ERGODICITY FOR INFINITE PERIODIC TRANSLATION SURFACES

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Abstract. For a $\mathbb{Z}$-cover $\widetilde{M} \to M$ of a translation surface, which is a lattice surface, and which admits infinite strips, we prove that almost every direction for the straightline flow is ergodic.

This paper is dedicated to Howard Masur whose work is a great source of inspiration for the authors.

1. Introduction

A translation surface is a topological surface equipped with a geometric structure which makes it possible to define a straightline flow on the surface in any direction θ. The study of the ergodic properties of straightline flows on compact translation surfaces is a classical subject which has been studied for nearly a century (see for instance [KS]). The dynamics of the straightline flows is best understood on so-called lattice surfaces, following celebrated work of Masur [Ma] and Veech [V]. See [MaTa, Vi, Zo] for detailed introductions to translation surfaces, including surveys of these and more recent developments.

For non-compact translation surfaces, while several examples of lattice surfaces have been given (see [Ho1, HoHuWe, HuSc]), there are not many results on the dynamics of the straightline flows. The only example which is well-understood is the infinite staircase surface, for which all ergodic invariant Radon measures were classified [HoHuWe]. The infinite staircase is an example of a lattice surface which arises as a $\mathbb{Z}$-cover of a compact translation surface, and a general theory for such surfaces was developed in [HoWe]. Note that for $\mathbb{Z}$-covers, the ergodicity question can be reduced to a question on the ergodicity of a $\mathbb{Z}$-valued skew product over a base dynamics. In the infinite staircase example the base dynamics is an irrational rotation of the circle; the skew products over rotations are well-understood, and in fact the results of [HoHuWe] are essentially just a reformulation of prior work on skew products [ANSS]. However for general $\mathbb{Z}$-covers, one is led to the study of skew products over interval exchange transformations. These are poorly understood and the reduction does not essentially simplify the analysis. Fortunately, the original arguments of Masur [Ma] can be adapted to this situation; using this approach we solve the ergodicity question in many cases and obtain new results about $\mathbb{Z}$-valued skew products over interval exchanges.

We now introduce the terminology needed for stating our results. We identify $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with the set of directions. For $\theta \in S^1$, throughout this paper we let $\mathbf{e}_\theta \in \mathbb{R}^2$ denote the vector $(\cos\theta, \sin\theta)$.

Let M be a compact translation surface, and $p : \widetilde{M} \to M$ a $\mathbb{Z}$-cover of translation surfaces; i.e. $\widetilde{M}$ is a (noncompact) translation surface, there is a finite set $P \subset M$ such that $p : \widetilde{M} \smallsetminus p^{-1}(P) \to M \smallsetminus P$ is a covering map which is a translation in each chart, and there is a translation automorphism $S : \widetilde{M} \to \widetilde{M}$ commuting with p, such that M is isomorphic to $\widetilde{M}/\langle S \rangle$.

An *infinite strip* in $\widetilde{M}$ is a subset isometric to an infinite strip in $\mathbb{R}^2$, i.e. isometric to $\mathbb{R} \times (-a, a)$ for some $a > 0$. As explained in Proposition 10 below, an infinite strip $\Sigma \subset \widetilde{M}$ projects to a cylinder $C \subset M$, and the lift to Σ of a core curve δ of C is not closed, so that its endpoints are x, $S^k x$ for some $k = k(\Sigma) \neq 0$. The holonomy vector of δ depends only on Σ and we denote it by $v(\Sigma)$. Also we denote by $A(\Sigma)$ the area of C. We say that a direction θ is *well-approximated by strips* if there are $\varepsilon > 0$, $k \neq 0$ and infinitely many strips $\Sigma \subset \widetilde{M}$ for which $k \equiv k(\Sigma)$, the $A(\Sigma)$ are bounded below by a uniform positive bound, and

$$|\mathbf{e}_\theta \wedge v(\Sigma)| \leq (1 - \varepsilon) \frac{A(\Sigma)}{2\|v(\Sigma)\|}. \tag{1.1}$$

We say that θ is an ergodic direction on $\widetilde{M}$ if the straightline flow in direction θ on $\widetilde{M}$ is ergodic with respect to Lebesgue measure. We say that θ is *ergodic on intermediate finite covers*, if for any $\widetilde{M} \to M' \to M$, where M' is a finite cover of M, θ is an ergodic direction on M'.

Theorem 1. *Let $\widetilde{M} \to M$ be a $\mathbb{Z}$-cover of translation surfaces. Suppose θ is a direction which is well-approximated by strips in $\widetilde{M}$, and ergodic on intermediate finite covers. Then θ is an ergodic direction on $\widetilde{M}$.*

Using Theorem 1 we provide examples of ergodic directions on infinite translation surfaces:

Theorem 2. *Suppose $\widetilde{M} \to M$ is a $\mathbb{Z}$-cover, such that $\widetilde{M}$ is a lattice and has an infinite strip. Then almost every direction on $\widetilde{M}$ is ergodic; in fact there is a closed set Θ of directions, of Hausdorff dimension less than 1, such that any $\theta \notin \Theta$ is ergodic.*

Remark 3. (1) *As we show at the end of the paper, there are recurrent $\mathbb{Z}$-covers which do not admit infinite strips, and for these our methods fail. However as we show in Proposition 19, these are quite rare.*

(2) *The set Θ admits an explicit description in terms of the behavior of geodesic trajectories in G/Γ_0, where Γ_0 is the Veech group of $\widetilde{M}$. Namely it is the set of geodesic trajectories which do not venture sufficiently far into the cusp corresponding to a cylinder in M which lifts to an infinite strip in $\widetilde{M}$.*

(3) *The deduction of Theorem 2 from Theorem 1 is inspired by ideas of Masur* [Ma]*. In his proof of unique ergodicity for translation surfaces, to prove the existence on M of long and thin rectangles in direction θ, Masur uses that a certain geodesic trajectory is not divergent* in the moduli space of compact translation surfaces*. Our approach to verifying that a.e. θ is well-approximable by cylinders is similar. To find long and thin rectangle on $\widetilde{M}$ in direction θ, we study the limiting points of the corresponding trajectory* within the G-orbit of $\widetilde{M}$*. This hints that a* moduli space of recurrent covers *might be very helpful for generalizing our results.*

(4) *In the special case in which $\widetilde{M}$ is the infinite staircase surface, the ergodicity of the straightline flow in every irrational direction follows from results of Conze* [Co] *on cylinder flows over irrational rotations. One can show that in this case the set Θ consists only of rational directions, thus Theorem 2 yields another proof of Conze's theorem.*

(5) *For compact translation surfaces, almost every direction is uniquely ergodic. For infinite translation surfaces, one does not usually expect unique ergodicity. In the infinite staircase* [HoHuWe] *there are uncountably many invariant measures, indexed by a positive real 'deformation' parameter. Hooper* [Ho2] *exhibits some uniquely ergodic cases but these are not typical.*

(6) *K. Fraczek and C. Ulcigrai* [FrUl] *provide examples of recurrent $\mathbb{Z}$-covers $\widetilde{M} \to M$ which do admit a strip, where $\widetilde{M}$ is* not *a lattice surface, and for which the straightline flow is* not *ergodic in almost every direction. In their examples M is a square-tiled surface of genus two, and the Veech group of $\widetilde{M}$ is infinitely generated of the first kind. It follows from Theorem 1 that in these examples, the set of ergodic directions is a dense G_δ, see Corollary 16.*

1.1. **Applications.** It is of interest to construct dense trajectories for vector fields on noncompact surfaces. For example Panov [Pa] constructed a dense trajectory for a vector field on the plane, which is a pullback under a covering map, of a constant slope field on a torus endowed with a quadratic differential. Since ergodic flows with respect to a measure of full support have dense trajectories, our theorem gives many new examples of infinite translation surfaces for which the straightline flow has dense trajectories.

