# Tremors and Horocycle Dynamics on the Moduli Space of Translation Surfaces

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Abstract. We introduce a ‘tremor’ deformation on strata of translation surfaces. Using it, we give new examples of behaviors of horocycle flow orbits $Uq$ in strata of translation surfaces. In the genus 2 stratum $H_{1,1}$ we find orbits $Uq$ which are generic for a measure whose support is strictly contained in $Uq$ and find orbits which are not generic for any measure. We also describe a horocycle orbit-closure whose Hausdorff dimension is not an integer.

1. Introduction

A surprisingly fruitful technique for studying mathematical objects is to study dynamics on their moduli spaces. Examples of this phenomenon occur in the study of integral values of indefinite quadratic forms (motivating the study of dynamics of Lie group actions on homogeneous spaces) and billiard flows on polygonal tables (motivating the study of the SL$_2(\mathbb{R})$-action on the moduli space of translation surfaces). In both cases, far-reaching results regarding the actions on the moduli spaces have been used to shed light on a wide range of problems in number theory, geometry, and ergodic theory. See [Zo, Wr2, KSS] for surveys of these developments.

Let $B \subset$ SL$_2(\mathbb{R})$ be the subgroup of upper triangular matrices, and let

$$U \overset{\text{def}}{=} \{u_s : s \in \mathbb{R}\}, \text{ where } u_s \overset{\text{def}}{=} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}. \quad (1.1)$$

The $U$-action is an example of a unipotent flow and, in the case of strata of translation surfaces, is also known as the horocycle flow. The actions of these groups on moduli spaces are fundamental in both dynamical settings. For homogeneous spaces of Lie groups, actions of subgroups such as SL$_2(\mathbb{R})$, $B$ and $U$ are strongly constrained and much is known about invariant measures and orbit-closures. For the action on a stratum $\mathcal{H}$ of translation surfaces, fundamental papers of McMullen,
Eskin, Mirzakhani and Mohammadi [McM1, EM, EMM] have shown that the invariant measures and orbit closures for the SL$_2(\mathbb{R})$-action and $B$-action on $\mathcal{H}$ are severely restricted and have remarkable geometric features; in particular orbit-closures are the image of a manifold under an immersion.

In this paper we examine the degree to which such regular behavior might hold for the $U$-action or horocycle flow on strata. We give examples showing that, with respect to orbit-closures and the asymptotic behavior of individual orbits, the $U$-action on $\mathcal{H}$ has features which are absent in homogeneous dynamics.

In order to set the stage for this comparison we first recall some results about the dynamics of unipotent flows on homogeneous spaces. Special cases of these results were proved by several authors and in complete generality the results were proved in celebrated work of Ratner (see [M] for a survey, and for the definitions used in the statement below).

**Theorem 1.1** (Ratner). Let $G$ be a connected Lie group, $\Gamma$ a lattice in $G$, $X = G/\Gamma$, and $U = \{u_s : s \in \mathbb{R}\}$ a one-parameter Ad-unipotent subgroup of $G$.

1. For any $x \in X$, $\overline{Ux} = Hx$ is the orbit of a group $H$ satisfying $U \subset H \subset G$, and $Hx$ is the support of an $H$-invariant probability measure $\mu_x$.
2. Any $x \in X$ is generic for $\mu_x$, i.e.
\[
\forall f \in C_c(X), \quad \lim_{T \to \infty} \frac{1}{T} \int_0^T f(u_s x)ds = \int_X f d\mu_x.
\]

Statement (1) is known as the orbit-closure theorem, and statement (2) is known as the genericity theorem.

1.1. **Main results.** We will introduce a method for constructing $U$-orbits with unexpected properties, and apply it in the genus two stratum $\mathcal{H}(1,1)$. Let $\mathcal{E}_4 \subset \mathcal{H}_1(1,1)$ denote the set of unit-area surfaces which can be presented as two identical tori glued along a slit (in the notation and terminology of McMullen [McM1], $\mathcal{E}_4$ is the subset of area-one surfaces in the eigenform locus of discriminant $D = 4$).

From now on we write $G \overset{\text{def}}{=} \text{SL}_2(\mathbb{R})$ and $\mathcal{E} \overset{\text{def}}{=} \mathcal{E}_4$. The locus $\mathcal{E}$ is 5-dimensional, is $G$-invariant, and is the support of a $G$-invariant ergodic probability measure $\mu_\mathcal{E}$.

**Theorem 1.2.** There is $q \in \mathcal{H}(1,1)$ which is not contained in $\mathcal{E}$ but which is generic for the measure $\mu_\mathcal{E}$ supported on $\mathcal{E}$. 
Since $E = \text{supp } \mu_E$ is strictly contained in $Uq$, this orbit does not satisfy the analogue of Theorem 1.1.

In the homogeneous setting, orbit-closures of unipotent flows are manifolds. It was known (see [SW]) that horocycle orbit-closures could be manifolds with boundary in the setting of translation surfaces. We show here that they can we considerably wilder.

**Theorem 1.3.** There is $q \in \mathcal{H}(1,1)$ for which the orbit-closure $Uq$ has non-integer Hausdorff dimension. In fact, by appropriately varying the initial surface $q$, we can construct an uncountable nested chain of distinct horocycle orbit-closures of fractional Hausdorff dimension.

We will give additional information about these orbit-closures in Theorems 1.8 and 1.9 below. The next result shows that the analogue of Ratner’s genericity theorem fails dramatically in $H(1,1)$:

**Theorem 1.4.** There is a dense $G_\delta$ subset of $q \in \mathcal{H}(1,1)$ and $f \in C_c(\mathcal{H}(1,1))$ so that

$$
\liminf_{T \to \infty} \frac{1}{T} \int_0^T f(u_s q) ds < \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(u_s q) ds.
$$

In particular such $q$ are not generic for any measure on $\mathcal{H}(1,1)$, and there are such $q$ whose geodesic orbit (i.e., $\{g_t q : t \in \mathbb{R}\}$ in the notation (2.4)) is dense.

One property of unipotent flows on homogeneous spaces which was crucially used in Ratner’s work is ‘controlled divergence of nearby trajectories’. The proof of Theorem 1.2 shows that in strata, divergence of nearby trajectories can be erratic. We make this precise in §8.3, see Theorem 8.6.

The proofs of Theorems 1.2, 1.3, and 1.4 rely on the tremor paths which we now introduce (the geological nomenclature is inspired by Thurston’s earthquake paths, see [T2]).

1.2. **Tremors.** We can describe the action of the horocycle flow on a translation surface geometrically as giving us a family of surfaces obtained by changing the flat structure on the original surface by shearing it horizontally. An interesting modification of this procedure was studied by Alex Wright [Wr1]. Let $q \in H$, let $M_q$ be the corresponding surface, and suppose $M_q$ contains a horizontal cylinder $C$. Then one can deform $M_q$ by shearing the flat structure on $C$ and leaving $M_q \setminus C$ unchanged. This cylinder shear operation defines a flow on the subset of the stratum consisting of surfaces containing a horizontal cylinder. This subset of $H$ is invariant under the horocycle flow and on it, the
flow defined by the cylinder shear commutes with the horocycle flow. The tremors we study in this paper also commute with the horocycle flow, are also defined on a subset of $\mathcal{H}$, and are a common generalization of cylinder shears and the horocycle flow. While Wright’s analysis of cylinder shears focused on shears that leave surfaces inside a $G$-invariant locus, we will study tremors that move a surface $q$ in a $G$-invariant locus away from that locus, and use these tremors to exhibit new behaviors of the horocycle flow.

