FINITENESS RESULTS FOR FLAT SURFACES: A SURVEY AND PROBLEM LIST

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Abstract. We state some of the results obtained in the recent papers [SmWe1, SmWe2, SmWe3], describe some of the ideas involved, and list some open problems related to this work.

1. Introduction

The purpose of this note is to decribe some of the results and ideas of [SmWe1, SmWe2, SmWe3] and to list some open questions which this work leaves unanswered. Besides describing these results, one of our goals in writing this survey is to emphasize the common point of view and similar ideas involved in the proof.

Our objects of study are flat surfaces and their Veech groups. Flat surfaces arose in different mathematical communities and were given different names. In complex analysis they are known as abelian and quadratic differentials over complex structures on a surface. In the study of billiards on rational polygons they have become known respectively as translation and half-translation surfaces. In the study of homeomorphisms of surfaces they were known as pairs of transverse measured foliations. All of these are essentially equivalent, and we refer to them as flat surfaces. We refer the reader to the surveys [MaTa, Sm, Vo, HuSc3] for definitions of these structures and discussions of some of the many mathematical problems in which they arise.

In writing this survey we have assumed that the reader has some familiarity with flat surfaces. In particular the paper is not self-contained, and we refer the reader to the surveys above for the relevant definitions.

We state our main results in §2–§4, describe some representative ideas in §5–§9, and conclude with some open questions.

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2. Organizing the study of Veech groups

Let M be a flat surface of area 1, let $\mathcal{M}$ be the stratum of flat surfaces containing M, and let $G = \mathrm{PSL}(2, \mathbb{R})$. There is an action of G on $\mathcal{M}$ and the stabilizer

$$\Gamma_M = \{g \in G : gM = M\}$$

is a Fuchsian group called the *Veech group* of M. Alternatively this group is the image of the affine automorphism group $\mathrm{Aff}(M)$ under the map $D : \mathrm{Aff}(M) \to G$ sending an affine map to its derivative. For generic $M \in \mathcal{M}$ this group is trivial, however non-trivial Veech groups do arise and are quite interesting. It is a challenging question to determine which Fuchsian groups may arise as Veech groups. Veech's discovery [Ve1] of Veech groups which are non-arithmetic lattices in G stimulated a lot of interest in this and related questions. Here is a partial list of what is known at present:

- Veech groups are never cocompact [Ve1].
- Lattices (i.e. subgroups of finite covolume) may arise, including non-arithmetic lattices [Gu, Ve1].
- A finite index subgroup of any Veech group is conjugate to a subgroup of $\mathrm{PSL}(2, \mathcal{O})$ where $\mathcal{O}$ is the ring of integers in a number field whose degree over $\mathbb{Q}$ is bounded above by the genus of the surface [KeSm].
- There are infinitely generated Veech groups [HuSc2, McM1].
- There are infinitely many non-commensurable non-arithmetic lattices among Veech groups in genus 2 [Cal, McM2].
- There are Veech groups which do not contain parabolic elements [HuLa].

This is certainly not an exhaustive list; we refer the reader to the survey [HuSc3] and the papers [HuSc1, GuHuSc, GuJu, McM2, BM] for additional information about Veech groups.

A successful, and quite nontrivial, classification of Veech groups was carried out in two restricted settings: lattices arising from acute and right-angled rational triangles [KeSm, Pu] and the lattices arising in genus 2 surfaces [Cal, McM2]. However, as is evident from the above (particularly from the existence of infinitely generated Veech groups), classifying all Veech groups is a formidable task.

As a first step, one would like to organize the list of Veech groups and the associated flat surfaces in a sensible way. Note that Veech groups are invariant under conjugation in G, i.e. $\Gamma_{gM} = g\Gamma_M g^{-1}$. Ideally one would like to associate a parameter or parameters to a flat surface such that for a fixed bound on the value of the parameter one has only finitely many flat surfaces (up to the G-action), and relate these parameters to the geometry of either M or the quotient orbifold $\mathbb{H}/\Gamma_M$ (here $\mathbb{H}$ is the upper half plane).

This point of view motivates our first set of results.

