BOUNDED GEODESICS IN MODULI SPACE

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Abstract. In the moduli space of quadratic differentials over complex structures on a surface, we construct a set of full Hausdorff dimension of points with bounded Teichmuller geodesic trajectories. The main tool is quantitative nondivergence of Teichmuller horocycles, due to Minsky and Weiss. This has an application to billiards in rational polygons.

1. Introduction

Let S be a compact orientable surface of genus g with n punctures, where $3g + n \geq 3$, and let $\mathcal{Q}$ be the moduli space of unit-area holomorphic quadratic differentials over complex structures on S. This is a noncompact orbifold, on which $G \stackrel{\mathrm{def}}{=} \mathrm{SL}(2, \mathbb{R})$ acts continuously with discrete stabilizers, preserving a smooth finite measure.

For $t \in \mathbb{R}$, let $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in G$, and denote the one-parameter subgroup $\{g_t : t \in \mathbb{R}\}$ by F. The restriction of the G-action to F defines a flow on $\mathcal{Q}$ called the *geodesic flow* (or sometimes the *Teichmüller geodesic flow*). Its dynamical properties have been extensively studied in connection with various problems in ergodic theory and geometry. See the surveys [MT], [Mo] for more details.

It is known [M1] that the geodesic flow is ergodic and hence a typical trajectory for the geodesic flow is dense. It is natural (and of importance for applications) to inquire as to the existence and abundance of atypical trajectories. Let us denote the Hausdorff dimension of a subset Y of a metric space X by $\dim(Y)$, and say that Y is *thick in* X if for any nonempty open subset W of X, $\dim(W \cap Y) = \dim(W)$ (i.e. Y has full Hausdorff dimension at any point of X).

In this note we discuss *bounded* trajectories, and prove the following:

Theorem 1. *The set $\{q \in \mathcal{Q} :$ the orbit Fq is bounded$\}$ is thick in $\mathcal{Q}$.*

Theorem 1 follows from a more precise result. Recall (see §2 for more details) that $\mathcal{Q}$ is partitioned into finitely many G-invariant suborbifolds called *strata*. Say that $X \subset \mathcal{Q}$ is *bounded in a stratum* if its closure is a compact subset of a single stratum.

Let

$$F^+ \stackrel{\mathrm{def}}{=} \{g_t : t \geq 0\} \subset F\,.$$

For s, $\theta \in \mathbb{R}$ let

$$h_s \stackrel{\mathrm{def}}{=} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad r_\theta \stackrel{\mathrm{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and

$$B \stackrel{\mathrm{def}}{=} \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} : b \in \mathbb{R},\ a \in \mathbb{R} \smallsetminus \{0\} \right\}.$$

With this notation, we have:

Theorem 2. *Let $q \in \mathcal{Q}$. Then*

(i) $\{s \in \mathbb{R} : F^+ h_s q$ *is bounded in a stratum*$\}$ *is thick in* $\mathbb{R}$.
(ii) *For any* $f : \mathbb{R} \to B$,
$\{s \in \mathbb{R} : F^+ f(s) h_s q$ *is bounded in a stratum*$\}$ *is thick in* $\mathbb{R}$.
(iii) $\{\theta \in \mathbb{R} : F^+ r_\theta q$ *is bounded in a stratum*$\}$ *is thick in* $\mathbb{R}$.
(iv) $\{x \in G : Fxq$ *is bounded in a stratum*$\}$ *is thick in* G.

It follows from a result of Masur [M2] that the set in (iii) has measure zero, and it follows easily that the same holds for the sets in (i), (ii), (iv).

Note that Theorem 1 follows immediately from Theorem 2(iv) by considering the foliation of $\mathcal{Q}$ into G-orbits. Theorem 2(iii) has an interpretation in terms of rational billiards, which we now describe. Let $\mathcal{P}$ be a rational billiard table. Fix any $\theta \in \mathbb{R}$, and for any $p \in \mathcal{P}$ consider a particle moving at a constant velocity along a ray at angle θ starting at p. The law of reflection (angle of incidence equals angle of return) gives rise to a dynamical system, whose phase space $\mathcal{P}_\theta$ is the complement of a countable set of lines in finitely many copies of $\mathcal{P}$. See [MT] for more details. Let d denote the restriction of the Euclidean metric to $\mathcal{P}_\theta$, and let $b_t^\theta : \mathcal{P}_\theta \to \mathcal{P}_\theta$ denote the time t map of this flow.