We can think of both the cylinder shear and the horocycle flow as arising from transverse invariant measures to the horizontal foliation $\mathcal{F}_q$ on the surface $M_q$, where the amount and location of shearing is determined by the transverse measure. If the cylinder shear flow takes $q$ to $q'$ then the relationship between their period coordinates (see §2.1, where we will explain the notation and make our discussion more precise) is given by

$$\text{hol}_{q'}(\gamma) = \text{hol}_q(\gamma) + t \cdot \tau(\gamma), \quad \text{hol}_{q'}(\tau) = \text{hol}_q(\tau).$$

(1.3)

Here $\text{hol}_q$ and $\text{hol}_q(\gamma)$ denote the cohomology classes corresponding to the transverse measures $dx$ and $dy$ on $M_q$ respectively, $\gamma$ is an oriented closed curve or path joining singularities on $M_q$, $t$ is the parameter for the cylinder shear flow, and $\tau$ is the cohomology class corresponding to the transverse measure which is the restriction of $dy$ to the cylinder.

The horocycle flow is given in period coordinates as

$$\text{hol}_{u_q}(\gamma) = \text{hol}_q(\gamma) + s \cdot \text{hol}_q(\gamma), \quad \text{hol}_{u_q}(\tau) = \text{hol}_q(\tau).$$

(1.4)

Recalling that $\text{hol}_q(\gamma)$ is the cohomology class corresponding to the transverse measure $dy$, and recalling that some surfaces may have additional transverse measures to the horizontal foliation $\mathcal{F}_q$, we define a surface $q'$ via the formula

$$\text{hol}_{q'}(\gamma) = \text{hol}_{u_q}(\gamma) + t \cdot \beta(\gamma), \quad \text{hol}_{q'}(\beta) = \text{hol}_q(\beta).$$

(1.5)

where $\beta$ is the cohomology class associated with a transverse measure on $M_q$. In §2 and §4, we will make this more precise and explain why this definition makes sense and why $q'$ is uniquely determined by $q, t,$ and $\beta$. We will write

$$\text{trem}_{q, \beta}(t) \overset{\text{def}}{=} q' \text{ or } q_\beta(t) \overset{\text{def}}{=} q'$$

(depending on the context). We will also write

$$\text{trem}_{q, \beta} \overset{\text{def}}{=} \text{trem}_{\beta}(q) \overset{\text{def}}{=} \text{trem}_{q, \beta}(1)$$

and refer to any surface of the form $\text{trem}_{q, \beta}(t)$ as a tremor of $q$. 

We now give some additional definitions needed for stating our results. If the transverse measure corresponding to \( \beta \) is absolutely continuous with respect to \( dy \) (see §4.1.3) we call \( \beta \) and the tremor \( \text{trem}_{q, \beta} \) absolutely continuous. If \( q \) has no horizontal saddle connections and the transverse measure is not a scalar multiple of \( dy \), we say they are essential. We will denote the subspace of cohomology corresponding to signed transverse measures at \( q \) by \( T_q \). This can be related to the tangent space to the stratum, see §2.3 and §4.1.1. If the transverse measure is non-atomic, i.e., assigns zero measure to all horizontal saddle connections, then the tremor path can be continued for all time, see Proposition 4.8. The case of atomic transverse measures presents some technical difficulties which will be discussed in §13.

1.3. More detailed results. The importance of tremor maps for the study of the horocycle flow is that, where they are defined, they commute with the horocycle flow, i.e., \( u_s \text{trem}_\beta(q) = \text{trem}_\beta(u_s q) \) (for this to make sense we need to explain how we can consider the same cohomology class \( \beta \) as an element of both \( T_q \) and \( T_{u_s q} \), a topic we will discuss in §5). In particular we will see that for many tremors, the surfaces \( u_s q, u_s \text{trem}_\beta(q) \) stay close to each other, and this leads to the following:

**Theorem 1.5.** Let \( \mathcal{H} \) be any stratum, let \( \mathcal{H}_1 \) be its subset of area-one surfaces, and let \( \mathcal{L} \subset \mathcal{H}_1 \) be a closed \( U \)-invariant set which is the support of a \( U \)-invariant ergodic measure \( \mu \). Let \( q \in \mathcal{L} \), \( \beta \in T_q \) and \( q_1 = \text{trem}_{q, \beta} \). Then:

(i) If \( \beta \) is absolutely continuous then for the sup-norm distance \( \text{dist} \) on \( \mathcal{H} \) (see §2.6), we have

\[
\sup_{s \in \mathbb{R}} \text{dist}(u_s q, u_s q_1) < \infty.
\]

(ii) If \( \beta \) is absolutely continuous then any surface in \( \overline{Uq_1} \setminus \mathcal{L} \) has a horizontal foliation which is not uniquely ergodic. In particular, if \( \mathcal{L} \neq \mathcal{H}_1 \) then \( Uq_1 \) is not dense in \( \mathcal{H}_1 \).

(iii) If \( \mu \)-a.e. surface in \( \mathcal{L} \) has no horizontal saddle connection and if \( q \) is generic for \( \mu \), then \( q_1 \) is also generic for \( \mu \).

We will give examples of loci \( \mathcal{L} \) and surfaces \( q \) for which the hypotheses of Theorem 1.5 are satisfied, namely we will find \( \mathcal{L} \) and \( q \) for which:

(I) The locus \( \mathcal{L} \) is \( G \)-invariant and is the support of a \( G \)-invariant ergodic measure \( \mu \), and the orbit \( Uq \) is generic for \( \mu \).
(II) The surface $M_q$ has no horizontal saddle connections and the transverse measure corresponding to $dy$ on $M_q$ is not ergodic (and hence $q$ admits essential absolutely continuous tremors).

(III) There is an essential absolutely continuous tremor $q_1$ of $q$ which is not in $\mathcal{L}$.

There are many examples of strata $\mathcal{H}$ and loci $\mathcal{L}$ for which these properties hold. One particular example which we will study in detail is $L = E \subset \mathcal{H}_1(1,1)$ (see §3.1 for more information on $E$). Namely we will prove the following result which, in conjunction with Theorem 1.5, immediately implies Theorem 1.2.

**Theorem 1.6.** There are surfaces $q \in E$ satisfying (I), (II), (III) above. Moreover, for any surface $q \in E$ which admits an essential tremor $\beta \in T_q$, the surfaces

$$q_r \overset{\text{def}}{=} q_\beta(r) \in \mathcal{H}(1,1) \ (\text{where } r > 0)$$

satisfy

$$0 < r_1 < r_2 \implies \overline{Uq_{r_1}} \neq \overline{Uq_{r_2}}. \quad (1.7)$$

**Remark 1.7.** Theorem 1.6 is also true if $E$ is replaced with any of the other eigenform loci $E_D \subset \mathcal{H}(1,1)$. See §8.2 for more details.

For certain $q \in E$ and $\beta \in T_q$, we can give a complete description of the closure of $Uq_1$ where $q_1 = \text{trem}_{q,\beta}$. To state this result we will need a measurement of the size of a tremor and to do this we introduce the total variation $|L|_{q,\beta}$ of $\beta \in T_q$, see §4.1.2 for the definition. Also we say that $q \in E$ is aperiodic if the horizontal foliation is not periodic, i.e. it is either minimal or contains a slit separating the surface into two tori on which the horizontal foliation is minimal.

**Theorem 1.8.** For any $a > 0$ there is $q \in E$ and an essential tremor $q_1 \in \mathcal{H}(1,1)$ of $q$ such that

$$\overline{Uq_1} = \{ \text{trem}_{q,\beta} : q \in E \text{ is aperiodic, } \beta \in T_q, \ |L|_{q,\beta} \leq a \} \subset \{ \text{trem}_{q,\beta} : q \in E, \ \beta \in T_q, \ |L|_{q,\beta} \leq a \}. \quad (1.8)$$

Moreover, writing $q_1 = \text{trem}_{q,\beta}$ for $q \in E$, and setting $q_r \overset{\text{def}}{=} \text{trem}_{q,r,\beta}$, we have that the orbit-closure $\overline{Uq_r}$ admits the description in (1.8) with $a$ replaced by $ra$, and the $q_r$ satisfy the following strengthening of (1.7):

$$0 < r_1 < r_2 \implies \overline{Uq_{r_1}} \not\subset \overline{Uq_{r_2}}. \quad (1.9)$$

The following more explicit result implies Theorem 1.3.
Theorem 1.9. Let \( q_1 \in \mathcal{H}(1,1) \) be the surface described in Theorem 1.8. Then the Hausdorff dimension of the horocycle orbit closure of \( q_1 \) satisfies
\[
5.5 \leq \dim \overline{Uq_1} < 6.
\]

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2. Basics

This section introduces basic concepts and sets up notation. Much of this material is standard but we will need to be careful and detailed regarding orbifold issues on strata.

