FRIENDLY MEASURES, HOMOGENEOUS FLOWS AND SINGULAR VECTORS

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1. Introduction

The theory of diophantine approximation studies how well $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ can be approximated by $(p_1/q_1, \dots, p_n/q_n) \in \mathbb{Q}^n$ of a given 'complexity', where this complexity is usually measured by the quantity $\mathrm{lcm}(q_1, \dots, q_n)$. Thus one is interested in minimizing the difference, in a suitable sense, between $q\mathbf{x}$ and a vector $\mathbf{p}$, where $\mathbf{p} \in \mathbb{Z}^n$ and $q \in \mathbb{N}$, with a given upper bound on q. Often one finds that certain approximation problems admit a solution for almost every $\mathbf{x}$, while others admit a solution for almost no $\mathbf{x}$; one is then interested in understanding whether the typical properties remain typical when additional restrictions are placed on $\mathbf{x}$.

As an example, consider the notion of a *singular* vector, introduced by A. Khintchine in the 1920s (see [Kh, Ca]). Say that $\mathbf{x}$ is *singular* if for any $\delta > 0$ there is T_0 such that for all $T \geq T_0$ one can find $\mathbf{p} \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ with

$$\|q\mathbf{x} - \mathbf{p}\| < \frac{\delta}{T^{1/n}} \quad \text{and} \quad q < T\,. \tag{1.1}$$

Clearly this definition is independent on the choice of the norm. Note also that by Dirichlet's Theorem, when $\delta > 1$ and $\|\cdot\|$ is chosen to be the supremum norm, the system (1.1) has a nonzero integer solution for any $T > 1$. Thus singular vectors are often referred to as those for which Dirichlet's theorem can be infinitely improved.

Let us say that $\mathbf{x}$ is *totally irrational* if $1, x_1, \dots, x_n$ are linearly independent over $\mathbb{Q}$. It is not hard to see that vectors which are not totally irrational are singular, and that the converse is true for $n = 1$. However, for $n > 1$ Khintchine [Kh] proved the existence of totally irrational singular vectors. On the other hand it is straightforward to verify [Ca, Ch. V, §7] that Lebesgue measure of the set of singular vectors is zero. In the late 1960s H. Davenport and W. Schmidt showed [DS] that $\mathbf{x} \in \mathbb{R}^2$ of the form $\mathbf{x} = (t, t^2)$ is not singular for Lebesgue-a.e. $t \in \mathbb{R}$. This was later extended to certain classes of smooth curves and higher-dimensional submanifolds of $\mathbb{R}^n$ by R. Baker [Ba1, Ba2] and

M. Dodson, B. Rynne and J. Vickers [DRV1] respectively; see §3 for precise statement of their results.

In this paper we consider several generalizations of the notion of a singular vector. Namely, following [Kl1, Kl2, PV], we attach different weights to different components of $\mathbf{x}$ by means of the $\mathbf{r}$-*quasinorm*

$$\|\mathbf{x}\|_{\mathbf{r}} \stackrel{\text{def}}{=} \max_{i=1,\dots,d} |x_i|^{1/r_i}\,,$$

where

$$\mathbf{r} = (r_1, \dots, r_n) \quad \text{with} \quad r_i > 0 \quad \text{and} \quad \sum_{i=1}^{n} r_i = 1\,. \tag{1.2}$$

Then say that $\mathbf{x}$ is $\mathbf{r}$-*singular* if for any $\delta > 0$ there is T_0 such that for all $T \geq T_0$ one can find $\mathbf{p} \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ with

$$\|q\mathbf{x} - \mathbf{p}\|_{\mathbf{r}} < \frac{\delta}{T} \quad \text{and} \quad q < \delta T\,. \tag{1.3}$$

Further, for $\mathbf{r}$ as above and an unbounded subset $\mathcal{T}$ of $\mathbb{R}_{\geq 1}$, say that $\mathbf{x}$ is $\mathbf{r}$-*singular along* $\mathcal{T}$ if for any $\delta > 0$ there is T_0 such that for all $T \in \mathcal{T} \cap [T_0, \infty)$ one can find $\mathbf{p} \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ satisfying (1.3). We will denote the set of $\mathbf{r}$-singular (along $\mathcal{T}$) vectors by $\mathrm{Sing}(\mathbf{r})$ and $\mathrm{Sing}\,(\mathbf{r}, \mathcal{T})$ respectively.

It is clear that $\mathbf{x}$ is singular if and only if $\mathbf{x} \in \mathrm{Sing}(\mathbf{n})$, where

$$\mathbf{n} \stackrel{\text{def}}{=} (1/n, \dots, 1/n)\,,$$

the vector assigning equal weights to each coordinate, and that $\mathrm{Sing}(\mathbf{r}) = \mathrm{Sing}\,(\mathbf{r}, \mathbb{R}_{\geq 1})$ is contained in $\mathrm{Sing}\,(\mathbf{r}, \mathcal{T})$ for any $\mathcal{T} \subset \mathbb{R}_{\geq 1}$. However an elementary modification of the proof in [Ca, Ch. V, §7] shows that Lebesgue measure of $\mathrm{Sing}\,(\mathbf{r}, \mathcal{T})$ is zero for any $\mathbf{r}$ as in (1.2) and any unbounded $\mathcal{T}$. In this paper we consider the class of *friendly* measures on $\mathbb{R}^n$, originally introduced in [KLW] and described in detail in §3, and prove

Theorem 1.1. *If μ is a friendly measure on $\mathbb{R}^n$, then for any $\mathbf{r}$ as in (1.2) and any unbounded $\mathcal{T}$, $\mu\big(\mathrm{Sing}(\mathbf{r}, \mathcal{T})\big) = 0$.*

A special case of this theorem, with $\mathbf{r} = \mathbf{n}$ and $\mathcal{T} = \mathbb{R}_{\geq 1}$, was announced in [KLW].