The question of ergodicity of a $\mathbb{Z}$-valued skew product over $f : X \to X$ is only well-understood for a limited class of X. For example the case in which $X = S^1$ is the circle and f is an irrational rotation is well-understood (see [ANSS] and the references therein). This is equivalent to the case in which X is an interval and f is an interval exchange on 2 intervals. On the other hand the question for interval exchange transformations on an arbitrary number of intervals is challenging and poorly understood. There are not many examples of $\mathbb{Z}$-valued skew products over interval exchanges which are known to be ergodic. Our Theorem 2 is a source of many new examples. Suppose $\widetilde{M} \to M$ is a recurrent $\mathbb{Z}$-cover which is a lattice surface containing an infinite strip, and θ is an ergodic direction on M. Let I be a segment in M transverse to direction θ and $\widetilde{I}$ a segment which is a preimage of I in $\widetilde{M}$. If $f : I \to I$ (resp. $\widetilde{f} : \widetilde{I} \to \widetilde{I}$) is the return map to I (resp. $\widetilde{I}$) along lines in direction θ in M (resp. $\widetilde{M}$) then $\widetilde{f}$ is well-defined in light of Proposition 8(ii), and is a $\mathbb{Z}$-valued skew product over f. If $\widetilde{M} \to M$ satisfies the conditions of Theorem 1 then $\widetilde{f}$ is ergodic. An alternative approach to the ergodicity question for $\mathbb{Z}$-valued skew products over *periodic* interval exchanges (i.e. those for which the geodesic flow is periodic), under an assumption on the gap in the Lyapunov spectrum, is developed in the recent work [CoFr].

A well-known result of Kesten [Ke] implies that on the infinite staircase surface, there are no bounded trajectories (i.e. trajectories for the straightline flow which in all positive times are confined between two levels of the staircase). By an argument of [Pe] which we recall in Proposition 18, Theorem 1 provides more examples of recurrent $\mathbb{Z}$-covers without bounded trajectories. Note that Ralston [Ra] has constructed trajectories on the infinite staircase which are bounded above.

2. Background

In this section, we briefly introduce our notation and state the results we will need.

2.1. **Translation surfaces.** A surface is called a *translation surface* if it can be obtained by edge-to-edge gluing of polygons in the plane, only using translations (the polygons need not be compact or finitely many but should be at most countably many, and the lengths of sides and diagonals should be bounded away from zero). Two translation surfaces are considered equivalent if the corresponding decompositions into polygons have a common locally finite refinement. The translation structure induces a flat metric with conical singularities; in the non-compact case, 'infinite angle singularities' may arise. The translation structure also induces horizontal and vertical transverse measures dx, dy, and a Lebesgue measure $dxdy$.

Let M be a compact translation surface. Sometimes we add finitely many marked points to the set of singularities, i.e. distinguished points at which the cone angle is 2π which we consider as singularities. To avoid uninteresting complications we always assume that the set of singularities is nonempty. A *saddle connection* is a geodesic segment for the flat metric starting and ending at a singularity, and containing no singularity in its interior. A *cylinder* on a translation surface is a maximal connected union of homotopic simple closed geodesics. We call the length of such a closed geodesic the *circumference* of the cylinder, and the length of a perpendicular segment going across the cylinder, its *height*. Let M be a compact translation surface, let $P \subset M$ be a finite set of points containing the singularities, and let δ be a curve on M which is either closed or connects points of P. We denote by $\mathrm{hol}(\delta)$ the holonomy of the translation structure on M along δ; i.e. the vector in $\mathbb{R}^2$ obtained by integrating dx and dy elements along δ. The holonomy map is well defined on $H_1(M, P; \mathbb{Z})$.

Given any translation surface M, an *affine diffeomorphism* is an orientation preserving homeomorphism of M that permutes the singularities of the flat metric and acts affinely on the polygons defining M. The group of affine diffeomorphisms is denoted by $\mathrm{Aff}(M)$. The image of the derivation map

$$d : \begin{cases} \mathrm{Aff}(M) \to \mathrm{GL}(2, \mathbb{R}) \\ \qquad f \mapsto df \end{cases}$$

is called the *Veech group*, and denoted by $\Gamma(M)$. If M is a compact translation surface, then $\Gamma(M)$ is a discrete subgroup of $G = \mathrm{SL}(2, \mathbb{R})$.

A translation surface is a *lattice surface* if its Veech group is a lattice in G. Note that lattice surfaces are sometimes referred to as *Veech surfaces*. We say that two subgroups Γ_1, Γ_2 of a group G are *commensurable* if they share a common finite index subgroup. Let $\overline{\phi}_t^{\theta}$ denote the straightline flow in direction θ on $\widetilde{M}_w$ and by ϕ_t^θ the flow on M. A straightline flow on a translation surface is *uniquely ergodic* if the only invariant probability measure is Lebesgue measure. If the linear flow ϕ_t^θ is uniquely ergodic, we say that the direction θ is uniquely ergodic.

Let $G = \mathrm{SL}(2, \mathbb{R})$, and let

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.1}$$

There is a moduli space or *stratum* of translation surfaces of a fixed genus and singularity pattern, and this space is equipped with a G-action. The flow induced by the action of $\{g_t\}$ is called the geodesic flow. For any flow on a topological space, we say that a trajectory $\{g_t x\}$ is divergent if for any compact subset K of the space there is t_0 such that $g_t x \notin K$ for all $t > t_0$. Masur's well-known criterion provides a link between the dynamics of the straightline flow on a surface and the dynamics of the geodesic flow on its stratum of translation surfaces. Throughout this paper we set $\theta' = \pi/2 - \theta$.

Proposition 4 (Masur condition [Ma])**.** *If M is a compact translation surface and $\{g_t r_{\theta'} M\}$ is not divergent in the moduli space of translation surfaces, then θ is a uniquely ergodic direction on M. In particular, denoting by Γ the Veech group of M and by $x \mapsto [x]$ the natural map $G \to G/\Gamma$, if $\{g_t[r_{\theta'}]\}$ is not divergent in G/Γ, then θ is a uniquely ergodic direction on M.*

2.2. **Fuchsian groups, geodesic flow, and approximation by cusps.** Let Γ be a Fuchsian group, i.e. a discrete subgroup of G. In this subsection we recall some facts we will need — see [Ka] for an introduction to Fuchsian groups.

We denote the upper half-plane by $\mathbb{H}$, so that G acts on $\mathbb{H}$ by Möbius transformations. For us it will be convenient to use the *right-action*

$$z.g = \frac{dz - b}{-cz + a}, \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \tag{2.2}$$

A horoball in $\mathbb{H}$ is the isometric image of

$$\mathbb{H}_c^+ = \{z \in \mathbb{H} : \Im z > c\}$$

for some $c > 0$, where $\Im z$ denotes the imaginary part of z. A parabolic element of G is any matrix of trace 2 other than the identity. If nontrivial, the stabilizer of a horoball in Γ is a maximal unipotent subgroup, i.e. an infinite cyclic group generated by a parabolic element which is maximal with respect to inclusion. A cusp in $\mathbb{H}/\Gamma$ is the image of the map

$$B/P \to \mathbb{H}/\Gamma, \tag{2.3}$$

where B is a horoball, P is the stabilizer of B in Γ and is nontrivial, and B is maximal with respect to the property that the map (2.3) is injective. Any maximal parabolic subgroup $P \subset \Gamma$ stabilizes a cusp, which is a proper subset of G/Γ when Γ is non-elementary. Γ is called a non-uniform lattice if G/Γ is not compact but has finite measure. In this case $\mathbb{H}/\Gamma$ has a finite nonzero number of cusps, and their complement is a compact subset of $\mathbb{H}/\Gamma$. The map sending a cusp in $\mathbb{H}/\Gamma$ to the conjugacy class of the group P as in (2.3) is well-defined and induces a bijection between cusps and conjugacy classes of maximal parabolic subgroups.