2.1. Strata and period coordinates. There are several possible approaches for defining the topology and geometric structure on strata, see [FM, MaTa, Wr2, Y, Zo]. For the most part we follow the approach of [BSW], where the reader can find additional details.

Let \( M \) be a compact oriented surface of genus \( g \) and let \( \Sigma \subset M \) be a non-empty finite set. We make the convention that the points of \( \Sigma \) are labeled. Let \( r = \{r_\sigma : \sigma \in \Sigma \} \) be a list of non-negative integers satisfying \( \sum r_\sigma = 2g - 2 \). A translation surface of type \( r \) is given by an atlas on \( M \) of orientation preserving charts \( \mathcal{A} = (\psi_\alpha, U_\alpha)_{\alpha \in \mathcal{A}} \), where the \( U_\alpha \subset M \setminus \Sigma \) are open and cover \( M \setminus \Sigma \), the transition maps \( \psi_\alpha \circ \psi_\beta^{-1} \) are restrictions of translations to the appropriate domains, and such that the planar structure in a neighborhood of each \( \sigma \in \Sigma \) completes to a cone angle singularity of total cone angle \( 2\pi(r_\sigma + 1) \). A translation equivalence between translation surfaces is a homeomorphism \( h \) which preserves the labels and the translation structure.

These charts determine a metric on \( M \) and a measure which we denote by Leb. These charts also allow us to define natural “coordinate” vector fields \( \partial_x \) and \( \partial_y \) and 1-forms \( dx \) and \( dy \) on \( M \). The (partially defined) flow corresponding to \( \partial_x \) will be called the horizontal straight-line flow, and we will denote the trajectory parallel to \( \partial_x \) starting at \( p \in M_q \) by \( t \mapsto \Upsilon^{(p)}(t) \). The corresponding foliation of \( M \setminus \Sigma \), which we denote by \( \mathcal{F} \), will be called the horizontal foliation. If we remove from \( M \) the horizontal trajectories that hit singular points, then the straightline flow becomes an actual flow defined on a dense \( G_\delta \) subset.
of full Lebesgue measure. If this flow is \textit{minimal}, i.e. all infinite horizontal straightline flow trajectories are dense, we will say that $\mathcal{F}$ is minimal or that $M$ is \textit{horizontally minimal}.

A \textit{saddle connection} on $M$ is a path with endpoints in $\Sigma$, which is a straight line in each planar chart, and does not contain singularities in its interior. A \textit{cylinder} on $q$ is the isometric image of a set of the form $\mathbb{R}/c\mathbb{Z} \times (0, h)$ for positive $c, h$, and which is maximal (not a proper subset of another such set). The number $c > 0$ is called the \textit{circumference} of the cylinder, $h$ is called the \textit{height}, and the image of a curve $\mathbb{R}/c\mathbb{Z} \times \{x\}$, $x \in (0, h)$, is called a \textit{core curve}. The maximality assumption implies that the boundary of a cylinder consists of one or two components made of saddle connections parallel to its core curves.

Fix $r, g$ and $k$ satisfying the relation $\sum r_\sigma = 2g - 2$. Choose a surface $S$ of genus $g$ and a labelled set $\Sigma \subset S$ of cardinality $k$ (note that we use the same symbol $\Sigma$ to denote finite subsets of $S$ and of $M$, labelled with the same set of labels; where confusion may arise we use the symbols $\Sigma_S, \Sigma_M$). We refer to $(S, \Sigma)$ as the \textit{model surface}. A \textit{marking map} of a translation surface $M$ is a homeomorphism $\varphi : (S, \Sigma) \to (M, \Sigma)$ which preserves labels on $\Sigma$. We say that two markings maps $\varphi : (S, \Sigma) \to (M, \Sigma_M)$ and $\varphi' : (S, \Sigma) \to (M', \Sigma_{M'})$ are equivalent if there is a translation equivalence $h : M \to M'$ so that $h \circ \varphi$ is isotopic to $\varphi'$ (via an isotopy which maps $\Sigma$ to $\Sigma_{M'}$ respecting the labels). An equivalence class of translation surfaces with marking maps is a \textit{marked translation surface}. There is a forgetful map which takes a marked translation surface, which is the equivalence class of $\varphi : S \to M$, to the translation equivalence class of $M$. We will denote this map by $\pi$ and usually denote an element of $\pi^{-1}(q)$ by $\tilde{q}$.

The set of translation self-equivalences of $M$ is a finite group which we denote by $\Gamma_M$. In particular we get a left action, by postcomposition, of $\Gamma_M$ on the set of marking maps of $M$. Note that a marking map determines a marked translation surface, but the marked translation surface need not uniquely determine the marking map $\varphi$. Indeed, if $h \in \Gamma_M$ is nontrivial, and $\tilde{q}$ is the equivalence class of a marking map $\varphi$, then $\varphi$ and $h \circ \varphi$ are different (in fact, non-isotopic) representatives of the same marked translation surface.

As we have seen a flat surface structure on $M$ determines two natural 1-forms $dx$ and $dy$ and these 1-forms determine cohomology classes in $H^1(M, \Sigma; \mathbb{R})$ which we denote by $\text{hol}^{(x)}$ and $\text{hol}^{(y)}$. Specifically for an oriented curve $\gamma$ we have $\text{hol}^{(x)}(\gamma) = \int_\gamma dx$ and $\text{hol}^{(y)}(\gamma) = \int_\gamma dy$. We can combine these classes to create an $\mathbb{R}^2$-valued cohomology class $\text{hol}_M = (\text{hol}^{(x)}, \text{hol}^{(y)})$ in $H^1(M, \Sigma; \mathbb{R}^2)$. Conversely, any $\mathbb{R}^2$-valued
cohomology class gives rise to two $\mathbb{R}$-valued cohomology classes via the identification $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$. We denote the corresponding direct sum decomposition by

$$H^1(M, \Sigma; \mathbb{R}^2) = H^1(M, \Sigma; \mathbb{R}_x) \oplus H^1(M, \Sigma; \mathbb{R}_y).$$  \hspace{1cm} (2.1)$$

Now consider a marked translation surface $\tilde{q}$ with choice of marking map $\varphi : (S, \Sigma) \rightarrow (M, \Sigma)$, where $M = M_{\tilde{q}}$ is the underlying translation surface. In this situation we have a distinguished element $\text{hol}_{\tilde{q}} = \varphi^* (\text{hol}_M) \in H^1(S, \Sigma; \mathbb{R}^2)$ given by using the map $\varphi$ to pull back the cohomology class $\text{hol}_M$ from $H^1(M, \Sigma; \mathbb{R}^2)$ to $H^1(S, \Sigma; \mathbb{R}^2)$. More concretely if $\gamma$ is an oriented curve in $S$ with endpoints in $\Sigma$ then $\text{hol}_{\tilde{q}}(\gamma) = \text{hol}_M(\varphi(\gamma))$. The cohomology class $\text{hol}_{\tilde{q}}$ is independent of the choice of the marking map and depends only on its equivalence class $\tilde{q}$. We write $\text{dev}(\tilde{q})$ for the cohomology class $\text{hol}_{\tilde{q}} \in H^1(S, \Sigma; \mathbb{R}^2)$.

