, and this gives the required result. □

We extend Corollary 5.1 to the case of equality:

Corollary 5.2. *With the above notation, suppose that μ is not one of the Haar-Siegel measures mentioned in Corollary 5.1, and suppose* $\dim V_{\mathrm{phys}} = \dim V_{\mathrm{int}}$. *Then either $d = 2$ and $H' = \mathrm{Sp}_4(\mathbb{R})$, or $d \geqslant 2$ and there is a real quadratic field $\mathbb{K}$ such that H' is (the group of real points of)* $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathrm{SL}_d)$.

Proof. If the strict inequality in (5.1) becomes non-strict, it is also possible that $H' = \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathrm{SL}_d)$ and $\mathbb{K}$ is a real quadratic field, or $\mathbb{K} = \mathbb{Q}$, $d = 2$ and $H' = \mathrm{Sp}_4(\mathbb{R})$. □

As shown by Pleasants [Ple03], an example of a cut-and-project set associated with a real quadratic field as in Corollary 5.2 is the vertex set of an Ammann-Beenker tiling, where in this case the associated field is $\mathbb{K} = \mathbb{Q}(\sqrt{2})$. Similarly, as discussed in [MS14, §2.2], the Penrose tiling vertex set can be described as a finite union of cut-and-project sets associated with the real quadratic field $\mathbb{Q}(\sqrt{5})$.

We record the following trivial but useful fact.

Proposition 5.3. *For any affine RMS measure μ, one can assume the window W contains the origin in its interior.*

Proof. Let W be the window in the construction of the RMS measure μ. By **(Reg)**, let $x_0 \in V_{\mathrm{int}}$ be a point in the interior of W. By assertion (i) of Theorem 3.1, the measure $\bar{\mu}$ is invariant under translations by the full group $\mathbb{R}^n$ of translations, and in particular by the translation by x_0. So we can replace any $\mathcal{L} \in \mathscr{Y}_n$ by $\mathcal{L} - x_0$ without affecting the measure $\bar{\mu}$. But clearly for $x_0 \in V_{\mathrm{int}}$ we have

$$\Lambda(\mathcal{L}, W) = \Lambda(\mathcal{L} - x_0, W - x_0).$$

So the measure μ can be obtained from $\bar{\mu}$ by using the window $W - x_0$, which contains the origin in its interior. □

Recall that we have an inclusion

$$\iota : \mathrm{SL}_n(\mathbb{R}) \to \mathrm{ASL}_n(\mathbb{R}),\ \iota(g) = (g, \mathbf{0}_n),$$

i.e., $\iota(\mathrm{SL}_n(\mathbb{R}))$ is the stabilizer of the origin in the affine action of $\mathrm{ASL}_n(\mathbb{R})$ on $\mathbb{R}^n$. This induces an inclusion $\bar{\iota} : \mathscr{X}_n \to \mathscr{Y}_n$, and these maps form right inverses to the maps appearing in (3.3):

$$\pi \circ \iota = \mathrm{Id}_{\mathrm{SL}_n(\mathbb{R})},\quad \underline{\pi} \circ \bar{\iota} = \mathrm{Id}_{\mathscr{X}_n}.$$

In the linear case, we can use these maps to understand the measures $\bar{\mu}$ on $\mathscr{Y}_n$ appearing in Theorem 3.1 in terms of measures on $\mathscr{X}_n$. Namely we have:

Proposition 5.4. *Let $F = \mathrm{SL}_d(\mathbb{R})$, embedded in $\mathrm{ASL}_n(\mathbb{R})$ via (2.3), and let $\bar{\mu}$ be a measure on $\mathscr{Y}_n$ projecting to a linear RMS measure on $\mathscr{C}(\mathbb{R}^d)$; i.e., $\bar{\mu}$ is F-invariant and ergodic, and not invariant under $\mathrm{ASL}_d(\mathbb{R})$. Let H, $\mathcal{L}_1$ be as in Theorem 3.1. Let $\underline{F} \overset{\mathrm{def}}{=} \pi(F)$. Then one of the following holds:*

(i) *We have $\operatorname{supp}\bar{\mu} \subset \bar{\iota}(\mathscr{X}_n)$ and $\underline{\pi}|_{\operatorname{supp}\bar{\mu}}$ is a homeomorphism which maps $\bar{\mu}$ to an $\underline{F}$-invariant ergodic measure on $\mathscr{X}_n$. In this case H is contained in $\underline{G} \overset{\mathrm{def}}{=} \iota(\mathrm{SL}_n(\mathbb{R}))$, i.e., $H = \iota \circ \pi(H)$.*

(ii) *We have $\bar{\mu}(\bar{\iota}(\mathscr{X}_n)) = 0$, and there are $D_1, D_2 \in \mathbb{N}$ such that $\underline{\pi}|_{\operatorname{supp}\bar{\mu}}$ is a closed map of degree D_1, and for every $\mathcal{L} \in \operatorname{supp}\bar{\mu}$ there is a lattice $\mathcal{L}' \in \mathscr{X}_n$, depending only on $\underline{\pi}(\mathcal{L})$, such that $\mathcal{L}'$ contains $\underline{\pi}(\mathcal{L})$ with index $[\mathcal{L}' : \underline{\pi}(\mathcal{L})] = D_2$, and such that $\mathcal{L}$ is a translate of $\underline{\pi}(\mathcal{L})$ by an element of $\mathcal{L}'$.*

Proof. The set of lattices $\bar{\iota}(\mathscr{X}_n) \subset \mathscr{Y}_n$ is clearly F-invariant, so by ergodicity is either null or conull for the measure $\bar{\mu}$. If it is conull then $\bar{\iota}(\mathscr{X}_n)$ is a closed subset of full measure, i.e., $\operatorname{supp}\bar{\mu} \subset \bar{\iota}(\mathscr{X}_n)$. Since

$\bar{\iota}$ is a right inverse for $\underline{\pi}$ we have that $\underline{\pi}|_{\operatorname{supp}\bar{\mu}}$ is a homeomorphism. Furthermore, since we have a containment of orbits

$$H\mathcal{L}_1 = \operatorname{supp}\bar{\mu} \subset \bar{\iota}(\mathscr{X}_n) = \underline{G}\mathbb{Z}^n = \underline{G}\mathcal{L}_1,$$

and the groups $H, \underline{G}$ are connected analytic submanifolds of G, we have a containment of groups $H \subset \underline{G}$. This proves (i).

Now suppose $\bar{\mu}\left(\bar{\iota}(\mathscr{X}_n)\right) = 0$, and let H, $\mathcal{L}_1$ be as in the statement of Theorem 3.1, so that $\operatorname{supp}\bar{\mu} = H\mathcal{L}_1$. Let $\mathbb{T}^n \stackrel{\text{def}}{=} \underline{\pi}^{-1}(\underline{\pi}(\mathcal{L}_1))$ be the orbit of $\mathcal{L}_1$ under translations. Since we are in the linear case, H is transverse to the group of translations $\mathbb{R}^n$ which moves along the fibers of $\underline{\pi}$, and since $H\mathcal{L}_1$ does not accumulate on itself and $\mathbb{T}^n$ is compact, the intersection $\Omega \stackrel{\text{def}}{=} \mathbb{T}^n \cap H\mathcal{L}_1$ is a finite set. Then by (3.4), for any $\mathcal{L} = h\mathcal{L}_1 \in \operatorname{supp}\bar{\mu}$ we have

$$h\Omega = \underline{\pi}^{-1}(\underline{\pi}(\mathcal{L})) \cap H\mathcal{L}_1,$$

and thus the map $\underline{\pi}|_{\operatorname{supp}\bar{\mu}}$ has fibers of a constant cardinality $D_1 \stackrel{\text{def}}{=} |\Omega|$.

Now denote

$$\Gamma_1 \stackrel{\text{def}}{=} \{h \in H : h\mathcal{L}_1 = \mathcal{L}_1\}, \qquad \Gamma_2 \stackrel{\text{def}}{=} \{h \in H : h\Omega = \Omega\}.$$

By equivariance we have $\Gamma_1 \subset \Gamma_2$ and the index of the inclusion is D_1 since Γ_2 acts transitively on Ω. The bijection

$$\mathbb{R}^n/\underline{\pi}(\mathcal{L}_1) \to \mathbb{T}^n,\ x \bmod \underline{\pi}(\mathcal{L}_1) \mapsto x + \mathcal{L}_1$$

endows $\mathbb{T}^n$ with the structure of a real torus, whose identity element corresponds to $\mathcal{L}_1$. In these coordinates Γ_2 acts by affine maps of $\mathbb{T}^n$ but Γ_1 acts by toral automorphisms, since it preserves $\mathcal{L}_1$. Thus, Ω is a finite invariant set for the action of an irreducible lattice in a group acting $\mathcal{L}_1$-irreducibly on $\mathbb{R}^n$, and thus by [GS04] consists of torsion points in $\mathbb{T}^n$. That is, there is $q \in \mathbb{N}$ so that they belong to the image of $\frac{1}{q} \cdot \mathcal{L}_1$ in $\mathbb{T}^n$. By equivariance the same statement holds, with the same q, for $h\mathcal{L}_1$ in place of $\mathcal{L}_1$. Thus, the second assertion holds if we let $\mathcal{L}' = \frac{1}{q} \cdot \mathcal{L},\ D_2 = q^n$. □

Example 5.5. *It is possible that in case (ii) we have* $\operatorname{supp}\bar{\mu} \cap \bar{\iota}(\mathscr{X}_n) \neq \varnothing$. *For example, take* $n = 3, d = 2$, *let* f *be the translation* $f(x) \stackrel{\text{def}}{=} x + \frac{1}{2}\mathbf{e}_3$, *where* $\mathbf{e}_3$ *is the unit vector in the third axis. Let* H *be the conjugate of* $\mathrm{SL}_3(\mathbb{R})$ *by* f *and let* $\mathcal{L}_1 = f(\mathbb{Z}^3)$. *Then* $F \subset H$ *and* $H\mathcal{L}_1$ *is a closed homogeneous orbit. Since* $\mathcal{L}_1 \notin \bar{\iota}(\mathscr{X}_3)$, *the corresponding homogeneous measure does not satisfy (i). But one can check that the lattice* $\operatorname{span}_{\mathbb{Z}}(\mathbf{e}_1, 2\mathbf{e}_2, \frac{1}{2}\mathbf{e}_3)$ *is contained in* $H\mathcal{L}_1$, *that is,* $H\mathcal{L}_1 \cap \bar{\iota}(\mathscr{X}_3) \neq \varnothing$.

6. Integrability of the Siegel-Veech transform

In this section we prove Theorem 1.2. Let μ be an RMS measure and let $\bar{\mu}$, H_1, $\mathcal{L}_1 = g_1\mathbb{Z}^n$ be as in Theorem 3.1. Recall that the function $\hat{f}$ defined in (1.3) is defined on $\operatorname{supp}\mu$. Also let $\pi : \mathrm{ASL}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$, $\underline{\pi} : \mathscr{Y}_n \to \mathscr{X}_n$, $\underline{H}_1 = \pi(H_1)$ be as in §3.2. Let $\Gamma_1 \stackrel{\text{def}}{=} H_1 \cap \mathrm{ASL}_n(\mathbb{Z})$, $\underline{\Gamma}_1 \stackrel{\text{def}}{=} \underline{H}_1 \cap \mathrm{SL}_n(\mathbb{Z})$ be the $\mathbb{Z}$-points of H_1 and $\underline{H}_1$, and let $\mathbf{X}_1 \stackrel{\text{def}}{=} H_1/\Gamma_1$, $\underline{\mathbf{X}}_1 \stackrel{\text{def}}{=} \underline{H}_1/\underline{\Gamma}_1$. We will use the results of §3.2 to lift $\hat{f}$ to a function on $\mathbf{X}_1$, and show that it is dominated by the pullback of a function on $\underline{\mathbf{X}}_1$. For the arithmetic homogeneous space $\underline{\mathbf{X}}_1$ we will develop the analogue of the Siegel summation formula and its properties. Specifically, we will describe a Siegel set $\mathfrak{S} \subset \underline{H}_1$, which is an easily described subset projecting onto $\underline{\mathbf{X}}_1$, and estimate the rate of decay of the Haar measure of the subset of $\mathfrak{S}$ covering the 'thin part' of $\underline{\mathbf{X}}_1$.

6.1. Reduction theory for some arithmetic homogeneous spaces.

We begin our discussion of Siegel sets. For more details on the terminology and statements given below, see [Bor19, Chaps. 11-13].

Let $\mathbf{H}$ be a semisimple $\mathbb{Q}$-algebraic group, let $\mathbf{P}$ be a minimal $\mathbb{Q}$-parabolic subgroup, and let $H = \mathbf{H}_{\mathbb{R}}$. Then $P = \mathbf{P}_{\mathbb{R}}$ has a decomposition $P = MAN$ (almost direct product), where:

- A is the group of $\mathbb{R}$-points of a maximal $\mathbb{Q}$-split torus $\mathbf{A}$ of $\mathbf{P}$;
- N is the unipotent radical of P;
- and M is the connected component of the identity in the group of $\mathbb{R}$-points of $\mathbf{M}$, a maximal $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of the centralizer of $\mathbf{A}$ in $\mathbf{P}$.

Furthermore, $H = KP$ for a maximal compact subgroup K of H.

As in §2.4, we think of $\mathbf{H}$ as concretely embedded in $\mathrm{SL}_{n_0}(\mathbb{R})$ for some $n_0 \in \mathbb{N}$, where we take this embedding to be defined over $\mathbb{Q}$ for the standard $\mathbb{Q}$-structure on $\mathrm{SL}_{n_0}(\mathbb{R})$. Let $\mathfrak{a}$ and $\mathfrak{n}$ denote respectively the Lie algebras of A and N, let $\Phi \subset \mathfrak{a}^*$ denote the $\mathbb{Q}$-roots of H and choose an order on Φ for which $\mathfrak{n}$ is generated by the positive root-spaces.

Every element of H can be written in the form

$$h = kman \quad (k \in K,\, m \in M,\, a \in A,\, n \in N), \tag{6.1}$$

and one can express the Haar volume element dh of H in these coordinates in the form

$$dh = dk\, dm\, dn\, \rho_0(a) da, \tag{6.2}$$

where dk, dm, dn, da denote respectively the volume elements corresponding to the Haar measures on the (unimodular) groups K, M, N, A,

and

$$\rho_0(a) = |\det(\mathrm{Ad}(a)|_{\mathfrak{n}})| = \exp(2\rho(X)), \tag{6.3}$$

where $a = \exp(X)$ and ρ is the character on $\mathfrak{a}$ given by $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} c_\alpha \alpha$, for Φ^+ the positive roots in Φ, and $c_\alpha = \dim \mathfrak{h}_\alpha$. We note that this formula for Haar measure is well-defined despite the fact that the decomposition (6.1) is not unique.

Let $\Delta \subset \Phi^+$ be a basis of simple $\mathbb{Q}$-roots. For fixed $t \in \mathbb{R}$, let

$$A_t \stackrel{\text{def}}{=} \{\exp(X) : X \in \mathfrak{a},\ \forall \chi \in \Delta,\ \chi(X) \leqslant t\} \tag{6.4}$$

and for a compact neighborhood of the identity $\omega \subset MN$, let

$$\mathfrak{S}_{t,\omega} \stackrel{\text{def}}{=} KA_t\omega.$$

These sets are referred to as *Siegel sets*, and by a fundamental result, a finite union of translates of Siegel sets contains a fundamental domain for the action of an arithmetic group; that is, there is a finite subset $F_0 \subset \mathbf{H}_{\mathbb{Q}}$ and there are t, ω such that $\mathfrak{S}_{t,\omega}F_0$ projects onto H/Γ_H, where $\Gamma_H = \mathbf{H}_{\mathbb{Z}}$; equivalently $H = \mathfrak{S}_{t,\omega}F_0\Gamma_H$. The sets $\mathfrak{S}_{t,\omega}F_0$ do not represent Γ_H-cosets uniquely, in fact the map $\mathfrak{S}_{t,\omega}F_0 \to H/\Gamma_H$ is far from being injective. Nevertheless the formulas (6.1) and (6.3) make it possible to make explicit computations with the restriction of Haar measure to $\mathfrak{S}_{t,\omega}F_0$, and in particular to show that Siegel sets have finite Haar measure.

An important observation is that the set $\bigcup_{a\in A_t} a\omega a^{-1}$ is bounded, because of the definition of M and N and because of the compactness of ω. This means that a Siegel set is contained in a set of the form $\omega' A_t$, where ω' is a bounded subset of H.

6.2. **The integrability exponent of an auxiliary function on $\mathscr{X}_n$.** We will specialize the discussion in §6.1 to the specific choices of H/Γ_H that arise in our application. Let H be as above, let $\mathfrak{S}_{t,\omega}$ be a Siegel set and let $F_0 \subset \mathbf{H}_{\mathbb{Q}}$ be a finite subset for which $\mathfrak{S}_{t,\omega}F_0\Gamma_H = H$. Given functions φ_1, φ_2 defined on H, we will write $\varphi_1 \ll \varphi_2$ if there is a constant c such that for all $x \in \mathfrak{S}_{t,\omega}F_0$ we have $\varphi_1(x) \leqslant c\varphi_2(x)$. The constant c is called the *implicit constant*. We will also write $\varphi_1 \asymp \varphi_2$ if $\varphi_1 \ll \varphi_2$ and $\varphi_2 \ll \varphi_1$. In general these relations on functions depend on the choice of Siegel set (i.e., the choice of t) and the choice of the finite set F_0, but in the case we will be interested in, when φ_1, φ_2 are actually lifts of function defined on H/Γ_H, this notion does not depend on choices.