It can be shown (see e.g. [Bo1, §2]) that every billiard trajectory is *recurrent*, that is, $\liminf_{t\to\infty} d(p, b_t^\theta p) = 0$ for any $p \in \mathcal{P}_\theta$. Furthermore, one can use [Bo2, Theorem 1.5] and the existence of one-dimensional Poincaré sections for the flow $(\mathcal{P}_\theta, b_t^\theta)$ (see a discussion in [MT, §1.7]) to derive a quantitative version for generic trajectories: namely, given $\mathcal{P}$ and θ, there is a constant C, such that for almost every $p \in \mathcal{P}_\theta$ one has

$$\liminf_{t\to\infty} t \cdot d(p, b_t^\theta p) \leq C\,.$$

On the other hand it turns out that for a thick set of angles θ, up to a change of the contstant, the opposite inequality holds for all $p \in \mathcal{P}_\theta$:

Corollary 3. *The set*

$$\{\theta : \exists\, c > 0 \text{ such that } \liminf_{t\to\infty} t\cdot d(p, b_t^\theta p) \geq c \quad \forall\, p \in \mathcal{P}_\theta\}$$

is thick in $\mathbb{R}$.

Note that if $\mathcal{P}$ is rectangular (or more generally *integrable*, see [MT]), then $b_t^\theta : \mathcal{P}_\theta \to \mathcal{P}_\theta$ is isomorphic to the flow along a ray in the torus $\mathbb{R}^2/\mathbb{Z}^2$, and Corollary 3 is equivalent to the well-known fact (see [Sc] and the references therein) that the set of *badly approximable* $\theta \in \mathbb{R}$ (that is, those for which there exists $c > 0$ with $|\theta - \frac{p}{q}| \geq \frac{c}{q^2}$ for all integers p, q) is thick.

Our work is part of a fruitful interaction between the study of dynamics on quadratic differential spaces and dynamics on homogeneous spaces of Lie groups, see the survey [We] and the references therein. Bounded trajectories on homogeneous spaces were extensively studied due to their relation to linear forms which are badly approximable by rationals, and results analogous to ours were obtained by W. Schmidt [Sc], S.G. Dani [Da], and, in most general form, in joint work of the first-named author and G.A. Margulis [KM1]. See [KSS, §4.1b] for references to other related results.

Our construction of bounded trajectories is similar in some respects to the one used in [KM1], but whereas in that paper a crucial estimate was obtained by using the mixing property of the flow, in this paper the estimate follows from quantitative nondivergence results for horocycles obtained in [MW]. This approach, powered by nondivergence estimates for unipotent trajectories of homogeneous flows [KM2], can also be used to reprove the abundance of bounded orbits for partially hyperbolic diagonalizable flows on $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{R})$, and to obtain new applications to number theory. It was originally introduced in the talk given by the first-named author in July 2000 at a Euroconference on Ergodic Theory, Number Theory and Geometric Rigidity (Issac Newton Institute, Cambridge, UK).

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2. Terminology

Let $\mathrm{Mod}(S)$ be the mapping class group, let $\widetilde{\mathcal{Q}}$ be the space of unit-area holomorphic quadratic differentials over complex structures on S, so that $\mathcal{Q} = \widetilde{\mathcal{Q}}/\operatorname{Mod}(S)$. Let $\pi : \widetilde{\mathcal{Q}} \to \mathcal{Q}$ be the natural quotient map. The space $\widetilde{\mathcal{Q}}$ is a manifold on which G acts continuously, and π is G-equivariant. Each $\mathbf{q} \in \widetilde{\mathcal{Q}}$ can be defined as an equivalence class of atlases of charts of the following type. Outside a finite singularity set $\Sigma = \Sigma(\mathbf{q}) \subset S$ we have charts $S \smallsetminus \Sigma \to \mathbb{C}$ for which the transition functions $\mathbb{C} \to \mathbb{C}$ are of the form $z \mapsto \pm z + c$. The charts are required to have k-pronged singularities around points of Σ, with $k \geq 3$. At the punctures they are required to extend to the 'filled-in' surface, and the punctures may have k-prongs with $k \geq 1$. The equivalence relation is given by the natural action of $\mathrm{Homeo}_0(S)$ by pre-composition on each chart. An atlas is said to be orientable if the transition functions can be taken to be of the form $z \mapsto z + c$. See [MT, §4] or [MW, §4] for details.