2.2. An atlas of charts on $\mathcal{H}_m$. Let $\mathcal{H}_m = \mathcal{H}_m(r)$ (respectively $\mathcal{H} = \mathcal{H}(r)$) denote the collection of marked translation surfaces (respectively, translation equivalence classes of translation surfaces) of a fixed type $r$. We will use the developing map defined above to equip these sets with a topology, via a local coordinate system which is referred to as period coordinates.

A geodesic triangulation of a translation surface is a decomposition of the surface into triangles whose sides are saddle connections, and whose vertices are singular points, which need not be distinct. The existence of a geodesic triangulation of any translation surface is proved in [MS, §4]. Let $\varphi : (S, \Sigma) \rightarrow (M, \Sigma)$ be a marking map, let $\tilde{q}$ be the corresponding marked translation surface, and let $\tau$ denote the pull-back of a geodesic triangulation with vertices in $\Sigma$, from $(M, \Sigma)$ to $(S, \Sigma)$. The cohomology class $\text{hol}_{\tilde{q}}$ assigns coordinates in $\mathbb{R}^2$ to edges of the triangulation and thus can be thought of as giving a map from the triangles of $\tau$ to triangles in $\mathbb{R}^2$ (well-defined up to translation), and so each triangle in $\tau$ has a Euclidean structure coming from $M$. Let $U_\tau$ be the collection of all cohomology classes which map each triangle of $\tau$ into a positively oriented non-degenerate triangle in $\mathbb{R}^2$. Each $\beta \in U_\tau$ gives a translation surface $M_{\tau, \beta}$ built by gluing together the corresponding triangles in $\mathbb{R}^2$ along parallel edges, as well as a distinguished marking map, which we denote by $\varphi_{\tau, \beta} : (S, \Sigma) \rightarrow (M_{\tau, \beta}, \Sigma)$, which is the unique map taking each triangle of the triangulation $\tau$ of $S$ to the corresponding triangle of the triangulation of $M_{\tau, \beta}$ and which is affine on each triangle (with respect to the Euclidean structure coming from $M$). Let $\tilde{q}_{r, \beta}$ denote the marked translation surface corresponding
to the marking map $\varphi_{\tau,\beta}$. Let

$$V_\tau \overset{\text{def}}{=} \{ \tilde{q}_{\tau,\beta} : \beta \in U_\tau \} \quad \text{and} \quad \Psi_\tau : U_\tau \to V_\tau, \quad \Psi_\tau(\beta) = \tilde{q}_{\tau,\beta}.$$  

By construction, $\beta$ agrees with $\text{dev}(\tilde{q}_{\tau,\beta})$ on edges of $\tau$, and these edges generate $H_1(S, \Sigma)$. Thus the map

$$\Phi_\tau : V_\tau \to U_\tau, \quad \Phi_\tau(\tilde{q}) = \text{dev}(\tilde{q})$$

is an inverse to $\Psi_\tau$ (and in particular $\Psi_\tau$ is injective). The collection of maps $\{\Phi_\tau\}$ gives an atlas of charts for $H_m$ and the collection of maps $\{\Psi_\tau\}$ gives an inverse atlas for $H_m$. These charts give $H_m$ a manifold structure for which the map $\text{dev}$ is a local diffeomorphism. In fact this atlas determines an affine structure on $H_m$ so that $\text{dev}$ is an affine map.

We denote the tangent space of $H_m$ at $\tilde{q} \in H_m$ by $T_{\tilde{q}}H_m$ and by $T_pH_m$ the tangent bundle of $H_m$. Using the fact that the developing map is a local diffeomorphism we can identify the tangent space at each point of $H_m$ with $H_1(S, \Sigma; \mathbb{R}^2)$ so $T(H_m) = H_m \times H^1(S, \Sigma; \mathbb{R}^2)$. We say that two tangent vectors $v_i \in T_{\tilde{q}}(H_m)$ $(i = 1, 2)$, or two subspaces $V_i \subset T_{\tilde{q}}(H_m)$ are parallel if they map to the same element or subspace of $H^1(S, \Sigma; \mathbb{R}^2)$. We say that a sub-bundle of $T(H_m)$ is flat if the fibers over different points are parallel, and that a sub-bundle of $T(H)$ is flat if each of the connected components of its pullback to $T(H_m)$ is flat.

Using the explicit marking maps $\varphi_{\tau,\beta} : (S, \Sigma) \to (M_{\tau,\beta}, \Sigma)$, we get explicit comparison maps between surfaces $M_{\tau,\beta}, M_{\tau,\beta'} \in U_\tau$, of the form

$$\varphi_{\tau,\beta,\beta'} \overset{\text{def}}{=} \varphi_{\tau,\beta} \circ \varphi_{\tau,\beta'}^{-1} : M_{\tau,\beta'} \to M_{\tau,\beta}.$$  

The maps $\varphi_{\tau,\beta,\beta'}$ are continuous and piecewise affine, and may have different derivatives on different triangles.

Let $\text{Mod}(S, \Sigma)$ be the group of isotopy classes of homeomorphisms $S$ which fix $\Sigma$ pointwise. We will call this group the mapping class group (although it is usually called the pure mapping class group). It acts on the right on marking maps by pre-composition, and this induces a well-defined action on $H_m$ (note that $\Gamma_M$ acts on the left). It also acts on $T(H_m) = H_m \times H^1(M, \Sigma; \mathbb{R}^2)$ by $\gamma : (\varphi, \beta) \mapsto (\varphi \circ \gamma, \gamma^*(\beta))$. The developing map is $\text{Mod}(S, \Sigma)$-equivariant with respect to these two right actions and thus the action of an element of $\text{Mod}(S, \Sigma)$ on $H_m$, when expressed in charts, is linear. This implies that the $\text{Mod}(S, \Sigma)$-action preserves the affine structure on $H_m$. This action is properly discontinuous, but not free. Elements with nontrivial stabilizer groups correspond to surfaces with nontrivial translation equivalences.

We caution the reader that different variants of these definitions can be found in the literature, and they might not be equivalent to our
definitions, specifically as regards the question of whether or not point of $\Sigma$ are labelled.

The group $\text{Mod}_{\Sigma}^{p \Sigma}$ acts transitively on isotopy classes of marking maps, and hence each fiber of the forgetful map $\pi : \mathcal{H}_m \to \mathcal{H}$ is a $\text{Mod}_{\Sigma}^{p \Sigma}$-orbit. We can thus view $\mathcal{H}$ as the quotient $\mathcal{H}_m/\text{Mod}_{\Sigma}^{p \Sigma}$, and equip it with the quotient topology. Viewed as a map between topological spaces the forgetful map is typically not a covering map due to the presence of translation surfaces in $\mathcal{H}$ with non-trivial translation equivalences (the same is true for the finite-to-one map $\mathcal{H} \to \mathcal{H}'$ discuss in the previous paragraph). To make this map behave more like a covering map we work in the category of orbifolds.

2.3. The orbifold structure of a stratum. An orbifold structure on a space $X$ is given by an atlas of inverse charts. This consists of a collection of open sets $W_j$ that cover $X$, a collection of maps $\phi_j : U_j \to W_j$ where $U_j$ are open sets in a vector space $V$, and a collection of finite groups $G_j$ acting linearly on the sets $U_j$ so that each $\phi_j$ induces a homeomorphism from $U_j/G_j$ to $W_j$. Furthermore we require that the transition maps on overlaps respect the affine structure and group actions. The local groups $G_j$ give rise to a local group $G_x$, depending only on $x \in X$, and well-defined up to a conjugation. More information is contained in [AK, Definitions 2.1 & 2.2].