We now define an auxiliary function, and compute its integrability exponent. Given a nonzero discrete subgroup $\mathcal{L}' \subset \mathbb{R}^n$ (not necessarily

of rank n), we denote by $\mathrm{covol}(\mathcal{L}')$ the volume of a fundamental domain for $\mathcal{L}'$ in $\mathrm{span}_{\mathbb{R}}(\mathcal{L}')$ (with respect to Lebesgue measure on $\mathrm{span}_{\mathbb{R}}(\mathcal{L}')$, normalized using the standard inner product on $\mathbb{R}^n$). For $g \in \mathrm{SL}_n(\mathbb{R})$ and $\mathcal{L} = g\mathbb{Z}^n \in \mathscr{X}_n$, define

$$\hat{\alpha}(g) = \alpha(\mathcal{L}) \stackrel{\mathrm{def}}{=} \max\left\{\mathrm{covol}(\mathcal{L}')^{-1} : \mathcal{L}' \subset \mathcal{L},\, \mathcal{L}' \neq \{0\}\right\}. \tag{6.5}$$

Recall that $\underline{\mathbf{X}}_1 = \underline{H}_1/\underline{\Gamma}_1$ is embedded in $\mathscr{X}_n$ as the closed orbit $\underline{\mathbf{X}}_1 = \underline{H}_1\mathbb{Z}^n$, and so we can consider the restrictions of α and $\hat{\alpha}$ to $\underline{\mathbf{X}}_1$ and to $\underline{H}_1$.

Proposition 6.1. *In the two cases* $\mathbf{G} \cong \mathrm{SL}_k$, $\mathbf{G} \cong \mathrm{Sp}_{2k}$, *let* $p < r_0 \stackrel{\mathrm{def}}{=} \mathrm{rank}_{\mathbb{K}}(\mathbf{G}) + 1$ *(see* (1.6)*). Then*

$$\alpha \in L^p\left(\underline{\mu}\right) \smallsetminus L^{r_0}\left(\underline{\mu}\right), \tag{6.6}$$

where $\underline{\mu}$ *is the* $\underline{H}_1$*-invariant probability measure on* $\underline{\mathbf{X}}_1$.

Proof. Let $\lambda_i = \lambda_i(\mathcal{L})$, $i = 1, \ldots, n$ be the successive minima of a lattice $\mathcal{L}$, and let $i_0 = i_0(\mathcal{L})$ be the index for which $\lambda_{i_0}(\mathcal{L}) \leqslant 1 < \lambda_{i_0+1}(\mathcal{L})$. Then it is easy to see using Minkowski's second theorem (see e.g. [Cas97, §VIII.2]) that (as functions on $\mathscr{X}_n$),

$$\alpha(\mathcal{L}) \asymp \left(\lambda_1 \cdots \lambda_{i_0(\mathcal{L})}(\mathcal{L})\right)^{-1}. \tag{6.7}$$

As a consequence, for any $C \subset \mathrm{SL}_n(\mathbb{R})$ bounded, we have

$$\forall u \in C, \quad \alpha(u\mathcal{L}) \asymp \alpha(\mathcal{L})$$

(with the implicit constant depending on C).

Let $\mathbf{T}$ denote the diagonal subgroup of $\mathrm{SL}_n(\mathbb{R})$, let $T = \mathbf{T}^\circ_{\mathbb{R}}$ and let $\mathfrak{t}$ be the Lie algebra of T. In what follows we will replace T by its conjugate over $\mathrm{SL}_n(\mathbb{Q})$, where the conjugate will be conveniently chosen with respect to $\underline{H}_1$ and its subgroups. The reader should note that the statements to follow about T are not affected by such conjugations in $\mathrm{SL}_n(\mathbb{Q})$.

It is easy to check that for the lattice $\mathbb{Z}^n$ and for $a = \exp(\mathrm{diag}(X_1, \ldots, X_n)) \in T$, we have $\lambda_i(a\mathbb{Z}^n) = e^{X_{j(i)}}$ where $i \mapsto j(i)$ is a permutation giving $X_{j(1)} \leqslant X_{j(2)} \leqslant \cdots \leqslant X_{j(n)}$, and hence

$$\hat{\alpha}(a) = \alpha(a\mathbb{Z}^n) \asymp \exp\left(-\sum_{X_i<0} X_i\right). \tag{6.8}$$

Furthermore, for an element $f_0 \in \mathrm{SL}_n(\mathbb{Q})$ we have that $\lambda_i(af_0\mathbb{Z}^n) \asymp e^{X_{j(i)}}$, where implicit constants depend on f_0, and thus $\hat{\alpha}(a) \asymp \hat{\alpha}(af_0)$.

Recall the notation $D = \deg(\mathbb{K}/\mathbb{Q})$ from Theorem 3.1. We first prove the proposition under the assumption $D = 1$. That is, we have $\mathbb{K} = \mathbb{Q}$, $\underline{H}_1 = \mathrm{SL}_k(\mathbb{R})$ and $n = k$ in case $\mathbf{G} \cong \mathrm{SL}_k$, and $n = 2k$, $\underline{H}_1 =$

$\mathrm{Sp}_{2k}(\mathbb{R})$ in case $\mathbf{G} \cong \mathrm{Sp}_{2k}$. Now consider a Siegel set for $H = \underline{H}_1$, and suppose A_t is the corresponding subset of the maximal $\mathbb{Q}$-split torus of $\underline{H}_1$. Since $\mathbf{T}$ is a maximal $\mathbb{Q}$-split torus of $\mathrm{SL}_n(\mathbb{R})$, by [Bor91, Thm. 15.14], applying a conjugation in $\mathrm{SL}_n(\mathbb{Q})$ we can assume that $\mathbf{A} \subset \mathbf{T}$ and the order on the roots Φ is consistent with the standard order on the group of characters on $\mathfrak{t}$; that is, $A_t \subset T_{t'}$ for some t', as can be observed by an elementary computation (see [Bor19, Ex. 11.15] for a description of $\mathbf{A}$ in the symplectic case). In particular, for $a = \exp(\mathrm{diag}(X_j)) \in A_t$ we have $\exp(X_j) \ll \exp(X_{j+1})$ for $j = 1, \dots, n-1$. Then from (6.8), for $a \in A_t$ and $f_0 \in F_0$, where F_0 is a finite subset of $(\underline{H}_1)_{\mathbb{Q}}$, we have

$$\hat{\alpha}(af_0) \asymp \max_{1 \leqslant j \leqslant n-1} \exp\left(-\beta_j(X)\right), \tag{6.9}$$

where

$$\beta_j(X) \stackrel{\text{def}}{=} \sum_{i=1}^{j} X_i, \qquad X = \mathrm{diag}(X_\ell). \tag{6.10}$$

Since a Siegel set $\mathfrak{S}_{t,\omega}$ is contained in a set of the form $\omega' A_t$, where ω' is a compact subset of H, this implies that

$$\hat{\alpha}(kmanf_0) \ll \max_{1 \leqslant j \leqslant n-1} \exp(-\beta_j(X)).$$

We will first show the following:

(i) For any j, and any $X \in \mathfrak{a}$ for which $\exp(X) \in A_t$, we have $(2\rho - r_0\beta_j)(X) \ll 1$.
(ii) The number r_0 is the largest number for which the conclusion of (i) remains valid.

For $\ell = 1, \dots, n-1$ let χ_ℓ denote the simple roots on $\mathfrak{t}$, that is,

$$\chi_\ell : \mathfrak{t} \to \mathbb{R}, \quad \chi_\ell\left(\mathrm{diag}\left(X_1, \dots, X_n\right)\right) \stackrel{\text{def}}{=} X_{\ell+1} - X_\ell. \tag{6.11}$$

In order to show (i), since the χ_ℓ are bounded above on A_t, it suffices to show that if we write $2\rho = \sum a_\ell \chi_\ell$ and $\beta_j = \sum b_\ell^{(j)} \chi_\ell$, then $r_0 b_\ell^{(j)} \leqslant a_\ell$. In order to show (ii) it suffices to check that there are some j, ℓ for which equality holds, i.e., $r_0 b_\ell^{(j)} = a_\ell$. This can be checked using the tables of [Bou02, pp. 265-270, Plates I & III] (note that the restrictions of the β_j to A are the fundamental weights in both cases). Namely, for $\mathbf{G} = \mathrm{SL}_k$ we have

$$a_\ell = \ell(k-\ell), \qquad r_0\, b_\ell^{(j)} = \begin{cases} \ell(k-j) & \text{if } \ell < j \\ j(k-\ell) & \text{if } \ell \geqslant j \end{cases}$$

and we have the desired inequality, with equality when $\ell = j$. If $\mathbf{G} = \mathrm{Sp}_{2k}$ we have

$$a_\ell = \begin{cases} \ell(2k-\ell+1) & \text{if } \ell < k \\ \frac{k(k+1)}{2} & \text{if } \ell = k \end{cases} \qquad r_0\, b_\ell^{(j)} = \begin{cases} \ell(k+1) & \text{if } \ell < j \\ j(k+1) & \text{if } j \leqslant \ell < k \\ \frac{j(k+1)}{2} & \text{if } \ell = k \end{cases}$$

and again the inequality holds, with equality when $\ell = j = k$.

Now to see that $\alpha \in L^p\left(\underline{\mu}\right)$, since a Siegel set is contained in $\omega' A_t$ with ω' bounded, and by (6.2), it suffices to prove that for $f_0 \in F_0$ we have $\int_{A_t} \hat{\alpha}^p(af_0)\rho_0(a)da < \infty$. Using the preceding discussion, if we let $\mathfrak{a}_t$ denote the cone in $\mathfrak{a}$ with $A_t = \exp(\mathfrak{a}_t)$ (where A_t is as in (6.4)), and use that da is the pushforward under the exponential map of dX, we have

$$\begin{aligned} &\int_{A_t} \hat{\alpha}^p(af_0)\rho_0(a)\, da \ll \int_{\mathfrak{a}_t} \max_j \exp\left(-p\beta_j(X)\right) \cdot \exp\left(2\rho(X)\right)\, dX \\ &= \int_{\mathfrak{a}_t} \max_j \exp\left[\left(\frac{p}{r_0}\left(2\rho - r_0\beta_j\right) + \left(1 - \frac{p}{r_0}\right) 2\rho\right)(X)\right] dX \\ &\overset{(i)}{\ll} \int_{\mathfrak{a}_t} \exp\left[2\rho(X)\right]^{1-\frac{p}{r_0}}\, dX < \infty, \end{aligned}$$

where the integral is finite as the integrand is the exponential of a linear functional which is strictly decreasing along the cone $\mathfrak{a}_t$. The same computation and (ii) show that we have a corresponding lower bound $\int_{A_t} \hat{\alpha}^{r_0}(af_0)\rho_0(a)da \gg \int_{\mathfrak{a}_t} \exp\left(\tau(X)\right)\, dX$, where τ is a linear functional which is constant along a face of $\mathfrak{a}_t$. We have shown (6.6) for $D = 1$.

Now suppose $D > 1$. Our strategy will be to show that we can repeat the computations used for the case $D = 1$, with the only difference being that in some of the formulas, the characters ρ and β_j are multiplied by a factor of D. Write $G_1 \overset{\text{def}}{=} \mathbf{G}_{\mathbb{R}}^{\sigma_1}$, let V be as in the statement of Theorem 3.1, a $\mathbb{K}$-subspace of $\mathbb{R}^n$. Let

$$t \overset{\text{def}}{=} \begin{cases} k & \text{if } \mathbf{G} \cong \mathrm{SL}_k \\ 2k & \text{if } \mathbf{G} \cong \mathrm{Sp}_{2k}, \end{cases} \tag{6.12}$$

so that $\dim V = t$. Let $\mathbf{A}_1$ denote a maximal $\mathbb{K}$-split torus in $\mathbf{G}$, and let $\mathfrak{a}_1$ denote its Lie algebra. Then, with respect to a suitable basis of $V_{\mathbb{K}}$, we can write elements of $\mathfrak{a}_1$ as matrices $\mathrm{diag}(X_1, \ldots, X_t)$, where $\sum X_i = 0$ when $\mathbf{G} \cong \mathrm{SL}_k$ and $X_{i+k} = -X_i$ when $\mathbf{G} \cong \mathrm{Sp}_{2k}$.

Let $\mathbf{B} \overset{\text{def}}{=} \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{A}_1)$, and let $\mathbf{A}$ denote a maximal $\mathbb{Q}$-split torus in $\underline{H}_1$. The dimension of $\mathbf{A}_1$ is the number of independent one-parameter

multiplicative $\mathbb{K}$-subgroups (morphisms $\mathbb{K}^\times \to \mathbf{A}_1$), and, applying restriction of scalars, each such one-parameter group gives rise to a one-parameter $\mathbb{Q}$-subgroup $\mathbb{Q}^\times \to \mathbf{B}$. This implies that $\mathbf{B}$ contains a $\mathbb{Q}$-split torus of dimension equal to $\dim \mathbf{A}_1$. Since the $\mathbb{Q}$-rank of $\mathbf{H}$ is the same as the $\mathbb{K}$-rank of $\mathbf{G}$, see [BT65, 6.21 (i)], the dimensions of these groups coincide. Since all maximal $\mathbb{Q}$-split tori in $\mathbf{H}$ are conjugate over $\mathbf{H}_\mathbb{Q}$, we can assume that $\mathbf{A} \subset \mathbf{B}$, and by conjugating $\mathrm{SL}_n(\mathbb{R})$ by an element of $\mathrm{SL}_n(\mathbb{Q})$, we can also assume that $\mathbf{A} \subset \mathbf{T}$ and the order on the roots Φ is consistent with the order on the roots of $\mathfrak{t}$. We claim that after these conjugations, the elements of $A = \mathbf{A}^\circ_\mathbb{R}$ are of the form

$$\mathrm{diag}\left(\underbrace{X_{j(1)},\dots,X_{j(1)}}_{D \text{ times}},\dots,\underbrace{X_{j(t)},\dots,X_{j(t)}}_{D \text{ times}}\right), \tag{6.13}$$

where $\mathrm{diag}(X_1,\dots,X_t)$ ranges over the elements of $\mathfrak{a}_1$ in the above-chosen basis, and $i \mapsto j(i)$ is a permutation guaranteeing $\exp\left(X_{j(1)}\right) \ll \dots \ll \exp\left(X_{j(t)}\right)$.

We first assume the validity of (6.13), and conclude the proof of the case $D > 1$. We will use (6.13) to compare characters on A_1 with characters on A. First, comparing the character ρ appearing in (6.3) for the two groups $\underline{H}_1, G_1$, we see that each real field embedding $\sigma_i, i \leqslant r$ contributes one dimension to the dimension of a root space, and each pair $\sigma_i, \bar{\sigma}_i$, $i > r$ of conjugate non-real embedding contributes two dimensions. Alternatively: in G_1 the root spaces are one dimensional and defined over $\mathbb{K}$, since G_1 is $\mathbb{K}$-split. The root spaces in H_1 are obtained from the root spaces in G_1 by applying the restriction of scalars operation to each one individually. This implies that the character ρ for $\underline{H}_1$ is obtained from the corresponding character for G_1 by a multiplication by D. Similarly, it is clear from (6.13) that the characters β_j appearing in (6.10) for $\underline{H}_1$ are obtained from the same characters β_j for G_1, multiplied by D. Thus, the computations guaranteeing (6.6) for $D = 1$, imply the same property for general D.