The class of friendly measures includes Hausdorff measures supported on various self-similar sets such as Cantor's ternary set, Koch snowflake, Sierpinski gasket, etc. It also includes volume measures on smooth nondegenerate submanifolds of $\mathbb{R}^n$. We recall that $M \subset \mathbb{R}^n$ is called *nondegenerate* if it is parameterized by a smooth map $\mathbf{f}$ from an open subset U of $\mathbb{R}^d$ to $\mathbb{R}^n$ such that for Lebesgue-a.e. $\mathbf{x} \in U$ there exists $\ell \in \mathbb{N}$ such that partial derivatives of $\mathbf{f}$ at $\mathbf{x}$ up to order ℓ span $\mathbb{R}^n$. If $\mathbf{f}$ is real analytic and U is connected, the latter condition is equivalent to $\mathbf{f}(U)$ not being contained in a proper affine hyperplane of $\mathbb{R}^n$. Thus Theorem 1.1 significantly generalizes the aforementioned result of [DS] about the curve $\{(t, t^2) : t \in \mathbb{R}\}$, as well as additional results obtained by several authors.

Note that it is not hard to construct a friendly measure whose support does not contain any singular vectors at all; for example, the set of badly approximable vectors supports friendly measures of arbitrarily small codimension [KW, Ur2]. The situation is however different for volume measures on real analytic nondegenerate manifolds. Namely, we prove

Theorem 1.2. *Let $M \subset \mathbb{R}^n$ be a real analytic submanifold of dimension at least 2 which is not contained in any proper rational affine hyperplane of $\mathbb{R}^n$, and let $\mathbf{r}$ be as in (1.2). Then there exists a totally irrational $\mathbf{x} \in M \cap \mathrm{Sing}(\mathbf{r})$.*

Our approach to Theorem 1.1 is modelled on [KM, KLW]: in §2 we translate the aforementioned diophantine properties of $\mathbf{x} \in \mathbb{R}^n$ into dynamical properties of certain trajectories in the homogeneous space G/Γ, where

$$G = \mathrm{SL}(n+1, \mathbb{R}) \quad \text{and} \quad \Gamma = \mathrm{SL}(n+1, \mathbb{Z})\,. \tag{1.4}$$

Namely, we show (Proposition 2.1) that $\mathbf{x} \in \mathbb{R}^n$ is $\mathbf{r}$-singular along $\mathcal{T}$ if and only if the corresponding trajectory leaves every compact subset of G/Γ. To control the measure of points with divergent trajectories we employ quantitative nondivergence estimates from [KLW], described in detail in §4. Theorem 1.2 is proved in §5; the argument is a modification of the proof of [We, Thm. 5.2], and is based on ideas going back to Khintchine [Kh].

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2. Dynamical interpretation of singular vectors

Let G and Γ be as in (1.4), and denote by π the quotient map from G onto G/Γ. G acts on G/Γ by left translations via the rule

$g\pi(h) = \pi(gh)$, $g, h \in G$. Define

$$\tau(\mathbf{x}) \stackrel{\text{def}}{=} \begin{pmatrix} I_n & \mathbf{x} \\ 0 & 1 \end{pmatrix}, \quad \bar\tau \stackrel{\text{def}}{=} \pi\circ\tau\,, \tag{2.1}$$

where I_n stands for the $n\times n$ identity matrix. Then, given $\mathbf{r}$ as in (1.2), consider the one-parameter subgroup $\{g_t^{(\mathbf{r})}\}$ of G given by

$$g_t^{(\mathbf{r})} \stackrel{\text{def}}{=} \operatorname{diag}(e^{r_1 t}, \dots, e^{r_n t}, e^{-t})\,.$$

Recall that G/Γ is noncompact. For an unbounded subset A of $\mathbb{R}_+$ and $x \in G/\Gamma$, say that a trajectory $\{g_t^{(\mathbf{r})}x : t \in A\}$ is *divergent* if the map $A \to G/\Gamma$, $t \mapsto g_t^{(\mathbf{r})}x$, is proper; that is, for any compact $K \subset G/\Gamma$ there exists t_0 such that $g_t^{(\mathbf{r})} \notin K$ for all $t \in A \cap [t_0, \infty)$.

It was proved in [Da, Proposition 2.12] that $\mathbf{x}$ is singular if and only if the trajectory $\{g_t^{(\mathbf{n})}\bar\tau(\mathbf{x}) : t \geq 0\}$ in G/Γ is divergent, and in [Kl1, Theorem 7.4] that $\mathbf{x}$ is $\mathbf{r}$-singular if and only if the trajectory $\{g_t^{(\mathbf{r})}\bar\tau(\mathbf{x}) : t \geq 0\}$ is divergent. We generalize this correspondence one step further:

Proposition 2.1. $\mathbf{x}$ *is* $\mathbf{r}$*-singular along* $\mathcal{T}$ *if and only if the trajectory*

$$\{g_t^{(\mathbf{r})}\bar\tau(\mathbf{x}) : t \in \log\mathcal{T}\} \subset G/\Gamma$$

is divergent, where $\log\mathcal{T} \stackrel{\text{def}}{=} \{\log T : T \in \mathcal{T}\}$.

To prove Proposition 2.1, we need an explicit description of compact subsets of G/Γ. Since Γ is the stabilizer of $\mathbb{Z}^{n+1}$ under the action of G on the set of lattices in $\mathbb{R}^{n+1}$, G/Γ can be identified with $G\mathbb{Z}^{n+1}$, that is, with the set of all unimodular lattices in $\mathbb{R}^{n+1}$. Fix a norm $\|\cdot\|$ on $\mathbb{R}^{n+1}$, and for $\varepsilon > 0$ let

$$K_\varepsilon \stackrel{\text{def}}{=} \pi\big(\{g \in G : \|g\mathbf{v}\| \geq \varepsilon \quad \forall\, \mathbf{v} \in \mathbb{Z}^{n+1} \smallsetminus \{0\}\}\big); \tag{2.2}$$

i.e., K_ε is the collection of all unimodular lattices in $\mathbb{R}^{n+1}$ which contain no nonzero vector with norm than ε. By Mahler's compactness criterion (see e.g. [Ra, Chapter 10]), each K_ε is compact, and for each compact $K \subset G/\Gamma$ there is $\varepsilon > 0$ such that $K \subset K_\varepsilon$.