We now modify our construction of the atlas for $\mathcal{H}_m$ to give an orbifold atlas for $\mathcal{H}$. Let $q \in \mathcal{H}$, let $M = M_q$ be the underlying translation surface, and let $\Gamma_q = \Gamma_M$ be the group of translation equivalences of $M_q$. Choose a marking map $\varphi : (S, \Sigma) \to (M, \Sigma)$. By pulling back a triangulation from the quotient of $M$ by $\Gamma_q$, we can find a geodesic triangulation $\tau'$ of $M$ which is $\Gamma_M$-invariant, and we let $\tau = \varphi^{-1}(\tau')$ be the pullback of this triangulation to $S$. As before, let $U_{\tau}$ be the set of cohomology classes compatible with $\tau$. Let $G_q$ be the (conjugacy class of the) subgroup of $\text{Mod}(S, \Sigma)$ corresponding to the isotopy classes of the elements $\{\varphi^{-1} \circ h \circ \varphi : h \in \Gamma_q\}$. Since $\tau'$ is $\Gamma_q$-invariant, the group $G_q$ acts on $U_{\tau}$, and the maps $\pi \circ \Psi : U_{\tau} \to \mathcal{H}$ induce maps from $U_{\tau}/G_q$ to $\mathcal{H}$. By possibly replacing $U_{\tau}$ by a smaller neighborhood $U_{\tau}' \subset U_{\tau}$ on which this induced map is injective, we get a collection of inverse charts for an orbifold atlas for $\mathcal{H}$.

An orbifold structure on a space $X$ determines a local group at a point $x \in X$. For $q \in \mathcal{H}$ this local group can be identified with $G_q$. The singular set of an orbifold is the set of points where the local group is not the identity. The singular set has a stratification into submanifolds which we will call orbifold substrata, defined as the connected components of the subsets of the stratum on which the local group is
constant. We will denote the orbifold substratum corresponding to $\mathcal{G}_q$ by $O_q$. Smooth maps between orbifolds are described in [AK, Definition 2.4]. The map $\pi : \mathcal{H}_m \to \mathcal{H}$ can be given the structure of a smooth orbifold map.

The tangent bundle of an orbifold is defined in [AK, Prop. 4.1]. It is itself an orbifold, and is equipped with a projection map $T(X) \to X$, such that the fiber over $x$ can be identified with the quotient of a vector space by a linear action of $G_x$. The projection map $T(X) \to X$ is a bundle map in the category of orbifolds. Note that its fibers can vary from point to point.

We denote the orbifold tangent space of $q$ at $\mathcal{H}$ by $T_q(\mathcal{H})$, and the tangent bundle of $\mathcal{H}$ by $T(\mathcal{H})$. We can identify $T(\mathcal{H})$ with the quotient of the tangent bundle of $\mathcal{H}_m$ under the action of the mapping class group. The bundle $T(\mathcal{H})$ has a canonical $\text{Mod}(S, \Sigma)-$invariant splitting coming from the decomposition

$$H^1(S, \Sigma; \mathbb{R}^2) = H^1(S, \Sigma; \mathbb{R}_x) \oplus H^1(S, \Sigma; \mathbb{R}_y)$$

(which is the analogue of (2.1) for the model surface $S$) and we refer to the summands as the horizontal and vertical sub-bundles.

Since $\mathcal{H}$ is the quotient of an affine manifold $\mathcal{H}_m$ by a group acting affinely and properly discontinuously it inherits the structure of an affine orbifold. A map between affine orbifolds is affine if it can be expressed by affine maps in local charts.

Affine structures do not give a metric geometry but some familiar notions from the theory of Riemannian manifolds have analogues for affine manifolds. Thus an affine geodesic is a path in an affine manifold $N$ parametrized by an open interval in the real line which has the property that in any affine chart the parametrization is linear. Affine geodesics are projections of orbits of a partially defined flow on the tangent bundle which we call the affine geodesic flow. An affine geodesic has a maximal domain of definition which is a connected open subset of $\mathbb{R}$, which may or may not coincide with $\mathbb{R}$. We denote by $\text{Dom}(\tilde{q}, v) \subset \mathbb{R}$ the maximal domain of definition of the affine geodesic which is tangent at time $t = 0$ to $v \in T_{\tilde{q}}(\mathcal{H}_m)$.

With the above description of the orbifold tangent bundle of $\mathcal{H}$, we obtain a description of the sub-bundle corresponding to the orbifold substrata.

**Proposition 2.1.** Let $q \in \mathcal{H}$ be a surface with a nontrivial local group and let $O_q$ be the corresponding orbifold substratum. Then $\partial q \overset{\text{def}}{=} \pi^{-1}(O_q)$ is an affine submanifold of $\mathcal{H}_m$, and its tangent space $T_{\tilde{q}}(\partial q)$ at $\tilde{q}$ is
identified via the developing map with the set of vectors in $H^1(S, \Sigma; \mathbb{R}^2)$ fixed by $G_q$.

The proof is left to the reader.

We will need explicit formulas for the projections onto the tangent space to an orbifold substratum, and onto a normal sub-bundle. Let $M_q$ be a surface with a non-trivial group of translation equivalences. Choose a marking map of $M_q$ and let $G_q$ be the corresponding local group acting on this chart. Define $P^+ : H^1(S, \Sigma; \mathbb{R}^2) \to H^1(S, \Sigma; \mathbb{R}^2)$ by

$$P^+ (\beta) \overset{\text{def}}{=} \frac{1}{|G_q|} \sum_{\gamma \in G_q} \gamma^* (\beta).$$

By Proposition 2.1, $P^+$ is a projection of $H^1(S, \Sigma; \mathbb{R}^2)$ onto the tangent space to the substratum. The kernel of $P^+$, which we denote by $\mathcal{N} (O_q)$, is a natural choice for a normal bundle to $O_q$. We denote by $P^- \overset{\text{def}}{=} \text{Id} - P^+$ the projection onto the normal space to the orbifold substratum. Note that $P^\pm$ depend on $G_q$ but this will be suppressed in the notation. It will also be useful to further decompose the normal bundle into its intersections with the horizontal and vertical sub-bundles, and we denote these sub-bundles by $\mathcal{N}_x (O_q)$ and $\mathcal{N}_y (O_q)$.

**Proposition 2.2.** Given an orbifold sub-locus $O$, the bundles $T(O), \mathcal{N}(O), \mathcal{N}_x(O)$ and $\mathcal{N}_y (O)$ are flat, and each has a volume form which is well-defined (independent of a marking).

**Proof.** The map $P^+$ respects the splitting of cohomology into horizontal and vertical factors, i.e., it commutes with the two projections onto the summands in (2.2). Moreover, since the $\text{Mod}(S, \Sigma)$-action on $H^1(S, \Sigma; \mathbb{R}^2)$ preserves $H^1(S, \Sigma; \mathbb{Z}^2)$, it takes integral classes to rational classes, i.e., is defined over $\mathbb{Q}$. It thus induces a map

$$H^1(S, \Sigma; \mathbb{R}_x) \supset H^1(S, \Sigma; \mathbb{Z}_x) \xrightarrow{P^+} H^1(S, \Sigma; \mathbb{Q}_x) \subset H^1(S, \Sigma; \mathbb{R}_x)$$

(with the obvious notations $\mathbb{Z}_x, \mathbb{Q}_x$ for the corresponding summands), and a corresponding map for the second summand $\mathbb{Z}_y, \mathbb{Q}_y, \mathbb{R}_y$. The kernels of these maps are lattices in $\mathcal{N}_x(O)$ and $\mathcal{N}_y (O)$ which are parallel. This means that the Lebesgue measure on $\mathcal{N}_x(O)$, coming from the affine structure of Proposition 2.1, has a natural normalization which does not depend on the choice of a particular lift $\tilde{O} \to O$. \hfill $\square$

The space of marked translation surfaces with area one is a submanifold $\mathcal{H}_{m,1}$ of $\mathcal{H}_m$, which is invariant under $\text{Mod}(S, \Sigma)$. We refer to the quotient orbifold as the normalized stratum and denote it by $\mathcal{H}_1$. The normalized stratum is a codimension one sub-orbifold of $\mathcal{H}$ but it is
not an affine sub-orbifold. The developing map dev maps \( \mathcal{H}_{m,1} \) into a quadric in \( H^1(S, \Sigma; \mathbb{R}^2) \), and the tangent \( T_{\tilde{q}}(\mathcal{H}_{m,1}) \) is a linear subspace of \( H^1(S, \Sigma; \mathbb{R}^2) \) on which area is constant to first order. This subspace varies with \( \tilde{q} \). Nevertheless it is often quite useful to use the ambient affine coordinates to discuss it. In general if we consider a vector tangent to \( \mathcal{H}_1 \), then the affine geodesic determined by this vector need not lie in \( \mathcal{H}_1 \) but in the particular cases of horocycles and tremors it will be the case that these paths lie in \( \mathcal{H}_1 \).