We have a map $G/\Gamma \to \mathbb{H}/\Gamma$ which maps $[g]$ to the image in $\mathbb{H}/\Gamma$ of $\mathbf{i}.g$, where $\mathbf{i} = \sqrt{-1}$. In case Γ is torsion-free, the quotient $\mathbb{H}/\Gamma$ is a noncompact finite-volume hyperbolic manifold, G/Γ can be identified with its unit tangent bundle, and the above projection is the projection mapping a unit tangent vector to its basepoint. The group G acts on G/Γ via left translations $g_1[g_2] = [g_1 g_2]$, and the action of the group $\{g_t\}$ gives rise to the geodesic flow on the unit tangent bundle. Let $\mathcal{C}$ be a cusp in $\mathbb{H}/\Gamma$, which is the image of a horoball under a map (2.3), where B is the isometric image of $\mathbb{H}^+_{c_0}$ for some $c_0 > 0$. For any $c > c_0$ let $\mathcal{C}_c$ be the image of $\mathbb{H}^+_c$ under the same maps. That is, the sets $\{\mathcal{C} \smallsetminus \mathcal{C}_c : c > c_0\}$, give an exhaustion of $\mathcal{C}$ by bounded sets. We say that a geodesic orbit $\{g_t x\}$ *penetrates infinitely often into* $\mathcal{C}_c$ if there is $t_n \to \infty$ such that $g_{t_n} x \in \mathcal{C}_c$ for all n.

Another action of G is its linear action on the punctured plane $\mathbb{R}^2 \smallsetminus \{0\}$. Under this action, a point x has a discrete orbit under a non-uniform lattice Γ if and only if its stabilizer contains a parabolic element. For any x, the set of directions of vectors in the orbit Γx is dense in S^1. To quantify such a density statement, for a discrete orbit Γx, $\theta \in S^1$ and $d > 0$, we say that θ is *d-well-approximable by Γx* if there are infinitely many $\gamma \in \Gamma$ for which

$$\|\gamma x\| \, |\mathbf{e}_\theta \wedge \gamma x| < d. \tag{2.4}$$

Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$ and $|u \wedge v| = \|u\| \|v\| |\sin\theta|$ where θ is the angle between the directions of u and v.

Proposition 5. *Let Γ be a non-uniform lattice, let $x \in \mathbb{R}^2 \smallsetminus \{0\}$ such that Γx is discrete, and let $\mathcal{C}$ denote the cusp in G/Γ corresponding to the stabilizer of x. For any c there is d such that if $\theta \in S^1$ is d-well-approximable by Γx then $\{g_t[r_{\theta'}]\}$ penetrates infinitely often into $\mathcal{C}_c$. Conversely, for any d there is c such that if $\{g_t[r_{\theta'}]\}$ penetrates infinitely often into $\mathcal{C}_c$ then θ is d-well-approximable by Γx.*

Proof. This is a standard computation (see e.g. [P]). To keep the paper self-contained, we prove the implication $\implies$, leaving the converse to the reader.

Let $g_0 \in G$ such that $g_0 x = \mathbf{e}$, where $\mathbf{e} = (1,0)$ is the first standard basis vector. Replacing Γ with $g_0 \Gamma g_0^{-1}$, we may assume that $x = \mathbf{e}$, so that the cusp corresponding to x is $\pi(\mathbb{H}^+_{c_0})$ for some c_0. Given $c > 0$, we may assume without loss of generality that $c > c_0$, and we let $d = 1/2c$. Suppose that for some sequence $\gamma_n \in \Gamma$, (2.4) holds for all $\gamma = \gamma_n$. Let T_n, θ_n denote respectively the length and angle of $\gamma_n x$, i.e. $\gamma_n x = T_n \mathbf{e}_{\theta_n}$, and write $\gamma_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, so that

$$\gamma_n x = \gamma_n \mathbf{e} = \begin{pmatrix} a_n \\ c_n \end{pmatrix} = T_n \mathbf{e}_{\theta_n}.$$

Then (2.4) is equivalent to

$$T_n^2 \, |\sin(\theta - \theta_n)| < d. \tag{2.5}$$

It follows easily from (2.2) that for any $g \in G$, $\Im(\mathbf{i}.g) = \frac{1}{a^2+c^2}$. Let $t_n = \log T_n - \log \sqrt{d}$. If we write

$$\begin{pmatrix} \bar{a}_n & \bar{b}_n \\ \bar{c}_n & \bar{d}_n \end{pmatrix} = g_{t_n} r_{\theta'} \gamma_n, \text{with } \theta' = \pi/2$$

$$\text{then } \begin{pmatrix} \bar{a}_n \\ \bar{c}_n \end{pmatrix} = g_{t_n} r_{\theta'} T_n \mathbf{e}_{\theta_n} = \begin{pmatrix} T_n^2 \sin(\theta - \theta_n)/\sqrt{d} \\ \sqrt{d}\cos(\theta - \theta_n) \end{pmatrix}.$$

This implies via (2.5) that

$$\Im(\mathbf{i}.(g_{t_n} r_{\theta'} \gamma_n)) = \frac{1}{\bar{a}_n^2 + \bar{c}_n^2} > \frac{1}{2d} = c.$$

This shows that for all n, $g_{t_n} r_{\theta'} \gamma_n \in \mathbb{H}^+_c$, as required. □

Proposition 6. *For any non-uniform lattice Γ, any $d > 0$, and any $x \in \mathbb{R}^2 \smallsetminus \{0\}$ for which Γx is discrete, the set*

$$\Theta_d = \left\{\theta \in S^1 : \theta \text{ is not } d\text{−well − approximable by } \Gamma x\right\}$$

has zero Lebesgue measure, and moreover its Hausdorff dimension is less than 1.

Proof. We first prove that Θ_d has zero Lebesgue measure. The geodesic flow on G/Γ is ergodic, and for any $c > 0$, the cusp $\mathcal{C}_c$ has positive measure, with respect to the G-invariant probability measure μ on G/Γ induced by Haar measure on G. This implies that for any $c > 0$,

$$\Omega_c = \{x \in G/\Gamma : \{g_t x\} \text{ does not penetrate into } \mathcal{C}_c \text{ infinitely often}\}$$

has measure zero. On the other hand, by Proposition 5, for some $c > 0$, and some cusp $\mathcal{C}$ in G/Γ, Ω_c contains $\{[r_{\theta'}] : \theta \in \Theta_d\}$. Let $h_s^- = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$. Then $g_t h_s^- g_{-t} \to_{t\to\infty} \mathrm{Id}$ in G and this implies that if $x \in \Omega_c$ then $h_s^- x \in \Omega_{c'}$ for any s and any $c' > c$. Haar measure is smooth on G and the map $(s, t, \theta) \mapsto g_t h_s^- r_\theta$ is a local homeomorphism. This implies that for any $c' > c$, the set $\Omega_{c'}$ contains

$$\{\pi(g_t h_s^- r_{\theta'}) : \theta \in \Theta_d\},$$

which has μ-positive measure whenever Θ_d has positive Lebesgue measure. This proves the claim.