Now take $\mathbf{r}$ as in (1.2) and write

$$\bar r = \min_{1\leq i\leq n} r_i\,.$$

Lemma 2.2. *Let* $\|\cdot\|$ *be the supremum norm, let* ε *and* t *be positive numbers with* $e^{\bar r t} \geq \varepsilon$, *and denote* $T = e^t$. *Then* (1.3) *with* $\delta = \varepsilon^{1/\bar r}$ *implies* $g_t^{(\mathbf{r})}\bar\tau(\mathbf{x}) \notin K_\varepsilon$, *which in turn implies* (1.3) *with* $\delta = \varepsilon$.

Proof. Suppose (1.3) holds with $\delta = \varepsilon^{1/\bar{r}}$ and with $\mathbf{p} \in \mathbb{Z}^n$, $q \in \mathbb{N}$. This implies that

$$e^{-t}q = q/T < \delta < \varepsilon$$

and for $i = 1, \ldots, d$,

$$e^{r_i t}|qx_i - p_i| < e^{r_i t}\delta^{r_i}/T^{r_i} = \varepsilon^{r_i/\bar{r}} \leq \varepsilon\,.$$

From this one concludes that for $\mathbf{v} = (-\mathbf{p}, q) \in \mathbb{Z}^{n+1} \smallsetminus \{0\}$,

$$\|g_t^{(\mathbf{r})}\tau(\mathbf{x})\mathbf{v}\| = \max\left\{e^{-t}q,\, e^{r_1 t}|-p_1 + qx_1|, \ldots, e^{r_n t}|-p_n + qx_n|\right\} < \varepsilon\,,$$

so $g_t^{(\mathbf{r})}\bar{\tau}(\mathbf{x}) \notin K_\varepsilon$. The proof of the second implication is similar and is omitted. □

Proof of Proposition 2.1. By the preceding lemma, $\mathbf{x}$ is $\mathbf{r}$-singular along $\mathcal{T}$ if and only if for any $\varepsilon > 0$ there is T_0 such that $g_t^{(\mathbf{r})}\bar{\tau}(\mathbf{x}) \notin K_\varepsilon$ whenever $e^t \in \mathcal{T} \cap [T_0, \infty)$. The latter, in view of Mahler's compactness criterion, is equivalent to the fact that the trajectory

$$\{g_t^{(\mathbf{r})}\bar{\tau}(\mathbf{x}) : t \in \log \mathcal{T}\}$$

eventually leaves every compact subset of G/Γ. □

As an application of this dynamical approach, we can state a condition on a measure μ on $\mathbb{R}^n$ guaranteeing that it assigns measure zero to singular vectors.

Proposition 2.3. *Let μ be a measure on $\mathbb{R}^n$, and suppose that for μ-a.e. $\mathbf{x}_0 \in \mathbb{R}^n$ there is a ball B centered at $\mathbf{x}_0$ such that $\forall\, \delta > 0$ there exist $\varepsilon > 0$ and a sequence $t_k \to \infty$, $t_k \in \log \mathcal{T}$, with*

$$\mu\big(\{\mathbf{x} \in B : g_{t_k}^{(\mathbf{r})}\bar{\tau}(\mathbf{x}) \notin K_\varepsilon\}\big) < \delta \text{ for every } k\,. \tag{2.3}$$

Then $\mu\big(\mathrm{Sing}(\mathbf{r}, \mathcal{T})\big) = 0$.

Proof. Indeed, if we take $\{t_k\}$ as above and let

$$B_\varepsilon \stackrel{\text{def}}{=} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \{\mathbf{x} \in B : g_{t_k}^{(\mathbf{r})}\bar{\tau}(\mathbf{x}) \notin K_\varepsilon\}\,,$$

then (2.3) implies that $\mu(B_\varepsilon) \leq \delta$. But the set $\bigcap_\varepsilon B_\varepsilon$, in view of Mahler's compactness criterion, coincides with

$$\{\mathbf{x} \in B : g_{t_k}^{(\mathbf{r})}\bar{\tau}(\mathbf{x}) \text{ is divergent}\}\,,$$

and therefore has measure zero. □

Remark 2.4. Note that even though the definition of K_ε depends on the choice of the norm $\|\cdot\|$ in (2.2), the assumption of Proposition 2.3 is clearly independent of this choice. Thus without loss of generality we may, and will, fix a Euclidean structure on $\mathbb{R}^{n+1}$ and choose $\|\cdot\|$ to be the Euclidean norm.

3. The inheritance problem and friendly measures

In general, given a property which holds for a typical point in $\mathbb{R}^n$, it is natural to inquire for which subsets (e.g. submanifolds, self-similar 'fractals', etc.) a typical point on the subset also satisfies the property. The prototype for such an 'inheritance question' was the famous conjecture of K. Mahler from the 1930s, settled three decades later by V. Sprindžuk, which led to the theory of diophantine approximation on manifolds.

As was mentioned in the introduction, the first inheritance result related to the notion of singular vectors is due to Davenport and Schmidt: they showed that almost no points (with respect to the smooth measure class) on M are singular, where

(a) $M = \{(t, t^2) : t \in \mathbb{R}\} \subset \mathbb{R}^2$ [DS, Thm. 3].

Later, R. Baker proved that almost no points on M are singular when:

(b) $M = \{(t, t^2, t^3) : t \in \mathbb{R}\} \subset \mathbb{R}^3$ [Ba1, Thm. 2];
(c) $M \subset \mathbb{R}^2$ is a curve with continuous third derivatives and non-vanishing curvature almost everywhere [Ba2, Thm. 2].