2.4. Action of \( G = \text{SL}_2(\mathbb{R}) \) on strata. Let \( M \) be a translation surface with an atlas \( \mathcal{A} = (\phi_\alpha, U_\alpha)_{\alpha \in \mathbb{A}} \). We can apply an element \( g \in G \) to the atlas \( \mathcal{A} \) by post-composing each chart with the map \( g \) viewed as a linear map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), i.e. \( g \) applied to \( \mathcal{A} \) is the atlas \( g\mathcal{A} = (g \circ \phi_\alpha, U_\alpha)_{\alpha \in \mathbb{A}} \). This gives rise to an action of \( G \) on \( \mathcal{H}_m \), which commutes with the action of \( \text{Mod}(S, \Sigma) \) and preserves the normalized stratum \( \mathcal{H}_m,1 \). Thus we have an induced \( G \)-action on \( \mathcal{H} \) and on \( \mathcal{H}_1 \), and the forgetful map \( \pi : \mathcal{H}_m \to \mathcal{H} \) is \( G \)-equivariant.

We now check that the action is affine in charts. There is a natural left action of \( G \) on \( H^1(S, \Sigma; \mathbb{R}^2) \) which is given by the action of \( G \) on the coefficient system, i.e. by postcomposition of \( \mathbb{R}^2 \) valued 1-cochains. Let \( \tau \) be a triangulation of \( S \), and let \( U_\tau \subset H^1(S, \Sigma; \mathbb{R}^2) \) be defined as in §2.2. For \( \beta \in U_\tau \) and \( g \in G \), we see that \( g\beta \overset{\text{def}}{=} g \circ \beta \in U_\tau \). Let \( \varphi_{\tau,\beta,g\beta} : M_\beta \to M_{g\beta} \) be the comparison map. Notice that it has the same derivative on each triangle, namely its derivative is everywhere equal to the linear map \( g \). In particular, the comparison map \( \varphi_{\tau,\beta,g\beta} \) does not depend on \( \tau \). We will call it the affine comparison map corresponding to \( g \) and denote it by \( \psi_g \). The action of \( g \) on \( \mathcal{H}_m \) can now be expressed as replacing a marking map \( \varphi : S \to M \) by \( \psi_g \circ \varphi : S \to gM \). Other affine maps \( M_q \to M_{gq} \) with derivative \( g \) can be obtained by composing \( \psi_q \) with translation equivalences. Since the \( G \)-action commutes with the \( \text{Mod}(S, \Sigma) \)-action, \( G \) preserves the orbifold stratification of \( \mathcal{H} \). Additionally, the normal and tangent bundles of Propositions 2.1 and 2.2 are \( G \)-equivariant.

We introduce some notation for subgroups of \( G \). Recall the group \( U = \{ u_s : s \in \mathbb{R} \} \) introduced in (1.1). We will also use the following notation for other subgroups:

\[
g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad \tilde{g}_t = g_{-t}, \quad r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]

and

\[
B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}.
\]
With this notation we note that the $U$-action is given in period coordinates by
\[
\text{hol}_{u,q}^{(x)}(\gamma) = \text{hol}_{q}^{(x)}(\gamma) + s \cdot \text{hol}_{q}^{(y)}(\gamma), \quad \text{hol}_{u,q}^{(y)}(\gamma) = \text{hol}_{q}^{(y)}(\gamma);
\]
this now gives a precise meaning to (1.4). Our next goal is to give a precise meaning to (1.5), by defining transverse measures and their associated cohomology class.

2.5. Transverse (signed) measures and foliation cocycles. In this section we define transverse measures and cocycles and cohomology classes associated with a non-atomic transverse measure. It will be useful to include signed transverse measures. In some settings it is useful to pass to limits of non-atomic transverse measures, and these limits may be certain atomic transverse measures. In \S\ 13 we will discuss the case of these atomic transverse measures.

Let $M$ be a translation surface, let $\theta \in S^1$ be a direction (i.e., a unit vector $(\cos \theta, \sin \theta) \in \mathbb{R}^2$), and let $\mathcal{F}_\theta$ denote the foliation of $M$ obtained by pulling back the foliation of $\mathbb{R}^2$ by lines parallel to $\theta$. A transverse arc to $\mathcal{F}_\theta$ is a piecewise smooth curve $\gamma : (a, b) \to M \setminus \Sigma$ of finite length which is everywhere transverse to leaves of $\mathcal{F}_\theta$. A transverse measure on $\mathcal{F}_\theta$ is a family $\{\nu_\gamma\}$ where $\gamma$ ranges over the transverse arcs, the $\nu_\gamma$ are finite regular Borel measures defined on $\gamma$ which are invariant under isotopy through transverse arcs and so that if $\gamma' \subset \gamma$ then $\nu_{\gamma'}$ is the restriction of $\nu_\gamma$ to $\gamma'$. Since transverse measures are defined via measures, the usual notions of measure theory (absolute continuity, Radon-Nikodym theorem, etc.) make sense for transverse measures (or a pair of transverse measures). In particular it makes sense to speak of atoms of a transverse measure, and we will say that $\nu$ is non-atomic if none of the $\nu_\gamma$ have atoms. In this paper, if transverse measures have atoms we require that the atoms be supported on closed loops, each of which is a closed leaf, or a union of saddle connections that meet at angles $\pm \pi$ (see \S\ 13 for a formal definition). These are the atomic transverse measures that can arise as limits of non-atomic transverse measures. We remark that in the literature, there are several different conventions regarding atomic transverse measures.

A (finite) signed measure on $X$ is a map from Borel subsets of $X$ to $\mathbb{R}$ satisfying all the properties satisfied by a measure. Recall that every signed measure has a canonical Hahn decomposition, i.e., a unique representation $\nu = \nu^+ - \nu^-$ as a difference of fine measures. A signed transverse measure is a system $\{\nu_\gamma\}$ of signed measures, satisfying the same hypotheses as a signed measure; or equivalently, the difference of
two transverse measures \( \{\nu^+_\gamma\}, \{\nu^-_\gamma\} \). In what follows, the words ‘measure’ and ‘transverse measure’ always refer to non-negative measures (i.e. measures for which \( \nu^- = 0 \)). When we want to allow general signed measures we will include the word ‘signed’. We say that \( \nu \) is non-atomic if \( \nu^\pm \) are both non-atomic. The sum \( \nu^+(X) + \nu^-(X) \) is called the total variation of \( \nu \).

If \( M \) is a translation surface, \( \mathcal{F}_\theta \) is a directional foliation on \( M \), and \( \nu \) is a non-atomic signed transverse measure on \( \mathcal{F}_\theta \), we have a map \( \beta_\nu \) from transverse line segments to real numbers, defined as follows. If \( \gamma \) is a transverse oriented line segment and the (counterclockwise) angle between the direction \( \theta \) and the direction of \( \gamma \) is in \((0, \pi)\), set \( \beta_\nu(\gamma) = \nu(\gamma) \). If the angle is in \((-\pi, 0)\) set \( \beta_\nu(\gamma) = -\nu(\gamma) \). We extend this to all straight line segments by stipulating that \( \beta_\nu(\gamma) = 0 \) for any line segment \( \gamma \) that is contained in a leaf of the foliation. By linearity we extend \( \beta_\nu \) to finite concatenations of oriented straight line segments. Similarly we can define \( \beta_\nu(\gamma) \) for an oriented piecewise smooth curve \( \gamma \), where the sign of an intersection is measured using the derivative of \( \gamma \).