2.1. **Finiteness results.** Recall that $g \in G$ is called *hyperbolic* (resp. *parabolic*) if $|\mathrm{tr}(g)| > 2$ (resp. $|\mathrm{tr}(g)| = 2$). Suppose Γ_M contains a hyperbolic element h which is the derivative of an affine automorphism φ of M; in Thurston's terminology [Th], φ is a pseudo-Anosov homeomorphism of S. We denote the larger eigenvalue of h by $\lambda(h)$. Associated to φ are Markov partitions of M. A *Markov partition* of M is a decomposition of M into finitely many parallelograms P_i, with sides parallel to the eigenvectors of h, such that the boundary of P_i in the contracting (resp. expanding) direction are mapped into the boundary of one of the P_j by φ (resp. φ^{-1}); i.e., if $\varphi(P_i) \cap P_j \neq \varnothing$ then $\varphi(P_i)$ extends all the way across P_j in the expanding direction. We let $p = p(M, h)$ be the minimal number of parallelograms in such a partition. These quantities are constant along G-orbits, that is, if $h \in \Gamma_M$ is hyperbolic then $ghg^{-1} \in \Gamma_{gM}$ is also hyperbolic with $p(M, h) = p(gM, ghg^{-1})$ and $\lambda(h) = \lambda(ghg^{-1})$. For $T > 0$ and $m \in \mathbb{N}$ we define

$$\mathrm{NSMP}(p, T) = \{(M, h) : p(M, h) = p, \lambda(h) < T\}$$

(NSMP stands for 'no simple Markov partition').

A well-known argument, probably dating back to Thurston, shows:

Proposition 2.1 ([SmWe3])**.** *For a fixed $T > 0$ and $p \in \mathbb{N}$, the number of G-orbits in* $\mathrm{NSMP}(p, T)$ *is finite.*

Associated with a parabolic element $g = D\varphi \in \Gamma_M$ is a cylinder decomposition of the surface, see [Ve1, Vo]. That is, M is decomposed into finitely many cylinders C_i, whose waist direction is fixed by g, such that the inverse moduli μ_i of C_i are linearly dependent over $\mathbb{Q}$. The action of φ on M amounts to n_i Dehn twists around the cylinder C_i, where $n_i \mu_i$ is a constant independent of i. Using this it is not hard to find, in any stratum, countably many M in different G-orbits such that Γ_M contains a parabolic.

The situation is different if $\Gamma = \Gamma_M$ is assumed to be *non-elementary*, i.e., Γ does not contain an abelian subgroup of finite index. Maximal subgroups P of Γ consisting of parabolics are in one to one correspondence with cusps in the quotient orbifold $\mathbb{H}/\Gamma$, where a cusp is a union of a continuous family of parallel closed horocycles. We can associate to a pair (Γ, P) the least upper bound of hyperbolic areas of a cusp associated to P, which we denote by $t_0(\Gamma, P)$. Also associated to P is the number $m(M, P)$ of cylinders in the decomposition of M corresponding to a generator of P. We let

$$\mathrm{NLC}(m,T) = \{(M,P) : t_0(\Gamma_M, P) \leq T,\ m(M,P) = m\}$$

(NLC stands for 'no large cusp'). Then we have:

Theorem 2.2 ([SmWe3]). *For any $T > 0$ and any $m \in \mathbb{N}$, the number of G-orbits in $\mathrm{NLC}(m,T)$ is finite.*

A flat surface M is equipped with a finite set of distinguished points $\Sigma = \Sigma_M \neq \varnothing$ (which includes all singularities and may or may not include 'marked points'). A *triangle* in M is the image under an affine map of a triangle in $\mathbb{R}^2$, such that map is an embedding on the interior of the triangle and takes the vertices of the triangle to points in Σ (in particular the vertices of the triangle need not be distinct). We will say that M has *no small triangles* if there is a uniform lower bound on the area of any triangle in M. In other words, let $\mathcal{T}(M)$ be the collection of areas of all triangles in M, and for $\alpha > 0$, write

$$\mathrm{NST}(\alpha) = \{M : \inf \mathcal{T}(M) \geq \alpha\}.$$

Then M has no small triangles if $M \in \mathrm{NST}(\alpha)$ for some $\alpha > 0$. Since elements of g preserve area and map triangles to triangles, we have $\mathcal{T}(M) = \mathcal{T}(gM)$. In particular $\mathrm{NST}(\alpha)$ is G-invariant.

Theorem 2.3 ([SmWe1]). *For any $\alpha > 0$, the number of G-orbits in $\mathrm{NST}(\alpha)$ is finite.*

The no small triangles property was introduced by Vorobets [Vo]. Answering a question of Vorobets we prove:

Theorem 2.4 ([SmWe1]). *M has no small triangles if and only if M is a lattice surface (i.e. Γ_M is a lattice in G).*

Thus our finiteness results give a way to list the following sets (up to the G-action):

(i) Pairs (M, h) where $h \in \Gamma_M$ is hyperbolic.
(ii) Pairs (M, P) where Γ_M is non-elementary and $P \subset \Gamma_M$ is maximal parabolic.
(iii) Lattice surfaces.