The data consisting of the number of singularities of each type, and the orientability of the atlas, assumes one of finitely many possibilities. A level set for this data is called a *stratum* in $\widetilde{\mathcal{Q}}$. If $\mathcal{M} \subset \widetilde{\mathcal{Q}}$ is a stratum we also call its projection $\pi(\mathcal{M}) \subset \mathcal{Q}$ a *stratum* in $\mathcal{Q}$. Any stratum in $\widetilde{\mathcal{Q}}$ is a submanifold and any stratum in $\mathcal{Q}$ is a sub-orbifold. The strata are invariant under the $\mathrm{SL}(2,\mathbb{R})$-action.

A *saddle connection* for $\mathbf{q}$ is a path $\delta : (0,1) \to S \smallsetminus \Sigma$ whose image in any chart is a Euclidean straight line and which extends continuously to a map from $[0,1]$ to the completion of S, mapping $\{0,1\}$ to singularities or punctures. In all cases except $g = 1$, $n = 0$, there are saddle connections on the surface. Since our results are immediate for this case, we assume from now on that $(g,n) \neq (1,0)$.

Integrating the local projections of $d\delta$ one obtains a vector, denoted by $\big(x(\delta,\mathbf{q}), y(\delta,\mathbf{q})\big)$, well-defined up to a multiple of ± 1. Let $\mathcal{L}_{\mathbf{q}}$ denote the set of all saddle connections for $\mathbf{q}$. Since linear maps do not change the property of being a straight line, we may identify $\mathcal{L}_{\mathbf{q}}$ with $\mathcal{L}_{A\mathbf{q}}$ for any $A \in G$. The action of G transforms the components of saddle connections linearly:

$$\text{for } A \in G\,, \ \begin{pmatrix} x(\delta, A\mathbf{q}) \\ y(\delta, A\mathbf{q}) \end{pmatrix} = \pm A \begin{pmatrix} x(\delta, \mathbf{q}) \\ y(\delta, \mathbf{q}) \end{pmatrix}. \tag{1}$$

For a saddle connection $\delta \in \mathcal{L}_{\mathbf{q}}$, define its 'length' $l(\delta,\mathbf{q})$ by

$$l(\delta,\mathbf{q}) \stackrel{\text{def}}{=} \max\big(|x(\delta,\mathbf{q})|, |y(\delta,\mathbf{q})|\big)\,.$$

For $\varepsilon > 0$, let

$$K_\varepsilon \stackrel{\text{def}}{=} \pi\left(\{\mathbf{q} \in \widetilde{\mathcal{Q}} : \forall \delta \in \mathcal{L}_{\mathbf{q}},\, l(\delta,\mathbf{q}) \geq \varepsilon\}\right).$$

It is known that each K_ε is compact. Moreover, for each stratum $\mathcal{M}$ in $\mathcal{Q}$, each compact $K \subset \mathcal{M}$ is contained in K_ε for some ε, and

$$\overline{\pi(\mathcal{M}) \cap K_\varepsilon} \subset \pi(\mathcal{M}).$$

Thus $X \subset \mathcal{M}$ is bounded in a stratum if and only if $X \subset K_\varepsilon$ for some $\varepsilon > 0$.

3. From nondivergent horocycles to bounded geodesics

Recall that the restriction of the G-action to the subgroup $\{h_s : s \in \mathbb{R}\}$ defines the so-called *horocycle flow*. It was proved in [Ve], using arguments of [KMS], that this flow does not admit divergent trajectories. The main ingredient for the proof of Theorem 2 is a quantitative strengthening of this result, along the lines of [KM2]. Here $|\cdot|$ stands for Lebesgue measure on $\mathbb{R}$.