By a polygon decomposition of a translation surface \( M \), we mean a decomposition into simply connected polygons for which all the vertices are singular points. As we saw every \( M \) admits a geodesic triangulation which is a special case of a polygon decomposition. Let \( \beta_\nu \) be as in the preceding paragraph. Any element \( \alpha \in H_1(M, \Sigma) \) has a representative \( \tilde{\alpha} \) that is a concatenation of edges of a polygon decomposition. The invariance property of a transverse measure ensures that the value \( \beta_\nu(\tilde{\alpha}) \) depends only on \( \alpha \) and not on the representative \( \tilde{\alpha} \); in particular it does not depend on the cell decomposition used, and \( \beta_\nu \) is a cochain and defines a cohomology class in \( H^1(M, \Sigma; \mathbb{R}) \). We have defined a mapping \( \nu \mapsto \beta_\nu \) from non-atomic signed transverse measures to \( H^1(M, \Sigma; \mathbb{R}^2) \), and in §13 we will explain how to extend this map to atomic transverse measures. We will only be interested in transverse measures to the horizontal foliation. Any element of cohomology which is the image under this map of a transverse measure (resp., a signed transverse measure) to the horizontal foliation will be called a foliation cocycle (respectively, signed foliation cocycle), and \( \beta_\nu \) will be called the (signed) foliation cocycle corresponding to \( \nu \).

Identifying \( \mathbb{R} \) with \( \mathbb{R}_x \) and \( H^1(M, \Sigma; \mathbb{R}) \) with the first summand in \((2.1)\), we identify the collection of all signed foliation cocycles with a subspace \( \mathcal{T}_q \subset H^1(M, \Sigma; \mathbb{R}_x) \), and the collection of all foliation cocycles with a cone \( C^-_q \subset \mathcal{T}_q \). We refer to these respectively as the space of signed foliation cocycles and the cone of foliation cocycles. Hahn decomposition of transverse measures implies that every \( \beta \in \mathcal{T}_q \) can be
written uniquely as $\beta = \beta^+ - \beta^-$ for $\beta^\pm \in C_q^\pm$. For every $q$, the 1-form $dy$ gives rise to a canonical transverse measure and to the corresponding cohomology class $\text{hol}_q^{(y)}$. When we want to think of this class as a foliation cocycle, we will denote it by $dy$ or $(dy)_q$, and refer to it as the canonical foliation cocycle.

Analogously to the horizontal straightline flow, we can define a (partially defined) flow in direction $\theta$ by using planar charts to lift the vector field on $\mathbb{R}^2$ in direction $\theta$. We say that a finite Borel measure $\mu$ on $M$ is $\mathcal{F}_\theta$-invariant if it is invariant under the straightline flow in direction $\theta$. We have the following well-known relationship between transverse measures and invariant measures.

**Proposition 2.3.** For each non-atomic transverse measure $\nu$ on $\mathcal{F}_\theta$ there exists an $\mathcal{F}_\theta$-invariant measure $\mu_\nu$ with

$$\mu_\nu(A) = \nu(v) \cdot \ell(h)$$

(2.6)

for every isometrically embedded rectangle $A$ with one side $h$ parallel to $\theta$, and another side $v$ orthogonal to $\theta$, where $\ell$ is the Euclidean length. The map $\nu \mapsto \mu_\nu$ is a bijection between non-atomic transverse measures and $\mathcal{F}_\theta$-invariant measures that assign zero measure to horizontal leaves. It extends to a bijection between non-atomic signed transverse measures and $\mathcal{F}_\theta$-invariant signed measures assigning zero measure to horizontal leaves.

Indeed, the formula (2.6) defines a pre-measure on the ring generated by rectangles of the above form. By Carathéodory’s extension theorem there is a measure $\mu_\nu$ on the $\sigma$-algebra generated by this ring, which is the Borel $\sigma$-algebra on $M$. By the invariance property of a transverse measure, the pre-measure is invariant and therefore so is the measure.

It is clear from (2.6) that two different transverse measures give different measures to some rectangle, and so the assignment is injective. To see that each $\mathcal{F}_\theta$-invariant measure arises from a transverse measure, partition $M$ into rectangles and use disintegration of measures to define a transverse measure on each rectangle. This transverse measure will be non-atomic if the invariant measure gives zero measure to every horizontal leaf.

The map $\nu \mapsto \beta_\nu$ is almost injective. More precisely, we have:

**Proposition 2.4 (Katok).** If $M_q$ has no horizontal cylinders and $\nu_1 \neq \nu_2$ are distinct non-atomic signed transverse measures to the horizontal foliation, then $\beta_{\nu_1} \neq \beta_{\nu_2}$, and moreover the restrictions of $\beta_\nu$ to the absolute period space $H_1(S)$ are different.
For a proof see [K]. Katok considered measures rather than signed measures, but the passage to signed measures follows from the uniqueness of the Hahn decomposition.

2.6. **The Sup-norm Finsler metric.** We now recall the sup-norm Finsler metric on $\mathcal{H}_m$ studied by Avila, Gouëzel and Yoccoz in [AGY]. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$. For a translation surface $q$, denote by $\Lambda_q$ the collection of saddle connections on $M_q$ and let $\ell_q(\sigma) = \|\text{hol}_q(\sigma)\|$ be the length of $\sigma \in \Lambda_q$. For $\beta \in H^1(M_q, \Sigma_q; \mathbb{R}^2)$ we set

$$\|\beta\|_q \overset{\text{def}}{=} \sup_{\sigma \in \Lambda_q} \frac{\|\beta(\sigma)\|}{\ell_q(\sigma)}. \quad (2.7)$$

We now define a Finsler metric for $\mathcal{H}_m$. Let $\varphi : (S, \Sigma) \to (M_q, \Sigma)$ be a marking map, which represents $\tilde{q} \in \mathcal{H}_m$. Recall that we can identify $T_{\tilde{q}}(\mathcal{H}_m)$ with $H^1(S, \Sigma; \mathbb{R}^2)$. Then $\|\varphi^*\beta\|_{\tilde{q}} = \|\beta\|_q$ is a norm on $H^1(S, \Sigma; \mathbb{R}^2)$, or equivalently:

$$\|\beta\|_{\tilde{q}} \overset{\text{def}}{=} \sup_{\tau \in \Lambda_{\tilde{q}}} \frac{\|\beta(\varphi(\tau))\|}{\ell_{\tilde{q}}(\varphi(\tau))}. \quad (2.8)$$

Note that $\Lambda_{\tilde{q}}$ varies as $\tilde{q}$ changes, and that $\|\theta\|_{\tilde{q}}$ is well-defined (i.e. depends on $\tilde{q}$ and not on the actual marking map $\varphi$). Recall that using period coordinates, the tangent bundle $T(\mathcal{H}_m)$ is a product $\mathcal{H}_m \times H^1(S, \Sigma; \mathbb{R}^2)$. As shown in [AGY, Prop. 2.11], the map

$$T(\mathcal{H}_m) \to \mathbb{R}, \quad (\tilde{q}, \beta) \mapsto \|\beta\|_{\tilde{q}}$$

is continuous.

The Finsler metric defines a distance function on $\mathcal{H}_m$ which we call the *sup-norm distance* and define as follows:

$$\text{dist}(\tilde{q}_0, \tilde{q}_1) \overset{\text{def}}{=} \inf_{\gamma} \int_0^1 \|\gamma'(\tau)\|_{\gamma(\tau)} d\tau, \quad (2.9)$$

where $\gamma$ ranges over smooth paths $\gamma : [0, 1] \to \mathcal{H}$ with $\gamma(0) = \tilde{q}_0$ and $\gamma(1) = \tilde{q}_1$. This distance is symmetric since $\|\beta\|_{\tilde{q}} = \|\beta\|_{\tilde{q}}$.