Moreover these lists are organized in order of increasing complexity, with respect to some natural geometric measures of complexity. In addition, the proofs of these results describe the finite sets in (i), (ii), (iii) in terms of elementary combinatorial quantities such as certain permutations or certain integer matrices. This makes it possible to obtain explicit bounds on the finite numbers in these results, and raises the possibility of conducting effective computer searches to investigate the three lists above.

3. The Veech dichotomy and characterizations of the lattice property

Combining Theorem 2.4 with previous results of Vorobets yields several characterizations of the lattice property. These involve dynamics of both the directional foliations on M, and the G-action on the stratum $\mathcal{M}$ containing M.

Let $\mathcal{L} = \mathcal{L}_M$ denote the set of all saddle connections on M, and for a direction $\theta \in [0, 2\pi)$ let $\mathcal{L}(\theta) = \mathcal{L}_M(\theta)$ denote the set of saddle connections in direction θ. We say that θ is a *periodic direction* if each connected component of $M \smallsetminus \mathcal{L}(\theta)$ is a cylinder (equivalently, all noncritical leaves of the foliation of M in direction θ are are closed). Veech [Ve1] showed that a lattice surface satisfies the following *Veech dichotomy*:

I. If $\mathcal{L}(\theta) \neq \varnothing$ then θ is a periodic direction.
II. If $\mathcal{L}(\theta) = \varnothing$ then $\mathcal{F}_\theta$ is uniquely ergodic.

Namely, say that M is *uniformly completely periodic* if there is $s > 0$ such that each θ for which $\mathcal{L}(\theta) \neq \varnothing$ is a periodic direction, and the ratio of lengths of any two segments in $\mathcal{L}(\theta)$ does not exceed s.

A periodic direction θ is called *parabolic* if the moduli of all the cylinders are commensurable, and M is called *uniformly completely parabolic* if it is uniformly completely periodic, with all periodic directions parabolic. The restriction of the action of G to the one-parameter diagonal subgroup $\{g_t\}$ is called the *geodesic flow.*

Theorem 3.1 ([Vo, SmWe1]). *The following are equivalent for a flat surface M:*

(i) *M is a lattice surface.*
(ii) *M is uniformly completely periodic.*
(iii) *M is uniformly completely parabolic.*
(iv) *The set of triangles for M consists of finitely many* $\mathrm{Aff}(M)$*-orbits.*
(v) *M has no small triangles.*
(vi) *The G-orbit of M is closed in the stratum $\mathcal{M}$ containing M.*
(vii) *There is a compact $K \subset \mathcal{M}$ such that for any $g \in G$, the geodesic orbit of gM intersects K.*

Additional characterizations of the lattice property, as well as properties of the billiard in a polygon which gives rise to a lattice surface, are obtained in [SmWe1].

A stronger form of Veech's dichotomy was established in [Vo]. Let us say that a surface M satisfies the *V-dichotomy* if it satisfies II above, in addition:

I'. M is uniformly completely periodic.

Combining Theorem 3.1(ii) with Veech's theorem one sees that the lattice property is equivalent to the V-dichotomy. In contrast, Veech's dichotomy is not equivalent to the lattice property:

Theorem 3.2 ([SmWe2]). *There is a surface M which satisfies the Veech dichotomy but is not a lattice surface.*

4. Restrictions on Veech groups

Recall that two Fuchsian groups Γ_1, Γ_2 are *commensurable* if $\Gamma_1 \cap \Gamma_2$ is of finite index in both. For a group Γ and $h \in \Gamma$ we write h^Γ for the conjugacy class of h in Γ. Our results easily imply:

Corollary 4.1 ([SmWe3]). *Suppose Γ is commensurable to Γ_M for some M. Then:*

a. *For any $T > 0$,*

$$\#\{P \subset \Gamma : P \text{ is a maximal parabolic, } t_0(\Gamma, P) \le T\} < \infty.$$

b. $\#\{h^\Gamma : h \in \Gamma \text{ is hyperbolic with } \lambda(h) < T\} < \infty$.

