ON THE ERGODIC THEORY OF THE REAL REL FOLIATION

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Abstract. Let $\mathcal{H}$ be a stratum of translation surfaces with at least two singularities, let $m_{\mathcal{H}}$ denote the Masur-Veech measure on $\mathcal{H}$, and let Z_0 be a flow on $(\mathcal{H}, m_{\mathcal{H}})$ obtained by integrating a Rel vector field. We prove that Z_0 is mixing of all orders, and in particular is ergodic. We also characterize the ergodicity of flows defined by Rel vector fields, for more general spaces $(\mathcal{L}, m_{\mathcal{L}})$, where $\mathcal{L} \subset \mathcal{H}$ is an orbit-closure for the action of $G = \mathrm{SL}_2(\mathbb{R})$ (i.e., an affine invariant subvariety) and $m_{\mathcal{L}}$ is the natural measure. These results are conditional on a forthcoming measure classification result of Brown, Eskin, Filip and Rodriguez-Hertz. We also prove that the entropy of Z_0 with respect to any of the measures $m_{\mathcal{L}}$ is zero.

1. Introduction

Let $\mathcal{H}$ be a stratum of area-one translation surfaces and let $G \stackrel{\mathrm{def}}{=} \mathrm{SL}_2(\mathbb{R})$. There is a G-invariant finite measure $m_{\mathcal{H}}$ on $\mathcal{H}$ known as the *Masur-Veech measure*, and the dynamics of the G-action on $(\mathcal{H}, m_{\mathcal{H}})$ have been intensively studied in recent years and are intimately connected to many problems in geometry and ergodic theory, see e.g. [MaTa, Zo]. Suppose that surfaces in $\mathcal{H}$ have k singularities, where $k \geq 2$. Then there is a $k-1$-dimensional foliation of $\mathcal{H}$, known as the *real Rel foliation*. A precise definition of the foliation and some of its properties will be given below in §2.2. Loosely speaking, two surfaces are in the same real Rel leaf if one can be obtained from the other by a surgery in which singular points are moved with respect to each other in the horizontal direction, without otherwise changing the geometry of the surface. A natural question, which we address here, is the ergodic properties of this foliation.

As we review in §2.2, by labeling the singularities and removing a set of leaves of measure zero, we can think of the real Rel leaves as being the orbits of an action of a group Z on $\mathcal{H}$, where $Z \cong \mathbb{R}^{k-1}$, and the restriction of this action to any one-dimensional subgroup of Z defines a flow. Our first main result is the following.

Theorem 1.1. *Let $\mathcal{H}$ be a connected component of a stratum $\mathcal{H}(a_1, \ldots, a_k)$ with all $a_i > 0$ (i.e., no marked points). Let $m_{\mathcal{H}}$ be the Masur-Veech measure on $\mathcal{H}$, let $Z \cong \mathbb{R}^{k-1}$ be the corresponding action given by translation along the leaves of the real Rel foliation, and let $Z_0 \subset Z$ be any one-dimensional connected subgroup of Z. Then the Z_0-flow on $(\mathcal{H}, m_{\mathcal{H}})$ is mixing of all orders (and in particular, ergodic).*

The definition of mixing of all orders is given in §3.3. For purposes of this introduction it is enough to note that it implies ergodicity of any nontrivial element. Note that when $\mathcal{H}$ has marked points, there will be subgroups Z_0 which only move the marked points on surfaces without otherwise changing the geometry, and the conclusion of Theorem 1.1 will not hold. This is the only obstruction to generalizing our results to strata with marked points, see Theorem 8.1.

The proof of Theorem 1.1, as well as most of the other results of this paper, relies crucially on measure-rigidity results of Eskin, Mirzakhani and Mohammadi [EM, EMM], and further forthcoming work extending these results, which we will describe in §5.

Theorem 1.1 improves on the results of several authors. In those results, ergodicity for the *full Rel foliation* was studied. The full Rel foliation (also referred to as the 'kernel foliation', 'isoperiodic foliation', or 'absolute period foliation') will also be defined in §2.2. Its leaves are of dimension $2(k-1)$, that is, twice the dimension of the real Rel leaves. Loosely speaking, two surfaces are in the same leaf for this foliation if one can be obtained from the other by moving the singularities (without otherwise affecting the geometry of the surface). That is, we relax the hypothesis that points can only be moved horizontally. The first ergodicity results for the full Rel foliation were obtained by McMullen [McM], who proved ergodicity in the two strata $\mathcal{H}(1,1)$ and $\mathcal{H}(1,1,1,1)$. Subsequently, Calsamiglia, Deroin and Francaviglia [CDF] proved ergodicity in all principal strata, and Hamenstädt [Ham] reproved their result by a simpler argument. Recently, Winsor [Wi1] proved ergodicity for most of the additional strata, and in [Wi2], showed that there are dense orbits for the Z_0-flow, for any Z_0 as in Theorem 1.1. Note that ergodicity for a foliation is implied by ergodicity for any of its subfoliations, and that ergodicity implies the existence of dense leaves, and thus Theorem 1.1 generalizes all of these results. Also note that full Rel is a foliation which is not given by a group action, and the notions of mixing and multiple mixing do not make sense in this case.

The papers [McM, CDF] go beyond ergodicity and obtain classifications of full Rel closed leaves and leaf-closures in their respective settings. We suspect that there is not a reasonable classification of real Rel leaf-closures, indeed it is already known (see [HW]) that there are real Rel trajectories that leave every compact set never to return.

The strata $\mathcal{H}$ support other interesting measures for which similar questions could be asked. Namely, by work of Eskin, Mirzakhani and Mohammadi [EM, EMM], for any $q \in \mathcal{H}$, the orbit-closure $\mathcal{L} \stackrel{\mathrm{def}}{=} \overline{Gq}$ is the support of a unique smooth G-invariant measure which we denote by $m_{\mathcal{L}}$. Let $Z_{\mathcal{L}}$ be the subgroup of Z leaving $\mathcal{L}$ invariant. Then $Z_{\mathcal{L}}$ also preserves $m_{\mathcal{L}}$ and for many choices of $\mathcal{L}$, we have $\dim Z_{\mathcal{L}} > 0$. In these cases, for any closed connected $Z_1 \subset Z_{\mathcal{L}}$, there is a *complexification* $\mathfrak{R}_1$, which gives a foliation of $\mathcal{L}$ whose leaves $\mathfrak{R}_1(q)$ have dimension $2\dim Z_1$ (see §2.2). The leaves $\mathfrak{R}_1(q)$ have a natural translation structure, and this induces a natural locally finite translation-invariant measure on each leaf. With this terminology we can now state the main result of this paper:

Theorem 1.2. *Let $\mathcal{L}$ be a G-orbit closure, and let $m_{\mathcal{L}}$, $Z_{\mathcal{L}}, \mathfrak{R}_{\mathcal{L}}$ be as above, where $\dim Z_{\mathcal{L}} > 0$. Let z_0 be a nontrivial element of $Z_{\mathcal{L}}$ and let $Z_0 = \mathrm{span}_{\mathbb{R}}(z_0)$. Then either*

(1) The action of Z_0 on $(\mathcal{L}, m_{\mathcal{L}})$ is mixing of all orders (and in particular, z_0 acts ergodically); or
(2) there is an intermediate closed connected subgroup Z_1 so that $Z_0 \subset Z_1 \subset Z_{\mathcal{L}}$, and the complexification $\mathfrak{R}_1$ of Z_1 satisfies

- *for every $q \in \mathcal{L}$, the leaf $\mathfrak{R}_1(q)$ is closed, and*
- *for $m_{\mathcal{L}}$-a.e. q, $\mathfrak{R}_1(q)$ is of finite volume with respect to its translation-invariant measure, and $\overline{Z_0 q} = \mathfrak{R}_1(q)$.*

Thus, in order to establish ergodicity of real Rel subfoliations on G-orbit-closures, it is enough to rule out Case (2). We will prove Proposition 7.1, which provides a simple way to achieve this, using cylinder circumferences of surfaces in $\mathcal{L}$. Theorems 1.1 and 8.1 are deduced from Theorem 1.2 using Proposition 7.1.

The following statement is an immediate consequence of Theorem 1.2.

Corollary 1.3. *Let $\mathcal{L}$ be a G-orbit-closure, let $m_{\mathcal{L}}, Z_{\mathcal{L}}$ be as above, and let $Z_1 \subset Z_{\mathcal{L}}$ be one-dimensional. Suppose that the foliation induced by the complexification $\mathfrak{R}_1$ has a dense leaf. Then the Z_1-flow on $(\mathcal{L}, m_{\mathcal{L}})$ is mixing of all orders (and in particular, ergodic).*

The density of certain leaves of the full Rel foliation in G-orbit-closures of rank one was obtained by Ygouf in [Y]. Using these results we obtain ergodicity of one-dimensional subgroups of the real Rel foliation in many cases. For instance, using [Y, Thm. A & Prop. 5.1] we have:

Corollary 1.4. *The real Rel foliation is mixing of all orders (and in particular, ergodic) in any eigenform locus in $\mathcal{H}(1,1)$ with a non-square discriminant.*

Recall that in [Wi2] Winsor proved the existence of dense real Rel leaves, and dense leaves for one-dimensional flows Z_0, in all strata. Using these results in conjunction with Corollary 1.3, one can obtain an alternative proof of Theorem 1.1 that avoids the use of Proposition 7.1.

We also consider the entropy of real-Rel flows, and show the following:

Theorem 1.5. *Let $\mathcal{L}, m_{\mathcal{L}}, Z_{\mathcal{L}}, z_0$ be as in the statement of Theorem 1.2. Then the entropy of the action of Rel_{z_0} on the measure space $(\mathcal{L}, m_{\mathcal{L}})$ is zero.*

Using the geodesic flow one easily shows that Rel_{z_0} is conjugate to Rel_{tz_0} for any $t > 0$, and from this it follows that the entropy is either zero or infinite. However, the Rel flow is not continuous, and we could not find a simple way to rule out infinite entropy. Our proof gives a more general result — see Theorem 9.1. However, the argument fails for Z_0-invariant measures for which the backward time geodesic flow diverges almost surely, and thus we do not settle the question of whether the topological entropy of real Rel flows is zero.

1.1. **Outline.** In §2 we give background material on translation surfaces, their moduli spaces, and the Rel foliation. In §3 we use standard facts about joinings to build a measure θ on the product of two strata (see (3.1)), depending on a real Rel flow Z_0, such that if θ is the product measure, then Z_0 is ergodic. In §3.3 we discuss a technique of Mozes that makes it possible to upgrade ergodicity to mixing of all orders. In §4 we show that θ is ergodic for the diagonal action of the upper triangular group $P \subset G$ on the product of two strata. In §5 we state a far-reaching measure rigidity result of Brown, Eskin, Filip and Rodriguez-Hertz for P-ergodic measures on products of two strata. In §6 we use this measure rigidity result, as well as prior results for the action on one stratum due to Wright, in order to characterize the situations in which θ is not a product measure, thus proving

Theorem 1.2. Proposition 7.1 is proved in §7, and we check its conditions to deduce Theorems 1.1 and 8.1 in §8. We prove Theorem 1.5 in §9.

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2. Preliminaries about translation surfaces

2.1. **Strata, period coordinates.** In this section we collect standard facts about translation surfaces, and fix our notation. For more details, we refer to reader to [Zo, Wr1, BSW]. Below we briefly summarize the treatment in [BSW, §2].

Let S be a compact oriented surface of genus g, $\Sigma = \{\xi_1, \ldots, \xi_k\} \subset S$ a finite set, $a_1, \ldots, a_k$ non-negative integers with $\sum a_i = 2g-2$, and $\mathcal{H} = \mathcal{H}(a_1, \ldots, a_k)$ the corresponding stratum of unit-area translation surfaces. We let $\mathcal{H}_{\mathrm{m}} = \mathcal{H}_{\mathrm{m}}(a_1, \ldots, a_k)$ denote the stratum of unit-area marked translation surfaces and $\pi : \mathcal{H}_{\mathrm{m}} \to \mathcal{H}$ the forgetful mapping. Our convention is that singular points are labeled, or equivalently, $\mathcal{H} = \mathcal{H}_{\mathrm{m}} / \operatorname{Mod}(S, \Sigma)$, where $\operatorname{Mod}(S, \Sigma)$ is the group of isotopy classes of orientation-preserving homeomorphisms of S fixing Σ, up to an isotopy fixing Σ.