Theorem 4. [MW, Thm. 6.3] *There are positive constants C, α, ρ_0, depending only on S, such that if $\mathbf{q} \in \widetilde{\mathcal{Q}}$, an interval $I \subset \mathbb{R}$, and $0 < \rho \leq \rho_0$ satisfy:*

$$\text{for any } \delta \in \mathcal{L}_{\mathbf{q}}, \ \sup_{s \in I} l(h_s \mathbf{q}, \delta) \geq \rho, \tag{2}$$

then for any $0 < \varepsilon < \rho$ we have:

$$|\{s \in I : h_s \pi(\mathbf{q}) \notin K_\varepsilon\}| \leq C \left(\frac{\varepsilon}{\rho}\right)^\alpha |I|. \tag{3}$$

Proof of Theorem 2. We start with part (i). Let $J \subset \mathbb{R}$ be an interval and let $q_0 = \pi(\mathbf{q}_0) \in \mathcal{Q}$. We need to prove that the Hausdorff dimension of the set

$$\{s \in J : \ F_+ h_s q_0 \text{ is bounded in a stratum}\} \tag{4}$$

is equal to 1. Denote the length of J by r, and assume, as we may without loss of generality, that $J = [0, r]$ and $r \leq 1$.

Let $\eta > 0$ be arbitrary, and let C, α, ρ_0 be as in Theorem 4. Choose $\varepsilon > 0$ so that $q_0 \in K_\varepsilon$, and also

$$C \left(\frac{\varepsilon}{\rho_0}\right)^\alpha < \eta\,,$$

and let

$$K' \stackrel{\text{def}}{=} K_\varepsilon, \qquad K'' \stackrel{\text{def}}{=} \bigcup_{0 \leq s \leq r} h_s K'\,.$$

Then there is ε_0, depending only on ε, such that $K'' \subset K_{\varepsilon_0}$. Hence for each $q \in K''$, each $\mathbf{q} \in \pi^{-1}(q)$, and each $\delta \in \mathcal{L}_{\mathbf{q}}$, we have

$$l(\delta, \mathbf{q}) = \max\big(|x(\delta,\mathbf{q})|, |y(\delta,\mathbf{q})|\big) \geq \varepsilon_0 \,. \tag{5}$$

Claim 1. *There exists $t_1 > 0$, depending only on ε_0 and r, such that for every $q \in K''$, $\mathbf{q} \in \pi^{-1}(q)$ and $\delta \in \mathcal{L}_{\mathbf{q}}$,*

$$\sup_{0 \leq s \leq r} |x(\delta, g_{t_1} h_s \mathbf{q})| \geq \rho_0 \,. \tag{6}$$

To prove the claim, let us first show that

$$\max\big(|x(\delta,\mathbf{q})|, |x(\delta, h_r\mathbf{q})|\big) \geq \varepsilon_0 r/2 \,. \tag{7}$$

Let $x = x(\delta,\mathbf{q})$, $y = y(\delta,\mathbf{q})$. If $|x| \geq \varepsilon_0 r/2$, there is nothing to prove. So by (5) we may assume that $|y| \geq \varepsilon_0$. But then

$$|x(\delta, h_r\mathbf{q})| \overset{(1)}{=} |x + ry| \geq \varepsilon_0 r - \varepsilon_0 r/2 = \varepsilon_0 r/2 \,.$$

Setting $t_1 \overset{\text{def}}{=} 2\log\frac{2\rho_0}{\varepsilon_0 r}$, one obtains (6) from (7) via (1). This proves the claim.

Enlarging t_1 if necessary, assume that

$$N \overset{\text{def}}{=} e^{t_1} \in \mathbb{N}.$$

Claim 2. *Suppose $q' \in \mathcal{Q}$ and $t_0 \geq 0$ are such that $g_{t_0} h_s q' \subset K''$ for any $0 \leq s \leq re^{-t_0}$. Subdivide $[0, re^{-t_0}]$ into N subintervals $J_1, \dots, J_N$ of length $re^{-t_0}/N = re^{-(t_0+t_1)}$, and let*

$$\mathcal{I} \overset{\text{def}}{=} \{i \in \{1,\dots,N\} : g_{t_1+t_0} h_s q' \in K'' \ \forall\, s \in J_i\} \,.$$

Then $\#\mathcal{I} \geq (1-\eta)N$.

In order to prove this claim, we apply Theorem 4 with $I = [0, re^{t_1}]$ and $q = g_{t_1+t_0} q'$, that is, to the horocycle

$$\{h_s g_{t_1+t_0} q' : 0 \leq s \leq re^{t_1}\} = \{g_{t_1} h_s g_{t_0} q' : 0 \leq s \leq r\} \,.$$

Assumption (2), with $\rho = \rho_0$, is valid by (6).