c. Γ *is of finite index in its normalizer.*

Given a Fuchsian group, it is natural to ask 'how often' it arises as a Veech group. Using torus covers and increasing the genus of the surface, one can construct infinitely many lattice surfaces which are not affinely equivalent. On the other hand for a fixed stratum the following holds:

Corollary 4.2. *For any stratum $\mathcal{M}$ and any non-elementary Γ, there are only finitely many $M \in \mathcal{M}$ for which $\Gamma = \Gamma_M$.*

5. Closed orbits have finite volume

Note that a closed G-orbit of a flat surface is naturally identified with the quotient G/Γ_M and thus carries a well-defined volume element. A key ingredient in the proof of Theorems 2.4 and 3.1 is the following:

Proposition 5.1. *If the G-orbit of M is a closed subset of the stratum $\mathcal{M}$ containing M, then it has finite G-invariant measure, i.e. M is a lattice surface.*

The result was proved by both authors independently. Smillie's proof was described in [Ve2]. Here we describe the proof in [MiWe], which is modelled on an argument of Dani [Da] and gives a stronger conclusion. Recall that the horocycle flow $\{h_s\}$ is the restriction of the G-action to the one parameter subgroup of upper triangular unipotent matrices.

Theorem 5.2 ([MiWe], Theorem 2.6). *Suppose μ is a locally finite measure on a stratum $\mathcal{M}$ which is invariant under either $\{h_s\}$-action or the full G-action. Then there is a sequence of invariant sets Q_n with $\mu(Q_n) < \infty$ and $\mu\left(\mathcal{M} \smallsetminus \bigcup_n Q_n\right) = 0$. In particular if μ is ergodic then μ is finite.*

Proof. First we claim that it suffices to prove the theorem for the horocycle flow. To see this, suppose μ is a locally finite G-invariant measure on $\mathcal{M}$. An elementary calculation (see e.g. [Mar, §1]) shows that for any unitary representation of G, any vector which is fixed by $\{h_s\}$ is fixed by G. Considering the natural action of G on the Hilbert space $L^2(\mathcal{M}, \mu)$, and the vectors which are indicator functions of the sets Q_n, we obtain that these $\{h_s\}$-invariant sets are also G-invariant, so if the result holds for the horocycle flow, it also holds for the G-action.

So let μ be an $\{h_s\}$-invariant measure, which we assume to be locally finite, so there is a continuous positive function f on $\mathcal{M}$ such that $\int f \, d\mu = 1$. The methods of [KeMaSm] were sharpened in [MiWe], showing that for every $x \in \mathcal{M}$ there is a compact $K \subset \mathcal{M}$ such that

$$\liminf_{T \to \infty} \frac{1}{T} \left| \{ s \in [0, T] : h_s x \in K \} \right| \geq \frac{1}{2}. \tag{1}$$

By Birkhoff's ergodic theorem there is a measurable $\{h_s\}$-invariant function F such that for μ-a.e. x,

$$\frac{1}{T} \int_0^T f(h_s x) \, ds \to F(x). \tag{2}$$

Moreover $F \in L^1(\mathcal{M}, \mu)$. It follows from (1) and (2) that for any x for which the limit in (2) exists,

$$F(x) \geq \frac{1}{2} \min_{x \in K} f(x) > 0.$$

Thus the sets $Q_n = \{x \in \mathcal{M} : F(x) \geq 1/n\}$ have the required properties. $\square$

6. Combinatorial descriptions and finiteness

The strategy for proving Proposition 2.1 and Theorems 2.2, 2.3 is the same: we show that the flat surfaces which are being counted have a certain geometric structure which admits a combinatorial description, and that the combinatorial description determines the G-orbit of the flat surface. In other words we construct an injective map from the G-orbits of flat surfaces under consideration to a certain set of combinatorial data. The finiteness of these sets of combinatorial data then implies the result, and moreover gives explicit bounds on the finite numbers involved. To illustrate this we now give a

Sketch of proof of Theorem 2.2. Associated with a pair (M, P), where M is a flat surface and P is a maximal parabolic subgroup of Γ_M, are two geometric structures: a cusp on $\mathbb{H}/\Gamma_M$, and a pair of transverse cylinder decompositions of M. A cusp is a continuous family of parallel closed horocycles, which are injectively embedded in $\mathbb{H}/\Gamma_M$. It follows from the fact that Γ_M is non-elementary that the cusp has finite area, which means that if we move a closed horocycle 'down the cusp', at some point it will develop a self-tangency and cease being injectively embedded. Considering the element of Γ_M which causes the horocycle to close up, as well the element responsible for the self-tangency, one finds that after applying an appropriate element of G, Γ_M contains the elements

$$\gamma_1 = \begin{pmatrix} 1 & t_0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ -t_0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

Note that $\gamma_2 = \tau\gamma_1\tau^{-1}$.