There is an $\mathbb{R}_{>0}$-action that dilates the atlas of a translation surface by $c \in \mathbb{R}_{>0}$. For a stratum $\mathcal{H}$ and marked stratum $\mathcal{H}_{\mathrm{m}}$, we denote the collection of surfaces of arbitrary area, obtained by applying such dilations, by $\bar{\mathcal{H}}$, $\bar{\mathcal{H}}_{\mathrm{m}}$. The marked stratum $\bar{\mathcal{H}}_{\mathrm{m}}$ is a linear manifold modeled on the vector space $H^1(S, \Sigma; \mathbb{R}^2)$. It has a developing map $\operatorname{dev} : \bar{\mathcal{H}}_{\mathrm{m}} \to H^1(S, \Sigma; \mathbb{R}^2)$, sending an element of $\bar{\mathcal{H}}_{\mathrm{m}}$ represented by $f : S \to M$, where M is a translation surface, to $f^*(\operatorname{hol}(M, \cdot))$, where for an oriented path γ in M which is either closed or has endpoints at singularities, $\operatorname{hol}(M, \gamma) = \begin{pmatrix} \int_\gamma dx \\ \int_\gamma dy \end{pmatrix}$, and dx, dy are the 1-forms on M inherited from the plane. Furthermore, there is an open cover $\{\mathcal{U}_\tau\}$ of $\mathcal{H}_{\mathrm{m}}$, indexed by triangulations τ of S with triangles whose vertices are in Σ, and maps $\operatorname{dev}|_{\mathcal{U}_\tau} : \mathcal{U}_\tau \to H^1(S, \Sigma; \mathbb{R}^2)$, which are homeomorphisms onto their image, and such that the transition maps on overlaps for this atlas are restrictions of linear automorphisms of $H^1(S, \Sigma; \mathbb{R}^2)$.

This atlas of charts $\{(\mathcal{U}_\tau, \operatorname{dev}|_{\mathcal{U}_\tau})\}$ is known as *period coordinates*. Since each $\mathcal{U}_\tau$ is identified via period coordinates with an open subset of the vector space $H^1(S, \Sigma; \mathbb{R}^2)$, the tangent space at each $\mathcal{U}_\tau$ is identified canonically with $H^1(S, \Sigma; \mathbb{R}^2)$, and thus the tangent bundle of $\mathcal{H}_{\mathrm{m}}$ is locally constant. A sub-bundle of the tangent bundle is called *locally constant* or *flat* if it is constant in the charts afforded by period coordinates. The $\operatorname{Mod}(S, \Sigma)$-action on $\mathcal{H}_{\mathrm{m}}$ is properly discontinuous, and hence $\mathcal{H}$ is an orbifold, and the map $\pi : \mathcal{H}_{\mathrm{m}} \to \mathcal{H}$ is an orbifold covering map.

The group G acts on translation surfaces in $\mathcal{H}$ by modifying planar charts, and acts on $H^1(S, \Sigma; \mathbb{R}^2)$ via its action on $\mathbb{R}^2$, thus inducing a G-action on $\mathcal{H}_{\mathrm{m}}$. The G-action commutes with the $\operatorname{Mod}(S, \Sigma)$-action, and thus the map π is G-equivariant for these actions. The G-action on $\mathcal{H}_{\mathrm{m}}$ is free, since $\operatorname{dev}(g\mathbf{q}) \neq \operatorname{dev}(\mathbf{q})$ for any nontrivial $g \in G$. We will use the following subgroups of G:

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$
$$U = \{u_s : s \in \mathbb{R}\}, \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0,\ b \in \mathbb{R} \right\}.$$

2.2. **Rel foliation and real Rel foliation.** We define and list some important properties of the Rel foliation, the real Rel foliation, and the corresponding action on the space of surfaces without horizontal saddle connections. See [MW, BSW] for more details. See also [Zo, McM], and references therein.

We have a canonical splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ and we write $\mathbb{R}^2 = \mathbb{R}_{\mathrm{x}} \oplus \mathbb{R}_{\mathrm{y}}$ to distinguish the two summands in this splitting. There is a corresponding splitting

$$H^1(S, \Sigma; \mathbb{R}^2) = H^1(S, \Sigma; \mathbb{R}_{\mathrm{x}}) \oplus H^1(S, \Sigma; \mathbb{R}_{\mathrm{y}}). \tag{2.1}$$

We also have a canonical restriction map $\mathrm{Res} : H^1(S, \Sigma; \mathbb{R}^2) \to H^1(S; \mathbb{R}^2)$ (given by restricting a cochain to absolute periods). Since Res is topologically defined, its kernel ker(Res) is $\mathrm{Mod}(S, \Sigma)$-invariant. Moreover, from our convention that singular points are marked, the $\mathrm{Mod}(S, \Sigma)$-action on ker(Res) is trivial.

Let

$$\mathfrak{R} \stackrel{\mathrm{def}}{=} \ker(\mathrm{Res}) \quad \text{and} \quad Z \stackrel{\mathrm{def}}{=} \mathfrak{R} \cap H^1(S, \Sigma; \mathbb{R}_{\mathrm{x}}). \tag{2.2}$$

Since $H^1(S, \Sigma; \mathbb{R}_{\mathrm{x}})$ and $H^1(S, \Sigma; \mathbb{R}_{\mathrm{y}})$ are naturally identified with each other via their identification with $H^1(S, \Sigma; \mathbb{R})$, for each $Z_1 \subset Z$ we can define the space $\mathfrak{R}_1$ spanned by the two copies of Z_1 in $H^1(S, \Sigma; \mathbb{R}_{\mathrm{x}})$ and $H^1(S, \Sigma; \mathbb{R}_{\mathrm{y}})$ respectively. The space $\mathfrak{R}_1$ is the *complexification* of Z_1. This terminology arises from viewing $H^1(S, \Sigma; \mathbb{R}^2)$ as $H^1(S, \Sigma; \mathbb{C})$, a vector space over $\mathbb{C}$, viewing $H^1(S, \Sigma; \mathbb{R}_{\mathrm{x}})$ and $H^1(S, \Sigma; \mathbb{R}_{\mathrm{y}})$ as the real and imaginary subspace of this complex vector space. With this viewpoint, $\mathfrak{R}_1$ is the $\mathbb{C}$-span of Z_1.

For any subspace $Z_1 \subset \mathfrak{R}$, we can foliate the vector space $H^1(S, \Sigma; \mathbb{R}^2)$ by affine subspaces parallel to Z_1. Pulling back this foliation using the period coordinate charts gives rise to a foliation of $\bar{\mathcal{H}}_{\mathrm{m}}$. Since monodromy acts trivially on $\mathfrak{R}$, this foliation descends to a well-defined foliation on $\bar{\mathcal{H}}$. It is known (see e.g. [BSW, Prop. 4.1]) that the area of a surface is constant on leaves of the Rel foliation, and thus the Rel foliation and any of its subfoliations descend to a foliation of $\mathcal{H}$. The foliation corresponding to $\mathfrak{R}$ (respectively, to Z) is known as the *Rel foliation* (respectively, the *real Rel foliation*).

Because the $\mathrm{Mod}(S, \Sigma)$-monodromy action fixes all points of $\mathfrak{R}$, the leaves of the Rel foliation, and any of its sub-foliations, acquire a translation structure. In particular, they are equipped with a natural measure.

For any $v \in Z$ we have a constant vector field, well-defined on $\mathcal{H}_{\mathrm{m}}$ and on $\mathcal{H}$, everywhere equal to v. Integrating this vector field we get a partially defined *real REL flow (corresponding to v)* $(t, q) \mapsto \mathrm{Rel}_{tv}(q)$; the flow may not be defined for all time due to possible 'collide of zeroes'. For every $q \in \mathcal{H}$ it is defined for $t \in I_q$, where the *domain of definition* $I_q = I_q(v)$ is an open subset of $\mathbb{R}$ which contains 0. The sets $I_q(v)$, are explicitly described in [BSW, Thm. 6.1]. Let $\hat{\mathcal{H}}$ denote the set of surfaces in $\mathcal{H}$ with no horizontal saddle connections. Then $I_q = \mathbb{R}$ for all $q \in \hat{\mathcal{H}}$.

If $q \in \mathcal{H}$, $s \in \mathbb{R}$ and $\tau \in I_q$ then

$$\tau \in I_{u_s q} \quad \text{and} \quad \mathrm{Rel}_{\tau v}(u_s q) = u_s \mathrm{Rel}_{\tau v}(q).$$

Similarly, if $q \in \mathcal{H}$, $t \in \mathbb{R}$ and $\tau \in I_q$ then

$$\tau' \stackrel{\mathrm{def}}{=} e^t \tau \in I_{g_t q} \quad \text{and} \quad \mathrm{Rel}_{\tau' v}(g_t q) = g_t \mathrm{Rel}_{\tau v}(q). \tag{2.3}$$

In particular, since P preserves $\hat{\mathcal{H}}$ and $P = \{g_t u_s : t, s \in \mathbb{R}\}$, there is an action of $P \ltimes Z$ on $\hat{\mathcal{H}}$, given by $(p, z).q = p\mathrm{Rel}_z(q)$.

3. Preliminaries from ergodic theory

3.1. **Ergodic decomposition.** We will use the notation $\mathbb{G} \circlearrowleft (X, \mu)$ to indicate that $\mathbb{G}$ is a locally compact second countable group, $(X, \mathcal{B})$ is a standard Borel space, and μ is a probability measure on $\mathcal{B}$ preserved by the $\mathbb{G}$-action. We say that $\mathbb{G} \circlearrowleft (Y, \nu)$ is a *factor* of (X, μ) if there is a measurable $\mathbb{G}$-invariant conull subset $X_0 \subset X$, and a measurable map $T : X_0 \to Y$ such that $T \circ g = g \circ T$ for all $g \in \mathbb{G}$, and $\nu = T_* \mu$. In this situation we refer to T as the factor map. Given a factor map, there is a (unique up to nullsets) measure disintegration $\mu = \int \mu_y \, d\nu(y)$, for a Borel mapping $y \mapsto \mu_y$ from Y to the space of Borel probability measures on X, such that $\mu_y(T^{-1}(y)) = 1$ for ν-a.e. y. Equivalently we can write $\mu = \int_x \mu'_x \, d\mu(x)$, where $\mu'_x \stackrel{\mathrm{def}}{=} \mu_{T(x)}$. For a closed subgroup $H \subset \mathbb{G}$, we say that μ is H-ergodic if any invariant set is null or conull. We have the following well-known ergodic decomposition theorem:

Proposition 3.1. *Suppose $\mathbb{G} \circlearrowleft (X, \mu)$, and H is a closed subgroup of $\mathbb{G}$. Then there is a factor of $H \circlearrowleft (X, \mu)$, called the* space of ergodic components *and denoted by $X /\!\!/ H$, with the following properties:*

- *(i) For ν-a.e. $y \in X /\!\!/ H$, μ_y is H-invariant and H-ergodic.*
- *(ii) H acts trivially on $X /\!\!/ H$.*
- *(iii) $H \circlearrowleft (X, \mu)$ is ergodic if and only if $X /\!\!/ H = \{pt.\}$.*
- *(iv) The properties (i)–(iii) uniquely determine the factor $X /\!\!/ H$ up to measurable isomorphism.*
- *(v) If $H \lhd \mathbb{G}$ then $\mathbb{G} \circlearrowleft (X /\!\!/ H, \nu)$.*

Proof. For (i) and (ii) see [Va, Thm. 4.4] (in the notation of [Va], these assertions follow from the fact that β is a map on points and is H-invariant). Assertion (iii) is immediate from definitions and (iv) follows from [Va, Lemma 4.4]. For (v), one can argue using the uniqueness property (iv), and the fact that the image of an H-invariant ergodic measures under any element $g \in \mathbb{G}$ is also H-invariant and ergodic. $\square$

Remark 3.2. *An action is called* prime *if it has no factors (besides the action itself, and the trivial action on a point). The construction above shows that if $H \lhd \mathbb{G}$, $\mathbb{G}'$ is a subgroup of $\mathbb{G}$ so that $\mathbb{G}' \circlearrowleft (X, \mu)$ is prime and $H \circlearrowleft (X, \mu)$ is not isomorphic to the trivial action, then $H \circlearrowleft (X, \mu)$ is ergodic. This is not the approach we will take for proving Theorem 1.1.*

3.2. **Joinings.** We recall some well-known facts about joinings, see [dlR] and references therein. Let $\mathbb{G} \circlearrowright (X_i, \mu_i)$ for $i = 1, 2$. A *joining* is a measure θ on $X_1 \times X_2$, invariant under the diagonal action of $\mathbb{G}$, such that $\pi_{i*}\theta = \mu_i$. A *self-joining* is a joining in case $X_1 = X_2$. If $(X_i, \mu_i) \to (Z, \nu)$ is a joint factor then the *relatively independent joining over* Z is the joining $\int_Z (\mu_1)_z \times (\mu_2)_z \, d\nu(z)$, where $\mu_i = \int_Z (\mu_i)_z \, d\nu(z)$ is the disintegration of μ_i. In case $X_1 = X_2 = X$, and $Z = X/\!\!/H$ is the space of ergodic components of the action of H on (X, μ) as in Proposition 3.1, we obtain the *relatively independent self-joining over* $X/\!\!/H$. This joining satisfies:

Proposition 3.3. *The following are equivalent:*

- $H \circlearrowright (X, \mu)$ *is ergodic.*
- *The relatively independent self-joining over* $X/\!\!/H$ *is* $\mu \times \mu$.