It remains to prove (6.13). Recall that $\mathbf{B} = \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{A}_1)$, which we wish to describe explicitly using the discussion in §2.4. For $\vec{y} \in \mathbb{K}^t$ we define $a_1(\vec{y}) \stackrel{\mathrm{def}}{=} \mathrm{diag}(y_1,\dots,y_t) \in A_1(\mathbb{K})$; that is, these are matrices acting on V which are diagonal with respect to a $\mathbb{K}$-basis of V, and the y_i satisfy $y_1 + \dots + y_t = 0$ for $\mathbf{G} \cong \mathrm{SL}_k$ and $y_i = -y_{2k-i+1}$ for $\mathbf{G} \cong \mathrm{Sp}_{2k}$. Each $y \in \mathbb{K}$ has a representative which is a matrix in $\mathrm{Mat}_{D\times D}(\mathbb{Q})$. If we take $y \in \mathbb{Q}$ then the corresponding representative matrix is the scalar matrix $y \cdot \mathrm{Id}_D$. The elements of $\mathbf{B}$ can be considered as $t \times t$ matrices, whose entries are elements of $\mathrm{Mat}_{D\times D}$. In

particular, for $\vec{y} \in \mathbb{Q}^t$, we get matrices $a_2(\vec{y}) \in \mathrm{Mat}_{n\times n}(\mathbb{Q})$, which are simultaneously diagonalizable, with each y_i appearing as an eigenvalue D times. That is, up to permuting the coordinates, the matrices $a_2(\vec{y})$ are as in (6.13), with $X_i \in \mathbb{Q}$. The map $a_1(\vec{y}) \mapsto a_2(\vec{y})$ is a polynomially defined group homomorphism. Letting $\mathbf{A}_2$ denote the Zariski closure of $\{a_2(\vec{y}) : \vec{y} \in \mathbb{Q}^t,\, a_1(\vec{y}) \in A_1\}$, we see that $\mathbf{A}_2$ is a torus in $\mathbf{B}$ whose group of real points $(\mathbf{A}_2)_{\mathbb{R}}$ satisfies the description (6.13), and with $\dim \mathbf{A}_2 = \dim \mathbf{A}_1 = \dim \mathbf{A}$. Also, $\mathbf{A}_2$ is $\mathbb{Q}$-split since the maps $a_2(\vec{y}) \mapsto y_i$ are $\mathbb{Q}$-characters. Thus, $\mathbf{A}_2$ is a maximal $\mathbb{Q}$-split torus of $\mathbf{H}$, and by the uniqueness of the maximal $\mathbb{Q}$-split torus in the torus $\mathbf{B}$ (see [Bor19, Prop. 10.6]), we must have $\mathbf{A} = \mathbf{A}_2$. (See also the related discussion in [PR94, Example, p. 54], giving an explicit description of a maximal $\mathbb{Q}$-anisotropic torus in $\mathbf{B}$ as a product of norm-tori.) □

6.3. **An upper bound for the Siegel transform.** We will now state and prove a result implying Theorem 1.2. For a function F on $\mathbb{R}^n$, a measure $\bar{\mu}$ on $\mathscr{Y}_n$, and $\mathcal{L} \in \mathscr{Y}_n$, in analogy with (1.3) we denote

$$\widehat{F}(\mathcal{L}) = \begin{cases} \sum_{x\in\mathcal{L}\smallsetminus\{0\}} F(x) & \bar{\mu} \text{ is linear} \\ \sum_{x\in\mathcal{L}} F(x) & \bar{\mu} \text{ is affine} \end{cases} \tag{6.14}$$

Theorem 6.2. *Let $\bar{\mu}$ be the H-homogeneous measure on $\mathscr{Y}_n$ as in Theorem 3.1, and let $q = q_{\bar{\mu}}$ be as in* (1.5)*. Then for any $F \in C_c(\mathbb{R}^n)$ and any $p < q$ we have $\widehat{F} \in L^p(\bar{\mu})$. Moreover, there are $F \in C_c(\mathbb{R}^n)$ for which $\widehat{F} \notin L^q(\bar{\mu})$.*

We will prove Theorem 6.2 separately in the linear and affine cases. In the linear case, we will first show, using Proposition 5.4, that the Siegel-Veech transform (6.14) can be bounded in terms of a Siegel transform of a function on $\mathscr{X}_n$. The latter can be bounded in terms of the function α considered in §6.2.

Proof of Theorem 6.2, linear case. Suppose that $\bar{\mu}$ satisfies (i) of Proposition 5.4, i.e., $\bar{\mu}$ is supported on $\bar{\iota}(\mathscr{X}_n)$. Then we can assume that the cut-and-project scheme involves lattices in $\mathscr{X}_n$, rather than grids. Moreover, $H = \iota \circ \pi(H)$, $\underline{g}_1 = g_1$, $\underline{H}_1 = \iota \circ \pi(H_1)$, and the function $\widehat{F}$ is a Siegel-Veech transform of a Riemann integrable function on $\mathbb{R}^n$, for a homogeneous subspace of $\mathscr{X}_n$. It is known that the function α defined in (6.5) describes the growth rate of the Siegel transforms of functions on $\mathscr{X}_n$. Namely (see [EMM98, Lemma 3.1] or [KSW17, Lemma 5.1]), for any Riemann integrable function F on $\mathbb{R}^n$, for any $\mathcal{L} \in \mathscr{X}_n$, $\widehat{F}(\mathcal{L}) \ll \alpha(\mathcal{L})$. Furthermore, if F is the indicator of a ball around the origin then $\widehat{F}(\mathcal{L}) \gg \alpha(\mathcal{L})$. Thus, the conclusion of Theorem 6.2 in this case follows from Proposition 6.1.

Now assume that case (ii) of Proposition 5.4 holds. We cannot use Proposition 6.1 since $\widehat{F}$ is a function on $\mathscr{Y}_n$. To remedy this, we define for each $\mathcal{L} \in H\mathcal{L}_1$ the lattice $\mathcal{L}' = \mathcal{L}'(\underline{\pi}(\mathcal{L}))$ appearing in assertion (ii) of Proposition 5.4, and set

$$\underline{\widehat{F}}(\underline{\pi}(\mathcal{L})) \stackrel{\text{def}}{=} \sum_{x \in \mathcal{L}'(\underline{\pi}(\mathcal{L})) \smallsetminus \{0\}} F(x).$$

Then the bounds given in Proposition 5.4 imply that $\widehat{F}(\mathcal{L}) \ll \underline{\widehat{F}}(\underline{\pi}(\mathcal{L}))$, with a reverse inequality $\underline{\widehat{F}}(\underline{\pi}(\mathcal{L})) \ll \widehat{F}(\mathcal{L})$ for positive F. Since $\underline{\widehat{F}}$ is the Siegel-Veech transform of a function on $\mathbb{R}^n$ with respect to a measure on $\mathscr{X}_n$, we can apply Proposition 6.1 to conclude the proof in this case as well. □

For the affine case, we will need the following additional interpretation of the function α defined in (6.5).

Proposition 6.3. *Let $\underline{\mathcal{L}} \in \mathscr{X}_n$, let $\mathbb{T}^n_{\underline{\mathcal{L}}} = \mathbb{T}^n = \underline{\pi}^{-1}(\underline{\mathcal{L}}) \cong \mathbb{R}^n/\underline{\mathcal{L}}$ be the quotient torus, equipped with its invariant measure element $d\mathcal{L}$. Then for any ball $B \subset \mathbb{R}^n$ and any $p > 1$ we have*

$$\int_{\mathbb{T}^n} |B \cap \mathcal{L}|^p \, d\mathcal{L} \asymp \alpha(\underline{\mathcal{L}})^{p-1}, \tag{6.15}$$

where the implicit constants depend on the dimension n, on p, and on the radius of B.

Proof. Let $\lambda_1, \dots, \lambda_n$ be the Minkowski successive minima of $\underline{\mathcal{L}}$. Using Korkine-Zolotarev reduction, let $v_1, \dots, v_n$ be a basis for $\underline{\mathcal{L}}$ satisfying $\|v_i\| \asymp \lambda_i$ (where implicit constants are allowed to depend on the dimension n), and let $u_i \stackrel{\text{def}}{=} \frac{v_i}{\|v_i\|}$. For a vector $\vec{s}$ of positive numbers $s_1, \dots, s_n$ define

$$P_{\vec{s}} \stackrel{\text{def}}{=} \left\{ \sum a_i u_i : |a_i| \leqslant \frac{s_i}{2} \right\}.$$

Setting $\vec{v}_0 = (\|v_1\|, \dots, \|v_n\|)$, we have that $P_{\vec{v}_0} = \left\{ \sum b_i v_i : |b_i| \leqslant \frac{1}{2} \right\}$ is a fundamental parallelepiped for $\mathcal{L}$, and we can identify $\mathbb{T}^n$ with this parallelpiped via the bijection

$$P_{\vec{v}_0} \to \mathbb{T}^n, \quad x \mapsto \mathcal{L}_x \stackrel{\text{def}}{=} \underline{\mathcal{L}} + x,$$

which sends the Lebesgue measure on $P_{\vec{v}_0}$ to the Haar measure $d\text{vol}$ on $\mathbb{T}^n$.

Now set

$$P_r \stackrel{\text{def}}{=} P_{\vec{r}} \text{ where } \vec{r} = (r, \dots, r).$$

We can translate B so that it is centered at the origin without affecting the integral in (6.15), and since there is a lower bound on the angles

between the v_i, there are $r_1 \asymp R \asymp r_2$ such that $P_{r_1} \subset B \subset P_{r_2}$. Thus, we can replace B with P_R. Furthermore, the lower bound on the angles between the u_i implies

$$d\mathrm{vol}(x) \asymp dx_1 \cdots dx_n, \quad \text{where } x = \sum x_i u_i.$$

Writing each vector $y \in \mathbb{R}^n$ in the form $y = \sum_i c_i u_i$, and reducing each c_i modulo $\|v_i\| \cdot \mathbb{Z}$, it is easy to verify that for $x \in P_{\vec{v}_0}$ we have:

- if $\frac{R}{2} < |x_i| < \frac{\|v_i\|-R}{2}$ for some i, then $P_R \cap \mathcal{L}_x = \varnothing$; and
- if $|x_i| \leqslant \frac{R}{2}$ or $|x_i| \geqslant \frac{\|v_i\|-R}{2}$ for all i, then $|P_R \cap \mathcal{L}_x| \asymp \prod_{\|v_i\|<R} \left(\frac{R}{\|v_i\|}\right)$.

Since

$$\prod_{\|v_i\|<R} \frac{R}{\|v_i\|} \asymp \prod_{\lambda_i(\underline{\mathcal{L}})<1} \frac{1}{\lambda_i(\underline{\mathcal{L}})} \overset{(6.7)}{\asymp} \alpha(\underline{\mathcal{L}}),$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^n} |B \cap \mathcal{L}|^p \, d\mathcal{L} \asymp \int_{P_{\vec{v}_0}} |P_R \cap \mathcal{L}_x|^p \, d\mathrm{vol}(x) \\
&\asymp \alpha(\underline{\mathcal{L}})^p \cdot \mathrm{vol}\left(\left\{x \in P_{\vec{v}_0} : |x_i| \leqslant \frac{R}{2}\right\}\right) \\
&\asymp \alpha(\underline{\mathcal{L}})^p \cdot \prod_{\|v_i\| \leqslant R} \|v_i\| \cdot \prod_{\|v_i\|>R} R \asymp \ \alpha(\underline{\mathcal{L}})^p \cdot \prod_{\lambda_i(\underline{\mathcal{L}})<1} \lambda_i(\underline{\mathcal{L}}) \overset{(6.7)}{\asymp} \alpha(\underline{\mathcal{L}})^{p-1}.
\end{aligned}$$

□

Proof of Theorem 6.2, affine case. By decomposing F into its positive and negative parts, we see that it suffices to prove $\widehat{F} \in L^p(\mu)$ when F is the indicator of a ball in $\mathbb{R}^n$. By Theorem 3.1 we have that in the affine case, the translation group $\mathbb{R}^n$ is contained in H_1, which implies that we can decompose the measure $\bar{\mu}$ as

$$\int_{\mathbf{X}_1} \varphi(\mathcal{L}) \, d\bar{\mu}(\mathcal{L}) = \int_{\underline{\mathbf{X}}_1} \int_{\mathbb{T}^n_{\underline{\mathcal{L}}}} \varphi(\mathcal{L}_x) \, d\mathrm{vol}(x) \, d\underline{\mu}(\underline{\mathcal{L}}), \quad \forall \varphi \in L^1(\mathbf{X}_1, \bar{\mu}).$$

Now the statement follows from Propositions 6.1 and 6.3. The case of equality $p = q_\mu$ follows similarly, taking for F the indicator of a ball in $\mathbb{R}^n$. □

Proof of Theorem 1.2. Let $f \in C_c(\mathbb{R}^d)$ and let $\hat{f}$ be as in (1.3). Let μ be an RMS measure on $\mathscr{C}(\mathbb{R}^d)$ associated with a cut-and-project scheme involving grids in $\mathscr{Y}_n$, a decomposition $\mathbb{R}^n = V_{\mathrm{phys}} \oplus V_{\mathrm{int}}$, and a window $W \subset V_{\mathrm{int}}$. Let $\mathbf{1}_W$ be the indicator function of W and let $\bar{\mu}$ be an H-homogeneous measure, supported on the orbit $H\mathcal{L}_1 \subset \mathscr{Y}_n$ such that $\mu = \Psi_* \bar{\mu}$ (where we have replaced μ by its image under a rescaling map to simplify notation).

Define

$$F : \mathbb{R}^n \to \mathbb{R}, \quad F(x) = \mathbf{1}_W(\pi_{\text{int}}(x)) \cdot f(\pi_{\text{phys}}(x)), \tag{6.16}$$

and define $\widehat{F}$ via (6.14). Then it is clear from the definition of Ψ and (1.3) that $\hat{f}(\Psi(\mathcal{L})) = \widehat{F}(\mathcal{L})$ provided $\mathcal{L}$ satisfies **(I)**, and, in the linear case, provided all nonzero vectors of $\mathcal{L}$ project to nonzero vectors in V_{phys}; the last assumption is equivalent to requiring that

$$\mathcal{L} \notin \mathcal{N} \stackrel{\text{def}}{=} \{\mathcal{L}' \in H\mathcal{L}_1 : \mathcal{L}' \cap V_{\text{int}} \subsetneq \{0\}\}.$$

The condition that $\mathcal{L}$ satisfies **(I)** is valid for $\bar{\mu}$-a.e. $\mathcal{L}$ by definition of an RMS measure. We claim further that in the linear case $\bar{\mu}(\mathcal{N}) = 0$. Indeed, since $\bar{\mu}$ is induced by the Haar measure of H, otherwise we would have some fixed $v \in \mathcal{L}_1 \smallsetminus \{0\}$ such that $H^{\mathcal{N},v} \stackrel{\text{def}}{=} \{h \in H : hv \in V_{\text{int}}\}$ has positive Haar measure. Recall that for analytic varieties $\mathcal{V}_1, \mathcal{V}_2$, with $\mathcal{V}_1$ connected, if $\mathcal{V}_1 \cap \mathcal{V}_2$ has positive measure with respect to the smooth measure on $\mathcal{V}_1$, then $\mathcal{V}_1 \subset \mathcal{V}_2$. Since $H^{\mathcal{N},v}$ is an analytic subvariety in H, if it has positive measure with respect to the Haar measure on H, it must coincide with H. This contradicts Lemma 4.2. This contradiction shows that $\bar{\mu}$-almost surely we have $\widehat{f} \circ \Psi = \widehat{F}$. Since $\mu = \Psi_* \bar{\mu}$, the first assertion that $\hat{f} \in L^p(\mu)$ for $p < q_\mu$ now follows from the first assertion of Theorem 6.2.

For the second assertion, let f be a nonnegative continuous function whose support contains a ball around the origin. Since we have assumed that W contains a ball around the origin in V_{int}, the support of the function F also contains a ball around the origin in $\mathbb{R}^n$, so $\hat{f}$ is bounded below by the Siegel-Veech transform of the indicator of a ball in $\mathbb{R}^n$, and we have that such functions do not belong to $L^{q_\mu}(\bar{\mu})$. □

7. Integral formulas for the Siegel-Veech transform

In this section we will prove Theorem 1.3. We begin with its special case $p = 1$, i.e., with a derivation of (1.4). This will illustrate the method of Weil [Wei82] which we will use. Note that (1.4) was first proved by Marklof and Strömbergsson in [MS14] following an argument of Veech [Vee98]. Their argument does not rely on an integrability bound such as our Theorem 1.2, and instead, uses the result of Shah [Sha96], Theorem 2.5.

7.1. **A derivation of a 'Siegel summation formula'.** Given $f \in C_c(\mathbb{R}^d)$, define F via (6.16), and define $\widehat{F}(\mathcal{L})$ via (6.14). We can bound F pointwise from above by a compactly supported continuous function on $\mathbb{R}^n$, and hence, by Theorem 6.2, $\widehat{F} \in L^1(\bar{\mu})$. Therefore $f \mapsto \int_{\mathbf{X}_1} \widehat{F}\, d\bar{\mu}$

is a positive linear functional on $C_c(\mathbb{R}^d)$. By the Riesz representation theorem, there is some Radon Borel measure ν on $\mathbb{R}^d$ such that $\int_{\mathbf{X}_1} \widehat{F}\, d\bar{\mu} = \int_{\mathbb{R}^d} f\, d\nu$. From the equivariance relation (3.2), ν is invariant under $\mathrm{ASL}_d(\mathbb{R})$ in the affine case and under $\mathrm{SL}_d(\mathbb{R})$ in the linear case. Lebesgue measure is the unique (up to scaling) $\mathrm{ASL}_d(\mathbb{R})$-invariant Radon Borel measure on $\mathbb{R}^d$, and for $\mathrm{SL}_d(\mathbb{R})$, the only additional invariant measure is δ_0, the Dirac mass at the origin. Thus, there are constants c_1, c_2 such that

$$\nu = \begin{cases} c_1 \mathrm{vol} & \bar{\mu} \text{ is affine} \\ c_1 \mathrm{vol} + c_2 \delta_0 & \bar{\mu} \text{ is linear.} \end{cases} \tag{7.1}$$

As we have seen in the proof of Theorem 1.2, we have that $\widehat{F} = \widehat{f} \circ \Psi$ holds $\bar{\mu}$-a.e. Since $\mu = \Psi_* \bar{\mu}$, this implies that

$$\int_{\mathscr{C}(\mathbb{R}^d)} \widehat{f}\, d\mu = \int_{\mathbf{X}_1} \widehat{F}\, d\bar{\mu} = \int_{\mathbb{R}^d} f\, d\nu.$$

In combination with (7.1), this establishes (1.4) in the affine case, and gives

$$\int_{\mathscr{C}(\mathbb{R}^d)} \hat{f}\, d\mu = c_1 \int_{\mathbb{R}^d} f\, d\mathrm{vol} + c_2 f(0), \quad \forall f \in C_c(\mathbb{R}^d) \tag{7.2}$$

in the linear case. It remains to show that $c_2 = 0$.