The stronger result about Hausdorff dimension would follows by a similar argument, provided we knew that for any $c > 0$, the Hausdorff dimension of Ω_c is less than 3, which is the dimension of G/Γ. In case G/Γ has one cusp, this follows from [KlWe, Prop. 8.5]. As explained to the authors by Manfred Einsiedler [Ei], the general case of the latter statement can be deduced from his recent work with Kadyrov and Pohl [EiKaPo]. □

For a geodesic trajectory $\{g_t x\}$, either its projection in $\mathbb{H}/\Gamma$ belongs to a cusp for all $t > t_0$, or it returns infinitely often to the complement of the cusps. This implies:

Proposition 7. *If Γ is a non-uniform lattice, for all but countably many directions θ, the trajectory $\{g_t[r_{\theta'}]\}$ does not diverge in G/Γ.*

2.3. **$\mathbb{Z}$-covers.** We recall some terminology from [HoWe]. Let M be a compact translation surface. We fix a finite set P of points containing the singularities. Denote the relative homology by $H_1(M, P; \mathbb{Z})$ and $H_1(M \smallsetminus P, \mathbb{Z})$ the absolute homology of M puctured at P. The intersection form is a nondegenerate bilinear form

$$i : H_1(M, P; \mathbb{Z}) \times H_1(M \smallsetminus P, \mathbb{Z}) \to \mathbb{Z}.$$

The $\mathbb{Z}$-cover $\widetilde{M}_w$ of $M \smallsetminus P$ associated to a non-zero w in $H_1(M, P, \mathbb{Z})$ is the cover associated to the kernel of the homomorphism

$$\phi_w : \pi_1(M \smallsetminus P) \to \mathbb{Z}, \quad \delta \mapsto i(w, [[\delta]]),$$

where $[[\delta]]$ denotes the homology class of δ.

For a translation surface M and $w \in H_1(M, P; \mathbb{Z})$, let $\mathrm{hol}(w) \in \mathbb{R}^2$ denote the holonomy along w; that is the vector whose coordinates are respectively the integrals of the elements dx, dy along any path representing w. We have:

Proposition 8. (i) *There is a bijective correspondence between $\mathbb{Z}$-covers $\widetilde{M} \to M$ and projective classes of cycles $w \in H_1(M, P; \mathbb{Z})$.*

(ii) *The holonomy* $\mathrm{hol}(w)$ *vanishes if and only if for any* θ *for which* ϕ_t^θ *is ergodic,* $\overline{\phi}_t^\theta$ *is recurrent.*
(iii) *The Veech group of* $\widetilde{M}_w$ *is Fuchsian. If it is non-elementary then* $\mathrm{hol}(w) = 0$.
(iv) $\Gamma_{\widetilde{M}}$ *contains a finite index subgroup which descends to* Γ_M.
(v) *If* $\{g_t[r_{\theta'}]\}$ *is not divergent in* $G/\Gamma_{\widetilde{M}}$*, and* $\widetilde{M} \to M' \to M$ *is an intermediate finite cover of* M*, then* $\{g_t[r_{\theta'}]\}$ *is not divergent in* $G/\Gamma_{M'}$.

In (ii), by recurrence we mean that for any measurable $A \subset \widetilde{M}$, for a.e. $x \in A$ there is $t_n \to \infty$ such that $\overline{\phi}_{t_n} x \in A$. If (ii) holds we say that $\widetilde{M}$ is a *recurrent* $\mathbb{Z}$*-cover.*

Proof. Items (i)–(iv) are proved in [HoWe], and (v) follows from (iv) and the fact that a subgroup of $\Gamma(\widetilde{M})$ which descends to $\Gamma(M)$ also descends to $\Gamma(M')$ for any intermediate finite cover M'. □

Propositions 4, 7 and 8(v) imply one of the hypotheses of Theorem 1:

Corollary 9. *Suppose* $\widetilde{M} \to M$ *is a* $\mathbb{Z}$*-cover and* θ *is a direction such that* $\{g_t[r_{\theta'}]\}$ *is not divergent in* G/Γ*, where* Γ *is the Veech group of* $\widetilde{M}$*. Then* θ *is ergodic on intermediate covers. In particular, if* $\widetilde{M}$ *is a lattice surface, then* θ *is ergodic on intermediate covers for all but countably many* θ*.*

□

We give a more concrete description of the construction of the cover $\widetilde{M}_w$. One can represent w as $\sum a_j \delta_j$, where $a_j \in \mathbb{Z}$, and the δ_j are finitely many disjoint oriented arcs on M which are either closed or have endpoints in P. Now form countably many copies $\{M_k : k \in \mathbb{Z}\}$ (each with its copies of the arcs δ_j), and glue each M_k to M_ℓ along the left (resp. right) of δ_j if $\ell = k + a_j$ (resp. $\ell = k - a_j$). We now give a necessary and sufficient condition for the existence of infinite strips on $\widetilde{M}$.

Proposition 10. *Let* $\widetilde{M}_w \to M$ *be a recurrent* $\mathbb{Z}$*-cover of translation surfaces associated to the homology class* w*.* $\widetilde{M}$ *has an infinite strip if and only if there is a cylinder* C *on* M*, such that the homology class* $[[\delta]]$ *represented by the core curve* δ *of* C *satisfies* $i(w, [[\delta]]) = k \neq 0$*. In this case* S^k *maps the strip to itself, and is a translation along the direction of the strip.*

Proof. Given a cylinder C on M, a connected component of its lift to $\widetilde{M}$ is a cylinder or a strip. Let δ be the core curve of C. The cylinder C lifts to a cylinder on $\widetilde{M}$ if and only if preimages of δ are compact loops. By definition of the covering, this holds if and only if $i(w, [[\delta]]) = 0$. Conversely, if $\Sigma \subset \widetilde{M}$ is an infinite strip, let ℓ be an infinite straight line in its interior. Note that the projection of Σ to M does not contain singularities, so that the image of ℓ in M is a straight line whose distance to singularities remains bounded below. A basic fact on compact translation surfaces (see e.g. [MaTa, §1.6]) asserts that the projection δ of ℓ must be periodic, i.e. is the core curve of a cylinder C. That is the image of Σ covers C, and by the above, $i(w, [[\delta]]) \neq 0$.

Assume now that $i(w, [[\delta]]) = k \neq 0$. This exactly means that a lift $\widetilde{\delta}$ of δ starting at level 0 on $\widetilde{M}$ ends at level k. Thus each connected component of $\bigcup_{n\in\mathbb{Z}} S^n(\widetilde{\delta})$ is an infinite line in $\widetilde{M}$ passing through an infinite strip, and the action of S^k is by translation along the line. $\square$

2.4. **Cocycles and essential values.** Suppose $F_t : X \to X$ is a probability measure preserving flow on a non-atomic measure space (X, μ), and suppose $\alpha : X \times \mathbb{R} \to \mathbb{Z}$ is a measurable cocycle, i.e., a function satisfying

$$\alpha(x, t+s) = \alpha(x,t) + \alpha(F_t x, s). \tag{2.6}$$

Form the space $X_\alpha = X \times \mathbb{Z}$ with the obvious measure, and the flow $\overline{F}_t : X_\alpha \to X_\alpha$ defined by

$$\overline{F}_t(x, n) = (F_t x, n + \alpha(x,t)) .$$

This flow is called a $\mathbb{Z}$-*valued skew product* (over X, corresponding to α). If two cocycles α and β are cohomologous, i.e. if there is a measurable $g : X \to \mathbb{Z}$ such that $\alpha(x,t) = \beta(x,t) + g(F_t x) - g(x)$, then the skew products are measurably equivalent flows. A number $k \in \mathbb{Z}$ is called an *essential value* for this skew product if for any measurable $A \subset X$ with positive measure, there is a set of t of positive measure for which

$$\mu \{x \in A : F_t x \in A, \, \alpha(x,t) = k\} > 0.$$

There is a $\mathbb{Z}$-action on X_α obtained from the action of $\mathbb{Z}$ on itself by addition in the second factor; clearly it commutes with the $\overline{F}_t$-action, so for every subgroup $k\mathbb{Z} \subset \mathbb{Z}$ we can form the space $X_\alpha / k\mathbb{Z}$, and we have

$$X_\alpha \to X_\alpha / k\mathbb{Z} \to X_\alpha / \mathbb{Z} \simeq X.$$

Proposition 11 (Schmidt [Sch, Chap. 3])**.** *Suppose k is an essential value. Then $\overline{F}_t$ is ergodic on X_α if and only if it is ergodic on $X_\alpha / k\mathbb{Z}$.*

3. From strips to essential values

In this section we prove Theorem 1. The main point will be to approximate the flow in direction θ with the flow in a strip, where it does not encounter discontinuity points. The condition that the direction θ is well-approximated by strips ensures that the flow in a strip remains close to the flow in direction θ for a sufficiently long time, enabling us to produce essential values.