We note two properties of this self-joining. We fix a topology on X which generates the σ-algebra, and denote by $\operatorname{supp} \mu$ the topological support of μ, i.e., the smallest closed set of full measure.

Proposition 3.4. *Let* θ *be the measure on* $X \times X$ *which is the relatively independent self-joining over* $X/\!\!/H$, *for some* H, *and let* $T : X \to X/\!\!/H$ *be the factor map. Then the following hold:*

- *We have*

$$\theta = \int_X \mu_{T(x)} \times \mu_{T(x)} \, d\mu(x). \tag{3.1}$$

- *The set* $\left\{x \in X : x \notin \operatorname{supp} \mu_{T(x)}\right\}$ *is of* μ*-measure zero.*
- *If* $X = \operatorname{supp} \mu$ *then* $\operatorname{supp} \theta$ *contains the diagonal* $\Delta_X \stackrel{\text{def}}{=} \{(x, x) : x \in X\}$.

Proof. Formula (3.1) is immediate from the definition of the relatively independent self-joining over $X/\!\!/H$. Since each $\mu'_x = \mu_{T(x)}$ is H-invariant and ergodic, and $\mu'_x(T^{-1}(T(x))) = 1$, the set $\{x \in X : x \notin \operatorname{supp} \mu'_x\}$ is a nullset. From this, and from (3.1) we obtain the last assertion. $\square$

3.3. **Ergodicity, mixing, and mixing of all orders.** For $\mathbb{G} \circlearrowright (X, \mu)$, let $L^2_0(\mu)$ denote the Hilbert space of L^2-functions on (X, μ) of integral zero, and let $k \geq 2$. The action is called *k-mixing* if for any $f_1, \ldots, f_k \in L^2_0(\mu)$ and for any $k-1$ sequences $\left(g_n^{(i)}\right)_{n \in \mathbb{N}} \in \mathbb{G}$, $i = 1, \ldots, k-1$, for which all of the sequences

$$\left(g_n^{(i)}\right)_{n \in \mathbb{N}} \quad (1 \leq i \leq k-1) \quad \text{ and } \quad \left(g_n^{(i)} (g_n^{(j)})^{-1}\right)_{n \in \mathbb{N}} \quad (1 \leq i < j \leq k-1)$$

eventually leave every compact subset of $\mathbb{G}$, we have

$$\int_X f_1 \left(g_n^{(1)} x\right) \cdots f_{k-1} \left(g_n^{(k-1)} x\right) f_k(x) \, d\mu(x) \stackrel{n \to \infty}{\longrightarrow} \prod_{i=1}^{k} \int_X f_i \, d\mu.$$

We say that the action is *mixing* if it is 2-mixing, and *mixing of all orders* if it is mixing of order k for any $k \geq 2$. It is easy to check that mixing implies ergodicity of any unbounded subgroup of $\mathbb{G}$. We have the following:

Proposition 3.5. *Let* $Z_0 \cong \mathbb{R}$ *and let* $\{g_t\}$ *be a one-parameter group acting on* Z_0 *by dilations, i.e., for all* $v \in Z_0$ *and* $t \in \mathbb{R}$ *we have* $g_t v = e^{\lambda t} v$ *for some* $\lambda \neq 0$.

Let $F = \{g_t\} \ltimes Z_0$ and let $F \circlearrowright (X, \mu)$ be a probability space. The following are equivalent:

(a) the restricted flow $Z_0 \circlearrowright (X, \mu)$ is ergodic;
(b) the restricted flow $Z_0 \circlearrowright (X, \mu)$ is mixing of all orders;
(c) the restricted flow $Z_0 \circlearrowright (X, \mu)$ is mixing;
(d) any nontrivial element of Z_0 acts ergodically.

Remark 3.6. *The group F appearing in Proposition 3.5 is isomorphic as a Lie group to the subgroup P of upper triangular matrices in G, but in our application we will use it for the group generated by a one-parameter real Rel flow Z_0 and the diagonal flow $\{g_t\}$.*

Proof. Clearly (b) $\implies$ (c) $\implies$ (d) $\implies$ (a). We assume that the Z_0-flow is ergodic. To see that it is mixing, it is enough by [P, Chap. 2, Prop. 5.9] to prove that it has countable Lebesgue spectrum, and for this, use [KT, Prop. 1.23 & Prop. 2.2]. The proof of mixing of all orders follows verbatim from an argument of Mozes [Mo], for mixing actions of Lie groups which are 'Ad-proper'. Since our group F is not Ad-proper, we cannot cite [Mo] directly, so we sketch the proof. For notational convenience we deduce 3-fold mixing from mixing (the proof that 'k-fold mixing $\implies$ $k+1$-fold mixing', for $k \geq 3$, is identical but requires more cumbersome notation).

We use additive notation in the group Z_0, and denote the action of Z_0 on X by $(z, x) \mapsto z.x$. Let $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be sequences in Z_0 such that each of the sequences $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(b_n + c_n)_{n\in\mathbb{N}}$ eventually leaves every compact set, and let f_1, f_2, f_3 be in $L^2_0(\mu)$. We need to prove that

$$\int_X f_1(x) f_2(b_n.x) f_3((b_n + c_n).x)\, d\mu(x) \overset{n\to\infty}{\longrightarrow} \int_X f_1\, d\mu \int_X f_2\, d\mu \int_X f_3\, d\mu.$$

For each n, define a measure μ_n on $X^3 \overset{\text{def}}{=} X \times X \times X$ by

$$\int_{X^3} f\, d\mu_n \overset{\text{def}}{=} \int_X f(x, b_n.x, (b_n + c_n).x)\, d\mu(x), \quad \forall f \in C_c(X^3).$$

That is, μ_n is the pushforward of the diagonal measure on X^3 by the triple $(0, b_n, b_n + c_n)$. It is easy to see that 3-mixing is equivalent to the fact that the weak-* limit of μ_n is the measure $\mu^3 \overset{\text{def}}{=} \mu\times\mu\times\mu$. The group $F^3 \overset{\text{def}}{=} F\times F\times F$ acts on X^3 by acting separately on each component, and as in [Mo], since Z_0 is mixing, it suffices to show that any measure ν on X^3 which is a weak-* limit of a subsequence of $(\mu_n)_{n\in\mathbb{N}}$, is invariant under $(0, u, v) \in \mathbb{R}^3 \subset F^3$, for some $(u, v) \in \mathbb{R}^2 \smallsetminus (0, 0)$. We claim that for any $s \in \mathbb{R}$ the measure μ_n is invariant under

$$h_n(s) \overset{\text{def}}{=} (g_s, b_n \cdot g_s \cdot (-b_n), (b_n + c_n) \cdot g_s \cdot (-b_n - c_n)),$$

where the multiplication is in the group F^3. Indeed, since μ is $\{g_s\}$-invariant,

$$\int_{X^3} f\, d\mu_n = \int_X f\left(g_s x, b_n.(g_s x), (b_n + c_n).(g_s x)\right)\, d\mu(x),$$

and

$$h_n(s) \cdot (\mathrm{id}_F, b_n, b_n + c_n) = (g_s, b_n \cdot g_s, (b_n + c_n) \cdot g_s).$$

That is, applying $h_n(s)$ changes one description of μ_n to another.

We embed F as a multiplicative group of matrices in $\mathrm{GL}_2(\mathbb{R})$ and let d_F be the metric on F induced by some norm on $\mathrm{GL}_2(\mathbb{R})$. By a straightforward computation we have

$$h_n(s) = \left(g_s,\ (1-e^{\lambda s})b_n \cdot g_s,\ (1-e^{\lambda s})(b_n+c_n)\cdot g_s\right),$$

and $d_F(\mathrm{id}_F, h_n(s_n))$ is a continuous function of s which goes to 0 as $s \to 0$ and for any fixed $s > 0$, increases to infinity as $n \to \infty$. Therefore we can choose $s_n \to 0$ so that $d_F(\mathrm{id}_F, h_n(s_n)) = 1$ for all large enough n. As in [Mo], ν is invariant under some subsequential limit of $h_n(s_n)$ which is of the form $(0, u, v)$ for some $(u, v) \in \mathbb{R}^2 \smallsetminus (0, 0)$. This establishes our sufficient condition. □

4. The relatively independent self-joining for a Rel flow

Recall that $\hat{\mathcal{L}} \subset \mathcal{L}$ is the set of surfaces without horizontal saddle connections, and this is a P-invariant set of full measure with respect to $m_{\mathcal{L}}$. We can combine the product action of $Z_{\mathcal{L}} \times Z_{\mathcal{L}}$ on $\hat{\mathcal{L}} \times \hat{\mathcal{L}}$ with the diagonal action of P to obtain an action of the semi-direct product $P \ltimes (Z_{\mathcal{L}} \times Z_{\mathcal{L}})$ on $\hat{\mathcal{L}} \times \hat{\mathcal{L}}$. Since $\hat{\mathcal{L}} \subset \mathcal{L}$ is of full measure, and the arguments of this section involve passing to sets of full measure, in the remainder of this section we will ignore the distinction between $\mathcal{L}$ and $\hat{\mathcal{L}}$.

Proposition 4.1. *Let $Z \subset Z_{\mathcal{L}}$ be a closed connected subgroup. If θ is an invariant probability measure for an action of the semidirect product $P \ltimes (Z \times Z)$ on $\mathcal{L} \times \mathcal{L}$, then any $f \in L^2(\theta)$ which is $\{g_t\}$-invariant is also $Z \times Z$-invariant.*

Proof. For any $z \in Z \times Z$, $g_t z g_{-t} \to_{t \to -\infty} 0$. So the claim follows from the Mautner phenomenon, see e.g. [EW, Prop 11.18]. □

Proposition 4.2. *Let $(\mathcal{L}, m_{\mathcal{L}})$ be a G-orbit-closure with a fully supported P-invariant ergodic measure, let $Z \subset Z_{\mathcal{L}}$ be a connected closed subgroup, and let θ on $\mathcal{L} \times \mathcal{L}$ be the relatively independent joining over $\mathcal{L} /\!\!/ Z$. Then θ is P-invariant and $\{g_t\}$-ergodic (and hence P-ergodic). Also $\Delta_{\mathcal{L}} \subset \operatorname{supp} \theta$.*

As we will see in §5, under the conditions of the Proposition, $m_{\mathcal{L}}$ is the so-called 'flat measure' on $\mathcal{L}$.

Proof. Let $\pi : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ be the projection onto the first factor, and let $\nu = \pi_* \theta$. For each $x \in \mathcal{L}$, let $\Omega_x \stackrel{\mathrm{def}}{=} \pi^{-1}(x) = \{x\} \times \mathcal{L}$ be the fiber, and let θ_x be the fiber measure appearing in the disintegration $\theta = \int_{\mathcal{L}} \theta_x \, d\nu(x)$. Then Z acts on Ω_x via the second factor in $Z \times Z$, and θ_x is Z-invariant and ergodic by the definition of the ergodic decomposition.

It follows from Proposition 3.1(v) that θ is P-invariant. To prove ergodicity, let $f \in L^2(\mathcal{L} \times \mathcal{L}, \theta)$ be a P-invariant function. By Proposition 4.1, f is $Z \times Z$-invariant. For each $x \in \mathcal{L}$, let $f_x \stackrel{\mathrm{def}}{=} f|_{\Omega_x}$. There is $\mathcal{L}_0 \subset \mathcal{L}$ such that $m_{\mathcal{L}}(\mathcal{L}_0) = 1$ and for every $x \in \mathcal{L}_0$, f_x belongs to $L^2(\Omega_x, \theta_x)$ and is Z-invariant. Hence, by ergodicity, there is $\bar{f} : \mathcal{L}_0 \to \mathbb{R}$ such that for every $x \in \mathcal{L}_0$, $\bar{f}(x)$ is the θ_x-almost-sure value of f_x. Since f is P-invariant for the diagonal action of P, $\bar{f}$ is P-invariant for the action of P on $\mathcal{L}$. By ergodicity of $P \circlearrowright (\mathcal{L}, m_{\mathcal{L}})$, $\bar{f}$ is ν-a.e. constant, and thus f is θ-a.e. constant.