Let $B_r = B(0, r)$ be the ball in $\mathbb{R}^d$ centered at the origin, let $f \in C_c(\mathbb{R}^d)$ satisfy $\mathbf{1}_{B_1} \leqslant f \leqslant \mathbf{1}_{B_2}$, and let $f_r = f\left(\frac{x}{r}\right)$. Thus, as $r \to 0$, the functions f_r have smaller and smaller support around the origin. By (1.3) and discreteness of Λ we have that $\widehat{f_r}(\Lambda) \to_{r \to 0} 0$ for any Λ. The functions f_r vanish outside the ball B_{2r}, and for $r \leqslant 1$, the functions $\widehat{f_r}$ are dominated by $\widehat{f_1}$. Therefore

$$0 = \lim_{r \to 0} \int_{\mathscr{C}(\mathbb{R}^d)} \widehat{f_r}\, d\mu \overset{(7.2)}{=} \lim_{r \to 0} \left[c_1 \int_{\mathbb{R}^d} f_r\, d\mathrm{vol} + c_2 \cdot 1 \right] = c_2.$$

□

7.2. **A formula following Siegel-Weil-Rogers.** In this section we state and prove a generalization of Theorem 1.3. Let the notation be as in 3.1, so that $\bar{\mu}$ is an H-homogeneous measure on $\mathscr{Y}_n$. Let $p \in \mathbb{N}$ and let $\mathbb{R}^{np} = \underbrace{\mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n}_{p \text{ copies}}$. For $f \in C_c(\mathbb{R}^{np})$ and $\mathcal{L} \in \mathscr{Y}_n$, define

$${}^p\widehat{f}(\mathcal{L}) \overset{\text{def}}{=} \begin{cases} \displaystyle\sum_{v_1, \ldots, v_p \in \mathcal{L} \smallsetminus \{0\}} f(v_1, \ldots, v_p) & \bar{\mu} \text{ is linear} \\ \displaystyle\sum_{v_1, \ldots, v_p \in \mathcal{L}} f(v_1, \ldots, v_p) & \bar{\mu} \text{ is affine.} \end{cases} \tag{7.3}$$

Let $J \subset \mathrm{ASL}(np, \mathbb{R})$ be a real algebraic group and let θ be a locally finite Borel measure on $\mathbb{R}^{np}$. We say that θ is *J-algebraic* if J preserves θ and has an orbit of full θ-measure (in this case θ can be described in terms of the Haar measure of J, see [Rag72, statement and proof of Lemma 1.4]).

Theorem 7.1. *Let $p \in \mathbb{N}$ and assume that $p < q_{\bar{\mu}}$ where $q_{\bar{\mu}}$ is as in (1.5). Then there is a countable collection $\{\bar{\tau}_{\mathfrak{e}} : \mathfrak{e} \in \mathfrak{E}\}$ of H-algebraic Borel measures on $\mathbb{R}^{np}$ such that $\bar{\tau} \stackrel{\text{def}}{=} \sum \bar{\tau}_{\mathfrak{e}}$ is locally finite and for every $f \in L^1(\bar{\tau})$ we have*

$$\int_{\mathscr{Y}_n} {}^p\widehat{f}\, d\bar{\mu} = \int_{\mathbb{R}^{np}} f\, d\bar{\tau}. \tag{7.4}$$

As we will see in the proof, in the affine (resp. linear) case, the indexing set $\mathfrak{E}$ is naturally identified with the set of Γ_{H_1}-orbits in the set of p-tuples of (nonzero) vectors in $\mathbb{Z}^n$.

We will need a by-now standard result of Weil, which is a generalization of the Siegel summation formula and is proved via an argument similar to the one used in §7.1. Let $G_1 \subset G_2$ be unimodular locally compact groups, let $\Gamma_2 \subset G_2$ be a lattice in G_2 and let m_{G_2/Γ_2} denote the unique G_2-invariant Borel probability measure on G_2/Γ_2. Since G_1, G_2 are unimodular, there is a unique (up to scaling) locally finite G_2-invariant measure on G_2/G_1, which we denote by m_{G_2/G_1} (see e.g. [Rag72, Chap. I]). Define $\Gamma_1 \stackrel{\text{def}}{=} \Gamma_2 \cap G_1$, and for any $\gamma \in \Gamma_2$, denote its coset $\gamma\Gamma_1 \in \Gamma_2/\Gamma_1$ by $[\gamma]$. With this notation, Weil showed the following:

Proposition 7.2 ([Wei46])**.** *Assume that Γ_1 is a lattice in G_1. Then we can rescale m_{G_2/G_1} so that the following holds. For any $F \in L^1(G_2/G_1, m_{G_2/G_1})$, define*

$$\widetilde{F}(g\Gamma_2) \stackrel{\text{def}}{=} \sum_{[\gamma] \in \Gamma_2/\Gamma_1} F(g\gamma). \tag{7.5}$$

Then $\widetilde{F} \in L^1(G_2/\Gamma_2, m_{G_2/\Gamma_2})$ and

$$\int_{G_2/\Gamma_2} \widetilde{F}\, dm_{G_2/\Gamma_2} = \int_{G_2/G_1} F\, dm_{G_2/G_1}.$$

Proof of Theorem 7.1. Consider the map which sends $f \in C_c(\mathbb{R}^{np})$ to $\int {}^p\widehat{f}\, d\bar{\mu}$. This is well-defined by Theorem 6.2, and defines a positive linear functional on $C_c(\mathbb{R}^{np})$. Thus, by the Riesz representation theorem, there is a locally finite measure $\bar{\tau}$ on $\mathbb{R}^{np}$ such that

$$\forall f \in C_c(\mathbb{R}^{np}), \quad \int_{\mathscr{Y}_n} {}^p\widehat{f}\, d\bar{\mu} = \int_{\mathbb{R}^{np}} f\, d\bar{\tau}. \tag{7.6}$$

Our goal will be to present $\bar\tau$ as a countable linear combination of H-algebraic measures. Note that since $C_c(\mathbb{R}^{np})$ is a dense linear subspace of $L^1(\bar\tau)$, for any locally finite measure $\bar\tau$, it suffices to prove (7.4) for functions in $C_c(\mathbb{R}^{np})$.

Let H, g_1, $\mathcal{L}_1 = g_1\mathbb{Z}^n$, $H_1 = g_1^{-1}Hg_1$, $\Gamma_{H_1} = H_1 \cap \mathrm{ASL}_n(\mathbb{Z})$ be as in §3.2, so that Γ_{H_1} is a lattice in H_1 and $\bar\mu$ is an H-homogeneous measure supported on $H\mathcal{L}_1 \cong H_1/\Gamma_{H_1}$. In the affine (respectively linear) case, let $\mathbb{Z}^{np}$ denote the countable collection of ordered p-tuples of vectors in $\mathbb{Z}^n$ (respectively, in $\mathbb{Z}^n \smallsetminus \{0\}$). Let $\mathfrak{E}$ denote the collection of Γ_{H_1}-orbits in $\mathbb{Z}^{np}$. For each $\mathfrak{e} \in \mathfrak{E}$, define the restriction of the sum (7.3) to the orbits $H\mathcal{L}_1 \subset \mathscr{Y}_n$ and to the orbit $\mathfrak{e}$, by

$$ {}^p\widehat{f}_{\mathfrak{e}}(h\mathcal{L}_1) \stackrel{\text{def}}{=} \sum_{(x_1,\dots,x_p)\in\mathfrak{e}} f\left(hg_1x_1,\dots,hg_1x_p\right), \tag{7.7} $$

so that on $H\mathcal{L}_1$ we have

$$ {}^p\widehat{f} = \sum_{\mathfrak{e}\in\mathfrak{E}} {}^p\widehat{f}_{\mathfrak{e}}. \tag{7.8} $$

If f is a non-negative function then ${}^p\widehat{f}_{\mathfrak{e}} \leqslant {}^p\widehat{f}$ everywhere on $H\mathcal{L}_1$, and in particular ${}^p\widehat{f}_{\mathfrak{e}} \in L^1(\bar\mu)$. Thus, the assignment sending $f \in C_c(\mathbb{R}^{np})$ to

$$ \int f\, d\bar\tau_{\mathfrak{e}} \stackrel{\text{def}}{=} \int {}^p\widehat{f}_{\mathfrak{e}}\, d\bar\mu \tag{7.9} $$

is a positive linear functional and hence, via the Riesz representation theorem, defines the locally finite Borel measure $\bar\tau_{\mathfrak{e}}$ on $\mathbb{R}^{np}$. By (7.8), $\sum_{\mathfrak{e}\in\mathfrak{E}} \bar\tau_{\mathfrak{e}} = \bar\tau$. It remains to show that each $\bar\tau_{\mathfrak{e}}$ is H-algebraic.

For each $\mathfrak{e} \in \mathfrak{E}$, choose a representative p-tuple $\vec{x}_{\mathfrak{e}} = (x_1,\dots,x_p) \in \mathfrak{e}$ and let

$$ G_{1,\mathfrak{e}} \stackrel{\text{def}}{=} \{h \in H_1 : hx_i = x_i,\ i = 1,\dots,p\}. $$

We will apply Proposition 7.2 with $G_2 = H_1$, $\Gamma_2 = \Gamma_{H_1}$, $G_1 = G_{1,\mathfrak{e}}$, $\Gamma_1 = \Gamma_2 \cap G_1$, and with $F(h_1G_1) \stackrel{\text{def}}{=} f(g_1h_1\vec{x}_{\mathfrak{e}})$. Comparing (7.5) and (7.7) we see that these choices imply that $\widetilde{F}(h_1\Gamma_2) = {}^p\widehat{f}_{\mathfrak{e}}(h\mathcal{L}_1)$, for $h = g_1h_1g_1^{-1} \in H$. We will see below that Γ_1 is a lattice in G_1. Assuming this, we apply Proposition 7.2 to obtain

$$ \int_{\mathbb{R}^{np}} f\, d\bar\tau_{\mathfrak{e}} = \int_{\mathscr{Y}_n} {}^p\widehat{f}_{\mathfrak{e}}\, d\bar\mu = \int_{G_2/\Gamma_2} \widetilde{F}\, dm_{G_2/\Gamma_2} $$

$$ = \int_{G_2/G_1} f(g_1h_1\vec{x}_{\mathfrak{e}})\, dm_{G_2/G_1}(h_1G_1). $$

This shows that $\bar\tau_{\mathfrak{e}}$ is the pushforward of m_{G_2/G_1} under the map

$$ G_2/G_1 \to \mathbb{R}^{np}, \quad h_1G_1 \mapsto g_1h_1\vec{x}_{\mathfrak{e}}. $$

In particular, since $H = g_1 H_1 g_1^{-1}$, $\bar{\tau}_{\mathfrak{e}}$ is H-algebraic.

It remains to show that Γ_1 is a lattice in G_1. To see this, note that G_2 is a real algebraic group defined over $\mathbb{Q}$, and G_1 is the stabilizer in G_2 of a finite collection of vectors in $\mathbb{Z}^n$. Thus, G_1 is also defined over $\mathbb{Q}$. By the theorem of Borel and Harish-Chandra (see [Bor19, §13]), if G_1 has no nontrivial characters then $\Gamma_1 = G_1 \cap \mathrm{ASL}_n(\mathbb{Z})$ is a lattice in G_1. Moreover, a real algebraic group generated by unipotents has no characters. Thus, to conclude the proof of the claim, it suffices to show that G_1 is generated by unipotents. We verify this by dividing into the various cases arising in Theorem 3.1.

We first reduce to the case that G_1 is a subgroup of $\mathrm{SL}_n(\mathbb{R})$. In the linear case we simply identify G_2 with its isomorphic image $\pi(G_2)$, where $\pi : \mathrm{ASL}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$ is the projection in (3.3), and thus we can assume $G_1 \subset \mathrm{SL}_n(\mathbb{R})$. In the affine case, since the property of being generated by unipotents is invariant under conjugations in $\mathrm{ASL}_n(\mathbb{R})$, we may conjugate by a translation to assume that one of the vectors in $\vec{x}_{\mathfrak{e}}$ is the zero vector, so that $G_1 \subset \mathrm{SL}_n(\mathbb{R})$. Thus, in both cases we may assume that $G_2 = \underline{H}_1$ is the group of real points of $\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$, and G_1 is the stabilizer in G_2 of the finite collection $x_1, \ldots, x_p$, where these are vectors in the standard representation on $\mathbb{R}^n$.

Suppose first that $\mathbf{G} = \mathrm{SL}_k$. Then, in the notation of (2.5), we have that $G_2 = \mathbf{G}_{\mathbb{R}}^{\sigma_1} \times \cdots \times \mathbf{G}_{\mathbb{R}}^{\sigma_{r+s}}$, where for $i = 1, \ldots, i$ (respectively, for $i = r+1, \ldots, r+s$) we have that $\mathbf{G}_{\mathbb{R}}^{\sigma_i}$ is isomorphic to $\mathrm{SL}_k(\mathbb{R})$ (respectively to $\mathrm{SL}_k(\mathbb{C})$ as a real algebraic group). Furthermore, as in §2.4, there is a decomposition

$$\mathbb{R}^n = V_1 \oplus \cdots \oplus V_{r+s},$$

where $V_i \cong \mathbb{R}^k$ (resp., $V_i \cong \mathbb{R}^{2k}$) for $i = 1, \ldots, r$ (resp., for $i = r+1, \ldots, r+s$), and such that the action of G_2 on $\mathbb{R}^n$ is the product of the standard action of each $\mathbf{G}_{\mathbb{R}}^{\sigma_i}$ on V_i. Let $P_i : \mathbb{R}^n \to V_i$ be the projection with respect to this direct sum decomposition. Then the stabilizer in G_2 of $x_1, \ldots, x_p$ is the direct product of the stabilizer, in $\mathbf{G}_{\mathbb{R}}^{\sigma_i}$, of $P_i(x_1), \ldots, P_i(x_p)$. So it suffices to show that each of these stabilizers is generated by unipotents. In other words, we are reduced to the well-known fact that for $\mathrm{SL}_k(\mathbb{R})$ acting on $\mathbb{R}^k$ in the standard action, and for $\mathrm{SL}_k(\mathbb{C})$ acting on $\mathbb{R}^{2k} \simeq \mathbb{C}^k$ in the standard action, the stabilizer of a finite collection of vectors is generated by unipotents.