Corresponding to any parabolic element γ of Γ_M is a decomposition of M into cylinders $C_1, \ldots, C_m$ whose waist curves are parallel to the direction fixed by γ, and (possibly after replacing γ with a power) such that the action of the corresponding affine automorphism of M is a multitwist, i.e. induces n_i Dehn twists around C_i. Moreover, since conjugation by $\tau = D\varphi$ maps γ_1 to γ_2, the cylinders in the decomposition corresponding to γ_2 are $C'_i = \varphi(C_i)$; thus M has two transverse cylinder decompositions with the same number of cylinders, such that the height of C_i is the circumference of C'_i and vice versa.

Now suppose $(M, P) \in \mathrm{NLC}(m, T)$. We define some combinatorial quantities describing the cusp and cylinder decompositions above, and show how to bound these in terms of m and T. First it is easy to compute that the number t_0 in (3) is the hyperbolic area of the cusp, so that $t_0 \leq T$. The inverse modulus μ of a cylinder is the ratio of its width to its height; for the parabolic γ_1 of (3) the inverse moduli satisfy

$$t_0 = n_i \mu_i. \tag{4}$$

Consider the rectangles which are the connected components of $C_i \cap C'_j$. These glue together to form the surface; specifying which rectangle glues to which leads to another combinatorial structure which we call a *gluing pattern*, and is described in detail in [SmWe1]. Although one can make a more precise statement, for the purpose of this summary it suffices to note that the number of gluing patterns for a fixed number of rectangles is finite. Let $A = (a_{ij})$ be the $m \times m$ matrix such that a_{ij} is the number of rectangles in $C_i \cap C'_j$ and let B be the diagonal matrix with n_i's on the diagonal. One can easily show that BA is a positive matrix (in the sense of the Perron-Frobenius theorem). Consider now the vector $\vec{a'}$ of heights of cylinders C'_j. They add up to form the circumferences w_i of the C_i, so that $\vec{w} = A\vec{a'}$. Multiplying by B and using (4) we find that $t_0\vec{a'} = t_0\vec{a} = BA\vec{a}'$, so that $\vec{a'}$ is a positive eigenvector of BA. By Perron-Frobenius, t_0 is the maximal eigenvalue of BA.

There are only finitely many positive $m \times m$ integer matrices with a fixed upper bound on their maximal eigenvalue. Thus the numbers m, T lead to explicit finite bounds on the possible matrices B and A, on the total number of rectangles $\sum a_{ij}$, and hence on the number of gluing patterns. The vector $\vec{a}$, being the positive eigenvector of a positive matrix, is uniquely determined by the matrix BA, and since $\vec{w} = A\vec{a}$, the dimensions of all rectangles are specified by the matrices B and A. The dimensions of the rectangles and the gluing pattern specify a unique flat surface structure, and this completes the proof. $\square$

The proofs of Proposition 2.1 and Theorem 2.3 employ similar considerations. The relevant geometric patterns are respectively Markov partitions and pairs of transverse cylinder decompositions. Again the given bounds in the definitions of $\mathrm{NSMP}(p, T)$ and $\mathrm{NST}(\alpha)$ bound the number of possibilities for the combinatorial data; and again in both cases the uniqueness of a positive eigenvector in the Perron-Frobenius theorem is used to show that the combinatorial data determines the flat surface uniquely.

7. A closed locus consisting of isolated orbits

The principle that the combinatorial description determines the flat surface uniquely, up to the G-action, implies more than finiteness. It also shows that different orbits with bounds on their combinatorics cannot accumulate on each other, and furthermore that a given orbit cannot accumulate on itself in a complicated manner. Namely:

Theorem 7.1 (Isolation). *Suppose $\mathcal{M}$ is a stratum of flat surfaces, and $M_n \to M_0$ is a convergent sequence in $\mathcal{M}$ such that one of the following holds for $n = 0, 1, 2, \ldots$ and $\Gamma_n = \Gamma_{M_n}$:*

(i) *There are* p, T *and* $h_n \in \Gamma_n$ *such that* $(M_n, h_n) \in \mathrm{NSMP}(p, T)$ *and* $h_n \to h_0$.
(ii) *There are* m, T *and maximal parabolic subgroups* $P_n \subset \Gamma_n$*, with generators* p_n*, such that* $(M_n, P_n) \in \mathrm{NLC}(m, T)$ *and* $p_n \to p_0$.
(iii) *There is* $\alpha > 0$ *such that* $M_n \in \mathrm{NST}(\alpha)$.