The only other paper on this topic known to us is [DRV2], where it was proved that almost all points on a C^3 submanifold M of $\mathbb{R}^n$ are not singular if

(d) M has 'two-dimensional definite curvature almost everywhere'.

The latter condition requires the dimension of M to be at least 2 (see [DRV1] for more detail).

Note that all of the above examples are special case of nondegenerate manifolds defined in the introduction. (See [KM, Remark 6.3] for a discussion of the relation between nondegeneracy of M and the conditions of [DRV1, DRV2].) However, it had not been previously established that almost all points on M are not singular when M is a nondegenerate submanifold of $\mathbb{R}^n$, even for the special case

$$M = \{(t, \dots, t^n) : t \in \mathbb{R}\}, \tag{3.1}$$

which is the subject of the original Mahler's Conjecture.

A more general framework for discussing the inheritance problem is to recast it in terms of measures. That is, given a property $\mathcal{P}$ which holds for Lebesgue-a.e. $\mathbf{x} \in \mathbb{R}^n$, one wants to describe measures μ such that $\mathcal{P}$ also holds for μ-a.e. $\mathbf{x}$. Let us recall certain properties of a measure on $\mathbb{R}^n$, introduced in [KLW].

Suppose μ is a locally finite Borel measure on $\mathbb{R}^n$. Let $B(\mathbf{x}, r)$ denote the open ball of radius r centered at $\mathbf{x}$. Suppose $U \subset \mathbb{R}^n$ is open. We say that μ is *D-Federer on U* if for all $\mathbf{x} \in \operatorname{supp} \mu \cap U$ one has

$$\frac{\mu\big(B(\mathbf{x}, 3r)\big)}{\mu\big(B(\mathbf{x}, r)\big)} < D$$

whenever $B(\mathbf{x}, 3r) \subset U$.

Say that μ is *nonplanar* if $\mu(\mathcal{L}) = 0$ for any affine hyperplane $\mathcal{L}$ of $\mathbb{R}^n$. For an affine subspace $\mathcal{L} \subset \mathbb{R}^n$ we denote by $d_{\mathcal{L}}(\mathbf{x})$ the (Euclidean) distance from $\mathbf{x}$ to $\mathcal{L}$, and let

$$\mathcal{L}^{(\varepsilon)} \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n : d_{\mathcal{L}}(\mathbf{x}) < \varepsilon\}\,.$$

If $B \subset \mathbb{R}^n$ with $\mu(B) > 0$ and f is a real-valued function on $\mathbb{R}^n$, let

$$\|f\|_{\mu,B} \stackrel{\text{def}}{=} \sup_{x \in B \cap \operatorname{supp} \mu} |f(x)|\,.$$

Given C, $\alpha > 0$ and an open subset U of $\mathbb{R}^n$, say that μ is *(C, α)-decaying on U* if for any non-empty open ball $B \subset U$ centered in $\operatorname{supp} \mu$, any affine hyperplane $\mathcal{L} \subset \mathbb{R}^n$, and any $\varepsilon > 0$ one has

$$\mu\left(B \cap \mathcal{L}^{(\varepsilon)}\right) \le C \left(\frac{\varepsilon}{\|d_{\mathcal{L}}\|_{\mu,B}}\right)^{\alpha} \mu(B)\,. \tag{3.2}$$

Finally, let us say that μ is *friendly* if it is nonplanar, and for μ-a.e. $x \in \mathbb{R}^n$ there exist a neighborhood U of x and positive $C, \alpha, D > 0$ such that μ is D-Federer and (C, α) decaying on U.

The class of friendly measures is rather large; its properties are discussed in [KLW] and examples are given in [KLW, KW, Ur1, SU]. For example, it is essentially proved in [KM] (see [KLW, Lemma 7.1 and Propositions 7.2, 7.3]) that the natural measure on a nondegenerate manifold obtained by pushing forward Lebesgue measure on $\mathbb{R}^n$ is friendly. Thus our main result (Theorem 1.1) supersedes entries (a–d) in the above list, and in particular is applicable to the curve (3.1). Further, there are many more possibilities of interesting choices of sets which can support friendly measures. A particularly nice choice is given by limit sets of finite irreducible systems of contracting similarities (or, more generally, self-conformal maps of $\mathbb{R}^n$) satisfying the open set condition, see [KLW, §8] and [Ur1] for more detail.

We can now state the main measure estimate from which Theorem 1.1 will easily follow.

Theorem 3.1. *Let μ be a friendly measure on $\mathbb{R}^n$. Then for μ-a.e. $\mathbf{z} \in \mathbb{R}^n$ there is a ball B centered at $\mathbf{z}$ and positive $\tilde{C}, \alpha$ with the following property: for any $\mathbf{r}$ as in* (1.2) *there exists $t_0 > 0$ such that for all $t > t_0$ and all $\varepsilon > 0$ one has*

$$\mu\big(\{\mathbf{x} \in B : g_t^{(\mathbf{r})}\bar{\tau}(\mathbf{x}) \notin K_\varepsilon\}\big) < \tilde{C}\varepsilon^\alpha \mu(B)\,. \tag{3.3}$$

Proof of Theorem 1.1 assuming Theorem 3.1. Take B, $\tilde{C}$, α as in Theorem 3.1, given $\delta > 0$ choose $\varepsilon > 0$ with $\tilde{C}\varepsilon^\alpha\mu(B) < \delta$, and let $\{t_k\}$ be any unbounded subsequence of $\log \mathcal{T} \cap (t_0, \infty)$. Then (2.3) becomes an immediate consequence of (3.3), and an application of Proposition 2.3 finishes the proof. $\square$

4. A quantitative nondivergence estimate

We will derive Theorem 3.1 from a quantitative nondivergence result. To state it we need some additional definitions.