Now suppose that $\mathbf{G} = \mathrm{Sp}_{2k}$, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Then by a similar argument, we are reduced to the statement that for the standard action of $\mathrm{Sp}_{2k}(\mathbb{F})$ on $\mathbb{F}^{2k}$, the stabilizer of a finite collection of vectors is generated by unipotents. This can be shown as follows. Let ω be the symplectic form preserved by Sp_{2k}, let $V = \mathrm{span}(x_1, \ldots, x_p) \subset \mathbb{F}^{2k}$,

and let

$$Q \stackrel{\text{def}}{=} \{g \in \mathrm{Sp}_{2k}(\mathbb{F}) : \forall v \in V,\, gv = v\}.$$

We need to show that Q is generated by unipotents. We can write $V = V_0 \oplus V_1$, where $V_0 = \ker(\omega|_V)$ is Lagrangian, and V_1 is symplectic. Let $2\ell = \dim V_1$, where $\ell \leqslant k$. Since any element of Q fixes V_1 pointwise, it leaves $V_1^\perp$ invariant, and it also fixes pointwise the subspace $V_0 \subset V_1^\perp$. Thus, Q is isomorphic to

$$\{g \in \mathrm{Sp}(V_1^\perp) : \forall v \in V_0,\, gv = v\} \subset \mathrm{Sp}(V_1^\perp) \cong \mathrm{Sp}_{2m}(\mathbb{F}),$$

where $m \stackrel{\text{def}}{=} k - \ell$. This means we can reduce the problem to the case in which $V_1 = \{0\}$, i.e., $\omega(x_i, x_j) = 0$ for all i, j. We can apply a symplectic version of the Gram-Schmidt orthogonalization procedure to assume that $x_1, y_1, \ldots, x_p, y_p, x_{p+1}, y_{p+1}, \ldots, x_m, y_m$ is a symplectic basis and $V_0 = \mathrm{span}(x_1, \ldots, x_p)$. Let

$$V_2 \stackrel{\text{def}}{=} \mathrm{span}(x_{p+1}, y_{p+1}, \ldots, x_m, y_m) \quad \text{and} \quad V_3 \stackrel{\text{def}}{=} V_0 \oplus V_2.$$

Then V_2 is symplectic and the subgroup of Q leaving V_2 invariant is isomorphic to $\mathrm{Sp}_{2m-2p}(\mathbb{F})$, hence generated by unipotents. Also, for $i = 1, \ldots, p$, by considering the identity

$$\omega(gy_i, x_j) = \omega(gy_i, gx_j) = \omega(y_i, x_j) \quad (j = 1, \ldots, p)$$

one sees that any $g \in Q$ must map the y_i to vectors in $y_i + V_3$. This implies that Q is generated by symplectic matrices leaving V_2 invariant, and transvections mapping y_i to elements of $y_i + V_3$. In particular, Q is generated by unipotents. □

Definition 7.3. *Given a real algebraic group $J \subset \mathrm{ASL}_n(\mathbb{R})$, we will say that a locally finite measure τ on $\mathbb{R}^{dp}$ is* J-c&p-algebraic *if there is a J-algebraic measure $\bar\tau$ on $\mathbb{R}^{np}$ such that for every $f \in C_c(\mathbb{R}^{dp})$ we have*

$$\int_{\mathbb{R}^{dp}} f\, d\tau = \int_{\mathbb{R}^{np}} F\, d\bar\tau,$$

where $F : \mathbb{R}^{np} \to \mathbb{R}$ is defined by

$$F(x_1, \ldots, x_p) \stackrel{\text{def}}{=} \begin{cases} f\left(\pi_{\mathrm{phys}}(x_1), \ldots, \pi_{\mathrm{phys}}(x_p)\right) & \forall i,\ \pi_{\mathrm{int}}(x_i) \in W \\ 0 & \textit{otherwise}\ . \end{cases} \tag{7.10}$$

We will say τ is c&p algebraic *if it is J-c&p algebraic for some J.*

It is easy to check that for $p = 1$, the measure τ in Definition 7.3 is the pushforward under π_{phys} of the restriction of $\bar\tau$ to $\pi_{\mathrm{int}}^{-1}(W)$. For general p, define projections

$${}^p\pi_{\mathrm{phys}} : \mathbb{R}^{np} \to \mathbb{R}^{dp}, \quad {}^p\pi_{\mathrm{phys}}(x_1, \ldots, x_p) \stackrel{\text{def}}{=} \left(\pi_{\mathrm{phys}}(x_1), \ldots, \pi_{\mathrm{phys}}(x_p)\right),$$

and

$$ {}^p\pi_{\mathrm{int}} : \mathbb{R}^{np} \to \mathbb{R}^{mp}, \quad {}^p\pi_{\mathrm{int}}(x_1, \dots, x_p) \stackrel{\mathrm{def}}{=} (\pi_{\mathrm{int}}(x_1), \dots, \pi_{\mathrm{int}}(x_p)) . $$

Then the measures τ, $\bar{\tau}$ satisfy

$$ \tau = {}^p\pi_{\mathrm{phys}*}\left(\bar{\tau}|_{\mathcal{S}}\right), \text{ where } \mathcal{S} \stackrel{\mathrm{def}}{=} {}^p\pi_{\mathrm{int}}^{-1}\left(\underbrace{W \times \cdots \times W}_{p \text{ copies}}\right). \tag{7.11} $$

Proof of Theorem 1.3. By Theorem 4.1, after a rescaling of $\mathbb{R}^d$, there is a homogeneous measure $\bar{\mu}$ on $\mathscr{Y}_n$ such that $\mu = \Psi_* \bar{\mu}$. Suppose $h \in H$ satisfies that $\pi_{\mathrm{phys}}|_{h\mathcal{L}_1}$ is injective, and in the linear case, assume also that $h\mathcal{L}_1 \cap V_{\mathrm{int}} \subset \{0\}$. Since μ is an RMS measure, and in the linear case, arguing as in the proof of Theorem 1.2 using Lemma 4.2, we see that this holds for a.e. $h \in H$. For such h, letting $\Lambda_h \stackrel{\mathrm{def}}{=} \Psi(h\mathcal{L}_1)$, we can rewrite the function ${}^p\widehat{f}$ defined in (1.7) more succinctly in the form

$$ {}^p\widehat{f}(\Lambda_h) = \sum_{(x_1,\dots,x_p) \in \mathcal{L}_1^p} F(hx_1, \dots, hx_p), $$

where F is as in (7.10). Thus, Theorem 1.3 is reduced to Theorem 7.1. □

Remark 7.4. *The assignment $\mathfrak{e} \mapsto \bar{\tau}_{\mathfrak{e}}$ implicit in the proof of Theorem 1.3 is not injective, nor is it finite-to-one. To see this, take $p = 1$ and consider the RMS measure corresponding to the Haar-Siegel measure on $\mathscr{X}_n$. Then $H_1 = \mathrm{SL}_n(\mathbb{R})$, $\Gamma_{H_1} = \mathrm{SL}_n(\mathbb{Z})$, and there are countably many Γ_{H_1}-orbits on $\mathbb{Z}^n$, where two integer vectors belong to the same orbit if and only if the greatest common divisor of their coefficients is the same. On the other hand, as the proof of formula* (1.4) *shows, there are two c&p-algebraic measures, namely Lebesgue measure on $\mathbb{R}^d$ and the Dirac measure at 0. The Dirac measure is associated with the orbit of $0 \in \mathbb{Z}^n$, and all the other orbits of nonzero vectors in $\mathbb{Z}^n$ give rise to multiples of Lebesgue measure on $\mathbb{R}^d$.*

Nevertheless, we will continue using the symbol $\mathfrak{E}$ for both the collection of Γ_{H_1}-orbits in $\mathbb{Z}^{np}$, and for the indexing set for the countable collection of measure arising in Theorem 1.3. This should cause at most mild confusion.

8. The Rogers inequality on moments

In this section we will prove Theorem 1.4. We will need more information about the measures $\tau_{\mathfrak{e}}$ appearing in Theorem 1.3, in case $p = 2$. We begin our discussion with some properties that are valid for all $p \leqslant d$. Some of the results of §8.1 will be given in a greater level of

generality than required for our counting results. They are likely to be of use in understanding higher moments for RMS measures.

8.1. **Normalizing the measures.** For any k, denote the normalized Lebesgue measure on $\mathbb{R}^k$ by $\mathrm{vol}^{(k)}$. Some of the c&p-algebraic measures τ on $\mathbb{R}^{dp}$ which arise in Theorem 1.3 are the globally supported Lebesgue measures on $\mathbb{R}^{dp}$, i.e., multiples of $\mathrm{vol}^{(dp)}$. Indeed, such a measure arises if in Definition 7.3 we take $\bar\tau$ equal to a multiple of Lebesgue measure on $\mathbb{R}^{np}$. These measures give a main term in the counting problem we will consider in §10. We write $\tau_1 \propto \tau_2$ if τ_1, τ_2 are proportional, recall the measures $\{\tau_{\mathfrak{e}}\}$ defined in the proofs of Theorems 1.3 and 7.1, and set

$$\mathfrak{E}^{\mathrm{main}} \stackrel{\mathrm{def}}{=} \left\{ \mathfrak{e} \in \mathfrak{E} : \tau_{\mathfrak{e}} \propto \mathrm{vol}^{(dp)} \right\}, \qquad \tau_{\mathrm{main}} \stackrel{\mathrm{def}}{=} \sum_{\mathfrak{e} \in \mathfrak{E}^{\mathrm{main}}} \tau_{\mathfrak{e}}.$$

We define constants $c_{\mu,p}$ by the condition

$$\tau_{\mathrm{main}} = c_{\mu,p}\, \mathrm{vol}^{(dp)}.$$

The next result identifies the normalizing constants $c_{\mu,p}$. Recall from Theorem 4.1 that an RMS measure μ is of the form $\mu = \rho_{c*}\bar\mu$ where $\bar\mu$ is a homogeneous measure on $\mathscr{Y}_n$, c is the constant of (4.1), and μ-a.e. Λ is of the form $\Lambda = \Lambda(\mathcal{L}, W)$ for a grid $\mathcal{L}$ with $\mathrm{covol}(\mathcal{L}) = c^n$. We denote this almost-sure value of $\mathrm{covol}(\mathcal{L})$ by $\mathrm{covol}(\mu)$. Recall also that the function $\Lambda \mapsto D(\Lambda)$ defined in (1.10) is measurable and invariant, and hence is a.e. constant, and denote its almost-sure value by $D(\mu)$.

Proposition 8.1. *For any RMS measure $\mu = \rho_{c*}\Psi_*\bar\mu$ satisfying* (1.8) *(i.e., $\mathbf{G} = \mathrm{SL}_k$ or μ is affine), we have*

$$c_{\mu,1} = D(\mu) = \frac{\mathrm{vol}^{(m)}(W)}{\mathrm{covol}(\mu)}, \tag{8.1}$$

and for $p \in \mathbb{N}$ satisfying $p < q_\mu$ and $p \leqslant d$ we have

$$c_{\mu,p} = c_{\mu,1}^p. \tag{8.2}$$

Note that the normalizing constant $c_{\mu,1}$ discussed here is the same as the constant denoted by c_1 in (7.2) and by c in (1.4).

With the identification $\mathbb{R}^{\ell p} \cong M_{\ell,p}(\mathbb{R})$ in mind, we say that a subspace $V \subset \mathbb{R}^{\ell p}$ is an *annihilator subspace* if it is the common annihilator of a collection of vectors in $\mathbb{R}^p$; that is, there is a collection $\mathrm{Ann} \subset \mathbb{R}^p$ such that

$$\begin{aligned} V &= \mathcal{Z}(\mathrm{Ann}) \\ &\stackrel{\mathrm{def}}{=} \left\{ (v_1, \dots, v_p) \in \mathbb{R}^{\ell p} : \forall i,\ v_i \in \mathbb{R}^\ell \,\&\, \forall (a_1, \dots, a_p) \in \mathrm{Ann},\ \sum a_i v_i = 0 \right\}. \end{aligned}$$

Note that the meaning of $\mathcal{Z}(\mathrm{Ann})$ depends on the choice of the ambient space $\mathbb{R}^\ell$ containing the vectors v_i; when confusion may arise we will specify the ambient space explicitly.

Suppose $\ell \in \mathbb{N}$ and $(v_1, \dots, v_p)$ is a p-tuple in $\mathbb{R}^{\ell p}$. In the linear case, let

$$\mathrm{Ann}(v_1, \dots, v_p) \stackrel{\mathrm{def}}{=} \{(a_1, \dots, a_p) \in \mathbb{R}^p : \sum a_i v_i = 0\},$$

and in the affine case, let

$$\mathrm{Ann}(v_1, \dots, v_p) \stackrel{\mathrm{def}}{=} \{(a_1, \dots, a_{p-1}) \in \mathbb{R}^{p-1} : \sum a_i (v_i - v_p) = 0\}.$$

Let

$$L(v_1, \dots, v_p) \stackrel{\mathrm{def}}{=} \mathcal{Z}(\mathrm{Ann}(v_1, \dots, v_p)),$$

an annihilator subspace in $\mathbb{R}^{\ell p}$, say that $v_1, \dots, v_p$ are *independent* if $\mathrm{Ann}(v_1, \dots, v_p) = \{0\}$, and let

$$\mathrm{rank}(v_1, \dots, v_p) \stackrel{\mathrm{def}}{=} \begin{cases} p - \dim \mathrm{Ann}(v_1, \dots, v_p) & \mu \text{ is linear} \\ p - 1 - \dim \mathrm{Ann}(v_1, \dots, v_p) & \mu \text{ is affine.} \end{cases}$$

Note that in the linear case, this is the usual relation between the rank of a matrix and the dimension of its kernel. The dimension of $L(v_1, \dots, v_p)$ is equal to $\ell \,\mathrm{rank}(v_1, \dots, v_p)$.

We recall some notation from §2.4 and from Step 3 of the proof of Lemma 3.2. Let $\mathbb{K}$ be a real number field of degree $D = r + 2s$, with $\sigma_1, \dots, \sigma_r$ being distinct real embeddings, and $\sigma_{r+1}, \dots, \sigma_s$ denoting representatives of conjugate pairs of non-real embeddings. Let $\mathbf{G}$ be isomorphic to either $\mathrm{SL}_k(\mathbb{R})$ or to $\mathrm{Sp}_{2k}(\mathbb{R})$, and let $\mathbf{H} = \mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{G})$. Let $\mathbf{V}$ be a $\mathbb{K}$-vector space of dimension t, where t is as in (6.12), and denote $V_j = \mathbf{V}_{\mathbb{R}}^{\sigma_j}$, that is, $V_j \cong \mathbb{R}^t$ if $j = 1, \dots, r$ and $V_j \cong \mathbb{C}^t \cong \mathbb{R}^{2t}$ if $j = r+1, \dots, s$. These vector spaces are chosen so that $\mathbf{V}$ is equipped with the standard action of $\mathbf{G}$, and taking into account the isomorphism

$$\mathbb{R}^n \cong \left(\mathrm{Res}_{\mathbb{K}/\mathbb{Q}}(\mathbf{V})\right)_{\mathbb{R}} = V_1 \oplus \cdots \oplus V_{r+s}. \tag{8.3}$$

Let $\pi^{\sigma_j} : \mathbb{R}^n \to V_j$ be the corresponding projections. In the notation (2.5), let $\pi^j : \mathbf{H}_{\mathbb{R}} \to \mathbf{G}_{\mathbb{R}}^{\sigma_j}$, so that the action of $\mathbf{H}_{\mathbb{R}}$ factors through the action of each $\mathbf{G}_{\mathbb{R}}^{\sigma_j}$ on V_j. We can assume without loss of generality (see §2.1) that $V_2 \oplus \cdots \oplus V_{r+s} \subset V_{\mathrm{int}}$ and $\pi_{\mathrm{phys}} = \pi_{\mathrm{phys}} \circ \pi^{\sigma_1}$.

Lemma 8.2. *Suppose μ is an RMS measure of higher rank, and let $\mathbf{G}$ be the group appearing in Theorem 1.1. Let $p < q_\mu$, let $\vec{x}_{\mathfrak{e}} = (x_1, \dots, x_p) \in \mathfrak{e}$, where $\mathfrak{e} \in \mathfrak{E}$ is as defined before (7.7), and let $v_i \stackrel{\mathrm{def}}{=} \pi^{\sigma_1}(x_i)$, $i = 1, \dots, p$. Assume that*

$$\mathrm{rank}(v_1, \dots, v_p) \leqslant \begin{cases} d & \text{if } \mathbf{G} = \mathrm{SL}_k \\ 1 & \text{if } \mathbf{G} = \mathrm{Sp}_{2k}. \end{cases}$$

Let $\bar{\tau} \stackrel{\text{def}}{=} \bar{\tau}_{\mathfrak{e}}$ be the algebraic measure on $\mathbb{R}^{np}$ as in (7.9) *and let τ be a c&p-algebraic measure obtained from $\bar{\tau}$ as in Definition 7.3. Then τ is (up to proportionality) the Lebesgue measure on some annihilator subspace of $\mathbb{R}^{dp}$. This subspace is equal to $\mathbb{R}^{dp}$ if and only if $v_1, \ldots, v_p$ are independent.*

Proof. Let $\bar{\tau}$ be as in Definition 7.3. As in the proof of Theorem 7.1), we have that $H(x_1, \ldots, x_p)$ is a dense subset of full measure in $\operatorname{supp} \bar{\tau}$. We will split the proof according to the various cases arising in Theorem 3.1.

Case 1: μ is linear, $\mathbf{G} = \mathrm{SL}_k$. In this case, our proof will also show that $\operatorname{supp} \bar{\tau}$ is a sum of annihilator subspaces, one in each V_j; in fact, we first establish this statement.

The action of H on $\mathbb{R}^n$ factors into a product of actions of each $\mathbf{G}_{\mathbb{R}}^{\sigma_j}$ on V_j. That is, H acts on $v_i^j \stackrel{\text{def}}{=} \pi^{\sigma_j}(x_i)$, $i = 1, \ldots, p$ via its mapping to $\mathbf{G}_{\mathbb{R}}^{\sigma_j}$, i.e., via the standard action of $\mathrm{SL}_k(\mathbb{R})$ or $\mathrm{SL}_k(\mathbb{C})$ on $\mathbb{R}^k$ or $\mathbb{C}^k$. It follows from (1.5) and (1.6) that $p < q_\mu = k$. Therefore for each j, the rank R_j of $\left\{v_i^j : i = 1, \ldots, p\right\}$ is less than k. For the standard action, $\mathbf{G}_{\mathbb{R}}^{\sigma_j}$ is transitive on linearly independent R_j-tuples. From this, by choosing a linearly independent subset $B_j \subset \{v_1^j, \ldots, v_p^j\}$ of cardinality R_j and expressing any $v_i^j \notin B_j$ as a linear combination of elements of B_j, one sees that if $(u_1, \ldots, u_p), (w_1, \ldots, w_p)$ are two p-tuples in V_j

$$\begin{aligned} &\text{there is } h \in \mathbf{G}_{\mathbb{R}}^{\sigma_j} \text{ such that } h(w_1, \ldots, w_p) = (u_1, \ldots, u_p) \\ \iff &\operatorname{Ann}(w_1, \ldots, w_p) = \operatorname{Ann}(u_1, \ldots, u_p). \end{aligned} \tag{8.4}$$

This implies that $\mathbf{G}_{\mathbb{R}}^{\sigma_j}(v_1^j, \ldots, v_p^j)$ is open and dense in $L(v_1^j, \ldots, v_p^j)$, and hence $H(x_1, \ldots, x_p)$ is open and dense in $L_1^{r+s} \stackrel{\text{def}}{=} \bigoplus_{j=1}^{r+s} L(v_1^j, \ldots, v_p^j)$. We have shown that $\operatorname{supp} \bar{\tau} = L_1^{r+s}$ and that $\bar{\tau}$ is a multiple of the Lebesgue measure on L_1^{r+s}.