The last assertion follows from Proposition 3.4. □

5. An upgraded magic wand theorem

The celebrated 'magic wand' Theorem of Eskin and Mirzakhani [EM], and ensuing work of Eskin, Mirzakhani and Mohammadi [EMM], classified P- and G-invariant probability measures and orbit-closures on strata of translation surfaces. These results can be summarized as follows (see [EM, Defs. 1.1 & 1.2, Thms. 1.4 & 1.5]):

Theorem 5.1. *Let $\mathcal{H}$, $\mathcal{H}_{\mathrm{m}}$, $\bar{\mathcal{H}}$, $\bar{\mathcal{H}}_{\mathrm{m}}$ be as in §2.1. Any P-invariant ergodic probability measure m has the following properties:*

(i) It is G-invariant.

(ii) There is a complex-affine manifold $\mathcal{N}$ and a proper immersion $\varphi : \mathcal{N} \to \bar{\mathcal{H}}$ such that
$$\mathcal{L} \stackrel{\mathrm{def}}{=} \operatorname{supp} m = \mathcal{H} \cap \varphi(\mathcal{N}).$$

(iii) There is an open G-invariant subset $U \subset \bar{\mathcal{H}}$ satisfying $m(U) = 1$, and for any $x \in U \cap \mathcal{L}$ there is an open set V containing x such that V is evenly covered by $\mathcal{V} \subset \mathcal{H}_{\mathrm{m}}$ under the map $\pi : \bar{\mathcal{H}}_{\mathrm{m}} \to \bar{\mathcal{H}}$, and $\psi \stackrel{\mathrm{def}}{=} \mathrm{dev} \circ (\pi|_{\mathcal{V}})^{-1} \circ \varphi$ coincides on its domain with a $\mathbb{C}$-linear map, with real coefficients.

(iv) The subspace $W \stackrel{\mathrm{def}}{=} \mathrm{Im}(\psi)$ is symplectic, and the measure m is obtained via the cone construction from the Lebesgue measure on W.

(v) The complement $\mathcal{L} \smallsetminus U$ is a finite union of supports of measures satisfying properties (i)–(iv), for which the manifolds $\mathcal{N}'$ appearing in (ii) satisfy $\dim \mathcal{N}' < \dim \mathcal{N}$.

Any orbit-closure for the P-action is a set $\mathcal{L}$ as above.

We will refer to $\mathcal{L}$ as an *orbit-closure* and to $m = m_{\mathcal{L}}$ as a *flat measure* on $\mathcal{L}$. Orbit-closures are referred to as *affine invariant manifolds* and also as *invariant subvarieties*. The use of an evenly covered neighborhood in item (iii) is a standard approach for defining period coordinates (see e.g. [MS]). We refer to [Wr1] for a survey containing more information on orbit-closures.

In a forthcoming work of Brown, Eskin, Filip and Rodriguez-Hertz, the same conclusion is obtained for the diagonal actions of P and G on a product of strata $\mathcal{H} \times \mathcal{H}'$. Namely, the following is shown:

Theorem 5.2. *Let $\mathcal{H}, \mathcal{H}'$ be strata of translation surfaces, and let P and G act on $\mathcal{H} \times \mathcal{H}'$ via their diagonal embeddings in $G \times G$. Then all of the conclusions of Theorem 5.1 hold for this action (with $\bar{\mathcal{H}} \times \bar{\mathcal{H}}'$ replacing $\bar{\mathcal{H}}$).*

6. Proof of main result

Using Theorem 5.2 and further work of Wright [Wr2], we can prove our main result.

Proof of Theorem 1.2. Let $Z_0 = \mathrm{span}_{\mathbb{R}}(z_0)$ be a one-dimensional connected real Rel subgroup. Assume that (1) fails, so that the action of Z_0 on $(\mathcal{L}, m_{\mathcal{L}})$ is not mixing of all orders. Then, by Proposition 3.5 it is not ergodic. Let θ be the relatively independent self-joining over $\mathcal{L} /\!\!/ Z_0$. Applying Propositions 3.3 and 3.4 we have that $\theta \neq m_{\mathcal{L}} \times m_{\mathcal{L}}$ and $\Delta_{\mathcal{L}} \subset \operatorname{supp} \theta$. Applying Proposition 4.2 and Theorem 5.2, we have that there is a G-invariant open subset U of full θ-measure such that $U \cap \operatorname{supp} \theta$ is the isomorphic image of an affine complex-linear manifold whose dimension is strictly smaller than $2 \dim \bar{\mathcal{H}}$, and θ is obtained from Lebesgue measure on this complex-linear manifold by the cone construction.

We claim that the set

$$U_1 \stackrel{\text{def}}{=} \{q \in \mathcal{H} : (q, q) \in U\}$$

is of full measure for $(\pi_1)_* \theta$, where $\pi_1 : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is the projection onto the first factor. Indeed, the measure θ is invariant under $Z_0 \times \{\mathrm{Id}\}$, and hence so is its support. Since Z_0 acts by homeomorphisms where defined, and using property (v) in Theorems 5.1 and 5.2, we have that the set U is also $Z_0 \times \{\mathrm{Id}\}$-invariant. Thus for any Z_0-ergodic measure, it is either null or conull. Thus if $q \notin U_1$ and q is generic for the measure $\mu_{T(q)}$ appearing in (3.1), then $\mu_{T(q),q}$ assigns measure zero to U, where $\mu_{T(q),q}$ is the measure on $\operatorname{supp} \theta$ defined by $\mu_{T(q),q}(A) = \mu_{T(q)}(\{q' : (q', q) \in A\})$. If this were to happen for a positive measure of q it would follow from (3.1) and the fact that $\mu_{T(q)} \times \mu_{T(q)} = \int \mu_{T(q'),q'} d\mu_{T(q)}$ that U does not have full measure for θ.

For $q \in U_1$, let N_q denote the connected component of $U \cap \pi_1^{-1}(q) \cap \operatorname{supp} \theta$ containing (q, q). Since the fibers $\pi_1^{-1}(q)$ are also affine submanifolds of $\mathcal{L} \times \mathcal{L}$, we have that the N_q are affine submanifolds contained in $\pi_1^{-1}(q) \cong \mathcal{L}$, so we can identify them with invariant submanifolds in $\mathcal{L}$ (which we continue to denote by N_q). With this notation we have $q \in N_q$.

The mapping $q \mapsto T(N_q)$ is locally constant; that is, letting $V \subset \bar{\mathcal{H}}$ and $\mathcal{V} \subset \bar{\mathcal{H}}_{\mathrm{m}}$ be open sets such that $\pi|_{\mathcal{V}} : \mathcal{V} \to V$ is a homeomorphism and $q \in V$, the map $q \mapsto \operatorname{dev} \circ \pi|_{\mathcal{V}}^{-1}(q)$ sends a neighborhood of q in N_q to an affine subspace W_q of $H^1(S, \Sigma; \mathbb{R}^2)$, and the corresponding linear spaces $W_q - W_q$ are the same for all $q \in V$. Since $m_{\mathcal{L}} \times m_{\mathcal{L}}$ is the unique P-invariant ergodic measure on $\mathcal{L} \times \mathcal{L}$ of full support, we have $\dim N_q < \dim \mathcal{L}$ for every $q \in U_1$.

Let $\bar{N}_q$ denote the set of surfaces (not necessarily of area one) which are obtained by rescaling surfaces in N_q, and let

$$\mathfrak{N}_q \stackrel{\text{def}}{=} T_q(\bar{N}_q)$$

(the tangent space to $\bar{N}_q$ at q, thought of as a subset of the tangent space $T_q(\bar{\mathcal{L}})$). The assignment $q \mapsto \mathfrak{N}_q$ defines a proper flat sub-bundle of the tangent bundle $T(\bar{\mathcal{L}})$. Flat sub-bundles of $T(\bar{\mathcal{L}})$ were classified in [Wr2]. According to [Wr2, Thm. 5.1], $\mathfrak{N}_q \subset \mathfrak{R}_{\mathcal{L}}$ for each q, and $\mathfrak{N}_q$ is a complex linear subspace which is locally constant. Since $\mathfrak{R}_{\mathcal{L}}$ is acted on trivially by monodromy, we in fact have that $\mathfrak{N}_q$ is independent of q, and we denote it by $\mathfrak{N}$. The leaves $\mathfrak{N}(q)$ are contained in $\bar{N}_q$ for each q, and of the same dimension. That is, $\mathfrak{N}(q)$ is the connected component of $\bar{N}_q$ containing q. Since Rel deformations do not affect the area of the surface, we see that $\bar{N}_q = N_q$. In particular $\mathfrak{N}(q)$ is closed for each q.

By Proposition 3.4, for a.e. q, N_q is the support of the ergodic component $(m_{\mathcal{L}})_q$, and in particular

$$(m_{\mathcal{L}})_q(N_q) < \infty, \quad \text{for a.e. } q.$$

Since Z_0 acts ergodically with respect to $(m_{\mathcal{L}})_q$, we have that almost surely $N_q = \mathfrak{N}(q)$. Since the measure $(m_{\mathcal{L}})_q$ is affine in charts, it is a scalar multiple of the translation-invariant measure on $\mathfrak{N}(q)$, and thus the volume V_q of $\mathfrak{N}(q)$ (with respect to its translation-invariant measure) is almost surely finite. It is clear that the function $q \mapsto V_q$ is U-invariant, and by ergodicity, it is constant almost surely. $\square$

Remark 6.1. *We note that the above argument works under much weaker conclusions than those given in Theorem 5.2. Indeed, in the first step of the argument, Theorem 5.2 was used simply to extract a G-invariant assignment $q \mapsto N_q$, where N_q is a subspace of $T_q(\mathcal{L})$, which is proper if θ is not the product joining. A fundamental fact about such G-invariant assignments is that they are very restricted – besides* [Wr2]*, see* [EFW] *and* [Fi]*. In particular,* [Fi] *gives strong restrictions on assignments that are only assumed to be defined almost everywhere and measurable.*

7. A topological condition for Rel ergodicity

Let $Z_0 \subset Z$ be a subspace. We say that a translation surface x is *Z_0-stably periodic* if it can be presented as a finite union of horizontal cylinders and the Z_0-orbit of x is well defined. Recall that a *horizontal separatrix* is a horizontal leaf whose closure contains at least one singularity, and it is a *horizontal saddle connection* if its closure contains two singularities. Then the condition of being Z_0-stably periodic is equivalent to requiring that all horizontal separatrices starting at singular points are on horizontal saddle connections, and Z_0 preserves the holonomy of every horizontal saddle connection on x. In case $Z = Z_0$ is the full real Rel group, we say that x is *fully stably periodic.* This is equivalent to saying that all horizontal separatrices starting at singular points are on saddle connections, and all horizontal saddle connections start and end at the same singularity. In particular, for any cylinder C on a fully stably periodic surface, each boundary component of C is made of saddle connections starting and ending at the same singular point ξ; we say that the boundary component *only sees singularity* ξ. For more information on the real Rel action on surfaces which are horizontally completely periodic, see [HW, §6.1].

Proposition 7.1. *Suppose x is a surface which is Z_0-stably periodic, and $v \in Z_0$ moves two singularities p and q with respect to each other. Suppose that x contains two cylinders C_1 and C_2 that both only see singularity p on one boundary component and only see singularity q on another boundary component. Finally suppose the circumferences c_1, c_2 of these cylinders satisfy $\frac{c_1}{c_2} \notin \mathbb{Q}$. Then Case* (2) *of Theorem 1.1 does not hold for x.*

Proof. Since $\frac{c_1}{c_2} \notin \mathbb{Q}$, the trajectory $\{\mathrm{Rel}_{tv}(x) : t \in \mathbb{R}\}$ is not closed, let $\mathcal{L}$ denote its closure. We claim that the tangent space to $\mathcal{L}$ is not contained in Z. Let σ_1 denote a saddle connection from p to q in C_1 and let σ_2 denote a saddle connection from q to p in C_2. Let σ be the concatenation. Then σ represents an absolute homology class because it goes from p back to p, and it is nontrivial because the vertical component of its holonomy on x is nonzero. If we consider the restriction of the rel-action to $C_1 \cup C_2$ then it only affects the twist parameters, which is a 2-dimensional space. This space can be generated by the horizontal holonomy of σ_1 and the horizontal holonomy of σ_2. Since $\frac{c_1}{c_2} \notin \mathbb{Q}$, this restricted action does not give a closed orbit. So the tangent space to $\mathcal{L}$ contains directions, which continuously affect the holonomy of σ. Since σ is an absolute period, we see that the tangent space to $\mathcal{L}$ is not contained in Z. □

8. Checking the condition for strata

Let $\mathcal{H} = \mathcal{H}(a_1, \ldots, a_k)$ and for $i, j \in \{1, \ldots, k\}$, let ξ_i, ξ_j be the corresponding singular points of a surface in $\mathcal{H}$. Let $z \in \mathfrak{R}$ be a Rel cohomology class. We say that z *moves* ξ_i, ξ_j *with respect to each other* if for some (equivalently, every) $\alpha \in H_1(S, \Sigma)$ represented by a path starting at ξ_i and ending at ξ_j, we have $z(\alpha) \neq 0$. Below when we discuss a stratum $\mathcal{H}(a_1, \ldots, a_k)$ we allow $a_i = 0$, that is we allow marked points. We call points with cone angle 2π (that is, with $a = 0$) *removable singularities*, and otherwise we call them *non-removable*. The following result, which clearly implies Theorem 1.1, allows strata with removable singularities.