Let $f : \mathbb{R}^n \to \mathbb{R}$. Given C, $\alpha > 0$, $U \subset \mathbb{R}^n$ and a measure μ on $\mathbb{R}^n$, say that f is (C, α)-*good on U with respect to* μ if for any ball $B \subset U$ centered in $\operatorname{supp}\mu$ and any $\varepsilon > 0$ one has

$$\mu\big(\{y \in B : |f(y)| < \varepsilon\}\big) \le C\left(\frac{\varepsilon}{\|f\|_{\mu,B}}\right)^\alpha \mu(B)\,. \tag{4.1}$$

We refer the reader to [KM, KLW] for various properties and examples. We are going to need two elementary observations, which we state below for convenience.

Lemma 4.1. *Let $U \subset \mathbb{R}^n$ be open, $C, \alpha > 0$, μ a measure on $\mathbb{R}^n$.*

(a) [KLW, Lemma 4.2] *μ is (C, α)-decaying on U if and only if any affine function (equivalently, any function of the form $d_{\mathcal{L}}$, where $\mathcal{L}$ is an affine hyperplane) is (C, α)-good on U.*
(b) [KLW, Lemma 4.1] *If $f_1, \dots, f_k$ are (C, α)-good on U w. r. t. μ, then the function $\mathbf{x} \mapsto \|\mathbf{f}(\mathbf{x})\|$, where $\mathbf{f} = (f_1, \dots, f_k)$ and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^k$, is $(k^{\alpha/2}C, \alpha)$-good on U w. r. t. μ.*

Let

$$\mathcal{W} \stackrel{\text{def}}{=} \text{the set of nonzero rational subspaces of } \mathbb{R}^{n+1}\,.$$

For $V \in \mathcal{W}$ and $g \in G$, let

$$\ell_V(g) \stackrel{\text{def}}{=} \|g(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k)\|\,,$$

where $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a generating set for $\mathbb{Z}^{n+1} \cap V$ and $\|\cdot\|$ is the extension of the Euclidean norm from $\mathbb{R}^{n+1}$ to its exterior algebra; note that $\ell_V(g)$ does not depend on the choice of $\{\mathbf{v}_i\}$.

We will use the following estimate, which is a special case of [KLW, Theorem 4.3]:

Theorem 4.2. *Given $n \in \mathbb{N}$ and positive constants C, D, α, there exists $\tilde{C} = \tilde{C}(n, C, D, \alpha) > 0$ with the following property. Suppose μ is D-Federer on an open subset U of $\mathbb{R}^n$, h is a continuous map $U \to G$, $0 < \rho \le 1$, $\mathbf{z} \in U \cap \operatorname{supp} \mu$, and $B = B(\mathbf{z}, r)$ is a ball such that $B(\mathbf{z}, 3^n r) \subset U$, and that for each $V \in \mathcal{W}$,*

(i) *the function $\ell_V \circ h$ is (C, α)-good on $B(\mathbf{z}, 3^n r)$ with respect to μ, and*

(ii) $\|\ell_V \circ h\|_{\mu, B} \ge \rho$.

Then for any $0 < \varepsilon \le \rho$,

$$\mu\big(\{\mathbf{x} \in B : \pi\big(h(\mathbf{x})\big) \notin K_\varepsilon\}\big) \le \tilde{C}(\varepsilon/\rho)^\alpha \mu(B)\,. \tag{4.2}$$

Proof of Theorem 3.1. Recall that we are given a friendly measure μ. For μ-almost every $\mathbf{z} \in \mathbb{R}^n$, choose a neighborhood U of $\mathbf{z}$, positive constants C', D, α such that μ is D-Federer and (C', α)-decaying on U, and a ball $B = B(\mathbf{z}, r)$ centered at $\mathbf{z}$ such that $B(\mathbf{z}, 3^n r)$ is contained in U. Clearly the desired estimate (3.3) will coincide with (4.2) if one takes $\rho = 1$ and lets $h = h_{\mathbf{r},t}$ where the latter is defined by

$$h_{\mathbf{r},t}(\mathbf{x}) \stackrel{\text{def}}{=} g_t^{(\mathbf{r})} \tau(\mathbf{x})\,.$$

Therefore it suffices to verify the assumptions of Theorem 4.2 for the above choice of ρ and h. This is done below in Lemmas 4.3 and 4.4. □

Lemma 4.3. *Suppose that μ is (C', α)-decaying on an open $U \subset \mathbb{R}^n$. Then for any $V \in \mathcal{W}$, any $\mathbf{r}$ as in (1.2) and any $t \ge 0$,*

$$\textit{the function } \ell_V \circ h_{\mathbf{r},t} \textit{ is } (C, \alpha)\textit{-good on } U \textit{ with respect to } \mu\,, \tag{4.3}$$

where $C = (n+1)^{\alpha/2} C'$.

Lemma 4.4. *Suppose that μ is nonplanar. Then for any $\mathbf{r}$ as in (1.2) and any ball B with $\mu(B) > 0$ there is $t_0 = t_0(\mu, \mathbf{r}, B)$ such that for any $V \in \mathcal{W}$ and any $t \ge t_0$ one has*

$$\|\ell_V \circ h_{\mathbf{r},t}\|_{\mu, B} \ge 1\,. \tag{4.4}$$

The proof of both lemmas hinges on a computation of the $h_{\mathbf{r},t}(\mathbf{x})$-action on the exterior powers of $\mathbb{R}^{n+1}$, as in [KM] or [KLW]. We include the argument for the sake of completeness. For the remainder of the section, to simplify notation we will write g_t instead of $g_t^{(\mathbf{r})}$ and h_t instead of $h_{\mathbf{r},t}$.

Denote by V_0 the subspace

$$V_0 \stackrel{\text{def}}{=} \{(x_1, \dots, x_{n+1}) : x_{n+1} = 0\}$$

of $\mathbb{R}^{n+1}$, and let $\mathbf{e}_0 \stackrel{\text{def}}{=} (0, \dots, 0, 1)$ be a vector orthonormal to V_0. Note that for any $\mathbf{x} \in \mathbb{R}^n$,

$$\tau(\mathbf{x}) \text{ acts trivially on } V_0 \quad \text{and} \quad \tau(\mathbf{x})\mathbf{e}_0 = \mathbf{e}_0 + \mathbf{x}\,, \tag{4.5}$$

where we with some abuse of notation identified V_0 with $\mathbb{R}^n$.