Since $\pi_{\text{phys}} = \pi_{\text{phys}} \circ \pi^{\sigma_1}$, we have

$${}^p\pi_{\text{phys}}\left(L_1^{r+s}\right) = {}^p\pi_{\text{phys}}\left(L(v_1, \ldots, v_p)\right).$$

To simplify notation, write $H^1 \stackrel{\text{def}}{=} \mathbf{G}_{\mathbb{R}}^{\sigma_1} \cong \mathrm{SL}_k(\mathbb{R})$, and $v_i \stackrel{\text{def}}{=} v_i^1 \in V_1$. Let

$$\operatorname{Ann}_1 \stackrel{\text{def}}{=} \operatorname{Ann}\left(v_1, \ldots, v_p\right).$$

We have

$${}^p\pi_{\text{phys}}(L(v_1, \ldots, v_p)) = \mathcal{Z}(\operatorname{Ann}_1), \tag{8.5}$$

seen as an annihilator subspace of $\mathbb{R}^{dp}$. Indeed, the inclusion $\subset$ follows from linearity of π_{phys}. For the opposite inclusion, recall that we have

an inclusion $V_{\text{phys}} \hookrightarrow V_1$, and this induces an inclusion $\iota : \mathbb{R}^{dp} \hookrightarrow \mathbb{R}^{np}$. We clearly have

$$\iota\left(\mathcal{Z}\left(\text{Ann}_1\right)\right) \subset L(v_1, \ldots, v_p),$$

which implies the inclusion $\supset$ in (8.5).

Replacing x_i with elements of $x_i + V_{\text{phys}}$ does not change the condition $(x_1, \ldots, x_p) \in \mathcal{S}$, where $\mathcal{S}$ is as in (7.11). This shows that

$$\operatorname{supp} \tau = {}^{p}\pi_{\text{phys}}\left(L_1^{r+s}\right) = {}^{p}\pi_{\text{phys}}\left(L(v_1, \ldots, v_p)\right)$$

is an annihilator subspace, and τ is a multiple of Lebesgue measure on this subspace. Moreover, the subspace is proper if and only if $\text{Ann}_1 \neq \{0\}$, or equivalently, $v_1, \ldots, v_p$ are dependent.

Case 2: μ is linear, $\mathbf{G} = \text{Sp}_{2k}$, $d = 2$. The action of H splits as a Cartesian product of actions of the groups $\mathbf{G}_{\mathbb{R}}^{\sigma_j}$ on the spaces V_j, for $j = 1, \ldots, r+s$. As in Case 1, we will pay attention to the action on the first summand V_1, where H acts via $H^1 \stackrel{\text{def}}{=} \mathbf{G}_{\mathbb{R}}^{\sigma_1} \cong \text{Sp}_{2k}(\mathbb{R})$. We denote by ω the symplectic form on V_1 preserved by H^1. Let $L \stackrel{\text{def}}{=} \overline{H(x_1, \ldots, x_p)} = \operatorname{supp} \bar{\tau}$, where $\bar{\tau}$ is the unique (up to scaling) H-invariant measure with support L, and let $L^1 \stackrel{\text{def}}{=} L \cap V_1 = \pi^{\sigma_1}(L) = \overline{H^1\left(v_1, \ldots, v_p\right)}$, where $v_i \stackrel{\text{def}}{=} \pi^{\sigma_1}(x_i)$, $i = 1, \ldots, p$.

Let $F \cong \text{SL}_2(\mathbb{R})$ be as in (3.1). Then $F \subset H^1$, and hence τ is F-invariant. Write

$$V_{\text{int}}^1 \stackrel{\text{def}}{=} V_{\text{int}} \cap V_1 = V_{\text{phys}}^{\perp},$$

and abusing notation slightly, let $\pi_{\text{phys}}, \pi_{\text{int}}$ denote the restrictions of these mappings to V_1, so they are the projections associated with the direct sum decomposition $V_1 = V_{\text{phys}} \oplus V_{\text{int}}^1$. Define $R \stackrel{\text{def}}{=} \text{rank}(v_1, \ldots, v_p)$, and define R' as the maximal rank of $\{\pi_{\text{phys}}(hv_1), \ldots, \pi_{\text{phys}}(hv_p)\}$, as h ranges over elements of H^1. Thus, $0 \leqslant R' \leqslant R \leqslant 1$.

If $R' = 0$ this means that $\pi_{\text{phys}}(hv_i) = 0$ for all $h \in H$ and all i, and then τ is the Dirac measure at 0, and there is nothing to prove. Now suppose $R' = R = 1$. Since $R = 1$, there is some v_i such that $\pi_{\text{phys}}(v_i) \neq 0$, and there are coefficients a_j, $j \neq i$ so that $v_j = a_j v_i$. This implies that for all h, $\pi_{\text{phys}}(hv_j) = a_j \pi_{\text{phys}}(hv_i)$, that is,

$$\operatorname{supp} \tau \subset {}^{p}\pi_{\text{phys}}(L) \subset L' \stackrel{\text{def}}{=} \{(u_1, \ldots, u_p) \in \mathbb{R}^{2p} : \forall j \neq i,\ u_j = a_j u_i\}.$$

Moreover, since F acts transitively on nonzero vectors in V_{phys}, and τ is F-invariant, we actually have equality and τ is a multiple of Lebesgue measure on the annihilator subspace L', and L' is a proper subspace of $\mathbb{R}^{2p}$, unless $p = 1$.

Case 3: μ is affine. The affine case can be reduced to the linear case. Note that the definition of the annihilator $\text{Ann}(v_1, \ldots, v_p)$ in the

affine case is such that it does not change under the diagonal action of the group of translations, and that the group of translations in H is the full group $\mathbb{R}^n$, so that $x_1, \dots, x_p$ can be moved so that $x_p = 0$. Moreover, by Proposition 5.3, we can assume that $0 \in W$. We leave the details to the diligent reader. □

Let

$$\mathfrak{E}^{\mathrm{rest}} \stackrel{\mathrm{def}}{=} \mathfrak{E} \smallsetminus \mathfrak{E}^{\mathrm{main}}, \quad \tau_{\mathrm{rest}} \stackrel{\mathrm{def}}{=} \sum_{\mathfrak{e} \in \mathfrak{E}^{\mathrm{rest}}} \tau_{\mathfrak{e}}. \tag{8.6}$$

The preceding discussion gives a description of the measures $\tau_{\mathfrak{e}}$ with $\mathfrak{e} \in \mathfrak{E}^{\mathrm{rest}}$.

Corollary 8.3. *Under the conditions of Lemma 8.2, any measure $\tau_{\mathfrak{e}}$, $\mathfrak{e} \in \mathfrak{E}^{\mathrm{rest}}$, is Lebesgue measure on a proper subspace of $\mathbb{R}^{dp}$.*

Proof of Proposition 8.1. Let B_r denote the Euclidean ball of radius r around the origin in $\mathbb{R}^d$, let $\mathbf{1}_{B_r}$ be its indicator function, and let $\widehat{\mathbf{1}_{B_r}}$ be the function obtained from the summation formula (1.3), so that

$$D(\Lambda) = \lim_{r \to \infty} \frac{\widehat{\mathbf{1}_{B_r}}(\Lambda)}{\mathrm{vol}^{(d)}(B_r)}.$$

Applying (1.4) we get that for any $r > 0$,

$$\int_{\mathscr{C}(\mathbb{R}^d)} \frac{\widehat{\mathbf{1}_{B_r}}}{\mathrm{vol}^{(d)}(B_r)} \, d\mu = \frac{c_{\mu,1}}{\mathrm{vol}^{(d)}(B_r)} \int_{\mathbb{R}^d} \mathbf{1}_{B_r} \, d\mathrm{vol} = c_{\mu,1}. \tag{8.7}$$

Suppose $\Lambda = \Lambda(\mathcal{L}, W)$. We claim that for $r \geqslant 1$,

$$\widehat{\mathbf{1}_{B_r}}(\Lambda) \ll \mathrm{vol}^{(d)}(B_r) \alpha(\underline{\mathcal{L}}), \tag{8.8}$$

where $\Lambda = \Lambda(\mathcal{L}, W)$ and $\underline{\mathcal{L}} = \underline{\pi}(\mathcal{L})$, and where the implicit constant depends on d, n and W. Indeed, we can replace W with a larger convex set containing it, so that $\widehat{\mathbf{1}_{B_r}}(\Lambda)$ is bounded from above by $\#\,(K \cap \mathcal{L})$, where $K \stackrel{\mathrm{def}}{=} B_r \times W$. It is known (see [GL87, Chap. 2 §9.4] or [Wid12, Prop. 2.9]) that for any dimension n, for any bounded convex set K' and any lattice $\mathcal{L}' \subset \mathbb{R}^n$, if $K' \cap \mathcal{L}'$ is not contained in a proper affine subspace of $\mathbb{R}^n$, then

$$\#\,(K' \cap \mathcal{L}') \leqslant n! \, \frac{\mathrm{vol}(K')}{\mathrm{covol}(\mathcal{L}')} + n.$$

For any $\mathcal{L}$ we let x_0 be a translation vector such that $\mathcal{L} + x_0 = \underline{\mathcal{L}}$, set

$$V \stackrel{\mathrm{def}}{=} \mathrm{span}(\underline{\mathcal{L}} \cap (K + x_0)), \quad \ell \stackrel{\mathrm{def}}{=} \dim V, \quad \mathcal{L}' \stackrel{\mathrm{def}}{=} \underline{\mathcal{L}} \cap V, \quad K' \stackrel{\mathrm{def}}{=} V \cap (K + x_0),$$

and apply this estimate in $V \cong \mathbb{R}^\ell$ with $\ell \leqslant n$. For $r \geqslant 1$ we have $\mathrm{vol}^{(\ell)}(K') \ll r^d$ and $\mathrm{covol}(\mathcal{L}') \gg \lambda_1(\underline{\mathcal{L}}) \cdots \lambda_\ell(\underline{\mathcal{L}})$. Thus

$$\#(K \cap \mathcal{L}) = \#(K' \cap \mathcal{L}') \ll \ell! \frac{r^d}{\lambda_1(\mathcal{L}) \cdots \lambda_\ell(\mathcal{L})} + \ell \ll \mathrm{vol}^{(d)}(B_r)\alpha(\underline{\mathcal{L}}),$$

establishing (8.8) and proving the claim. Therefore, using Proposition 6.1 and the dominated convergence theorem, we are justified in taking a limit $r \to \infty$ inside the integral (8.7), finding that $c_{\mu,1} = D(\mu)$. Combining this with (1.10) gives (8.1). See [MS14, Proof of Thm. 1.5] for a different proof of (8.1).

Now to prove (8.2), let Q_r and Q_r^p denote the unit cube of sidelength r in $\mathbb{R}^d$ and $\mathbb{R}^{dp}$ respectively, let $\mathbf{1}_{Q_r}$ and $\mathbf{1}_{Q_r^p}$ be the indicator functions, and define ${}^p\widehat{\mathbf{1}_{Q_r^p}}$ via (1.7). Then we have

$${}^p\widehat{\mathbf{1}_{Q_r^p}}(\Lambda) = \#\bigtimes^p (Q_r \cap \Lambda);$$

that is, the number of p-tuples of elements of Λ in the p-fold Cartesian product Q_r^p. This implies that for μ-a.e. Λ,

$$\lim_{r\to\infty} \frac{{}^p\widehat{\mathbf{1}_{Q_r^p}}(\Lambda)}{r^{dp}} = \left(\lim_{r\to\infty} \frac{\#(Q_r \cap \Lambda)}{\mathrm{vol}^{(d)}(Q_r)}\right)^p = D(\Lambda)^p = c_{\mu,1}^p. \tag{8.9}$$

By Theorem 1.3 we have:

$$\begin{aligned} c_{\mu,p} =& \frac{1}{r^{dp}} \int_{\mathbb{R}^{dp}} \mathbf{1}_{Q_r^p}\, d\tau_{\mathrm{main}} = \frac{1}{r^{dp}} \left[\int_{\mathbb{R}^{dp}} \mathbf{1}_{Q_r^p}\, d\tau - \int_{\mathbb{R}^{dp}} \mathbf{1}_{Q_r^p}\, d\tau_{\mathrm{rest}}\right] \\ =& \int \frac{{}^p\widehat{\mathbf{1}_{Q_r^p}}(\Lambda)}{r^{dp}}\, d\mu - \frac{1}{r^{dp}} \sum_{\mathfrak{e}\in\mathfrak{E}^{\mathrm{rest}}} \int_{\mathbb{R}^{dp}} \mathbf{1}_{Q_r^p}\, d\tau_{\mathfrak{e}}. \end{aligned} \tag{8.10}$$

Repeating the argument establishing (8.8), we find

$${}^p\widehat{\mathbf{1}_{Q_r}}(\Lambda) \ll \left(\mathrm{vol}^{(d)}(Q_r)\right)^p \alpha(\underline{\mathcal{L}})^p,$$

and thus the integrable function α^p dominates the integral in the second line of (8.10), independently of r. Moreover, since they differ by a constant, α^p also dominates the series in the second line of (8.10). Using (8.9), the first integral gives $c_{\mu,1}^p$, and thus it remains to show that

$$\lim_{r\to\infty} \frac{1}{r^{dp}} \int_{\mathbb{R}^{dp}} \mathbf{1}_{Q_r^p}\, d\tau_{\mathfrak{e}} = 0, \quad \text{for every } \mathfrak{e} \in \mathfrak{E}^{\mathrm{rest}}. \tag{8.11}$$

From (1.8) and Corollary 8.3 we have that $\tau_{\mathfrak{e}}$ is (up to proportionality) equal to Lebesgue measure on a subspace $V' \subset \mathbb{R}^{dp}$, and we have $V' \neq \mathbb{R}^{dp}$ since $\mathfrak{e} \in \mathfrak{E}^{\mathrm{rest}}$. This implies (8.11). □

Remark 8.4. *One can also work in $\mathbb{R}^{np}$ rather than $\mathbb{R}^{dp}$, and define analogous normalization constants $\bar{c}_{\bar{\mu},p}$ by the formula $\bar{\tau}_{\mathrm{main}} = \bar{c}_{\bar{\mu},p}\,\mathrm{vol}^{(np)}$. Then one can show that $\bar{c}_{\bar{\mu},p} = 1$ for all $p < q_\mu$. We will not need the values of these constants and leave the proofs to the interested reader.*

8.2. **More details for $p = 2$.** We will need to describe the measure τ_{rest} in the case $p = 2$.