We will need some additional notation. Let M be a compact translation surface and $\widetilde{M} \to M$ a recurrent $\mathbb{Z}$-cover. For a direction θ, ϕ_t and $\overline{\phi}_t$ denote the straightline flows in direction θ on M and $\widetilde{M}$ respectively (note our notation does not reflect the dependence on θ). It is easy to see that $\overline{\phi}_t : \widetilde{M} \to \widetilde{M}$ is a skew product over $\phi_t : M \to M$. Namely, fix some x_0 in M and for each $x \in M$, a continuous path $\beta_{x_0,x}$ from x_0 to x, and let β_{x,x_0} denote the path with the same trace in the opposite direction. Let $w \in H_1(M, P; \mathbb{Z})$ such that $\widetilde{M} = \widetilde{M}_w$ as in Proposition 8, and let i be the intersection form on M. Now define a cocycle

$$\alpha(x,t) = i(w, \delta), \tag{3.1}$$

where δ is the path from x_0 to x along $\beta_{x_0,x}$, followed by moving from x to $\phi_t x$ along the line in direction θ, followed by $\beta_{\phi_t x, x_0}$. It is not hard to show that α is a cocycle and that making a different choice for x_0 and the $\beta_{x_0,x}$ would result in a cohomologous cocycle. Moreover it follows from the description of $\widetilde{M}_w$ that the straightline flow $\overline{\phi}_t : \widetilde{M} \to \widetilde{M}$ is measurably equivalent to the lift of ϕ_t to this skew product (see [HoWe, Proof of Proposition 15]).

Since θ is well-approximable by cylinders, there are $k \neq 0$, $\varepsilon > 0$, and for $n = 1, 2, \ldots$ there are strips Σ_n in $\widetilde{M}$, such that the corresponding $v_n = v(\Sigma_n)$ and $A_n = A(\Sigma_n)$ satisfy $k \equiv k(\Sigma_n)$ and

$$|\mathbf{e}_\theta \wedge v_n| \leq (1-\varepsilon)\,\frac{A_n}{2\|v_n\|} \qquad \text{and} \quad A_n \geq \varepsilon. \tag{3.2}$$

In light of Proposition 11, Theorem 1 follows immediately from:

Claim 12. *k is an essential value for the flow $\{\bar{\phi}_t\}$.*

The rest of this section is devoted to proving Claim 12. Let R be a rectangle, and for $c > 0$, let cR be a concentric rectangle which is obtained from R by dilating it by a factor of c. By saying that $\widetilde{M}$ contains a rectangle R we mean that there is an isometry which is a translation in charts, mapping R to $\widetilde{M}$; in particular the image of R under this isometry does not contain a singularity. When we refer to the corners of R, we mean the images of the corners under the above isometry.

Let $p : \widetilde{M} \to M$ be the covering map, and let $C_n = p(\Sigma_n)$. Note that $p^{-1}(C_n)$ is the union of images of Σ_n under the deck group. Let $\theta^* = \theta + \pi/2$ be the direction perpendicular to θ.

Definition 13. *We will say that $x \in \widetilde{M}$ admits a rectangle at stage n if $p^{-1}(C_n)$ contains cR, where R is the closed rectangle with sides in directions θ, θ^* and opposite corners at x, $S^k x$, and $c = (1-\varepsilon/2)(1-\varepsilon)^{-1} > 1$.*

We denote by $\widetilde{\mathcal{A}}$ the set of x which admit a rectangle at stage n, for infinitely many n and by $\mathcal{A}$, its projection to M.

Lemma 14. *The set $\mathcal{A}$ is conull (i.e. its complement has measure zero).*

Proof. On the compact surface M, the number of cylinders with a given bound on the length of their core curve is bounded. Therefore $\|v_n\| \to \infty$, so by (3.2), the direction of v_n tends to θ.

Let ℓ_n be a core curve of C_n at equal distances from its two boundaries, let $\bar{\ell}_n = p^{-1}(\ell_n)$, and let Σ'_n be the points of $p^{-1}(C_n)$ which are within distance $\eta_n = \dfrac{\varepsilon^2}{8\|v_n\|}$ of $\bar{\ell}_n$. Note that v_n is the diagonal of a rectangle in Σ_n with corners at $x, S^k x$ and $\|v_n\|$ is the circumference of C_n. This implies that

$$h_n = \frac{A_n}{2\|v_n\|} \geq \frac{\varepsilon}{2\|v_n\|} \tag{3.3}$$

is the distance from ℓ_n to the boundary edges of C_n, hence from $\bar{\ell}_n$ to the boundary edges of $p^{-1}(C_n)$. Therefore $c|\mathbf{e}_\theta \wedge v_n|$ is the length of the side of a rectangle cR as in Definition 13, in direction θ^*. Inequality (3.2) says that this side has length at most $(1-\varepsilon/2)h_n$, so that any point which is of distance

$$2\eta_n = \frac{\varepsilon^2}{4\|v_n\|} \overset{(3.3)}{\leq} \frac{\varepsilon h_n}{2}$$

from $\bar{\ell}_n$ admits a rectangle at stage n.

In particular any point of Σ'_n admits a rectangle at stage n. Let $C'_n = p(\Sigma'_n)$. By construction, each Σ'_n is deck group invariant, i.e. $\Sigma'_n = p^{-1}(C'_n)$. Now we let $\Sigma' = \limsup \Sigma'_n$ denote the set of points in $\widetilde{M}$ which belong to infinitely many of the Σ'_n, and $C' = \limsup C'_n$. Clearly $\Sigma' \subset \widetilde{\mathcal{A}}$, and by construction, $\Sigma' = p^{-1}(C')$. For each n, the measure of each C'_n in M is at least $\varepsilon^2/4$, so C' has positive measure and is contained in $\mathcal{A}$. To prove the lemma, by ergodicity of $\{\phi_t\}$ on M, it suffices to show that $\bigcup_t \phi_t(C')$, which is a $\{\phi_t\}$-invariant set, is contained in $\mathcal{A}$. So we fix $x \in C'$ and $t \in \mathbb{R}$, and consider $x' = \phi_t x$. We have shown that there are infinitely many n for which x is within η_n of ℓ_n, which is a curve parallel to v_n. Since the direction of v_n tends to θ, for all large enough n, x' is within $2\eta_n$ of ℓ_n, so admits a rectangle at stage n. That is, $x' \in \mathcal{A}$. $\square$

Continuing with the proof of Claim 12, let $A \subset M$ be a measurable set of positive measure, and let x_0 be a Lebesgue density point of $A \cap \mathcal{A}$. Let x be any of the preimages of x_0 in $\widetilde{M}$. There is a sequence of rectangles R_n in $\widetilde{M}$ which have x, $S^k x$ at opposite corners, have sides in directions θ, θ^*, and such that cR_n is embedded in $\widetilde{M}$. Note that the side of R_n in direction θ is getting longer with n and the side in direction θ^* is getting shorter; we will refer to these as the long and short side of R_n respectively, and denote their lengths by a_n, b_n. We choose a fundamental domain measurably isomorphic to M such that x belongs to level 0 and $S^k(x)$ belongs to level k (here we may use the concrete description of the covering given in §2.3), and we identify A with a subset of this fundamental domain.

Let Q_0 be the square centered at x with sides in directions θ and θ^*, such that the sidelength of Q_0 is equal to b_n. Denote

$$Q_1 = S^k(Q_0) \quad \text{and} \quad Q_2(t) = \bar{\phi}_t(Q_0),$$

where t is a parameter (note that Q_0, Q_1, Q_2 all depend on n but we omit this to simplify notation).