Now suppose that V is a k-dimensional subspace of $\mathbb{R}^{n+1}$, $k \geq 1$, spanned by integer vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, and denote $\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k$ by $\mathbf{w}$. By applying Gaussian elimination over the integers to $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ one can write $\mathbf{w}$ in the form

$$\mathbf{w} = \mathbf{w}_0 \wedge (q\mathbf{e}_0 - \mathbf{p})\,, \tag{4.6}$$

where $q \in \mathbb{Z}$, $\mathbf{p} \in V_0(\mathbb{Z})$ and $\mathbf{w}_0 \in \bigwedge^{k-1}\big(V_0(\mathbb{Z})\big)$. Using (4.5) and (4.6), one writes

$$\tau(\mathbf{x})\mathbf{w} = \mathbf{w}_0 \wedge \big(q\mathbf{e}_0 + q\mathbf{x} - \mathbf{p}\big) = \mathbf{w}_0 \wedge (q\mathbf{x} - \mathbf{p}) + q\mathbf{w}_0 \wedge \mathbf{e}_0\,,$$

and hence

$$h_t(\mathbf{x})\mathbf{w} = g_t\big(\mathbf{w}_0 \wedge (q\mathbf{x} - \mathbf{p})\big) + qg_t(\mathbf{w}_0 \wedge \mathbf{e}_0)\,. \tag{4.7}$$

Note that the two summands in (4.7) are orthogonal, therefore

$$\begin{aligned} \big(\ell_V \circ h_t(\mathbf{x})\big)^2 &= \|h_t(\mathbf{x})\mathbf{w}\|^2 \\ &= q^2\|g_t(\mathbf{w}_0 \wedge \mathbf{e}_0)\|^2 + \|g_t\big(\mathbf{w}_0 \wedge (q\mathbf{x} - \mathbf{p})\big)\|^2\,. \end{aligned} \tag{4.8}$$

Now we can return to the lemmas.

Proof of Lemma 4.3. Write the second summand in (4.8) in the form

$$\|g_t\mathbf{w}_0 \wedge g_t(q\mathbf{x} - \mathbf{p})\|^2 = \|g_t\mathbf{w}_0\|^2 d_{g_t\mathcal{P}}\big(g_t(q\mathbf{x} - \mathbf{p})\big)^2\,,$$

where $\mathcal{P}$ stands for the linear subspace of V_0 corresponding to $\mathbf{w}_0$. For any t, q and $\mathbf{p}$, the function $\mathbf{x} \mapsto d^2_{g_t\mathcal{P}}\big(g_t(q\mathbf{x} - \mathbf{p})\big)$ is the sum of squares of at most n affine functions, each of which is (C', α)-good on U with respect to μ in view of Lemma 4.1(a) and the assumption of Lemma 4.3. Therefore, by Lemma 4.1(b), the function $\ell_V \circ h_t$ is $\big((n+1)^{\alpha/2}C', \alpha\big)$-good on U with respect to μ. $\square$

Proof of Lemma 4.4. Let us denote by $e^{\gamma t}$ the smallest eigenvalue of the induced action of g_t on $\bigwedge^k(V_0)$ (here $\gamma > 0$ depends on $\mathbf{r}$). If $q = 0$, in view of (4.6) we have $\mathbf{w} \in \bigwedge^k \big(V_0(\mathbb{Z})\big)$, hence

$$\ell_V \circ h_t(\mathbf{x}) = \|g_t \tau(\mathbf{x})\mathbf{w}\| = \|g_t \mathbf{w}\| \geq e^{\gamma t}\|\mathbf{w}\| \geq 1$$

for all $t \geq 0$. Thus the conclusion of the lemma holds with e.g. $t_0 = 0$. Otherwise, using (4.7), one can write

$$\begin{aligned} \ell_V \circ h_t(\mathbf{x}) &\geq \|g_t\big(\mathbf{w}_0 \wedge (q\mathbf{x} - \mathbf{p})\big)\| \geq e^{\gamma t}\|\mathbf{w}_0 \wedge (q\mathbf{x} - \mathbf{p})\| \\ &= e^{\gamma t}\|\mathbf{w}_0\| d_{\mathcal{P}}(q\mathbf{x} - \mathbf{p}) = |q| e^{\gamma t}\|\mathbf{w}_0\| d_{(\mathcal{P}+\mathbf{p}/q)}(\mathbf{x}) \geq e^{\gamma t} d_{(\mathcal{P}+\mathbf{p}/q)}(\mathbf{x}) \end{aligned}$$

(the last inequality holds since both q and all the coordinates of $\mathbf{w}_0$ are integers).

If B is a ball with $\mu(B) > 0$, an easy compactness argument using the assumption that μ is nonplanar shows the existence of $c = c(B) > 0$ such that $\|d_{\mathcal{L}}\|_{\mu,B} \geq c$ for any proper affine subspace $\mathcal{L}$ of $\mathbb{R}^n$. Hence

$$\|\ell_V \circ h_t\|_{\mu,B} \geq c|q|e^{\gamma t}\|\mathbf{w}_0\| \geq c e^{\gamma t},$$

and the conclusion of the lemma holds with $t_0 = \frac{1}{\gamma}\log\frac{1}{c}$. □

5. Constructing singular vectors on submanifolds

In this section we adapt the methods of the paper [We] and exhibit points with divergent trajectories on certain proper subsets of G/Γ.