Proposition 8.5. *Let μ be an RMS measure so that* (1.8) *holds. Let $p = 2$, and let $\mathfrak{E}^{\mathrm{rest}}$, τ_{rest} be as in* (8.6)*. Then there is a partition $\mathfrak{E}^{\mathrm{rest}} = \mathfrak{E}_1^{\mathrm{rest}} \sqcup \mathfrak{E}_2^{\mathrm{rest}}$, and constants $\{a_{\mathfrak{e}} : \mathfrak{e} \in \mathfrak{E}_2^{\mathrm{rest}}\}$, $\{b_{\mathfrak{e}} : \mathfrak{e} \in \mathfrak{E}_1^{\mathrm{rest}}\}$, $\{c_{\mathfrak{e}} : \mathfrak{e} \in \mathfrak{E}^{\mathrm{rest}}\}$, such that the following hold.*

(1) *For all $f \in C_c(\mathbb{R}^{2d})$, we have*

$$\int_{\mathbb{R}^{2d}} f\,d\tau_{\mathrm{rest}} = \sum_{\mathfrak{e}\in\mathfrak{E}_1^{\mathrm{rest}}} c_{\mathfrak{e}} \int_{\mathbb{R}^d} f(x, b_{\mathfrak{e}}x)\,d\mathrm{vol}^{(d)}(x) + \sum_{\mathfrak{e}\in\mathfrak{E}_2^{\mathrm{rest}}} c_{\mathfrak{e}} \int_{\mathbb{R}^d} f(a_{\mathfrak{e}}x, x)\,d\mathrm{vol}^{(d)}(x). \tag{8.12}$$

(2) *$c_{\mathfrak{e}} > 0$ for all $\mathfrak{e} \in \mathfrak{E}^{\mathrm{rest}}$ and $\sum_{\mathfrak{e}\in\mathfrak{E}^{\mathrm{rest}}} c_{\mathfrak{e}} < \infty$.*

(3) *$|a_{\mathfrak{e}}| \leqslant 1$ for all $\mathfrak{e} \in \mathfrak{E}_2^{\mathrm{rest}}$ and $|b_{\mathfrak{e}}| \leqslant 1$ for all $\mathfrak{e} \in \mathfrak{E}_1^{\mathrm{rest}}$.*

Proof. Lemma 8.2 is applicable in view of (1.8); indeed, when $\mathbf{G} = \mathrm{SL}_k$, we have $p = 2 \leqslant d$, and when $\mathbf{G} = \mathrm{Sp}_{2k}$ and μ is affine, we have $\mathrm{rank}(v_1, v_2) \leqslant 1$. Therefore, for each $\mathfrak{e} \in \mathfrak{E}^{\mathrm{rest}}$, there is an annihilator subspace $V_{\mathfrak{e}} \subsetneq \mathbb{R}^{dp}$ such that $\tau_{\mathfrak{e}}$ is proportional to Lebesgue measure on $V_{\mathfrak{e}}$. Repeating the argument of §7.1 we can see that $\tau_{\mathfrak{e}}$ is not the Dirac mass at the origin. In other words $V_{\mathfrak{e}}$ has positive dimension. Since $p = 2$, this means we can find α, β, not both zero, such that $V_{\mathfrak{e}} = \mathcal{Z}(\alpha, \beta)$. We can rescale so that $\max(|\alpha|, |\beta|) = \max(\alpha, \beta) = 1$ and we define

$$\mathfrak{E}_1^{\mathrm{rest}} \overset{\mathrm{def}}{=} \{\mathfrak{e} \in \mathfrak{E}^{\mathrm{rest}} : \beta = 1\}, \quad \mathfrak{E}_2^{\mathrm{rest}} \overset{\mathrm{def}}{=} \mathfrak{E}^{\mathrm{rest}} \smallsetminus \mathfrak{E}_1^{\mathrm{rest}}.$$

Then if we set $b_{\mathfrak{e}} = -\alpha$ for $\mathfrak{e} \in \mathfrak{E}_1^{\mathrm{rest}}$ and $a_{\mathfrak{e}} = -\beta$ for $\mathfrak{e} \in \mathfrak{E}_2^{\mathrm{rest}}$, then the bounds in (3) hold and we have

$$V_{\mathfrak{e}} = \begin{cases} \{(x, b_{\mathfrak{e}}x) : x \in \mathbb{R}^d\} & \text{for } \mathfrak{e} \in \mathfrak{E}_1^{\mathrm{rest}} \\ \{(a_{\mathfrak{e}}x, x) : x \in \mathbb{R}^d\} & \text{for } \mathfrak{e} \in \mathfrak{E}_2^{\mathrm{rest}}. \end{cases}$$

We now define $c_{\mathfrak{e}}$ by the formula

$$\forall f \in C_c(\mathbb{R}^{2d}), \quad \int_{\mathbb{R}^{2d}} f\,d\tau_{\mathfrak{e}} = \begin{cases} c_{\mathfrak{e}} \int_{\mathbb{R}^d} f(a_{\mathfrak{e}}x, x)\,d\mathrm{vol}^{(d)}(x) & \text{for } \mathfrak{e} \in \mathfrak{E}_1^{\mathrm{rest}} \\ c_{\mathfrak{e}} \int_{\mathbb{R}^d} f(x, b_{\mathfrak{e}}x)\,d\mathrm{vol}^{(d)}(x) & \text{for } \mathfrak{e} \in \mathfrak{E}_2^{\mathrm{rest}}. \end{cases}$$

Then clearly (8.12) holds, and $c_{\mathfrak{e}} > 0$ for all $\mathfrak{e} \in \mathfrak{E}^{\mathrm{rest}}$.

It remains to show $\sum c_{\mathfrak{e}} < \infty$. Let $\mathbf{1}_B$ be the indicator of a ball in $\mathbb{R}^{2d}$ centered at the origin. Then there is a positive number λ which bounds from below all the numbers

$$\left\{ \int_{\mathbb{R}^d} \mathbf{1}_B(ax, x)\, d\mathrm{vol}^{(d)}(x) : |a| \leqslant 1 \right\} \bigcup \left\{ \int_{\mathbb{R}^d} \mathbf{1}_B(x, bx)\, d\mathrm{vol}^{(d)}(x) : |b| \leqslant 1 \right\}.$$

Since τ_{rest} is a locally finite measure, we have $\int_{\mathbb{R}^{2d}} \mathbf{1}_B\, d\tau_{\mathrm{rest}} < \infty$. But (8.12) implies that $\lambda \sum_{\mathfrak{e}\in\mathfrak{E}^{\mathrm{rest}}} c_{\mathfrak{e}} \leqslant \int_{\mathbb{R}^{2d}} \mathbf{1}_B\, d\tau_{\mathrm{rest}}$. Therefore $\sum_{\mathfrak{e}\in\mathfrak{E}^{\mathrm{rest}}} c_{\mathfrak{e}} < \infty$. □

Proof of Theorem 1.4. Given $f : \mathbb{R}^d \to [0,1]$ as in Theorem 1.4, define

$$\varphi : \mathbb{R}^{2d} \to [0,1] \quad \text{by } \varphi(x,y) \stackrel{\mathrm{def}}{=} f(x)f(y).$$

Clearly $\left(\int_{\mathbb{R}^d} f\, d\mathrm{vol}^{(d)}\right)^2 = \int_{\mathbb{R}^{2d}} \varphi\, d\mathrm{vol}^{(2d)}$, and it follows easily from (1.3) and (1.7) that

$$^2\widehat{\varphi}(\Lambda) = \hat{f}(\Lambda)^2. \tag{8.13}$$

Using (8.13), Theorem 1.3 with $p = 2$, (1.4), and (8.2) we have that

$$\begin{aligned}
&\int_{\mathscr{C}(\mathbb{R}^d)} \left| \hat{f}(\Lambda) - \int_{\mathscr{C}(\mathbb{R}^d)} \hat{f}\, d\mu \right|^2 d\mu(\Lambda) \\
=&\int_{\mathscr{C}(\mathbb{R}^d)} \hat{f}^2\, d\mu - \left[\int_{\mathscr{C}(\mathbb{R}^d)} \hat{f}(\Lambda)\, d\mu \right]^2 = \int_{\mathbb{R}^{2d}} \varphi\, d\tau - \left[c_{\mu,1} \int_{\mathbb{R}^d} f\, d\mathrm{vol}^{(d)} \right]^2 \\
=&c_{\mu,2} \int_{\mathbb{R}^{2d}} \varphi\, d\mathrm{vol}^{(2d)} + \int_{\mathbb{R}^{2d}} \varphi\, d\tau_{\mathrm{rest}} - c_{\mu,1}^2 \left[\int_{\mathbb{R}^d} f\, d\mathrm{vol}^{(d)} \right]^2 = \int_{\mathbb{R}^{2d}} \varphi\, d\tau_{\mathrm{rest}}.
\end{aligned}$$

It remains to show that

$$\int_{\mathbb{R}^{2d}} \varphi\, d\tau_{\mathrm{rest}} \ll \int_{\mathbb{R}^d} f\, d\mathrm{vol}^{(d)}, \tag{8.14}$$

where the implicit constant is allowed to depend on μ. And indeed, by Proposition 8.5, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \varphi\, d\tau_{\mathrm{rest}} \\
\stackrel{(8.12)}{=}& \sum_{\mathfrak{e}\in\mathfrak{E}_1^{\mathrm{rest}}} c_{\mathfrak{e}} \int_{\mathbb{R}^d} f(a_{\mathfrak{e}}x) f(x)\, d\mathrm{vol}(x) + \sum_{\mathfrak{e}\in\mathfrak{E}_2^{\mathrm{rest}}} c_{\mathfrak{e}} \int_{\mathbb{R}^d} f(x) f(b_{\mathfrak{e}}x)\, d\mathrm{vol}(x) \\
\stackrel{f\leqslant 1}{\leqslant}& \sum_{\mathfrak{e}\in\mathfrak{E}_1^{\mathrm{rest}}} c_{\mathfrak{e}} \int_{\mathbb{R}^d} f(x)\, d\mathrm{vol}(x) + \sum_{\mathfrak{e}\in\mathfrak{E}_2^{\mathrm{rest}}} c_{\mathfrak{e}} \int_{\mathbb{R}^d} f(x)\, d\mathrm{vol}(x) \\
=& \left(\sum_{\mathfrak{e}\in\mathfrak{E}^{\mathrm{rest}}} c_{\mathfrak{e}} \right) \int_{\mathbb{R}^d} f\, d\mathrm{vol}^{(d)}.
\end{aligned}$$

□

9. From bounds on correlations to a.e. effective counting

In this section we present two results which we will use for counting. The first is due to Schmidt [Sch60] but we recast it in a slightly more general form (see also [KS19, Thm. 2.9]). To simplify notation, for measurable $S \subset \mathbb{R}^n$, we will write $V_S \stackrel{\text{def}}{=} \text{vol}^{(n)}(S)$.

Theorem 9.1. *Let $n \in \mathbb{N}$ and let μ be a probability measure on $\mathscr{C}(\mathbb{R}^n)$. Let $\kappa \in [1,2)$, let $\Phi = \{B_\alpha : \alpha \in \mathbb{R}_+\}$ be an unbounded ordered family of Borel subsets of $\mathbb{R}^n$, and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. Suppose the following hypotheses are satisfied:*

(a) *The measure μ is supported on discrete sets, and for each $f \in L^1(\mathbb{R}^n, \text{vol})$, a Siegel-Veech transform as in (1.3) satisfies that $\widehat{f} \in L^2(\mu)$. Furthermore, there are positive a, b such that for any function $f : \mathbb{R}^n \to [0,1]$, $f \in L^1(\mathbb{R}^n, \text{vol})$, we have*

$$\int \widehat{f}\, d\mu = a \int_{\mathbb{R}^n} f \, d\text{vol} \tag{9.1}$$

and

$$\text{Var}_\mu(\widehat{f}) \stackrel{\text{def}}{=} \int \left| \widehat{f} - \int \widehat{f}\, d\mu \right|^2 d\mu \leqslant b \left(\int_{\mathbb{R}^n} f \, d\text{vol} \right)^\kappa. \tag{9.2}$$

(b) *The function ψ is non-decreasing, and satisfies $\int_0^\infty \frac{1}{\psi(x)}\, dx < \infty$.*

Then for μ-a.e. Λ, for every $S \in \Phi$

$$\#(S \cap \Lambda) = aV_S + O\left(V_S^{\frac{\kappa}{2}} \log(V_S)\, \psi(\log V_S)^{\frac{1}{2}} \right) \quad \text{as } V_S \to \infty. \tag{9.3}$$

Note that we allow defining $\widehat{f}$ as in either one of the linear or affine cases of (1.3), as long as the conditions in (a) are satisfied. For definiteness we will use the affine case, namely $\widehat{f} = \sum_{v \in \Lambda} f(v)$, so that $\widehat{\mathbf{1}_S}(\Lambda) = |S \cap \Lambda|$ for any subset $S \subset \mathbb{R}^n$ with indicator function $\mathbf{1}_S$. In the linear case we may have $\widehat{\mathbf{1}_S}(\Lambda) = |S \cap \Lambda| - 1$ or $\widehat{\mathbf{1}_S}(\Lambda) = |S \cap \Lambda|$ (depending on whether or not S contains 0), and the reader will have no difficulty adjusting the proof in this case.

Proof of Theorem 1.5 assuming Theorem 9.1. Taking $\kappa = 1$ and $\psi(t) = t^{1+\varepsilon}$, (9.3) becomes

$$\#(S \cap \Lambda) = aV_S + O\left(V_S^{\frac{1}{2}} (\log V_S)^{\frac{3}{2}+\varepsilon} \right) \quad \text{as } V_S \to \infty,$$

which implies (1.11). The hypotheses of Theorem 9.1 hold in the higher rank case by (1.4) and Theorem 1.4. □

Before giving the proof of Theorem 9.1 we will state the following more general result.

Theorem 9.2. *Let $d, m, n \in \mathbb{N}$ with $n = d + m$, let μ be a probability measure on $\mathscr{C}(\mathbb{R}^n)$, let $\lambda \in [0, 1)$, $\kappa \in [1, 2)$, let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, let $\Phi = \{B_\alpha : \alpha \in \mathbb{R}_+\}$ be an unbounded ordered family of Borel subsets of $\mathbb{R}^d$, and let $\{W_\alpha : \alpha \in \mathbb{R}_+\}$ be a collection of subsets of $\mathbb{R}^m$. Suppose that (a) and (b) of Theorem 9.1 are satisfied, and in addition:*

(c) *For any $N \in \mathbb{N}$ there is α such that $\mathrm{vol}^{(d)}(B_\alpha) = N$.*

(d) *Each W_α can be partitioned as a disjoint union $W_\alpha = \bigsqcup_{\ell=1}^{L_\alpha} C_\alpha(\ell)$, where $L_\alpha \asymp \left(\mathrm{vol}^{(d)}(B_\alpha)\right)^\lambda$, and where $w_\alpha \stackrel{\mathrm{def}}{=} \mathrm{vol}^{(m)}(C_\alpha(\ell))$ is the same for $\ell = 1, \dots, L_\alpha$, and is of order $\asymp \left(\mathrm{vol}^{(d)}(B_\alpha)\right)^{-\lambda}$.*

Denote $\bar{\Phi} \stackrel{\mathrm{def}}{=} \{B_\alpha \times W_\alpha : \alpha \in \mathbb{R}_+\}$ and for $S \in \bar{\Phi}$, denote $V_S \stackrel{\mathrm{def}}{=} \mathrm{vol}^{(n)}(S)$. Then for μ-a.e. Λ, for every $S \in \bar{\Phi}$

$$\#(S \cap \Lambda) = aV_S + O\left(V_S^{\frac{\kappa(1-\lambda)}{2}+\lambda} \log(V_S)\, \psi(\log V_S)^{\frac{1}{2}}\right), \quad \text{as } V_S \to \infty. \tag{9.4}$$

Note that for $\kappa = 1$ and $\psi(t) = t^{1+\varepsilon}$, (9.4) becomes

$$\#(S \cap \Lambda) = aV_S + O\left(V_S^{\frac{1+\lambda}{2}} \log {V_S}^{\frac{3}{2}+\varepsilon}\right). \tag{9.5}$$

Theorems 9.1 and 9.2 both follow from ideas developed by Schmidt in [Sch60]. We begin with Theorem 9.1, for which we need the following Lemmas.

By the definition of an unbounded ordered family, we can assume that for each $V > 0$ there is $\Omega \in \Phi$ such that $\mathrm{vol}(\Omega) = V$. For each $N \in \mathbb{N}$, let $S_N \in \Phi$ with $\mathrm{vol}(S_N) = N$ and let $\rho_N \stackrel{\mathrm{def}}{=} \mathbf{1}_{S_N}$ denote its indicator function. Given two integers $N_1 < N_2$, let

$$_{N_1}\rho_{N_2} \stackrel{\mathrm{def}}{=} \rho_{N_2} - \rho_{N_1}.$$

Since the S_N are nested, we have $_{N_1}\rho_{N_2} = \mathbf{1}_{S_{N_2} \smallsetminus S_{N_1}}$.

Lemma 9.3 (cf. [Sch60], Lemma 2)**.** *Let $T \in \mathbb{N}$ and let K_T be the set of all pairs of integers N_1, N_2 satisfying $0 \leqslant N_1 < N_2 \leqslant 2^T$, $N_1 = u2^t$, $N_2 = (u+1)2^t$, for integers u and $t \geqslant 0$. Then there exists $c > 0$ such that*

$$\sum_{(N_1,N_2)\in K_T} \mathrm{Var}_\mu(\widehat{{}_{N_1}\rho_{N_2}}) \leqslant c(T+1)2^{\kappa T}. \tag{9.6}$$

Proof. Indeed, (9.2) yields $\mathrm{Var}_\mu(\widehat{\mathbf{1}_{N_1}\rho_{N_2}}) \leqslant b(N_2-N_1)^\kappa$. Each value of $N_2-N_1=2^t$ for $0\leqslant t\leqslant T$ occurs 2^{T-t} times, hence

$$\sum_{(N_1,N_2)\in K_T}(N_2-N_1)^\kappa=\sum_{0\leqslant t\leqslant T}2^{T+(\kappa-1)t}\leqslant (T+1)2^{\kappa T}.$$

□

Lemma 9.4 (cf. [Sch60], Lemma 3)**.** *For all $T\in\mathbb{N}$ there exists a subset $\mathrm{Bad}_T\subset\operatorname{supp}\mu$ of measure*

$$\mu(\mathrm{Bad}_T)\leqslant c\psi(T\log 2-1)^{-1} \tag{9.7}$$

such that

$$(\widehat{\rho_N}(\Lambda)-aN)^2\leqslant T(T+1)2^{\kappa T}\psi(T\log 2-1) \tag{9.8}$$

for every $N\leqslant 2^T$ and all $\Lambda\in\operatorname{supp}\mu\smallsetminus\mathrm{Bad}_T$.