Theorem 8.1. *Let $\mathcal{H}$ be a connected component of a stratum $\mathcal{H}(a_1, \ldots, a_k)$. Let $m_{\mathcal{H}}$ be the Masur-Veech measure on $\mathcal{H}$, let Z be the corresponding real Rel foliation, and let $Z_0 \subset Z$ be a one-dimensional connected subgroup of Z. Suppose that there are $1 \leq i < j \leq k$ with corresponding singular points ξ_i, ξ_j, such that $a_i > 0$, $a_j > 0$ and such that some element of Z_0 moves ξ_i, ξ_j with respect to each other. Then the Z_0-flow on $(\mathcal{H}, m_{\mathcal{H}})$ is mixing of all orders (and in particular, ergodic).*

Clearly, Theorem 8.1 follows from Theorem 1.2, Proposition 7.1, and the following result.

Proposition 8.2. *Let $\mathcal{H} \subset \mathcal{H}(a_1, \ldots, a_k)$ be a connected component of a stratum of translation surfaces with at least two non-removable singular points. If $p \neq q$ is any pair of non-removable singularities then there exists $M \in \mathcal{H}$, which has cylinders C_1, C_2 with circumferences c_1, c_2 so that*

(1) M is fully stably periodic.
(2) $\frac{c_1}{c_2} \notin \mathbb{Q}$.
(3) Both C_1 and C_2 only see singularity p on one boundary component and only see singularity q on the other boundary component.

For the proof of Proposition 8.2 we will also need the following:

Proposition 8.3. *Let $\mathcal{H} = \mathcal{H}(a_1, \ldots, a_k)$ be a stratum of translation surfaces with at least two singular points (that is $k \geq 2$). If $p \neq q$ is any pair of distinct singularities (possibly removable), then there exists $M \in \mathcal{H}$, so that M is fully stably periodic and there exists a cylinder on M that only sees singularity p on one boundary component, and only sees singularity q on the other boundary component.*

Propositions 8.2 and 8.3 will both be proved by induction, after some preparations.

Lemma 8.4 (The basic surgery – gluing in a torus)**.** *Let $\mathcal{H} = \mathcal{H}(b_1, \ldots, b_\ell)$ be a stratum of translation surfaces, and let $M \in \mathcal{H}$, with singularities labeled by $\xi_1, \ldots, \xi_\ell$, so that the order of ξ_i is b_i. Suppose M has a horizontal cylinder C, with circumference c, where one boundary component is made of saddle connections that begin and end at ξ_i, and the other is made of saddle connections that begin and end at ξ_j, where $b_i \geq 0$ and $b_j \geq 0$ (so that ξ_i, ξ_j might be removable). Then for all $w > 0$ there exists $M' \in \mathcal{H}(b_1, \ldots, b_i+1, \ldots, b_j+1, \ldots, b_\ell)$, with singularities labeled $\xi'_1, \ldots, \xi'_\ell$, which has two horizontal cylinders C'_1, C'_2, where C'_1 has circumference $c + w$ and C'_2 has circumference w. The complements $M \smallsetminus C$ and $M' \smallsetminus (C_1 \cup C_2)$ are isometric, by an isometry mapping ξ'_i to ξ_i for all i. The cylinders C_1 and C_2*

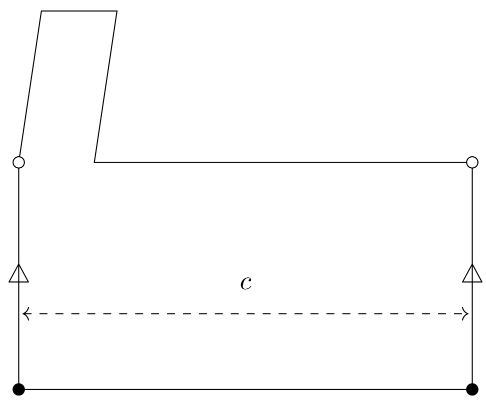

FIGURE 1. The surface M has a cylinder of circumference c, and its boundary components see only the singularities ξ_i and ξ_j (denoted by $\circ$ and $\bullet$). The edges not labeled by $\triangle$ are connected to $M \smallsetminus C$.

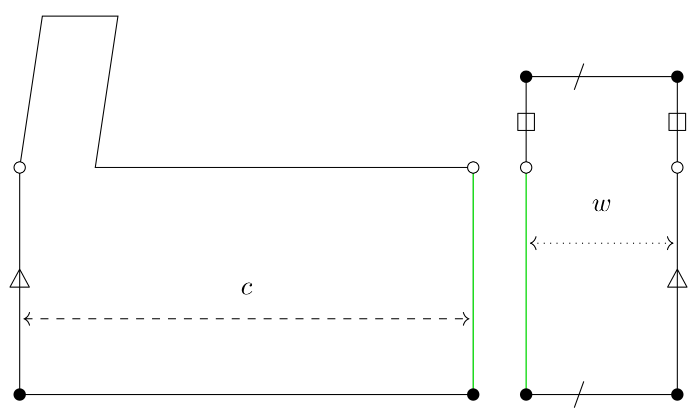

FIGURE 2. To obtain M' from M, glue in a torus (rectangle on the right). This transforms C into a cylinder C'_1 of circumference $c+w$, and adds a horizontal cylinder C'_2 of circumference w. Edges not labeled by $\triangle$, $\square$, / or the color green are attached to $M' \smallsetminus (C'_1 \cup C'_2)$.

only see singularity ξ'_i on one boundary component, and ξ'_j on another. Moreover, if M is fully stably periodic then so is M'.

Proof. It will be easier to follow the proof while consulting Figures 1 (before) and 2 (after). Given a polygonal presentation for M, we give a polygonal presentation for M'. Let M be a polygon representation for M in which the cylinder C is represented by a parallelogram P (in Figure 1, the large rectangle in the center of the presentation), with two horizontal sides of length c, non-horizontal sides identified to each other, and the singular points ξ_i, ξ_j on adjacent corners of P. Thus the non-horizontal sides of P represent a saddle connection σ on M connecting ξ_i to ξ_j. We consider the two non-horizontal sides of P as distinct and label them by σ_1, σ_2. Let P' be a parallelogram with sides parallel to those of P, where the horizontal sides have length w and the nonhorizontal sides are longer than the ones on P (in Figure 2, P' is to the right of P).

Label the two horizontal sides of P' by h'_1 and h'_2, and identify them by a translation. Partition the non-horizontal sides of P' into two segments. The segments σ'_1, σ'_2 are parallel to each other and have the same length as σ_1, σ_2, and start at a corner of P. The segments γ'_1, γ'_2 comprise the remainder of the non-horizontal sides of P' (and in particular, have the same length). Identify γ'_1 to γ'_2 by a translation, and identify σ'_1, σ'_2 to σ_1, σ_2 by a translation so that each σ'_i is attached to the σ_j with the opposite orientation. Let M' be the translation surface corresponding to this presentation. It is clear that M' has the required properties.

□

Proof of Proposition 8.3. The proof is by induction on $\sum a_i$.

Base of induction: The base case is the stratum $\mathcal{H}(a_1, 0^s)$, that is, one singular point (removable or non-removable) of order a_1, and some number $s \geq 1$ of removable singular points. In this case we take a surface in $\mathcal{H}(a_1)$ which is made of one horizontal cylinder. We label the singular point by ξ_1 and place additional removable singular points $\xi_2, \ldots, \xi_{s+1}$ in the interior of the cylinder, at different heights (so that the resulting surface has no horizontal saddle connections between distinct singularities) and so that ξ_i and ξ_j are on opposite sides of a cylinder.

Inductive step: Suppose $\mathcal{H}' = \mathcal{H}(a_1, \ldots, a_k)$ is our stratum, where at least two of the singularities are non-removable. Let p', q' be labels of singular points for surfaces in $\mathcal{H}'$, corresponding to indices $i \neq j$. To simplify notation assume $i = 1, j = 2$. There are three cases to consider: $a_i = a_j = 0$, or one of a_i, a_j are positive, or both are positive.

If $a_i = a_j = 0$ then by assumption $k \geq 4$. We take a cylinder C on a fully stably completely periodic surface M in $\mathcal{H} = \mathcal{H}(a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_k)$. The notation $\hat{a}_i$ means that the symbol should be ignored; that is on a stratum of the same genus with $k - 2 \geq 2$ singular points obtained by removing two removable singular points. We place two singular points marked p', q' in the interior of C at different heights. If $a_i > 0$ and $a_j = 0$ is zero we take a fully stably periodic surface M in $\mathcal{H}(a_1, \ldots, a_i - 1, \ldots, \hat{a}_j, \ldots, a_k)$, find a cylinder C on M whose boundary component is made of saddle connections starting and ending at ξ_i, place a marked point labeled ξ_j in the interior of C. If a_i and a_j are both positive we use the induction hypothesis to find a surface $M \in \mathcal{H}(a_1, \ldots, a_i - 1, \ldots, a_j - 1, \ldots, a_k)$ with a cylinder whose boundary components see ξ_i and ξ_j, and we perform the surgery in Lemma 8.4 to this cylinder. □

Lemma 8.5 (Two surgeries involving genus two surfaces)**.** *Let $\mathcal{H} = \mathcal{H}(b_1, \ldots, b_k)$ be a stratum of translation surfaces and let $M \in \mathcal{H}$ have a horizontal cylinder C, with circumference c. Let p and q be singular points with order b_i, b_j respectively, such that one boundary component of C only sees singularity p and the other only sees singularity q. Then for any $w_1, w_2 > 0$ there exists $M' \in \mathcal{H}' = \mathcal{H}(b_1, \ldots, b_i + 2, \ldots, b_j + 2, \ldots, b_k)$ which has three cylinders C_1, C_2, C_3 with circumferences $c + w_1 + w_2, w_1$ and w_2 respectively. The complements $M \smallsetminus C$ and $M' \smallsetminus (C_1 \cup C_2 \cup C_3)$ are isometric by an isometry preserving the labels of singular points, and C_1, C_2, C_3 all have one boundary component that sees only p, and another that sees only q. Thus, if M is fully stably periodic so is M'. Moreover, if the b_i are all even, so that*

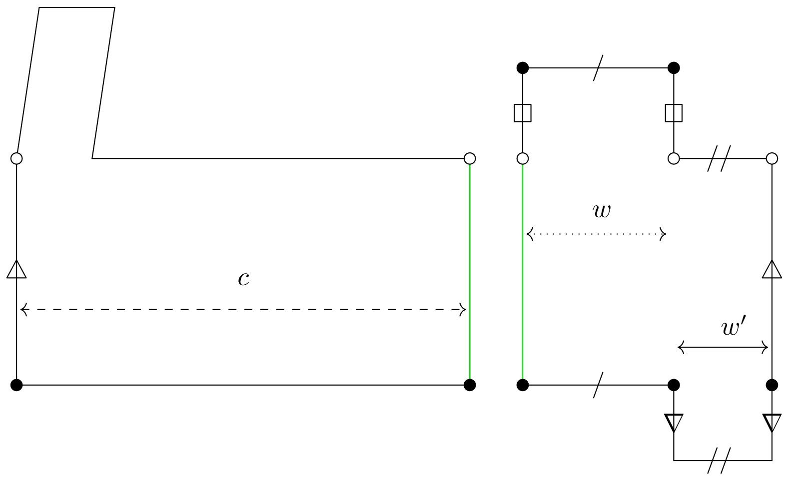

FIGURE 3. First option for M' in Lemma 8.5. Attaching the subsurface on the right increases the genus by 2. Unlabeled edges are attached to $M' \smallsetminus (C_1 \cup C_2 \cup C_3)$.