Theorem 5.1. *Let $\{g_t : t \in \mathbb{R}\}$ be a one-parameter subgroup of G. Suppose X is a closed subset of G, and $\{X_i : i \in \mathbb{N}\}$ and $\{X'_j : j \in \mathbb{N}\}$ are two lists of subsets of X, such that for some strictly decreasing continuous $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, the following conditions are satisfied.*

1. **Density.** *For every j, $X_j = \overline{X_j \cap \bigcup_{i \neq j} X_i}$.*
2. **Transversality I.** *For every $i \neq j$, $X_i = \overline{X_i \smallsetminus X_j}$.*
3. **Transversality II.** *For any i, j, $X_i = \overline{X_i \smallsetminus X'_j}$.*
4. **Local Uniformity w.r.t. $\{K_{\psi(t)}\}$.** *For every i and every $x \in X_i$ there is a neighborhood U in G and t_0 such that for all $t \geq t_0$ and all $z \in U \cap X_i$,*

$$g_t \pi(z) \notin K_{\psi(t)}.$$

Then there is $x \in X \smallsetminus \left(\bigcup_i X_i \cup \bigcup_j X'_j\right)$ and $t_0 > 0$ such that for all $t \geq t_0$, $g_t \pi(x) \notin K_{\psi(t)}$.

Proof. Equip X with the relative topology inherited from G, and write

$$K(t) = K_{\psi(t)}.$$

We will construct inductively a sequence of open sets with compact closure $\Omega_0, \Omega_1, \Omega_2, \ldots$ in X, an increasing sequence of indices $i_1, i_2, \ldots$, and an increasing sequence of times $T_0, T_1, \ldots$, such that the following hold for $k = 1, 2, \ldots$:

a. $\overline{\Omega_k} \subset \Omega_{k-1}$.
b. For every $j < i_k$, $X_j \cap \Omega_k = \varnothing$ and $X'_j \cap \Omega_k = \varnothing$.
c. $X_{i_k} \cap \Omega_k$ is nonempty and for every $z \in X_{i_k} \cap \Omega_k$ and every $t \geq T_k$ we have $g_t\pi(z) \notin K(t)$.

We will also have for $k = 2, 3, \ldots$:

d. For every $z \in \Omega_k$ and every $t \in [T_{k-1}, T_k]$, $g_t\pi(z) \notin K(t)$.

To see that such sequences suffice, note that by condition a, $\bigcap_k \Omega_k$ is nonempty, and for $z \in \bigcap \Omega_k$ we have by condition b that $z \notin \bigcup_i X_i \cup \bigcup_j X'_j$ and by condition d that $g_t\pi(z) \notin K(t)$ for $t \geq T_1$.

Now let us construct the sequences inductively. Choose $T_0 = 0$, $i_1 = 1$. Let $x \in X_1$ and using the local uniformity hypothesis, let Ω_1 be a small enough open neighborhood of x, and T_1 large enough, so that for all $z \in X_1 \cap \Omega_1$ and all $t \geq T_1$, we have $g_t\pi(z) \notin K(t)$. Now letting T_0 be arbitrary and Ω_0 be any open set containing $\overline{\Omega_1}$, we see that a, b and c hold for $k = 1$.

Suppose we have chosen i_s, Ω_s, T_s for $s = 1, \ldots, k$. By the density condition there are $\ell \neq i_k$ such that

$$X_\ell \cap \Omega_k \cap X_{i_k} \neq \varnothing.$$

Choose for i_{k+1} any such ℓ. Note that $i_{k+1} > i_k$ by b. Let $x \in X_{i_k} \cap \Omega_k \cap X_{i_{k+1}}$. By the local uniformity assumption, there is a small enough open neighborhood U of x and a large enough T_{k+1} such that for all $z \in U \cap X_{i_{k+1}}$ and all $t \geq T_{k+1}$, $g_t\pi(z) \notin K(t)$. In addition let U be small enough so that $\overline{U} \subset \Omega_k$. Since $x \in X_{i_k}$, $g_t\pi(x) \notin K(t)$ for $t \in [T_k, T_{k+1}]$, hence by continuity of ψ and the action, there is a small enough neighborhood $\widetilde{\Omega}$ of x contained in U so that

$$z \in \widetilde{\Omega},\ t \in [T_{k-1}, T_k] \quad \Longrightarrow \quad g_t\pi(z) \notin K(t).$$

Now we can define Ω_{k+1} by

$$\Omega_{k+1} = \widetilde{\Omega} \smallsetminus \bigcup_{j < i_{k+1}} \left(X_j \cup X'_j\right).$$

We now verify that $i_{k+1}, \Omega_{k+1}, T_{k+1}$ satisfy the required conditions. Condition a holds by our choice of U. Condition b follows from the definition of Ω_{k+1}. In condition c, $\Omega_{k+1} \cap X_{i_{k+1}} \neq \varnothing$ because $x \in \widetilde{\Omega} \cap X_{i_{k+1}}$, and because of the transversality assumptions. The second assertion in condition c holds because of the choice of T_{k+1} and U. Condition d holds because of the choice of $\widetilde{\Omega}$. $\square$

We now derive a consequence of this theorem. This requires some notation. Fix $\mathbf{r}$, let M be a submanifold of $\mathbb{R}^n$, and choose $1 \leq k < \ell \leq n$. For $\mathbf{v} \in \mathbb{Z}^n$ and $s \in \mathbb{Q}$ let

$$L_{\mathbf{v}}(s) = \{\mathbf{x} \in M : \langle \mathbf{x}, \mathbf{v} \rangle = s\}.$$

Let $\mathbf{e}_k, \mathbf{e}_\ell$ be the k-th and ℓ-th standard basis vectors and let $\{X_i\}$ be a list of the distinct connected components of the sets $\{L_{\mathbf{e}_r}(s) : r \in \{k, \ell\},\ s \in \mathbb{Q}\}$. Also let $\{X'_j\}$ be a list of the distinct connected components of the sets $\{L_{\mathbf{v}}(s) : \mathbf{v} \in \mathbb{Z}^n,\ s \in \mathbb{Q}\}$ which are not in the list $\{X_i\}$. In case $\{X'_j\} = \{X'_1, \ldots, X'_m\}$ happens to be a finite list, we put $X'_j = \varnothing$ for $j > m$, i.e., we may assume that $\{X'_j\}$ is indexed by $\mathbb{N}$ as well.