Then for all sufficiently large n*, there are* $g_n \in G$ *such that* $g_n \to e$ *and* $M_n = g_n M_0$*. Moreover in case (i),* $h_n = g_n h_0 g_n^{-1}$*, and in case (ii),* $p_n = g_n p_0 g_n^{-1}$.

As an application we now prove:

Proof of direction $\implies$ *in Theorem 2.4.* We need to show that the no small triangle property implies the lattice property. It is clear from the topology on $\mathcal{M}$ that $\mathrm{NST}(\alpha)$ is a closed subset of $\mathcal{M}$, and using Theorem 7.1(iii) this implies that each individual orbit in $\mathrm{NST}(\alpha)$ is closed. Using Proposition 5.1 we obtain that any $M \in \mathrm{NST}(\alpha)$ is a lattice surface. □

This answers a question of Vorobets, who introduced the no small triangles property in [Vo] and showed the implication $\Longleftarrow$.

8. Marked points and branched covers

Two natural operations which one can perform on a flat surface are marking points and constructing a branched cover. Recall that a flat structure M is equipped with a finite set of points on the underlying surface called the *singularities.* A small neighborhood of a singularity x is metrically isomorphic to $r = r_x$ half-planes glued together cyclically around x. Thus a parallel line field on M acquires at x an r-pronged singularity. Most authors only permit for r the values $1, 3, 4, 5, \ldots,$ since if $r = 2$ the flat surface structure can be completed to remove the singularity. However for various applications it is convenient to allow singularities x for which $r_x = 2$; to distinguish these removable singularities from non-removable ones, they are often called *marked points.*

Marking a point does not affect the foliations on M or their dynamical properties. However it can have a dramatic effect on $\mathrm{Aff}(M)$ and Γ_M, which by definition are required to map singularities to singularities. Thus marking a point on M we create a new flat surface M' which has identical geometry but for which $\Gamma_{M'}$ could be significantly smaller than Γ_M.

For example, in [HuSc2] Hubert and Schmidt described so-called *aperiodic connection points* on a genus 2 lattice surface M_0 such that if M' denotes the surface obtained from M_0 by marking such points, then $\Gamma_{M'}$ is of infinite index in Γ_{M_0}, but is infinitely generated.

One can also consider the effect of marking points on a stratum $\mathcal{M}$ of flat surfaces (without removable singularities). Recall that $\mathcal{M}$ is equipped with the structure of an orbifold, carrying a smooth finite G-invariant measure. Denote by $\mathcal{M}_b$ the set of all flat surfaces M' obtained by adding b marked points to flat surfaces $M \in \mathcal{M}$. There is a natural map $\Phi : \mathcal{M}_b \to \mathcal{M}$ which 'forgets marked points'. Then one has:

Proposition 8.1. *There is an orbifold structure on $\mathcal{M}_b$ such that $\Phi : \mathcal{M}_b \to \mathcal{M}$ is a G-equivariant orbifold cover. If $b = 1$ then Φ is a proper map whose generic fiber can be identified with S.*

This statement is well-known to experts but hard to locate in the literature. To remedy this we have provided a detailed exposition in [SmWe2, Appendix].

A branched cover is a surjective continuous map $\pi : \widetilde{S} \to S$ of connected compact surfaces, such there are finite set $\widetilde{\Sigma} \subset \widetilde{S}$, $\Sigma \subset S$ with $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$ such that the restriction of π to $\widetilde{S} \smallsetminus \widetilde{\Sigma}$ is a covering map. Points in Σ are called *branch points.* Given a branched cover and a flat surface structure M on the topological surface S, one can form a flat structure on $\widetilde{M}$ on $\widetilde{S}$, called the *pullback* of M. On $\widetilde{S} \smallsetminus \widetilde{\Sigma}$ the flat structure is pulled back via the covering π, and $\widetilde{\Sigma}$ becomes a subset of the set of singularities of $\widetilde{M}$. Using the branched cover construction, the following was proved:

Proposition 8.2 (Hubert-Schmidt)**.** *Let M_0 be as in the previous discussion. If $\widetilde{M}$ is the pullback of M_0 via a branched cover π branched over aperiodic connection points, then $\widetilde{M}$ is not a lattice surface, has an infinitely generated Veech group, and satisfies property I of the Veech dichotomy.*

Given strata $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ of flat surface structures, on the topological surfaces S, $\widetilde{S}$ respectively, and a branched cover $\pi : \widetilde{S} \to S$ with b branch points, one can consider the subset $\Omega(\pi)$ of $\widetilde{\mathcal{M}}$ consisting of all $\widetilde{M}$ which arise by pullback via a branched cover θ topologically equivalent to π (i.e. there are self-homeomorphisms $\widetilde{h}$, h such that $\pi \circ h = \widetilde{h} \circ \theta$). An imprecise folklore theorem asserts that up to finite ambiguity, $\widetilde{M} \in \Omega(\pi)$ is uniquely determined by M, π and the choice of branch points. Since $\mathcal{M}_b$ is equipped with the structure of an orbifold via Proposition 8.1, one can use this to equip $\Omega(\pi)$ with an orbifold structure as well:

Proposition 8.3. *There is an orbifold $\widetilde{\Omega}(\pi)$ equipped with a G-action, and finite-to-one G-equivariant surjective orbifold maps $R : \widetilde{\Omega}(\pi) \to \Omega(\pi) \subset \widetilde{\mathcal{M}}$ and $P : \widetilde{\Omega}(\pi) \to \mathcal{M}_b$, such that for $x \in \widetilde{\Omega}(\pi)$, $R(x)$ is a pullback of $\Phi \circ P(x)$ via a branched cover topologically equivalent to π and branched at the marked points of $P(x)$.*

$$\begin{tikzcd}
\widetilde{\Omega}(\pi) \arrow[d, "P"'] \arrow[dr, "R"] & \\
\mathcal{M}_b \arrow[d, "\Phi"'] & \Omega(\pi) \subset \widetilde{\mathcal{M}} \\
\mathcal{M} &
\end{tikzcd}$$

A detailed exposition of this result is also provided in [SmWe2, Appendix].

9. A non-lattice surface with the Veech dichotomy

The simplest example of a flat surface which is not a lattice surface, but satisfies the Veech dichotomy, is obtained by marking an aperiodic connection point on the surface M_0 described by Hubert and Schmidt. As shown in [HuSc2], after marking this point, the lattice property is destroyed, but property I of the Veech dichotomy continues to hold. Clearly property II depends only on the flat surface and cannot be destroyed by marking a point, so the resulting surface satisfies the conclusion of Theorem 3.2. Note however that the example has a removable singularity, so may be considered 'artificial'. Hubert and Schmidt ask [HuSc3] whether non-lattice surfaces, without removable singularities, can satisfy the Veech dichotomy. We answer this question affirmatively by transforming the marked point into a non-removable singularity using a branched cover construction. The main point is doing so without destroying property II. It will develop below that property II survives in the pulled back flat surface if the branch locus consists of a single point.

Proof of Theorem 3.2. We will show that the flat surface $\widetilde{M}$ of Proposition 8.2 satisfies the Veech dichotomy, as long as π has a single branch point. Property I is contained in Proposition 8.2, so suppose θ is such that $\mathcal{L}(\theta) = \varnothing$ and suppose by contradiction that θ is not a uniquely ergodic direction for $\widetilde{M}$. Let $r_\theta \in G$ be the rotation matrix which moves the direction θ to the vertical direction. By a well-known result of Masur [Mas], the forward trajectory of $r_\theta \widetilde{M}$ under the geodesic flow is divergent, i.e. eventually leaves every compact subset of $\widetilde{\mathcal{M}}$. Let $x \in \widetilde{\Omega}(\pi)$ such that $R(x) = \widetilde{M}$ and $\Phi \circ P(x) = M_0$. Since R is G-equivariant, the forward geodesic trajectory of $r_\theta x$ is divergent in $\widetilde{\Omega}(\pi)$, and since, by Propositions 8.1 and 8.3, $\Phi \circ P$ is proper, the geodesic trajectory of $r_\theta M_0$ is divergent. Since M_0 is a lattice surface, θ is a parabolic direction for M_0, and hence is periodic for $\widetilde{M}$. This implies that $\mathcal{L}_{\widetilde{M}}(\theta) \neq \varnothing$, a contradiction. □

10. A problem list

10.1. **From effective bounds to effective searches.** As we have seen, Proposition 2.1 and Theorems 2.2, 2.3 yield a list of certain flat surfaces with large Veech groups, in order of increasing complexity. For example, for a lattice surface M one could define

$$\alpha(M) = \inf \mathcal{T}(M),$$

and list all lattice surfaces as $M_1, M_2, \ldots$, where $\alpha(M_1) \geq \alpha(M_2) \geq \cdots$. The first few elements of this list were found in [SmWe1]. Thus $\mathrm{NST}(\frac{1}{6})$ consists of seven arithmetic surfaces.