Following the notations in figure 1, for n large enough, the right half of Q_0 denoted by P_0 is contained cR_n and thus embedded in $\widetilde{M}$. Moreover, by construction cR_n does not contain any singular point in its closure. This implies that if $t < t_n = ca_n - (c-1)b_n$, then the restriction of $\bar{\phi}_t$ to P_0 is continuous, i.e. the right half of $Q_2(t)$ denoted by $P_2(t)$ is also a rectangle embedded in $\widetilde{M}$. By the same reasoning the left hand side of Q_1 is a rectangle embedded in $\widetilde{M}$.

Let λ denote Lebesgue measure on $\widetilde{M}$. We have $\lambda(P_0) = \lambda(P_1) = \lambda(P_2)$. Moreover when $t = a_n$ we have P_1 and $P_2 = P_2(t)$, and obtain $\lambda\left(P_1 \cap P_2\right) = \lambda(P_0)$. See figure 1.

As x_0 is a density point of A, for every square Q centered at x_0 of sufficiently small diameter, $\frac{\lambda(Q \cap A)}{\lambda(Q)} \geq 15/16$. This inequality is fulfilled for Q_0 for large enough n, since $b_n \to 0$. A straightforward computation shows that the same is true for P_0.

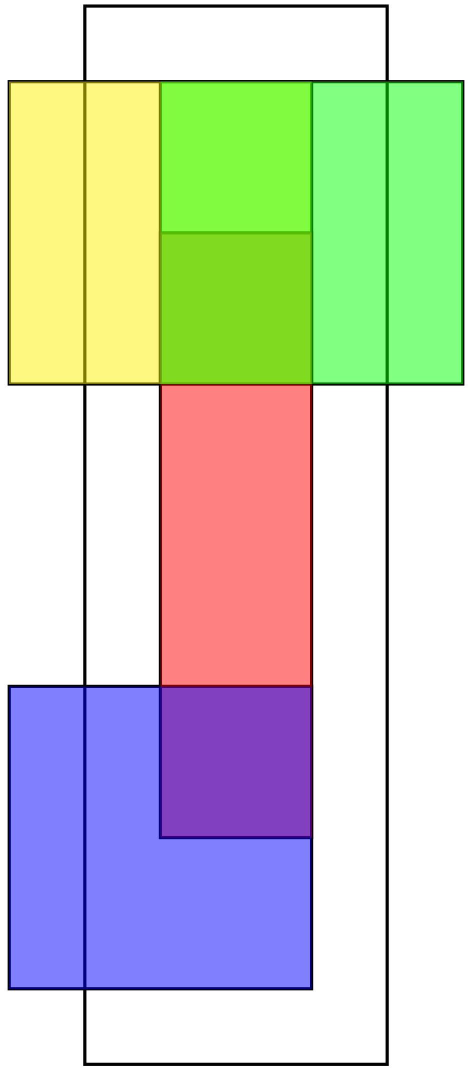

FIGURE 1. Rectangles. The direction θ is vertical. The red rectangle is R_n, its bottom left corner is x, its upper right corner is $S^k(x)$. The blue rectangle is Q_0, the upper right rectangle is Q_1, and the upper left rectangle is Q_2.

Note that $t_n > a_n$ for all large n. Therefore, there is an interval I centered at a_n such that if $t \in I$,

$$\frac{1}{2} < b \stackrel{\text{def}}{=} \frac{\lambda\,(P_1 \cap P_2(t))}{\lambda\,(P_0)} \le 1.$$

In order to show that k is an essential value, it will suffice to show that $A_k \cap A^{(t)}$ has positive measure for all $t \in I$, where

$$A_k = A \times \{k\} \ \text{ and } A^{(t)} = \bar{\phi}_t(A_0).$$

Denote $B^c = \widetilde{M} \smallsetminus B$ for every set B. We have the obvious relation

$$\begin{aligned} &\lambda\left(P_1 \cap P_2(t) \cap A^{(t)}\right) + \lambda\left(P_1^c \cap P_2(t) \cap A^{(t)}\right) \\ =&\lambda\left(P_2(t) \cap A^{(t)}\right) = \lambda\left(P_0 \cap A_0\right) \geq \frac{15}{16}\lambda\left(P_0\right). \end{aligned} \tag{3.4}$$

We have

$$\lambda\left(P_1^c \cap P_2(t) \cap A^{(t)}\right) \leq \lambda\left(P_1^c \cap P_2(t)\right) = (1-b)\lambda(P_0),$$

so from (3.4) and the definition of b:

$$\begin{aligned} \lambda\left(P_1 \cap P_2(t) \cap A^{(t)}\right) &\geq \left(\frac{15}{16} - 1 + b\right)\lambda\left(P_0\right) \\ &= \frac{1}{b}\left(\frac{15}{16} - 1 + b\right)\lambda(P_1 \cap P_2(t)). \end{aligned}$$

By the same reasoning,

$$\lambda\left(P_1 \cap P_2(t) \cap A_k\right) \geq \frac{1}{b}\left(\frac{15}{16} - 1 + b\right)\lambda\left(P_1 \cap P_2(t)\right).$$

If, by contradiction, $A_k \cap A^{(t)}$ were of measure zero, from the two preceding formulae we would have

$$\begin{aligned} &\frac{2}{b}\left(\frac{15}{16} - 1 + b\right)\lambda\left(P_1 \cap P_2(t)\right) \\ &\leq \lambda\left(P_1 \cap P_2(t) \cap A^{(t)}\right) + \lambda\left(P_1 \cap P_2(t) \cap A_k\right) \\ &= \lambda\left(\left(P_1 \cap P_2(t) \cap A^{(t)}\right) \cup \left(P_1 \cap P_2(t) \cap A_k\right)\right) \leq \lambda\left(P_1 \cap P_2(t)\right). \end{aligned}$$

As $1/2 < b \leq 1$, the proportion $\frac{2}{b}(\frac{15}{16} - 1 + b) > 1$ which leads to a contradiction. Therefore $\lambda(A^{(t)} \cap A_k) > 0$, as required. $\square$

4. Geodesic excursions and approximation by strips

Proof of Theorem 2. We will deduce the result from Theorem 1. Using Proposition 8(iv) and passing to finite-index subgroups, we can assume that the Veech groups of M and $\widetilde{M} = \widetilde{M}_w$ are the same lattice Γ in G, and this lattice fixes w. As observed by Veech [V], Γ is non-uniform, and the cusps of G/Γ correspond to cylinder decompositions of M. More precisely, cylinder decompositions in directions θ_1, θ_2 on M correspond to the same cusp in G/Γ if and only if they are stabilized by conjugate parabolic elements $p_1, p_2 \in \Gamma$, where p_i stabilizes direction

θ_i. Now suppose $\widetilde{M}$ has an infinite strip Σ in direction θ_0. It follows from Proposition 10, that to any infinite strip Σ on $\widetilde{M}_w$, there is a corresponding cylinder C on M with core curve δ, such that $k(\Sigma) = i(w, [[\delta]])$, $A(\Sigma) = \text{area}(C)$ and $v(\Sigma) = \text{hol}(\delta)$.

We say that strips Σ_1, Σ_2 are Γ-equivalent if there is an affine automorphism of $\widetilde{M}$ mapping Σ_1 to Σ_2. It follows from Proposition 8(iv), and the fact that affine automorphisms preserve the intersection pairing, that if Σ_1 and Σ_2 are equivalent then

$$k(\Sigma_1) = k(\Sigma_2), \quad A(\Sigma_1) = A(\Sigma_2).$$

Also $v(\Sigma)$ is fixed by a parabolic element of Γ so has a discrete orbit under Γ.