Corollary 5.2. *Suppose* $\mathbf{r}$, M, k, ℓ, $\{X_i\}$, $\{X'_j\}$ *are as above, and that the density and two transversality assumptions of Theorem 5.1 are satisfied. Suppose also that* $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ *is a function such that*

$$T^\rho \delta(T) \to_{T\to\infty} \infty, \quad \text{where } \rho = \min\{r_k, r_\ell\}/n.$$

Then there is a totally irrational $\mathbf{x} \in M$ *and* T_0 *such that for all* $T \geq T_0$ *there is* $\mathbf{p} \in \mathbb{Z}^n$, $q \in \mathbb{N}$ *satisfying* (1.3) *for* $\delta = \delta(T)$.

Proof. Set $\psi(t) \stackrel{\text{def}}{=} \delta(e^t)$, so that

$$e^{\rho t}\psi(t) \to \infty. \tag{5.1}$$

Suppose without loss of generality that ψ is a decreasing function. Using Lemma 2.2, it suffices to show that there are $x = \tau(\mathbf{x}) \in \tau(M)$ and t_0 such that $\mathbf{x}$ is totally irrational and for all $t \geq t_0$,

$$g_t^{(\mathbf{r})}\pi(x) \notin K_{\psi(t)}.$$

To this end, we apply Theorem 5.1 to the lists $\{X_i\}$, $\{X'_j\}$ (which we identify with their images under τ, and thus consider them as subsets of G). Since we have assumed the density and two transversality conditions, we need only check the locally uniform escape condition.

To verify the condition of local uniformity w.r.t. $\{K_{\psi(t)}\}$, fix X_i so that (possibly after exchanging k and ℓ), for some $p/q \in \mathbb{Q}$ and for all $\mathbf{z} = (z_1, \ldots, z_n) \in X_i$ we have $z_k = p/q$. Let

$$W = \{(w_1, \ldots, w_{n+1}) \in \mathbb{R}^{n+1} : qw_k + pw_{n+1} = 0\}.$$

This is a rational linear subspace of $\mathbb{R}^{n+1}$ of dimension n, with the property that for all $\mathbf{z} \in X_i$, all vectors in $\tau(\mathbf{z}) \cdot W$ have their k-th coordinate equal to zero. It follows that for any bounded neighborhood

U intersecting X_i there is a constant C, such that for all $\mathbf{z} \in U \cap X_i$ we have

$$\ell_W \left(g_t^{(\mathbf{r})} \tau(\mathbf{z}) \right) \le C e^{-r_k t}.$$

Using Minkowski's convex body theorem, we find a constant C' so that for all t and all $\mathbf{z} \in U \cap X_i$, $g_t^{(\mathbf{r})} \tau(\mathbf{z}) \left(W \cap \mathbb{Z}^{n+1} \right)$ contains a non-zero vector of length at most

$$C' e^{-r_k t/n} \le C' e^{-\rho t}.$$

Now from (5.1) if follows that there is t_0 such that for all $t \ge t_0$, the length of such a vector is less than $\psi(t)$. This concludes the proof. $\square$

Proof of Theorem 1.2. For $\mathbf{z} \in M$ let $T_{\mathbf{z}} M \subset \mathbb{R}^n$ be the tangent space to M at $\mathbf{z}$. Since $\dim(M) \ge 2$, there are distinct indices $k, \ell \in \{1, \ldots, n\}$ and an open $V \subset M$ such that if P is the projection

$$P(\mathbf{x}) = (x_k, x_\ell) \quad \text{for } \mathbf{x} = (x_1, \ldots, x_n),$$

then the derivative $D_{\mathbf{z}} \left(P|_M \right) : T_{\mathbf{z}} M \to \mathbb{R}^2$ is surjective for all $\mathbf{z} \in V$. With no loss of generality, replace M with V and define $\{X_i\}$, $\{X'_j\}$ as in the paragraph preceding Corollary 5.2. In view of the Corollary, it remain to check that the density and two transversality hypotheses hold for $\{X_i\}$, $\{X'_j\}$.

Since these hypotheses hold for horizontal and vertical lines in an open subset of the plane, and since the X_i's are the pre-images of these lines under P, we see that the density and the first transversality hypothesis hold. Now for i and j, suppose that $X_i \cap X'_j \neq \varnothing$ (otherwise there is nothing to prove). Then X_i and $X_i \cap X'_j$ are connected analytic submanifolds of M. Suppose if possible that $X_i \cap X'_j$ contains an open subset of X_i. Then, since they are analytic and X_i is connected, $X_i \subset X'_j$. Since M is not contained in a rational affine hyperplane, both X_i and X'_j are submanifolds of M of codimension one, and since X'_j is also connected we must have $X_i = X'_j$, contrary to the construction. This implies that $X_i = \overline{X_i \smallsetminus X'_j}$, as required. $\square$

Remark 5.3. The hypotheses of Theorem 1.2 are clearly satisfied when M is an analytic nondegenerate submanifold of dimension at least 2, and also when M is an affine subspace of dimension at least 2 not contained in any rational affine hyperplane.

Remark 5.4. The proof of Theorem 1.2 actually shows that $M \cap \mathrm{Sing}(\mathbf{r})$ contains *uncountably many* totally irrational vectors. Indeed, given any countable subset $A = \{\mathbf{z}_1, \mathbf{z}_2, \ldots\}$ of M, replace the list $\{X'_j\}$ with the list $\left\{ X'_1, \{\mathbf{z}_1\}, X'_2, \{\mathbf{z}_2\}, \ldots \right\}$. Applying the same argument yields an element of $M \cap \mathrm{Sing}(\mathbf{r})$ which is totally irrational and does not belong to A.

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