COUNTEREXAMPLES TO A CONJECTURE OF WOODS

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Abstract. A conjecture of Woods from 1972 is disproved.

A lattice in $\mathbb{R}^d$ is called *well-rounded* if its shortest nonzero vectors span $\mathbb{R}^d$, is called *unimodular* if its covolume is equal to one, and the *covering radius* of a lattice Λ is the least r such that $\mathbb{R}^d = \Lambda + B_r$, where B_r is the closed Euclidean ball of radius r. Let N_d denote the greatest value of the covering radius over all well-rounded unimodular lattices in $\mathbb{R}^d$. In [Woo72], A. C. Woods conjectured that $N_d = \sqrt{d}/2$, i.e., that the lattice $\mathbb{Z}^d$ realizes the largest covering radius among well-rounded unimodular lattices. Moreover, Woods proved this statement for $d \leq 6$. In [McM05], McMullen proved that Woods's conjecture implies a celebrated conjecture of Minkowski. Spurred by this result, Woods's conjecture has been proved for $d \leq 9$ by Hans-Gill, Kathuria, Raka, and Sehmi (see [KR14] and references therein), thus yielding Minkowski's conjecture in those dimensions. In this note we prove:

Theorem. *There is $c > 0$ such that $N_d > c\frac{d}{\sqrt{\log d}}$. For all $d \geq 30$, $N_d > \frac{\sqrt{d}}{2}$.*

Proof. Our examples will all be of the form

$$\Lambda = \alpha_1\Lambda_1 \oplus \alpha_2\mathbb{Z}^m$$

for some choices of $\Lambda_1, \alpha_1, \alpha_2, m$. It will be more convenient to work with the quantity $C(\Lambda) = 4r(\Lambda)^2$, where $r(\Lambda)$ is the covering radius of Λ. Clearly $C(\alpha\Lambda) = \alpha^2 C(\Lambda)$, and the Pythagorean theorem shows that $C(\Lambda_1 \oplus \Lambda_2) = C(\Lambda_1) + C(\Lambda_2)$. Let $\lambda_1(L)$ denote the length of the shortest nonzero vector of L, and suppose Λ_1, Λ_2 are well-rounded. If the α_i satisfy $\lambda_1(\alpha_1\Lambda_1) = \lambda_1(\alpha_2\Lambda_2)$, then $\alpha_1\Lambda_1 \oplus \alpha_2\Lambda_2$ is well-rounded. Moreover, there is a unique choice of α_i for which it is also unimodular. Namely, if Λ_1 is well-rounded and unimodular of dimension n, and $\Lambda_2 = \mathbb{Z}^m$, in order for Λ to be well-rounded and unimodular we must take $\alpha_1 = \lambda^{-\frac{m}{n+m}}$ and $\alpha_2 = \lambda^{\frac{n}{n+m}}$, where $\lambda = \lambda_1(\Lambda_1)$. Thus

$$C(\Lambda) = C(\Lambda_1)\,\lambda^{-\frac{2m}{n+m}} + m\,\lambda^{\frac{2n}{n+m}}.$$

For each $d > 3$, let $m = \left\lfloor \frac{d}{\log d} \right\rfloor$, $n = d - m$. Let Λ_1 be any lattice in $\mathbb{R}^n$ for which λ_1 is maximal, that is, Λ_1 is a lattice giving the densest lattice packing in dimension n. Although Λ_1 is only known in very few dimensions, it is a well-known result of Minkowski (see [GL87, Chapter 2] or [CS88, §1.1.5]) that there is $c_1 > 0$ such that for all n,

$$\lambda = \lambda_1(\Lambda_1) \geq c_1\sqrt{n}.$$

Recall that a lattice L_0 is called *critical* if the function $L \mapsto \lambda_1(L)$, considered as a function on the space of unimodular lattices, attains a local maximum at L_0. Then clearly Λ_1 is critical, and a theorem of Voronoi (whose proof is not difficult; see, e.g., [GL87, Chapter 6]) implies that Λ_1 is well-rounded. Now let α_1, α_2 be the unique positive numbers for which $\Lambda = \alpha_1\Lambda_1 \oplus \alpha_2\mathbb{Z}^m$ is well-rounded and unimodular. Then

$$C(\Lambda) \geq m\,\lambda^{\frac{2n}{m+n}} \geq c_2\,m\,n^{\frac{n}{d}} \geq c_3\frac{d^2}{\log d}$$

for positive c_2, c_3, and the first assertion follows.

Taking Λ_1 to be the laminated lattice Λ_{15} (see [CS88, Chapter 6]), we have[1] $C(\Lambda_1) \geq 7 \cdot 2^{\frac{2}{5}}$, $\lambda = 2^{\frac{7}{10}}$, $n = 15$ and so

$$C(\Lambda) \geq 7 \cdot 2^{\frac{2}{5}} \left(2^{\frac{7}{10}}\right)^{-\frac{2m}{15+m}} + m\left(2^{\frac{7}{10}}\right)^{\frac{30}{15+m}},$$

which is greater than $d = m + 15$ for all $m \geq 15$, as can be seen by evaluating the function at $m = 15$ and showing that its derivative is positive for $m \geq 15$. Note that Λ_{15} is generated by its shortest nonzero vectors and is in particular well-rounded. See [CS88, Chapter 6] or [Bar58]. □

Remark. *A similar construction using different choices of* Λ_1 *(instead of* Λ_{15}*) also works, giving slightly weaker results: the Leech lattice (for* $d \geq 38$*), the 'shorter Leech lattice'* O_{23} *(for* $d \geq 36$*), the 16-dimensional Barnes-Wall lattice (for* $d \geq 33$*), and the laminated lattice* Λ_{23} *will work (for* $d \geq 31$*). We are grateful to M. Dutour-Sikirić for suggesting the use of various lattices, and in particular the laminated lattice* Λ_{15} *for this problem. The covering radii of these and other lattices can be computed using his publicly available program* [DS].

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[1] Actually $C(\Lambda_{15}) = 7 \cdot 2^{\frac{2}{5}}$, as was shown in [DSSV09]; we only need the lower bound.

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