We obtain:

$$\begin{aligned} |\{0 \leq s \leq re^{-t_0} : g_{t_1+t_0} h_s q' \notin K'\}| &= e^{-(t_1+t_0)} |\{0 \leq s \leq re^{t_1} : h_s g_{t_1+t_0} q' \notin K'\}| \\ &\leq e^{-(t_1+t_0)} C \left(\frac{\varepsilon}{\rho_0}\right)^{\alpha} re^{t_1} < \eta r e^{-t_0} \,. \end{aligned} \tag{8}$$

Now let

$$\mathcal{I}_0 \overset{\text{def}}{=} \{i \in \{1,\dots,N\} : \ g_{t_1+t_0} h_s q' \in K' \text{ for some } s \in J_i\} \,.$$

Then (8) implies that $\#\mathcal{I}_0 \geq (1-\eta)N$. A simple calculation using the definition of K'' shows that $\mathcal{I}_0 \subset \mathcal{I}$, and the claim follows.

We now construct a subset $D \subset J$, of large Hausdorff dimension, by iterating Claim 2. At stage m of the construction we have a subdivision of J into N^m subintervals of length N^{-m}, and a collection of $[(1-\eta)N]^m$ of these subintervals with the property that if I is one of these intervals then

$$g_{kt_1} h_s q_0 \subset K'' \text{ for any } s \in I \text{ and } \; k = 0, \dots, m.$$

For each subinterval I belonging to this collection, we apply Claim 2 to I shifted by its left endpoint, with $t_0 = mt_1$ and an appropriate $q' \in \pi(\mathcal{M})$, and obtain that there are at least $(1-\eta)N$ subintervals of I which satisfy the conclusion of the claim. The union of these subintervals over all subintervals I comprises the collection for the $(m+1)$-st step.

Let D be the set of intersection points of all sequences of nested subintervals in the above construction.

A well-known result (see [Mc, Proposition 2.2]) implies that the Hausdorff dimension of D is at least

$$1 - \frac{\log(1-\eta)}{-\log(N)} \geq 1 + \log(1-\eta)\,.$$

If $s \in D$, then $g_{kt_1} h_s q \in K''$ for any $k \in \mathbb{Z}_+$. This implies that

$$F_+ h_s q_0 \subset K \stackrel{\text{def}}{=} \bigcup_{0 \leq t \leq t_1} g_t K''\,,$$

where K is compact.

Since η was chosen arbitrarily and since $1 + \log(1-\eta)$ tends to 1 as $\eta \to 0$, we obtain that the set (4) has Hausdorff dimension 1, as required.

For part (ii), note that for every $b \in B$, the set

$$\{g_t b g_{-t} : t \geq 0\} \subset G$$

is precompact, hence

$$F^+ f(s) h_s q = \left\{ \left(g_t f(s) g_{-t}\right) g_t h_s q : t \geq 0 \right\}$$

is bounded if and only if so is $F^+ h_s q$; thus (ii) follows from (i). The claim (iii) is a special case of (ii): it is easy to represent r_θ in the form $r_\theta = f(\theta) h_\theta$ where $f(\theta) \in B$.

Finally, to prove (iv), we follow the argument of [KM1, §1.5]. Take $U \subset G$ of the form $V^- V V^0$, where V^-, V and V^0 are neighborhoods of identity in

$$H^- \stackrel{\mathrm{def}}{=} \left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} : s \in \mathbb{R} \right\}, \quad H \stackrel{\mathrm{def}}{=} \{h_s : s \in \mathbb{R}\} \text{ and } F \text{ respectively,}$$

such that the multiplication map $V^- \times V \times V^0 \to U$ is bi-Lipschitz. It is enough to show that for any q,

$$\dim\big(\{h \in U : Fhq \text{ is bounded}\}\big) = 3\,. \tag{9}$$

Let C be the set $\{h \in U : F^+hq \text{ is bounded}\}$. From part (i) it follows that for any $h^0 \in V^0$, $\dim(C \cap Vh^0) = 1$. In view of the slicing properties of the Hausdorff dimension (see [Fa] or [KM1, Lemma 1.4]), one deduces that

$$\dim(C \cap VV^0) = 2\,. \tag{10}$$

Now for any $h \in C$ choose a neighborhood $V^-(h)$ of identity in H^- such that $V^-(h)h \subset U$. Then, by the argument used in the proof of (ii), one necessarily has $V^-(h)h \subset C$. Applying part (i) of the theorem with