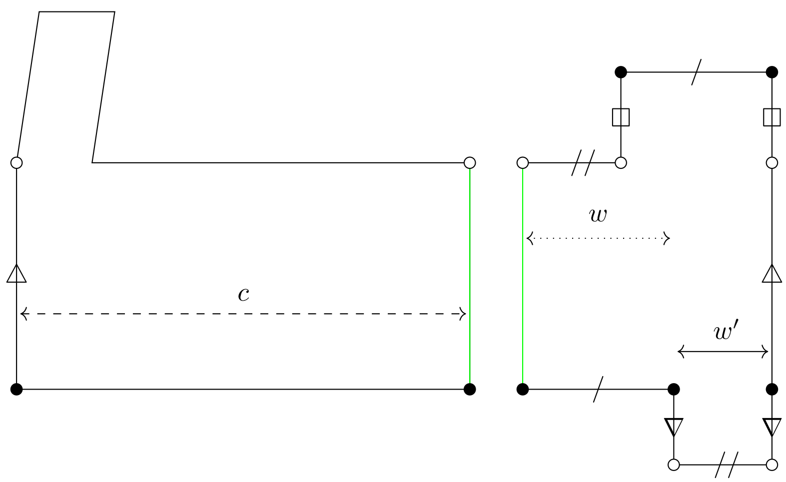

FIGURE 4. Second option for M', with a different spin.

$\mathcal{H}'$ has even and odd spin components, we can choose M' to be in either the even or odd connected component.

Proof. Once again we encourage the reader to consult Figures 3 and 4.

In Lemma 8.4 we made a slit in M, running through P from top to bottom, and glued in a torus with a slit. In this case we make an identical slit, this time gluing in a genus two surface with a slit. This surface is presented in Figures 3 and 4 as made up of three rectangles. It is straightforward to check that $M' \in \mathcal{H}'$ and that it has cylinders satisfying the desired properties. It remains to check the final assertion about the parity of the spin structure.

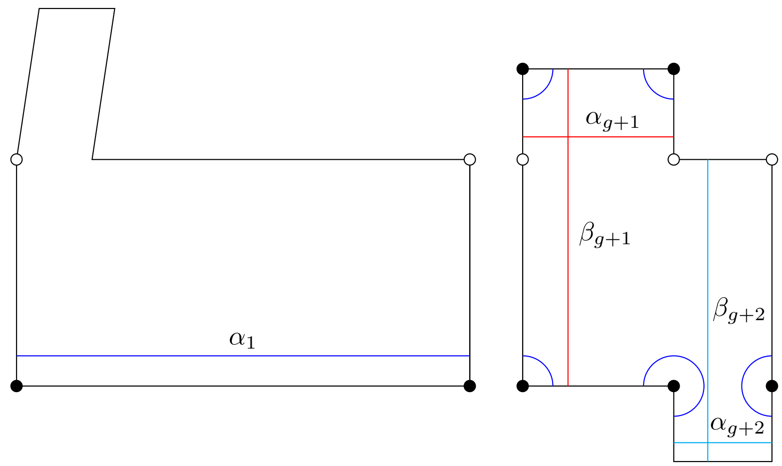

FIGURE 5. Modifying the symplectic basis. Gluings as in Figure 3.

Recall from [KZ, eqn. (4)] that where defined, the spin structure of a surface M of genus g can be computed as follows. Let α_i, β_j (where $1 \leq i, j \leq g$) be a symplectic basis for $H_1(M)$, realized explicitly as smooth curves on M. This means that all of these curves are disjoint, except for α_i and β_i which intersect once. For each curve γ, let $\mathrm{ind}(\gamma)$ be the turning index, that is the total number of circles made by the tangent vector to γ, as one goes around γ. The parity of M is then the parity of the integer $\sum_{i=1}^{g}(1 + \mathrm{ind}(\alpha_i))(1 + \mathrm{ind}(\beta_i))$. It is shown in [KZ] that this number is well-defined (independent of the choice of the symplectic basis) when all the singular points have even order.

Suppose M has genus g and is equipped with a symplectic basis. Since any non-separating simple closed curve can be completed to a symplectic basis, we can assume that α_1 is the core curve of C, and the other curves in the basis do not intersect the saddle connection from p to q passing through C. We construct a symplectic basis for M' in both cases, by modifying α_1, keeping $\alpha_2, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, and adding new curves $\alpha_{g+1}, \alpha_{g+2}, \beta_{g+1}, \beta_{g+2}$. The modified curves are shown in Figures 5, 6, and the reader can easily check that these new curves still form a symplectic basis, and that these two choices add two numbers of different parities to the spin structure. $\square$

Note that in Proposition 8.2 we care about all connected components of strata. We need to record some information about the classification of connected components of strata, due to Kontsevich and Zorich. A translation surface is *hyperelliptic* if it admits an involution which acts on absolute homology as $-\mathrm{Id}$ (see [FM] or [KZ, §2.1] for more details). A connected component of a stratum is *hyperelliptic* if all surfaces in the component are hyperelliptic.

Proposition 8.6 ([KZ], Theorems 1 & 5 and Corollary 5 of Appendix B)**.** *Let $\mathcal{H}(a_1, \ldots, a_k)$ be a stratum with $a_i > 0$ for all i. The following holds:*

- *$\mathcal{H}$ has three connected components in the following cases:*
  - $k = 1, a_1 = 2g - 2, g \geq 4$.

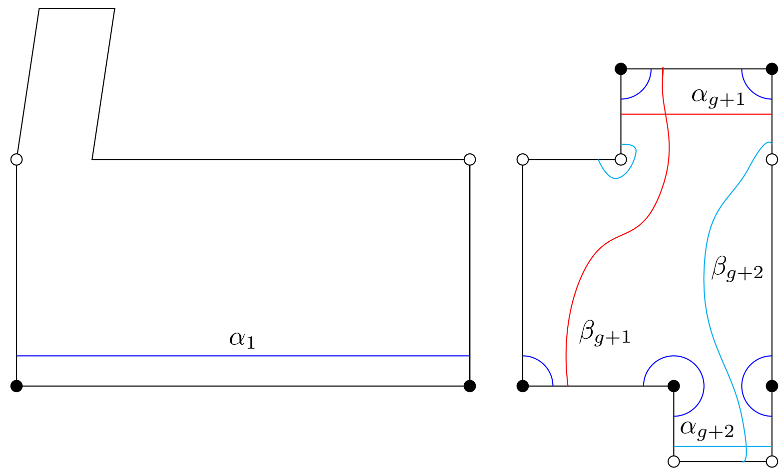

FIGURE 6. Modifying the symplectic basis, second case. Gluings as in Figure 4. Note the change in the rotation number of β_{g+2}.

  - *$k = 2$, $a_1 = a_2 = g - 1$, $g \geq 5$ is odd. One is hyperelliptic, and the two non-hyperelliptic strata are distinguished by the spin invariant.*
- *$\mathcal{H}$ has two connected components in the following cases:*
  - *All of the a_i are even, $g \geq 4$, and either $k \geq 3$ or $a_1 > a_2$. The components are distinguished by their spin.*
  - *$a_1 = a_2$ and g is either 3 or is even. One of the components is hyperelliptic and the other is not. When $g = 3$ the hyperelliptic component is even, and the other one is odd.*
- *$\mathcal{H}$ is connected in all other cases.*

Proof of Proposition 8.2. The proof will be case-by-case. Here are the cases:

(i) $\mathcal{H}(1, 1)$.
(ii) All the a_i are nonzero and $\mathcal{H}$ is connected.
(iii) All the a_i are nonzero and $\mathcal{H}$ has two connected components distinguished by spin.
(iv) All the a_i are nonzero and $\mathcal{H}$ has two connected components distinguished by hyperellipticity.
(v) All the a_i are nonzero and $\mathcal{H}$ has three connected components.
(vi) Some of the a_i are zero.

Case (i). There is just one connected component and the desired surface is a Z-shaped surface, with three horizontal cylinders C_1, C_2, C_3 of circumferences $c_1, c_1 + c_3, c_3$, where C_1, C_3 are simple. We put all of the removable singular points in the interior of C_3, and choose choose c_1, c_3 so that $c_1/(c_1 + c_3) \notin \mathbb{Q}$. It is clear that with these choices the conditions are satisfied.

Case (ii). The stratum $\mathcal{H}$ is connected, and we have at least two singularities of positive order. So with no loss of generality that they are labelled 1 and 2. The result follows from Lemma 8.4, applied to a surface in $\mathcal{H}(a_1 - 1, a_2 - 1, a_3, \ldots, a_k)$, and taking $w \notin c\mathbb{Q}$, so that $w/(c + w) \notin \mathbb{Q}$.

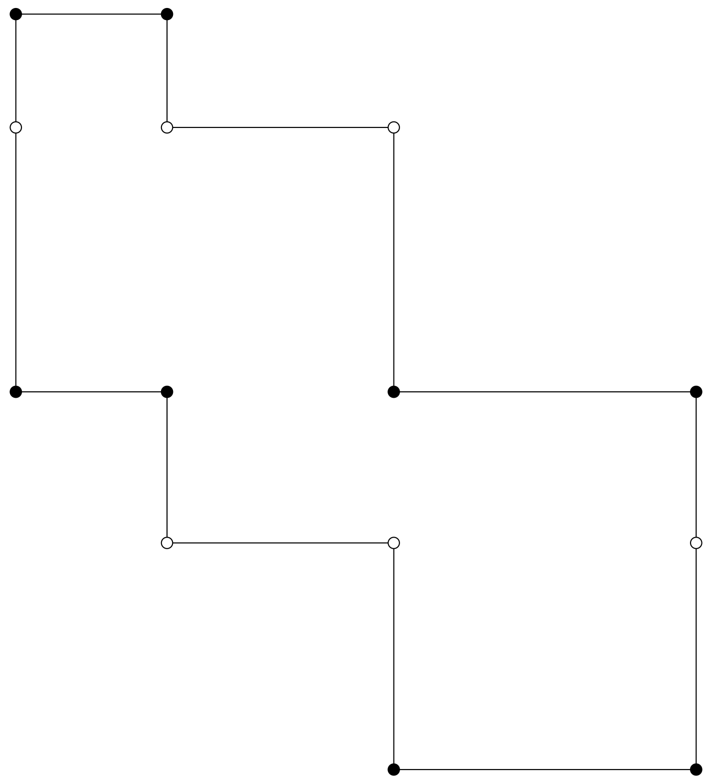

FIGURE 7. A surface in $\mathcal{H}^{hyp}(2,2)$.

Case (iii). We apply the surgery in Lemma 8.5, with $w_1/w_2 \notin \mathbb{Q}$. Namely if p and q are labelled i, j, we let $b_i = a_i - 2$, $b_j = a_j - 2$ and $b_\ell = a_\ell$ for $\ell \neq i, j$.

Case (iv). There are two connected components. One is hyperelliptic, one is not. This means that $a_1 = a_2$ and either $g = 3$ (in which case $a_1 = a_2 = 2$) or $g \geq 4$ is even (in which case $a_1 = a_2 = g - 1$). In this case we give explicit surfaces, one in each connected component. The first surface (the $\mathcal{H}(2,2)$ case is shown in Figure 7) is a 'staircase' surface made of gluing $2g$ rectangles to each other. The rectangles are labelled (k, B) and (k, T) for $k = 1, \ldots, g$. The top (respectively, bottom) of (k, B) is glued to the bottom (resp., top) of (k, T) for $k = 1, \ldots, g$, and the left (resp., right) of (k, T) is glued to the right (resp., left) of $(k + 1, B)$ for $k = 1, \ldots, g - 1$. The horizontal sides of $(1, B)$ are glued to each other, as are the horizontal sides of (g, T). This surface is hyperelliptic since it has a hyperelliptic involution rotating each rectangle around its midpoint, and this involution swaps the singularities (see [KZ, Remark 3]). The second surface is obtained as follows. We first construct a hyperelliptic surface in $\mathcal{H}(a_1 - 2, a_2 - 2)$ as in the previous paragraph. Then we perform the surgery described in Lemma 8.5. The resulting surface has a horizontal cylinder intersecting three vertical cylinders, and thus, by [Li, Lemma 2.1], is not hyperelliptic. See Figure 8 for an example in $\mathcal{H}(2,2)$. In both of these constructions there are no restrictions on the sidelengths of the rectangles, and we can easily arrange that two of the circumferences are incommensurable.

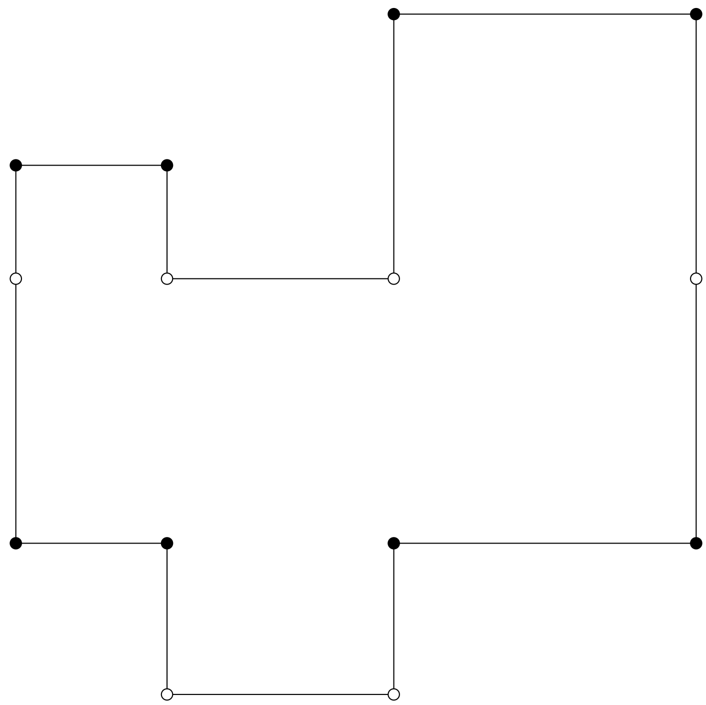

FIGURE 8. A surface in $\mathcal{H}^{nonhyp}(2,2)$.