Let $\{x_1, \ldots, x_r\} \subset \mathbb{R}^2 \setminus \{0\}$ be the vectors $v(\Sigma_i)$, where Σ_i range over representatives of the Γ-equivalence classes of strips in $\widetilde{M}$. By assumption $r \geq 1$ and by the above discussion, r is at most the number of cusps in $\mathbb{H}/\Gamma$. We obtain that there is $d > 0$ such that θ is well-approximated by strips if and only if there is i such that θ is d-well-approximated by Γx_i. Theorem 2 is now a direct consequence of Theorem 1 and Corollary 9 and Proposition 6. $\square$

Let $\partial\mathbb{H} = \mathbb{R} \cup \{0\}$ be the boundary of $\mathbb{H}$ in the one-point compactification of the complex plane. Recall that there is a Γ-equivariant bijection $\text{Vis} : S^1 \to \partial\mathbb{H}$ known as the 'visibility map'. The *limit set* of a Fuchsian group Γ is the set of accumulation points of any of its orbits in $\mathbb{H} \cup \partial\mathbb{H}$.

Proposition 15. *If $\widetilde{M} \to M$ is a $\mathbb{Z}$-cover with an infinite strip, and such that the Veech group Γ of $\widetilde{M}$ is of the first kind, then the set of well approximated directions is a dense G_δ subset of S^1.*

Proof. Let $\theta_0 \in S^1$ be the direction of a strip $\Sigma \subset \widetilde{M}$. Let $A = A(\Sigma) > 0$, $k = k(\Sigma) \in \mathbb{Z} \setminus \{0\}$, $x = v(\Sigma) \in \mathbb{R}^2$ be the corresponding elements as in the preceding proof. Let $d < A/2$, and enumerate the elements of Γ as $\{\gamma_1, \gamma_2, \ldots\}$. For each n let G_n be the set of θ satisfying (2.4) for some $\gamma \in \Gamma \setminus \{\gamma_1, \ldots, \gamma_n\}$. Then G_n is clearly open. Let $\theta_1 \in G_n$ since G_n contains all but a finite subset of the orbit $\Gamma\theta_1$, it is dense in S^1.

Therefore the set $\Omega = \bigcap G_n$ is a dense G_δ subset of Λ. If θ belongs to Ω it is well approximable by an infinite number of strips directions. $\square$

Corollary 16. *If M is a lattice surface, if $\widetilde{M} \to M$ is a $\mathbb{Z}$-cover with an infinite strip, and Veech group of $\widetilde{M}$ is Fuchsian of the first kind, then the set of ergodic directions on $\widetilde{M}$ is a dense G_δ.*

Proof. By construction, the points in Ω are well approximated by strips, namely the strips in the orbit of Σ, as in the preceding proof. So in order to apply Theorem 1, it suffices to show that the directions in Ω are ergodic on finite covers of M. This is certainly true since M is a Veech surface. $\square$

Remark 17. *As we already mention, As mentioned above, Corollary 16 applies to examples studied by Fraczek and Ulcigrai* [FrUl]*. They consider a genus 2 square tiled surface M and assume that the cocycle defining the $\mathbb{Z}$-cover belongs to the intersection of the absolute homology and the kernel of the holonomy. By* [HoWe]*, Cor. 18, the Veech group of the cover $\widetilde{M}$ is of the first kind. By the result of Fraczek and Ulcigrai, the set of ergodic directions for the linear flow on $\widetilde{M}$ has zero measure, but by Corollary 16, it is a dense G_δ.*

5. Examples

There are several known constructions of examples satisfying the assumptions of Theorem 2. The easiest is the infinite staircase, and there are other $\mathbb{Z}$-covers of the torus which satisfy the hypotheses of Theorem 2. In these examples the proof of ergodicity can be reduced to known results about skew products over an irrational rotation.

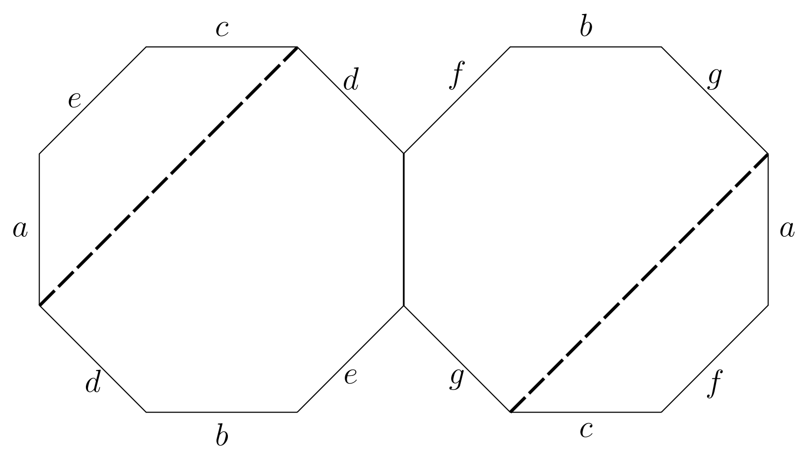

Figure 2. Double octagon.

5.1. **Surfaces satisfying our conditions.** We now present some examples which cannot be reduced to skew products over rotations, and to which our methods apply. An example of a $\mathbb{Z}$-cover which is a lattice surface is given at the end of [HoWe]. The compact lattice surface M

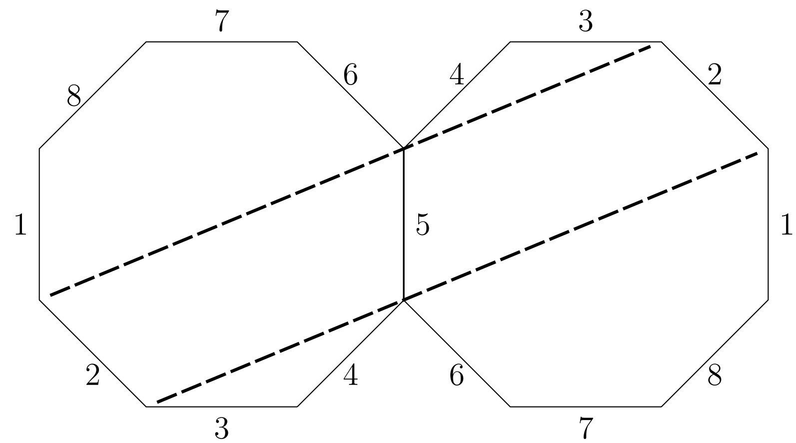

FIGURE 3. Veech's double octagon.

is obtained from two octagons with pairs of sides glued together (the gluings are different from the example of Veech in [V]). Hooper and Weiss proved that one homology class w_0 ($(b)-(c)$ in Figure 2) induces a $\mathbb{Z}$-cover $\widetilde{M}_{w_0}$ which is a lattice surface. The direction of slope 1 has an infinite strip on $\widetilde{M}_{w_0}$ since the cylinder of slope 1 which intersects (b) does not intersect (c) in M. Thus a.e. direction on $\widetilde{M}_{w_0}$ is ergodic.

Many other examples can be constructed. We recall that Veech's original examples [V] of lattice surfaces are obtained from two regular n-gons, where a side of one n-gon is glued to the opposite side of the other n-gon. We will denote this translation surface by Reg_n — see figure 3 for the case $n = 8$. Assume that $n = 4m$, and denote by $1, 2, \dots 4m$ the sides of one of the $4m$-gons with counter clockwise orientation. Denote by w the homology class of $1 + 3 + \cdots + 4m - 1$ and by $\mathrm{Reg}_{n,w}$ the associated $\mathbb{Z}$-cover. We have $\mathrm{hol}(w) = 0$ since each segment appears twice with opposite orientation. We recall that the Veech group of Reg_n is generated by the horizontal Dehn twist and by the rotation r of angle $\pi/2m$. The horizontal Dehn twist lifts to $\widetilde{\mathrm{Reg}}_{n,w}$ since w either intersects a horizontal cylinder once with a positive orientation and once with a negative one or it does not intersect it, see [HoWe, Proposition 24]. The image of w by r is the homology class $w*$ which is the sum of the segments $2, 4, \dots, 4m$. Their union partitions Reg_n into two connected components (the two n-gons), and $w = -w^*$, so by [HoWe, Proposition 8], the rotation r lifts to an affine automorphism of $\widetilde{\mathrm{Reg}}_{n,w}$. In particular $\widetilde{\mathrm{Reg}}_{n,w}$ is a lattice surface. Moreover the direction of slope $\tan(\pi/n)$ has a strip since one cylinder intersects the boundary of the n-gon along the sides $2, 2m+1$ (see figure 3).