$$F^- \stackrel{\mathrm{def}}{=} \{g_t : t \le 0\}$$

in place of F^+ and H^- in place of H (this can be justified e.g. by considering an automorphism $q \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} q$ of $\mathcal{Q}$, which leaves G-orbits and lengths of saddle connections invariant, and sends the action of H and F^+ to that of H^- and F^- respectively), one concludes that

$$\forall\, h \in C \cap VV^0 \quad \dim\big(\{h^- \in V^- : Fh^-hx \text{ is bounded}\}\big) = 1\,,$$

which, together with (10), yields (9). ☐

4. Application to rational billiards

Recall that any $\mathbf{q} \in \widetilde{\mathcal{Q}}$ defines a pair of transverse measured foliations on $S \smallsetminus \Sigma(\mathbf{q})$, called the *horizontal* and *vertical* foliations respectively, and that g_t acts by contracting (resp. expanding) the measure transverse to horizontal (resp. vertical) leaves by a factor of $e^{t/2}$. The Euclidean metric on a billiard table $\mathcal{P}$ determines a flat Riemannian metric on $S \smallsetminus \Sigma(\mathbf{q})$. The length of $\delta \in \mathcal{L}_{\mathbf{q}}$ with respect to this metric is at least as large as $l(\delta, \mathbf{q})$. Recall also that any line segment is a length-minimizing path in its homotopy class with endpoints fixed, and any free homotopy class has a length-minimizing representative which is a concatenation of saddle connections.

Given $\mathcal{P}$, there is a standard construction (see [KMS] or [MT]) of a closed surface S of genus $g \ge 1$, with a quadratic differential $\mathbf{q}$, such that, for each $\theta \in \mathbb{R}$, the flow $b_t^\theta : \mathcal{P}_\theta \to \mathcal{P}_\theta$ is isomorphic to the unit-speed flow in $S \smallsetminus \Sigma(\mathbf{q})$ along vertical leaves for $r_\theta \mathbf{q}$. The case $g = 1$ occurs if and only if $\mathcal{P}$ is integrable, and in this case Corollary 3 follows from the fact that badly approximable numbers form a thick subset of the reals. So assume $g \geq 2$.

Proof of Corollary 3. Suppose θ belongs to the thick subset of $\mathbb{R}$ for which $F^+ r_\theta \pi(\mathbf{q})$ is bounded in a stratum. Then there exists $0 < \varepsilon < 1$ with

$$F^+ r_\theta \pi(\mathbf{q}) \subset K_\varepsilon \,. \tag{11}$$

Let $c \stackrel{\mathrm{def}}{=} \varepsilon^2/2 < 1$; we claim that $d(b_t^\theta p, p) \geq c/t$ for any $p \in \mathcal{P}_\theta$ and $t \geq 1$. Indeed, assuming that there are $p \in \mathcal{P}_\theta$ and $t \geq 1$ with $d(b_t^\theta p, p) < c/t$, let

$$t_0 \stackrel{\mathrm{def}}{=} 2\log t - \log c \geq 0 \,.$$

With respect to $r_\theta \mathbf{q}$, let $\mathcal{C}_1$ be the vertical segment from p to $b_t^\theta p$, let $\mathcal{C}_2$ be a length-minimizing path from $b_t^\theta p$ to p, and let $\mathcal{C}$ be the concatenation of $\mathcal{C}_1$ and $\mathcal{C}_2$. The length of $\mathcal{C}_1$ with respect to $r_\theta \mathbf{q}$ is $t \geq 1$ and the length of $\mathcal{C}_2$ is less than $c/t < 1$. Since $\mathcal{C}_1$ is length-minimizing in its homotopy class with fixed endpoints, $\mathcal{C}$ is homotopically nontrivial.

With respect to $g_{t_0} r_\theta \mathbf{q}$, the length of $\mathcal{C}_1$ is $te^{-t_0/2} = \sqrt{c}$, and the length of $\mathcal{C}_2$ is less than $e^{t_0/2} c/t = \sqrt{c}$. Thus the length of $\mathcal{C}$ with respect to $g_{t_0} r_\theta \mathbf{q}$ is less than ε. Since $\mathcal{C}$ is not homotopically trivial, the shortest representative in its free homotopy class is a nontrivial concatenation of saddle connections. In particular there is $\delta \in \mathcal{L}_{\mathbf{q}}$ with $l(\delta, g_{t_0} r_\theta \mathbf{q}) < \varepsilon$, which contradicts (11). $\square$

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