Case (v). In this case $a_1 = a_2 = g - 1$ for $g \geq 5$ odd. Applying the argument in Case (iii), we obtain the required surfaces in the odd and even connected components. To obtain the required surface in the hyperelliptic component we use the 'staircase surface' describe in Case (iv).

Case (vi). Assume with no loss of generality that the removable singularities are labelled $k'+1, \ldots, k$ for some $k' \geq 2$, and let $\mathcal{H}' = \mathcal{H}(a_1, \ldots, a_{k'})$. Note that the singularities p and q have label in $\{1, \ldots, k'\}$. Apply the preceding considerations to obtain a surface in $\mathcal{H}'$ with the required cylinders. By examining the proof in all preceding case one sees that the number of horizontal cylinders on this surface is at least three, that is there is at least one cylinder C_3 which is distinct from the cylinders C_1, C_2, and we modify M by adding $k - k'$ in general position in the interior of C_3, to obtain the desired surface.

□

9. Zero entropy

In this section we prove the following result:

Theorem 9.1. *Let $\mathcal{H}$ be a stratum for which* $\dim Z > 0$*, let $z \in Z \smallsetminus \{0\}$, and let μ be a probability measure on $\mathcal{H}$ such that* $\mathrm{Rel}_z(q)$ *is defined for μ-a.e. q. Assume that μ is* Rel_z*-invariant and ergodic, and assume in addition that*

(9.1) *there is $t_n \to \infty$ so that $(g_{-t_n})_*\mu$ converges to a probability measure*

(in the weak- topology). Then the entropy of* Rel_z *acting on $(\mathcal{H}, \mu)$ is zero.*

For the proof of Theorem 9.1, we will need an estimate showing that points stay close to each other for times up to L, provided their initial distance is polynomially small (as a function of L). To make this precise we will use the sup-norm Finsler metric on $\mathcal{H}$, which was introduced by Avila, Gouëzel and Yoccoz [AGY] and whose definition we now recall. For q_0, q_1 belonging to the same connected component of a stratum $\mathcal{H}$, we write

$$\mathrm{dist}(q_0, q_1) = \inf_{\gamma} \int_0^1 \|\gamma'(t)\|_{\gamma(t)} dt, \tag{9.2}$$

where $\gamma : [0,1] \to \mathcal{H}_{\mathrm{m}}$ ranges over all C^1 curves with $\gamma(0) \in \pi^{-1}(q_0)$, $\gamma(1) \in \pi^{-1}(q_1)$, and $\|\cdot\|_{\mathbf{q}}$ is a pointwise norm on the tangent space to $\mathcal{H}_{\mathrm{m}}$ at $\mathbf{q}$, identified via the developing map with $H^1(S, \Sigma; \mathbb{R}^2)$. Below, balls, diameters of sets, and ε-neighborhoods of sets will be defined using this metric. We can now state our estimate.

Proposition 9.2. *Let $\mathcal{H}$ be a stratum of translation surfaces with at least two singularities, let Z be its real Rel space, let $z_0 \in Z$, and let T be the map of $\mathcal{H}$ defined by applying Rel_{z_0} (where defined). Then for every compact subset $K \subset \mathcal{H}$, there is $L_0 > 0$, such that if $q \in \mathcal{H}$, $L \in \mathbb{N}$, $L > L_0$, satisfy*

$$q \in K \quad \text{and} \quad g_{-\ell} q \in K, \quad \text{where } \ell \stackrel{\mathrm{def}}{=} 2 \log L, \tag{9.3}$$

then the maps $T, \ldots, T^L$ are all defined on $B\left(q, \frac{1}{L^5}\right)$, and we have

$$\max_{j=1,\ldots,L} \mathrm{diam}\left(T^j\left(B\left(q, \frac{1}{L^5}\right)\right)\right) \to_{L\to\infty} 0.$$

We have made no attempt to optimize the power 5 in this statement.

Our proof of Proposition 9.2 will use some properties of the sup-norm metric. They are proved in [AGY], see also [AG] and [CSW, §2]. Our notation will follow the one used in [CSW].

Proposition 9.3. *The following hold:*

- (a) *For all q_0, q_1 and all $t \in \mathbb{R}$, $\mathrm{dist}(g_t q_0, g_t q_1) \le e^{2|t|}\mathrm{dist}(q_0, q_1)$.*
- (b) *The metric dist is proper; that is, for any fixed basepoint q_0, the map $q \mapsto \mathrm{dist}(q, q_0)$ is proper. In particular, the ε-neighborhood of a compact set is pre-compact, for any $\varepsilon > 0$.*
- (c) *The map $\mathbf{q} \mapsto \|\cdot\|_{\mathbf{q}}$ is continuous, and hence bounded on compact sets. This means that for any compact $K \subset \mathcal{H}_{\mathrm{m}}$ there is $C > 0$ such that for any $\mathbf{q}_0, \mathbf{q}_1$ in K, the norms $\|\cdot\|_{\mathbf{q}_0}, \|\cdot\|_{\mathbf{q}_1}$ are bi-Lipschitz equivalent with constant C.*
- (d) *The infimum in (9.2) is actually a minimum, that is attained by some curve γ.*

With these preparations, we can give the

Proof of Proposition 9.2. Write $B \stackrel{\mathrm{def}}{=} B\left(q, \frac{1}{L^5}\right)$, $A \stackrel{\mathrm{def}}{=} g_{-\ell}(B)$ (note that A and B both depend on L and q but we suppress this from the notation). Let K' be the 1-neighborhood of K, which is a pre-compact subset of $\mathcal{H}$ by Proposition 9.3(b). Since $\mathrm{diam}(B) \le \frac{2}{L^5}$, Proposition 9.3(a) implies that

$$\mathrm{diam}(A) \le \frac{2}{L^3}. \tag{9.4}$$

It follows from (9.3) that $A \cap K \neq \varnothing$ and therefore $A \subset K'$. Since

$$\max_{j=1,\ldots,L} \|je^{-\ell} z_0\| \leq \frac{1}{L}\|z_0\| \to_{L\to\infty} 0, \tag{9.5}$$

for all large enough L (depending on K') we have that $\mathrm{Rel}_{je^{-\ell}z_0}(q')$ is defined for $q' \in K'$. Since $q_1' \stackrel{\mathrm{def}}{=} \mathrm{Rel}_{je^{-\ell}z_0} \circ g_{-\ell}(q_1)$ is defined for $q_1 \in B$, we have from (2.3) that $T^j(q_1) = \mathrm{Rel}_{jz_0}(q_1) = g_\ell(q_1')$ is also defined. This proves that the maps $T, T^2, \ldots, T^L$ are all defined on B.

Furthermore, this computation shows that $T^j(B) = g_\ell(\mathrm{Rel}_{je^{-\ell}z_0}(A))$, and so by Proposition 9.3(a), it suffices to show that

$$L^2 \cdot \mathrm{diam}\left(\mathrm{Rel}_{je^{-\ell}z_0}(A)\right) \to_{L\to\infty} 0.$$

Taking into account (9.4) and (9.5), it suffices to show that for any compact K' there are positive ε, C such that for any $q_0, q_1 \in K'$ with $\mathrm{dist}(q_0, q_1) < \varepsilon$, and any $z \in Z$ with $\|z\| < \varepsilon$, we have

$$\mathrm{dist}(\mathrm{Rel}_z(q_0, q_1)) \leq C\,\mathrm{dist}(q_0, q_1). \tag{9.6}$$

Informally, this is a uniform local Lipschitz estimate for the family of maps defined by small elements of Z.

To see (9.6), let ε_1 be small enough so that for any $q \in K'$, the ball $B(q, 2\varepsilon_1)$ is contained in a neighborhood which is evenly covered by the map $\pi : \mathcal{H}_{\mathrm{m}} \to \mathcal{H}$, and let C be a bound as in Proposition 9.3(c), corresponding to the compact set which is the $2\varepsilon_1$-neighborhood of K'. Let $\varepsilon < \varepsilon_1$ so that for any $z \in Z$ with $\|z\| < \varepsilon$ and any $q \in \mathcal{H}$, $\mathrm{dist}(q, \mathrm{Rel}_z(q)) < \varepsilon_1$. If $\mathrm{dist}(q_0, q_1) < \varepsilon$ then the path γ realizing their distance (see Proposition 9.3(d)) is contained in a connected component $\mathcal{V}$ of $\pi^{-1}(B(q_0, \varepsilon_1))$. Let

$$\bar\gamma : [0,1] \to H^1(S, \Sigma; \mathbb{R}^2), \quad \bar\gamma(t) \stackrel{\mathrm{def}}{=} \mathrm{dev}(\gamma(t)) - \mathrm{dev}(\gamma(0)),$$

let

$$\gamma_1 \stackrel{\mathrm{def}}{=} \mathrm{Rel}_z \circ \gamma,$$

and analogously define

$$\bar\gamma_1 : [0,1] \to H^1(S, \Sigma; \mathbb{R}^2), \quad \bar\gamma_1(t) \stackrel{\mathrm{def}}{=} \mathrm{dev}(\gamma_1(t)) - \mathrm{dev}(\gamma_1(0)).$$

By choice of ε and ε_1, the curve γ_1 also has its image in $\mathcal{V}$. Since Rel_z is expressed by $\mathrm{dev}|_{\mathcal{V}}$ as a translation map, the curves $\bar\gamma, \bar\gamma_1$ are identical maps. When computing $\mathrm{dist}(\mathrm{Rel}_z(q_0), \mathrm{Rel}_z(q_1))$ via (9.2), an upper bound is given by computing the path integral along the curve γ_1. We compare this path integral along γ_1, with the path integral along γ giving $\mathrm{dist}(q_0, q_1)$. In these two integrals, for any t, the tangent vectors $\gamma'(t), \gamma_1'(t)$ are identical elements of $H^1(S, \Sigma; \mathbb{R}^2))$ for all t, but the norms are evaluated using different basepoints. Since these basepoints are all in the $2\varepsilon_1$-neighborhood of K', by choice of C, we have $\|\gamma_1'(t)\|_{\gamma_1(t)} \leq C\|\gamma'(t)\|_{\gamma(t)}$ for all t. This implies (9.6). □

We now list a few additional results we will need. The first is the following weak Besicovitch-type covering Lemma, for balls of equal size.

Proposition 9.4. *For any compact $K \subset \mathcal{H}$ there are positive N, r_0 so that for any $r \in (0, r_0)$, for any $G \subset K$ the collection $\mathcal{C} \stackrel{\mathrm{def}}{=} \{B(q, r) : q \in G\}$ contains N finite* *subcollections* $\mathcal{F}_1, \ldots, \mathcal{F}_N$ *satisfying* $G \subset \bigcup_{i=1}^N \bigcup \mathcal{F}_i$, *and each collection* $\mathcal{F}_i$ *consists of disjoint balls.*

Proof. The argument is standard, we sketch it for lack of a suitable reference.

We first claim that given a compact K and $r \in (0, r_0)$, there is N so that the largest r-separated subset of any ball of radius $2r$, has cardinality at most N. Indeed, this property is true for Euclidean space by a simple volume argument, and is invariant under biLipschitz maps. Thus the claim holds by Proposition 9.3(c).