Question 1. *What is the first non-arithmetic example?*

We have also seen that the proofs of the finiteness theorems above yield effective bounds on the size of these sets. Moreover they set up a correspondence between the sets of G-orbits of flat surfaces and explicit combinatorial data. It seems likely that this correspondence can be used to conduct effective computer searches. Thus for example:

Question 2. *Program a computer to produce, for a given $\alpha > 0$, all lattice surfaces in* $\mathrm{NST}(\alpha)$.

10.2. **From triangles to other configurations.** The discussion in [SmWe1] focuses on the 'no small triangles' property introduced by Vorobets. One could extend the discussion by defining, for any n, the set of all flat surfaces with a uniform lower bound on the area of all their n-gons. For a fixed bound α let us denote this set by $\mathrm{NSnG}(\alpha)$. One could add additional restrictions on the configurations defining $\mathrm{NSnG}(\alpha)$, e.g. require them to be convex, require them to exist in the set $\mathrm{hol}(M)$ of holonomy vectors of saddle connections in M, etc.

Question 3. *Given $n \geq 4$ and α, describe all flat surfaces in* $\mathrm{NSnG}(\alpha)$.

It is interesting to note that for certain configurations, non-lattice examples can arise. For example, a surface constructed from gluing tori along a slit (see [MaTa]) has a lower bound on the areas of convex pentagons, although it need not be a lattice surface. This can be seen by noting that in this case

$$\mathrm{hol}(M) = \mathcal{P} \oplus \left(\mathbb{Z}^2 + v\right),$$

where $\mathcal{P}$ is the set of primitive vectors in $\mathbb{Z}^2$ and $v \in \mathbb{R}^2$ is the holonomy vector of the slit. Any pentagon has at least 3 vertices in either $\mathcal{P}$ or $\mathbb{Z}^2 + v$ and this leads to a lower bound on its area.

One reason why Question 3 might be interesting regards the dynamics of the G-action on a stratum. There has been a lot of research on understanding closed G-invariant subsets of a stratum $\mathcal{M}$, and a nice feature of $\mathrm{NST}(\alpha)$ is that its intersection with $\mathcal{M}$ is closed and G-invariant; moreover the same holds for $\mathrm{NSnG}(\alpha)$. There are many questions one could ask about the sets $\mathrm{NSnG}(\alpha)$ but so far not many answers. Here is a special case of Question 3:

Question 4. *Are there $\alpha_1 \neq \alpha_2$, a stratum $\mathcal{M}$, and n such that*

$$\dim\left(\mathcal{M} \cap \mathrm{NSnG}(\alpha_1)\right) \neq \dim\left(\mathcal{M} \cap \mathrm{NSnG}(\alpha_2)\right)?$$

Here dim *denotes Hausdorff dimension.*

10.3. **Variants of the Veech dichotomy.** As we have seen, the V-dichotomy, in which property I is replaced with a uniform version I', is equivalent to the lattice property. In fact I' by itself is equivalent to the lattice property. There are various refinements of these properties one could consider. This requires two more definitions. We say that M is *completely parabolic* if any direction containing a saddle connection is parabolic. We say that M is *completely parabolic with a uniform bound on twists* if there is $T > 0$ such that for any direction containing a saddle connection, M has a cylinder decomposition in that direction with commensurable moduli, such that the corresponding number of Dehn twists around each cylinder is bounded above by T. It was shown by Veech [Ve1] that a lattice surface is completely parabolic, and by Vorobets [Vo] that it is completely parabolic with a uniform bound on twists.

Question 5. *Do any of the following conditions imply that M is a lattice surface?*

- *M is completely parabolic.*
- *M is completely parabolic and any non-saddle direction for M is uniquely ergodic.*
- *M is completely parabolic with a uniform bound on twists.*
- *M is completely parabolic with a uniform bound on twists, and any non-saddle direction for M is uniquely ergodic.*

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