Proof. Let Bad_T be the set of $\Lambda\in\operatorname{supp}\mu$ for which it is not true that

$$\sum_{(N_1,N_2)\in K_T}\left(\widehat{\mathbf{1}_{N_1}\rho_{N_2}}(\Lambda)-a(N_2-N_1)\right)^2\leqslant (T+1)2^{\kappa T}\psi(T\log 2-1). \tag{9.9}$$

Then the bound (9.7) follows from Lemma 9.3 by Markov's inequality. Assume $N\leqslant 2^T$ and $\Lambda\in\operatorname{supp}\mu\smallsetminus\mathrm{Bad}_T$. The interval $[0,N)$ can be expressed as a union of intervals of the type $[N_1,N_2)$, where $(N_1,N_2)\in\mathcal{I}_N\subset K_T$ and $|\mathcal{I}_N|\leqslant T$. Therefore, $\widehat{\rho_N(\Lambda)}-aN=\sum(\widehat{\mathbf{1}_{N_1}\rho_{N_2}}(\Lambda)-a(N_2-N_1))$, where the sum is over $(N_1,N_2)\in\mathcal{I}_N$. Applying the Cauchy-Schwarz inequality to the square of this sum together with the bound from (9.9) we obtain (9.8). □

Proof of Theorem 9.1. Let Bad_T be the sets from Lemma 9.4. Since ψ^{-1} is integrable and monotone, we find by Borel-Cantelli and (9.7) that for μ-a.e. Λ there is T_Λ such that for any $T\geqslant T_\Lambda$, $\Lambda\notin\mathrm{Bad}_T$. Assume now $N\geqslant N_\Lambda=2^{T_\Lambda}$ and let T be the unique integer for which $2^{T-1}\leqslant N<2^T$. By Lemma 9.4,

$$(\widehat{\rho_N}(\Lambda)-aN)^2\leqslant T(T+1)2^{\kappa T}\psi(T\log 2-1)=O\left(N^\kappa(\log N)^2\psi(\log N)\right). \tag{9.10}$$

Given arbitrary $S\in\Phi$, let N be such that $N\leqslant V_S<N+1$, and let $S_N,S_{N+1}\in\Phi$ with $S_N\subset S\subset S_{N+1}$ and $\mathrm{vol}(S_N)=N$, $\mathrm{vol}(S_{N+1})=N+1$. Then

$$\#\left(S_N\cap\Lambda\right)-a(N+1)\leqslant\#\left(S\cap\Lambda\right)-aV_S\leqslant\#\left(S_{N+1}\cap\Lambda\right)-aN. \tag{9.11}$$

From (9.10), the LHS of (9.11) is $O\left(N^{\frac{\kappa}{2}}\log N\,\psi(\log N)^{\frac12}\right)$ and the RHS is $O\left((N+1)^{\frac{\kappa}{2}}\log(N+1)\,\psi(\log N+1)^{\frac12}\right)$, and these quantities

are of the same order $O\left(V_S^{\frac{\kappa}{2}}\log(V_S)\,\psi(\log V_S)^{\frac{1}{2}}\right)$. A similar upper bound for $aV_S - \#(S\cap\Lambda)$ is proved analogously. □

We turn to the proof of Theorem 9.2. Note that the collection $\bar{\Phi}$ is not ordered; nevertheless one can apply similar arguments to each ℓ separately, before applying Borel-Cantelli. We turn to the details.

Proof of Theorem 9.2. Given N, using assumption (c), for each N there is $\alpha = \alpha(N)$ so that $\mathrm{vol}^{(d)}(B_\alpha) = N$. It follows that $\mathrm{vol}^{(n)}(B_\alpha \times W_\alpha) = NL_\alpha w_\alpha \asymp N$. We let ρ_N^ℓ be the characteristic function of $B_\alpha \times C_\alpha(\ell)$, which is of volume $Nw_\alpha \asymp N^{1-\lambda}$. We will take ${}_{N_1}\rho_{N_2}^\ell$ to be the characteristic function of $\left(B_{\alpha(N_1)}\backslash B_{\alpha(N_2)}\right)\times C_{\alpha(N)}(\ell)$. Note that the dependence of the function ${}_{N_1}\rho_{N_2}^\ell$ on N is suppressed from the notation.

The argument proving Lemma 9.3 therefore yields (9.6), with κ replaced by $\kappa' \stackrel{\mathrm{def}}{=} \kappa(1-\lambda)$, i.e.,

$$\sum_\ell \sum_{(N_1,N_2)\in K_T} \mathrm{Var}_\mu(\widehat{{}_{N_1}\rho_{N_2}^\ell}) \leqslant cL_\alpha(T+1)2^{\kappa' T}. \tag{9.12}$$

For $S = B_{\alpha(N)}\times W_{\alpha(N)}$, $N \leqslant 2^T$, by the definition of $\widehat{{}_{N_1}\rho_{N_2}^\ell}(\Lambda)$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(\#(S\cap\Lambda) - aV_S)^2 &= \left(\sum_\ell \#\left((B_\alpha\times C_\alpha(\ell))\cap\Lambda\right) - a\mathrm{vol}^{(n)}(B_\alpha\times C_\alpha(\ell))\right)^2\\
&= \left(\sum_\ell\left(\widehat{\rho_N^\ell}(\Lambda) - aNw_\alpha\right)\right)^2\\
&= \left(\sum_\ell\sum_{(N_1,N_2)\in\mathcal{I}_N}\left(\widehat{{}_{N_1}\rho_{N_2}^\ell}(\Lambda) - a(N_2-N_1)w_\alpha\right)\right)^2\\
&\leqslant T\,L_\alpha\sum_\ell\sum_{(N_1,N_2)\in K_T}\left(\widehat{{}_{N_1}\rho_{N_2}^\ell}(\Lambda) - a(N_2-N_1)w_\alpha\right)^2.
\end{aligned}$$

As in the proof of Lemma 9.4, we denote by Bad_T the points Λ not satisfying the bound

$$\sum_\ell\sum_{(N_1,N_2)\in K_T}\left(\widehat{{}_{N_1}\rho_{N_2}^\ell}(\Lambda) - a(N_2-N_1)w_\alpha\right)^2 \leqslant L_\alpha(T+1)2^{\kappa' T}\psi(T\log 2-1).$$

Then applying (9.12) we get $\mu(\mathrm{Bad}_T)\leqslant c'\psi(T\log 2-1)^{-1}$, so that by Borel-Cantelli, a.e. Λ belongs to at most finitely many sets Bad_T. Also

for $\Lambda \notin \mathrm{Bad}_T$, we have

$$|\#(S \cap \Lambda) - aV_S|^2 \leqslant L_\alpha^2 T(T+1)2^{\kappa' T}\psi(T\log 2 - 1),$$

which replaces (9.8), and we proceed as before. □

10. Counting patches à la Schmidt

In this section we prove Theorem 1.6. We recall some notation and terminology from the introduction and the statement of the theorem. For a cut-and-project set $\Lambda \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $R > 0$, $\mathcal{P}_{\Lambda,R}(x) = B(0,R) \cap (\Lambda - x)$ is called the R-patch of Λ at x, and

$$D(\Lambda, \mathcal{P}_0) = \lim_{T\to\infty} \frac{\#\{x \in \Lambda \cap B(0,T) : \mathcal{P}_{\Lambda,R}(x) = \mathcal{P}_0\}}{\mathrm{vol}(B(0,T))}$$

is called the *frequency* of $\mathcal{P}_0$. Suppose Λ arises from a cut-and-project construction with associated dimensions $n = d + m$ and window $W \subset \mathbb{R}^m$, and is chosen according to an RMS measure μ of higher rank. The *upper box dimension* of the window in the cut-and-project construction is

$$\dim_B(W) \stackrel{\mathrm{def}}{=} \limsup_{r\to 0} \frac{\log N(W,r)}{-\log r}, \tag{10.1}$$

where $N(W,r)$ is the minimal number of balls of radius r needed to cover W. Set

$$\lambda_0 \stackrel{\mathrm{def}}{=} \frac{m}{m+2\delta} \tag{10.2}$$

where $\delta = m - \dim_B(\partial W) > 0$. Our goal is to show that for any $\lambda \in (\lambda_0, 1)$, any unbounded ordered family $\{B_\alpha : \alpha \in \mathbb{R}\}$, for μ-a.e. Λ, for any patch $\mathcal{P}_0 = \mathcal{P}_{\Lambda,R}(x_0)$,

$$\begin{aligned} &\#\{x \in B_\alpha \cap \Lambda : \mathcal{P}_{\Lambda,R}(x) = \mathcal{P}_0\} \\ =&D(\Lambda,\mathcal{P}_0)\,\mathrm{vol}(B_\alpha) + O\left(\mathrm{vol}(B_\alpha)^{\frac{1+\lambda}{2}}\right) \quad \text{as } \mathrm{vol}(B_\alpha) \to \infty, \end{aligned} \tag{10.3}$$

where the implicit constant depends on $\varepsilon, W, \Lambda, \mathcal{P}_0$. Note that (10.3) implies (1.13).

The strategy we will use is similar to that of [HKW14, Proof of Cor. 4.1].

Proof of Theorem 1.6. For every $K \in \mathbb{N}$ and $\ell \in \mathbb{Z}^m$ define the box

$$Q_K(\ell) = \left[\frac{\ell_1}{K}, \frac{\ell_1+1}{K}\right) \times \cdots \times \left[\frac{\ell_m}{K}, \frac{\ell_m+1}{K}\right).$$

It is well-known (see e.g. [Mat95, Chap. 5]) that in (10.1), we are free to replace $N(W,r)$ with the minimal number of cubes $Q_K(\ell)$ needed to cover W, where $K = \left\lfloor \frac{1}{r} \right\rfloor$. We consider cut-and-project sets of the form $\Lambda = \Lambda(W, \mathcal{L})$, with $\mathcal{L} \in \mathscr{Y}_n$. Here $W \subset \mathbb{R}^m$ is fixed and satisfies

$\dim_B(W) < m$, and Λ is chosen at random, according to a homogeneous measure $\bar{\mu}$ on $\mathscr{Y}_n$. Let Δ be an R-patch equivalence class in Λ, that is

$$\Delta = \{x \in \Lambda : \mathcal{P}_{\Lambda,R}(x) = \mathcal{P}_0\}$$

for some $R > 0$ and some $\mathcal{P}_0 = \mathcal{P}_{\Lambda,R}(x_0)$. By a well-known observation (see [BG13, Cor. 7.3]), Δ is itself a cut-and-project set, and in fact arises from the same lattice via a smaller window, i.e., there is $W_\Delta \subset W$ such that

$$\Delta = \Lambda(W_\Delta, \mathcal{L}).$$

In particular, for irreducible cut-and-project sets (which is a property satisfied by $\bar{\mu}$-a.e. $\mathcal{L}$), we have

$$D(\Lambda, \mathcal{P}_0) = D(\Delta) = \frac{\mathrm{vol}(W_\Delta)}{\mathrm{vol}(W)}\, D(\Lambda). \tag{10.4}$$

In addition, it is shown in [KW21, §2] that W_Δ is the intersection of finitely many translations of W and its complement. Since

$$\partial W_\Delta \subset F + \partial W,$$

for some finite $F \subset \mathbb{R}^m$, we deduce that the upper box dimension of ∂W_Δ is bounded from above by that of ∂W.

Let $\lambda \in (\lambda_0, 1)$, and let $\eta > 0$ be small enough so that

$$\max\left(\frac{1+\lambda_0}{2} + \eta, 1 - \frac{\lambda_0(\delta - \eta)}{m}\right) < \frac{1+\lambda}{2}. \tag{10.5}$$

Such η exists in light of (10.2). Given α, we let $K_\alpha \in \mathbb{N}$ so that $\mathrm{vol}(B_\alpha)^{\lambda_0} \asymp K_\alpha^m$. Define

$$A_\alpha^{(1)} \stackrel{\mathrm{def}}{=} \bigcup_{Q_{K_\alpha}(\ell) \subset W_\Delta} Q_{K_\alpha}(\ell), \quad A_\alpha^{(2)} \stackrel{\mathrm{def}}{=} \bigcup_{Q_{K_\alpha}(\ell) \cap W_\Delta \neq \varnothing} Q_{K_\alpha}(\ell),$$

and let $\mathcal{L} \in \operatorname{supp} \bar{\mu}$ satisfy **(D)** and **(I)**. Since $A_\alpha^{(1)} \subset W_\Delta \subset A_\alpha^{(2)}$, the associated cut-and-project sets

$$\Lambda_\alpha^{(i)} \stackrel{\mathrm{def}}{=} \Lambda\left(A_\alpha^{(i)}, \mathcal{L}\right) \quad (i = 1, 2)$$

satisfy that for all α,

$$\#\left(\Lambda_\alpha^{(1)} \cap B_\alpha\right) \leqslant \#\left(\Delta \cap B_\alpha\right) \leqslant \#\left(\Lambda_\alpha^{(2)} \cap B_\alpha\right)$$

and

$$D\left(\Lambda_\alpha^{(1)}\right) \leqslant D\left(\Delta\right) \leqslant D\left(\Lambda_\alpha^{(2)}\right).$$

Moreover, by (10.4),

$$D\left(\Lambda_\alpha^{(2)}\right) - D\left(\Lambda_\alpha^{(1)}\right) = \frac{D(\Lambda)}{\mathrm{vol}(W)}\left(\mathrm{vol}\left(A_\alpha^{(2)}\right) - \mathrm{vol}\left(A_\alpha^{(1)}\right)\right). \tag{10.6}$$

Using the triangle inequality we have

$$\begin{aligned}\left|\#\left(\Delta\cap B_\alpha\right)-D(\Delta)\operatorname{vol}(B_\alpha)\right| &\leqslant \max_{i=1,2}\left|\#\left(\Lambda_\alpha^{(i)}\cap B_\alpha\right)-D(\Delta)\operatorname{vol}(B_\alpha)\right| \\ \leqslant \max_{i=1,2}\left|\#\left(\Lambda_\alpha^{(i)}\cap B_\alpha\right)-D\left(\Lambda_\alpha^{(i)}\right)\operatorname{vol}\left(B_\alpha\right)\right| &+ \left(D\left(\Lambda_\alpha^{(2)}\right)-D\left(\Lambda_\alpha^{(1)}\right)\right)\operatorname{vol}(B_\alpha).\end{aligned} \tag{10.7}$$

We bound separately the two summands on the RHS of (10.7). For the first summand we use the case (9.5) of Theorem 9.2, with $W_\alpha = A_\alpha^{(i)}$ and $C_\alpha(\ell) = Q_{K_\alpha}(\ell)$. Note that assumption (d) is satisfied by our choice of K_α, with implicit constants depending on $\mathcal{P}_0$. We obtain, for $\bar{\mu}$-a.e. $\mathcal{L}$, that $\Lambda_\alpha^{(i)} = \Lambda(A_\alpha^{(i)}, \mathcal{L})$ satisfies

$$\left|\#\left(\Lambda_\alpha^{(i)}\cap B_\alpha\right)-D\left(\Lambda_\alpha^{(i)}\right)\operatorname{vol}\left(B_\alpha\right)\right| \leqslant c_1\left(\operatorname{vol}(B_\alpha)^{\frac{1+\lambda_0}{2}}\ (\log(\operatorname{vol}(B_\alpha)))^{\frac{3}{2}+\varepsilon}\right),$$

where c_1, as well as the constants appearing in the following inequalities, depends only on $\bar{\Phi} = \left\{B_\alpha \times A_\alpha^{(i)}\right\}$ and on $\mathcal{L}$.

For the second summand, recall that $\dim_B(\partial W_\Delta) \leqslant m-\delta$. This implies that the number of $\ell \in \mathbb{Z}^n$ with $Q_{K_\alpha}(\ell)\cap\partial W_\Delta \neq \varnothing$ is $\ll K_\alpha^{m-\delta+\eta}$. Therefore

$$\operatorname{vol}\left(A_\alpha^{(2)} \smallsetminus A_\alpha^{(1)}\right) = \sum_{Q_{K_\alpha}(\ell)\cap\partial W_\Delta\neq\varnothing} \operatorname{vol}\left(Q_{K_\alpha}(\ell)\right) \ll K_\alpha^{m-\delta+\eta}\, K_\alpha^{-m} = K_\alpha^{-\delta+\eta}.$$

This implies via (10.6) that

$$\begin{aligned}\left(D\left(\Lambda_\alpha^{(2)}\right)-D\left(\Lambda_\alpha^{(1)}\right)\right)\operatorname{vol}(B_\alpha) =& \frac{D(\Lambda)\operatorname{vol}(B_\alpha)}{\operatorname{vol}(W)}\left(\operatorname{vol}\left(A_\alpha^{(2)}\right)-\operatorname{vol}\left(A_\alpha^{(1)}\right)\right) \\ \ll& \operatorname{vol}(B_\alpha)\, K_\alpha^{-\delta+\eta} \ll \operatorname{vol}(B_\alpha)^{1-\frac{\lambda_0(\delta-\eta)}{m}}.\end{aligned}$$

Plugging these two estimates into (10.7), and using (10.5) and the fact that $(\log(\operatorname{vol}(B_\alpha)))^{\frac{3}{2}+\varepsilon} \leqslant \operatorname{vol}(B_\alpha)^\eta$ for large enough $\operatorname{vol}(B_\alpha)$, we have that for $\bar{\mu}$ -a.e. $\mathcal{L}$

$$\left|\#\left(\Delta\cap B_\alpha\right)-D(\Delta)\operatorname{vol}(B_\alpha)\right| \ll \operatorname{vol}(B_\alpha)^{\frac{1+\lambda}{2}},$$

with implicit constants depending on η, $\mathcal{L}$ and ε. This shows (10.3) and completes the proof. □

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