A similar argument works for the surface $\mathrm{Reg}_{n=2m+2}$. We use the same class w (sum of the odd-labelled segments), and in the argument, exchange the roles of the horizontal direction and the direction of slope $\tan(\pi/n)$ (see figure 4). We leave the details to the reader.

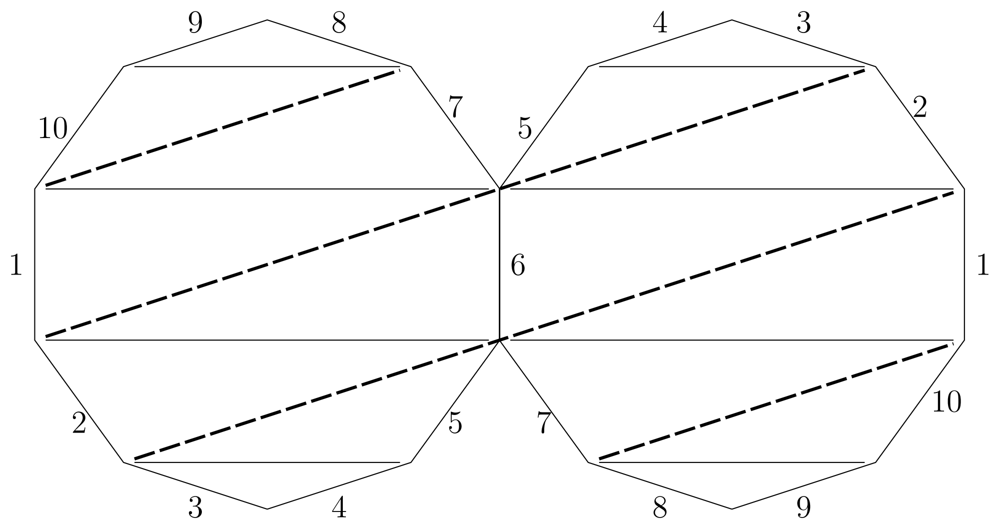

FIGURE 4. Veech's double decagon.

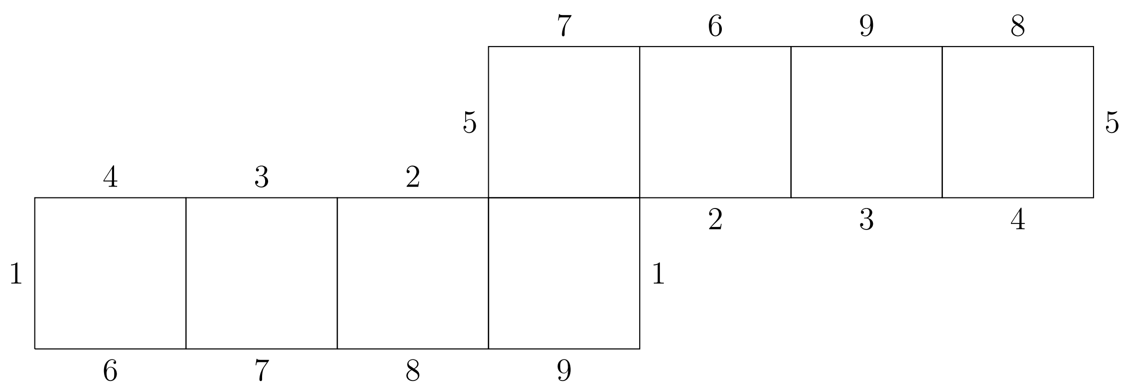

FIGURE 5. Wollmichsau.

5.2. **An extension of Kesten's theorem.** Kesten [Ke] showed that if θ is an irrational direction on the infinite staircase, then there is no x

for which $\{\overline{\phi}_t x : t \in \mathbb{R}\}$ is bounded. Adapting an argument of Petersen [Pe], we show the following:

Proposition 18. *Let $\widetilde{M} \to M$ be a recurrent $\mathbb{Z}$-cover and let θ be a direction which is ergodic on $\widetilde{M}$. Then there is no bounded trajectory on $\widetilde{M}$.*

Proof. Suppose by contradiction that $\widetilde{x}_1 \in \widetilde{M}$ is a point with a bounded trajectory. Since θ is an ergodic direction on $\widetilde{M}$, there are points in $\widetilde{M}$ whose trajectory visits every subset of $\widetilde{M}$ of positive measure; suppose $\widetilde{x}_2$ is such a point. Let x_1, x_2 denote the projections of $\widetilde{x}_1, \widetilde{x}_2$ in M. Then the choice of $\widetilde{x}_2$ implies that for the cocycle α defined by (3.1) we have

$$\{\alpha(x_2, t) : t > 0\} = \mathbb{Z}. \tag{5.1}$$

Since the orbit of $\widetilde{x}_1$ is bounded there is some N such that $|\alpha(x_1, t)| \leq N$ for all $t > 0$. Let $\{\beta_{x_0,x}\}$ be the family of curves described in §3, let $\widetilde{w}$ be a smooth representative of the homology class w, and fix $T \in \mathbb{R}$. There is a neighborhood U of x_2 in M such that $\phi_T|_U$ is an isometry, and such that the number of essential intersections of the curve $s \mapsto \phi_s x$ with $\widetilde{w}$, for s between 0 and T, is the same for all $x \in U$. This implies via (3.1) that for all $x_3 \in U$, $|\alpha(x_2, T) - \alpha(x_3, T)| \leq K$, where K is a constant depending only on the choice of paths $\{\beta_{x_0,x}\}$. Since θ is ergodic on $\widetilde{M}$, it is also ergodic on M, and since M is compact, this implies that the straightline flow in direction θ on M is minimal. Thus there is $t_0 > 0$ such that $T + t_0 > 0$ and $x_3 = \phi_{t_0}(x_1) \in U$, and (2.6) implies

$$|\alpha(x_3, T)| = |\alpha(\phi_{t_0}(x_1), T)| = |\alpha(x_1, T + t_0) - \alpha(x_1, t_0)| \leq 2N.$$

Therefore $|\alpha(x_2, T)| \leq 2N + K$, contradicting (5.1). $\square$

5.3. **An interesting non-example.** The existence of a strip is not automatic. On the Wollmilchsau W, depicted on figure 5 there is a relative homology class w_1 of zero holonomy such that *every* cylinder in W lifts to $\widetilde{W}_{w_1}$ as a union of cylinders. Indeed one can check that the class w_1 which is the difference of the segments marked 2 and 4 on figure 5 has this property. Thus, there is no infinite strip on $\widetilde{W}_{w_1}$ and our method does not apply to this case. On the other hand, by [HoWe, Proposition 30], $\widetilde{W}_{w_1}$ is a lattice surface. In [AvHuMa] it is shown, using a different method, that every irrational direction in $\widetilde{W}_{w_1}$ is ergodic.

In a certain sense, it is not easy to construct recurrent $\mathbb{Z}$-covers without infinite strips. We use the following observation:

Proposition 19. *If the cylinder core curves of M generate a finite-index subgroup of $H_1(M;\mathbb{Z})$ then for any holonomy-free $w \in H_1(M;\mathbb{Z})$, the recurrent $\mathbb{Z}$-cover $\widetilde{M_w}$ has infinite strips.*

Proof. This is immediate from the non-degeneracy of the intersection pairing. □

The condition that the cylinder core curves generate $H_1(M;\mathbb{Z})$ was studied in [Mo], where it is shown that it holds for almost every surface M.

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