We now inductively choose the $\mathcal{F}_i$. Let $\mathcal{F}_1$ be a maximal collection of disjoint balls of radius r with centers in G. For $i \geq 2$, suppose $\mathcal{F}_1, \ldots, \mathcal{F}_{i-1}$ have been chosen, let $G_i \stackrel{\text{def}}{=} G \smallsetminus \bigcup_{j=1}^{i-1} \bigcup \mathcal{F}_j$, and let $\mathcal{F}_i$ be the maximal collection of disjoint balls of radius r with centers in G_i. Clearly $G \supset G_1 \supset \cdots \supset G_N$, and we need to show that $G_{N+1} = \varnothing$. Since $\mathcal{F}_i$ is maximal, for any $x \in G_i$ there is x' which is the center of one of the balls of $\mathcal{F}_i$, so that $d(x, x') < 2r$. If $x \in G_{N+1} \neq \varnothing$, then the ball $B(x, 2r)$ contains $x_1, \ldots, x_N$ such that x_i is the center of one of the balls of $\mathcal{F}_i$. For $i' > i$, $d(x_i, x_{i'}) \geq r$ since $x_{i'} \in G_{i'}$. This contradicts the property of N from the preceding paragraph. $\square$

We will need to know that volumes of balls do not decay exponentially:

Lemma 9.5. *For any probability measure* μ *on* $\mathcal{H}$, *for* μ*-a.e.* q, *we have*

$$\lim_{r \to 0+} \frac{-\log(\mu(B(q, r)))}{r} = 0. \tag{9.7}$$

Proof. If we replace r in the denominator of (9.7) with $\log r$, and replace lim with lim sup, we get the definition of the upper pointwise dimension of μ at q. It is known that the upper pointwise Hausdorff dimension of a measure is at most the Hausdorff dimension of the ambient space. This implies the result. $\square$

We also need some standard facts about entropy. In the following proposition, X is a standard Borel space, $T : X \to X$ is a measurable map, $\mathrm{Prob}(X)^T$ denotes the T-invariant Borel probability measures on X, μ is a measure in $\mathrm{Prob}(X)^T$, $\mathcal{P}$ is a measurable partition of X, and $h_\mu(T, \mathcal{P})$ is the entropy of T with respect to μ and $\mathcal{P}$. Then the entropy of T with respect to μ is $\sup_{\mathcal{P}} h_\mu(T, \mathcal{P})$, where the supremum ranges over all finite $\mathcal{P}$. For $x \in X$, $\mathcal{P}_n(x)$ is the atom of the finite refinement $\bigvee_{i=0}^n T^{-i}\mathcal{P}$ containing x.

Proposition 9.6. *We have the following:*

(1) [Shannon-McMillan-Breiman Theorem.] *If* μ *is ergodic then for* μ*-a.e.* x *we have*
$$\lim_{n \to \infty} \frac{-\log(\mu(\mathcal{P}_n(x)))}{n} = h_\mu(T, \mathcal{P}).$$
(2) [Entropy and convex combinations.] *If* $\mu = \int_{\mathrm{Prob}(X)^T} \nu \, d\theta$, *for some probability measure* θ *on* $\mathrm{Prob}(X)^T$, *then*
$$h_\mu(T, \mathcal{P}) = \int_{\mathrm{Prob}(X)^T} h_\nu(T, \mathcal{P}) \, d\theta.$$
(3) [Partitions with small boundary.] *Let* X *be a locally compact, separable metrizable space. Then* $h_\mu(T) = \sup_{\mathcal{P} \in \mathrm{Part}_0} h_\mu(T, \mathcal{P})$, *where* Part_0 *denotes the finite partitions of* X *into sets* P_i *satisfy* $\mu(\partial P_i) = 0$ *for all* i.

For items (1) and (2) see e.g. [Gl, Thms. 14.35 & 15.12] or [ELW, Chaps. 2 & 3]. Item (3) is left as an exercise (see [ELW, Pf. of Thm. 2.2]).

Proof of Theorem 9.1. We assume that the entropy $h = h_\mu(T)$ satisfies $h > 0$, and we will derive a contradiction. Using Proposition 9.6(3), we choose a partition $\mathcal{P} = \{P_i\}_{i=1}^k$ so that $\mu(\partial P_i) = 0$ for each i and $h_\mu(T, \mathcal{P}) > \frac{h}{2}$. Choose K compact so that $\mu(K) > \frac{3}{4}$ and

$$\limsup_{t\to\infty} \mu\left(g_t(K)\right) > \frac{3}{4}. \tag{9.8}$$

A compact set with this property exists by the nondivergence assumption (9.1). Let N and r_0 be as in Proposition 9.4, for this choice of K. Using the Shannon-McMillan-Breiman theorem, let L_0 be large enough so that for all $L > L_0$, the set

$$W \stackrel{\text{def}}{=} \left\{ q : \frac{-\log(\mu(P_L(q)))}{L} \leq \frac{h}{2} \right\}$$

satisfies

$$\mu(W) < \frac{1}{6N}. \tag{9.9}$$

Our goal will be choose some $L > L_0$ for which we have a contradiction to (9.9).

Below we will simplify notation by writing $B\left(q, \frac{1}{L^5}\right)$ as $B_{q,L}$ or simply as B. Let

$$G_L \stackrel{\text{def}}{=} \left\{ q : \mu\left(B_{q,L} \cap W\right) > \frac{\mu(B_{q,L})}{2} \right\}.$$

We will show below that

$$\text{there are arbitrarily large } L \text{ for which } \mu(K \cap G_L) > \frac{1}{3}. \tag{9.10}$$

We first explain why (9.10) leads to a contradiction with (9.9). Let $L > L_0$ be large enough so that $\mathrm{diam}(B) \leq \frac{2}{L^5} < r_0$ and $\mu(K \cap G_L) > \frac{1}{3}$, and let

$$\mathcal{C} \stackrel{\text{def}}{=} \left\{B_{q,L} : q \in G_L\right\}.$$

By Proposition 9.4, there is a subcollection $\mathcal{F} \subset \mathcal{C}$, consisting of disjoint balls, so that

$$\mu\left(K \cap G_L \cap \bigcup \mathcal{F}\right) \geq \frac{1}{N}\, \mu(K \cap G_L) > \frac{1}{3N}.$$

Then we have

$$\mu(W) \geq \mu\left(W \cap \bigcup \mathcal{F}\right) = \sum_{B \in \mathcal{F}} \mu(W \cap B) > \sum_{B \in \mathcal{F}} \frac{\mu(B)}{2} \geq \frac{\mu\left(\bigcup \mathcal{F}\right)}{2} \geq \frac{1}{6N},$$

where the equality follows from the disjointness of $\mathcal{F}$ and the strict inequality follows from the definitions of G_L and $\mathcal{C}$. This gives the desired contradiction to (9.9).

It remains to show (9.10). Choose $\varepsilon > 0$ so that

$$21\,\varepsilon\,\log(k) < \frac{h}{2}. \tag{9.11}$$

Given any L_1 let $L > L_1$ and let $X_0 = X_0(L) \subset K$ such that $\mu(X_0) \geq \frac{1}{2}$, and so that for any $q \in X_0$ we have (9.3). Such L and X_0 exist by (9.8). Using Lemma 9.5 we can take L large enough so that

$$\mu\left(X_1\right) > \frac{99}{100}, \quad \text{where } X_1 \stackrel{\text{def}}{=} \left\{q : \mu\left(B_{q,L}\right) > k^{-\varepsilon L}\right\}, \tag{9.12}$$

and by making L even larger we can assume that

$$k^{-10\varepsilon L} < \frac{1}{2}. \tag{9.13}$$

Now choose $r > 0$ so that

$$\mu(V) < \varepsilon, \quad \text{where } V \stackrel{\text{def}}{=} \left\{ y : \operatorname{dist}\left(y, \bigcup_{i=1}^{k} \partial P_i \right) < r \right\}. \tag{9.14}$$

This is possible because $\mu\left(\bigcup_i \partial P_i\right) = 0$.

We claim that

$$\mu(X_2) > \frac{2}{5}, \qquad \text{where } X_2 \stackrel{\text{def}}{=} \left\{ q \in X_0 : |\{0 \le i \le L : T^i q \in V\}| < 10\varepsilon L \right\}. \tag{9.15}$$

To see this, define

$$E \stackrel{\text{def}}{=} \{q : |\{0 \le i \le L : T^i q \in V\}| \ge 10\varepsilon L\},$$

and let $\mathbf{1}_V$ denote the indicator function of V. Using (9.14), and since μ is T-invariant,

$$\varepsilon L > L\mu(V) = \sum_{i=1}^{L} \int \mathbf{1}_V\left(T^i q\right) d\mu \ge 10\varepsilon L \mu(E).$$

Dividing through by $10L\varepsilon$ we have $\mu(E) < \frac{1}{10}$, giving (9.15).

Let

$$\beta \stackrel{\text{def}}{=} k^{-20\varepsilon L}, \qquad \text{and write} \quad \mathcal{P}^{(L)} \stackrel{\text{def}}{=} \bigvee_{i=1}^{L} T^{-i}(\mathcal{P}).$$

For each q we let $\mathcal{P}^{(L,B)}$ be the elements of $\mathcal{P}^{(L)}$ which intersect $B = B_{q,L}$, and partition $\mathcal{P}^{(L,B)}$ into two subcollections defined by

$$\mathcal{P}_{\text{big}}^{(q,L)} \stackrel{\text{def}}{=} \left\{ P \in \mathcal{P}^{(L,B)} : \mu(P) \ge \beta\mu(B) \right\} \quad \text{and} \quad \mathcal{P}_{\text{small}}^{(q,L)} \stackrel{\text{def}}{=} \mathcal{P}^{(L,B)} \smallsetminus \mathcal{P}_{\text{big}}^{(q,L)}.$$

We claim that if $q \in X_2$ then

$$\mu\left(B \cap \bigcup \mathcal{P}_{\text{big}}^{(q,L)} \right) > \left(1 - k^{-10\varepsilon L}\right) \mu\left(B\right) \stackrel{(9.13)}{>} \frac{\mu(B)}{2}. \tag{9.16}$$

To see this, we note that for q satisfying the conclusion of Proposition 9.2, the cardinality of $\mathcal{P}^{(L,B)}$ is at most $k^{|\{0 \le i \le L \,:\, T^i q \in V\}|}$. Indeed, for such q, whenever $T^i q \notin V$, $T^i\left(B\right)$ is contained in one of the P_i (and for the other i we use the obvious bound that $T^i q \in V$, $T^i\left(B\right)$ could intersect all of the P_i). For $q \in X_2$ we also have that $\beta^{-1/2} \ge k^{|\{0 \le i \le L \,:\, T^i q \in V\}|}$, and this implies that

$$\mu\left(B \cap \bigcup \mathcal{P}_{\text{small}}^{(q,L)} \right) < \beta^{-1/2} \beta \mu(B) = k^{-10\varepsilon L} \mu(B),$$

and this proves (9.16).

If $q \in X_1 \cap X_2$ and $q' \in B_{q,L} \cap \bigcup \mathcal{P}_{\text{big}}^{(q,L)}$ then we have

$$\mu(P_L(q')) \ge \beta\mu(B_{q,L}) \ge k^{-21\varepsilon L},$$

and this implies via (9.11) that $q' \in W$. This and (9.16) shows that $X_1 \cap X_2 \subset G_L$. Thus

$$\mu(G_L) \ge \mu(X_1 \cap X_2) \ge \frac{2}{5} - \frac{1}{100} > \frac{1}{3}, \tag{9.17}$$

and we have shown (9.10). □

Proof of Theorem 1.5. Denote by T the map defined by Rel_{z_0} (where defined). Since $m_{\mathcal{L}}$ is G-invariant, it is rotation invariant, and thus $m_{\mathcal{L}}$-a.e. q has no horizontal saddle connections. In particular for such q, $\mathrm{Rel}_z(q)$ is defined for all $z \in Z$.

Assume first that $m_{\mathcal{L}}$ is ergodic. By G-invariance of $m_{\mathcal{L}}$ we have (9.1), so the hypotheses of Theorem 9.1 are satisfied for $\mu = m_{\mathcal{L}}$. Now suppose $m_{\mathcal{L}}$ is not ergodic, and let $\mu = \int_{\mathrm{Prob}(X)^T} \nu \, d\theta$ be the ergodic decomposition of μ, where θ is a probability measure on $\mathrm{Prob}(X)^T$ such that θ-a.e. ν is ergodic for T. By Proposition 9.6(2), it suffices to show that the entropy of ν is zero for θ-a.e. ν, and thus we only need to show that assumption (9.1) holds for θ-a.e. ν. This follows from the g_t-invariance of $m_{\mathcal{L}}$. Indeed, by invariance and regularity of $m_{\mathcal{L}}$, for any $\varepsilon > 0$ there exists a compact K, so that for all t, $m_{\mathcal{L}}(g_t(K)) = m_{\mathcal{L}}(K) > 1 - \varepsilon^2$. Thus for every t,

$$\theta(\{\nu : g_{-t*}\nu(K) \geq 1 - \varepsilon\}) \geq 1 - \varepsilon.$$

Thus for any $\varepsilon > 0$ there is K so that the set of ν for which $(g_{-t_i})_*\nu(K) \geq 1 - \varepsilon$ for a sequence $t_i \to \infty$, has θ-measure at least $1 - \varepsilon$. Since ε was arbitrary, we have (9.1) for θ-a.e. ν. □

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