The following was shown in [AGY, §2.2.2]:

**Proposition 2.5.** The metric $\text{dist}$ is proper, complete, and induces the topology on $\mathcal{H}_m$ given by period coordinates.

**Proof.** The fact that the sup-norm distance is a Finsler metric giving the topology on period coordinates is [AGY, proof of Proposition 2.11]. The fact that the metric is proper is [AGY, Lemma 2.12]. Completeness is [AGY, Corollary 2.13]. \qed
We will now compute the deviation of nearby $G$-orbits with respect to the sup-norm distance. Let $\|g\|_{op}$, $g^t$ and $\text{tr}(g)$ denote respectively the operator norm, transpose, and trace of $g \in G$. The operator norm can be calculated in terms of the singular values of $g$. Specifically the operator norm is the square root of the the largest eigenvalue of $g^t g$.

For a 2 by 2 matrix this eigenvalue can be expressed in terms of the trace and determinant of $g^t g$:

$$\|g\|_{op} = \sqrt{\frac{\text{tr}(g^t g) + \sqrt{\text{tr}^2(g^t g) - 4}}{2}} \quad (2.10)$$

Recall the affine comparison map $\psi_g : M_q \to M_{gq}$ with derivative $g$, from §2.4. For this map we have $\text{hol}(\psi(\sigma)) = g(\text{hol}(\sigma))$ and hence $\|\sigma\|_{gq} = \|g(\text{hol}(\sigma))\|_q$. From this it is not hard to deduce that

$$\|g^\beta\|_{qg} \leq \|g\|_{op} \cdot \|g^{-1}\|_{op} \cdot \|\beta\|_{q}.$$  

**Corollary 2.6** (See [AGY], equation (2.13)). For all $s, t \in \mathbb{R}$ we have

$$\|u_s(\theta)\|_{u_s q} \leq \left(1 + \frac{s^2 + |s|\sqrt{s^2 + 4}}{2}\right)\|\theta\|_{\bar{q}}$$

and

$$\|g_t(\theta)\|_{g_t \bar{q}} \leq e^{|t|}\|\theta\|_{\bar{q}}.$$  

Integrating these pointwise bounds and using the definition of the sup-norm distance, we find that nearby horocycle (geodesic) trajectories diverge from each other at most quadratically (exponentially). Namely:

**Corollary 2.7.** For any $\tilde{q}_0, \tilde{q}_1 \in \mathcal{H}_m$, and any $s, t \in \mathbb{R}$,

$$\left(1 + \frac{s^2 + |s|\sqrt{s^2 + 4}}{2}\right)^{-1} \text{dist}(\tilde{q}_0, \tilde{q}_1) \leq \text{dist}(u_s \tilde{q}_0, u_s \tilde{q}_1) \leq \left(1 + \frac{s^2 + |s|\sqrt{s^2 + 4}}{2}\right) \text{dist}(\tilde{q}_0, \tilde{q}_1)$$

and

$$e^{-2|t|}\text{dist}(\tilde{q}_0, \tilde{q}_1) \leq \text{dist}(g_t \tilde{q}_0, g_t \tilde{q}_1) \leq e^{2|t|}\text{dist}(\tilde{q}_0, \tilde{q}_1). \quad (2.11)$$

In the case of unipotent flows in homogeneous dynamics nearby orbits diverge at most polynomially with respect to an appropriate metric. Corollary 2.7 shows that on strata, nearby horocycles orbits diverge from each other at most quadratically. In §8.3 we will discuss the more delicate question of lower bounds for the rate of divergence of horocycles, and show that erratic divergence is possible.
3. The space of two tori glued along a slit

In this section we collect some information we will need regarding the structure of $\mathcal{E}$ and the dynamics of the straightline flow on surfaces in $\mathcal{E}$. We also prove Proposition 3.5, which plays an important role in §10. It shows that for surfaces in $\mathcal{E}$, the ergodic measures in directions which are not uniquely ergodic have good approximations by splittings of the surface into two tori. This may be considered as a converse to a construction of Masur and Smillie [MaTa, §3.1].

3.1. The locus $\mathcal{E}$. McMullen studied the eigenform loci $\mathcal{E}_D$, which are affine $G$-invariant suborbifolds of $\mathcal{H}(1,1)$ and have several equivalent descriptions (see [McM1] and references therein). The description which will be convenient for us is the following. Let $\mathcal{H}(0,0)$ be the stratum of tori with two marked points, then $\mathcal{E}$ is the collection of surfaces $q \in \mathcal{H}(1,1)$ for which there is a branched 2 to 1 translation cover onto a surface in $\mathcal{H}(0,0)$. To avoid confusion with different conventions used in the literature, we remind the reader that we take the marked points in $\mathcal{H}(0,0)$ and $\mathcal{H}(1,1)$ to be labelled. See [BSW, §7] for additional information.

The following proposition shows that, with respect to the terminology of §2.3, $\mathcal{E}$ consists of points in $\mathcal{H}(1,1)$ where the local orbifold group is non-trivial; namely, it is the group of order two generated by an involution in $\text{Mod}(S, \Sigma)$.

Proposition 3.1. The locus $\mathcal{E}$ is connected. It admits a four to one covering map $P : \mathcal{E} \to \mathcal{H}(0,0)$ which is characterized by the following property: for every $q \in \mathcal{E}$ there is an order 2 translation equivalence $\iota = \iota_q : M_q \to M_q$, such that the quotient surface $M_q/\langle \iota \rangle$ is a translation surface which is translation equivalent to the torus $T_{P(q)}$.

Proof. By definition, if $q \in \mathcal{E}$ then $M_q$ has a translation automorphism $\iota$ such that $M_q/\langle \iota \rangle$ is a torus in $\mathcal{H}(0,0)$. Since nontrivial translation automorphisms cannot fix nonsingular points, the only possible fixed points are singularities, and singularities cannot be interchanged because the quotient surface is in $\mathcal{H}(0,0)$. In particular $\iota$ represents a conjugacy class of elements of $\text{Mod}(S, \Sigma)$.

Connectedness of $\mathcal{E}$ is proved in [EMS, Theorem 4.4]. It remains to show that $P$ is four to one. Let $T \in \mathcal{H}(0,0)$ and denote $\Sigma = \{\xi_0, \xi_1\}$. Any $q \in \mathcal{E}$ for which $P(q) = T$ gives an unbranched cover $M_q \setminus P^{-1}(\Sigma) \to T \setminus \Sigma$. Conversely any unbranched cover can be completed to a branched cover, and this cover is ramified at $\xi \in \Sigma$ precisely if a small loop $\ell_\xi$ around $\xi$ in $T$ does not lift as a closed loop in $M_q$. So the cardinality of $P^{-1}(T)$ is the number of topologically
distinct degree 2 covers of $T \setminus \Sigma$ for which the loops $\ell_\xi$ do not lift as closed loops. Equivalently, it is the number of conjugacy classes of homomorphisms $\pi_1(T \setminus \Sigma) \to \mathbb{Z}/2\mathbb{Z}$ for which the image of the class of each $\ell_\xi$ is nontrivial. Since $\mathbb{Z}/2\mathbb{Z}$ is abelian, the covering spaces are determined uniquely by elements $\theta \in H^1(T \setminus \Sigma; \mathbb{Z}/2\mathbb{Z})$ and we are counting those $\theta$ for which both $\theta(\ell_\xi) \neq 0$. Since the two $\ell_\xi$ are homologous to each other, this gives one linear equation on a vector space of dimension 3 over $\mathbb{Z}/2\mathbb{Z}$, so we have four solutions. □

Given a torus $T \in \mathcal{H}(0,0)$ and a saddle connection $\delta$ joining the two marked points we can build a surface $M \in \mathcal{H}(1,1)$ by cutting $T$ along $\delta$, viewing the resulting surface as a surface with boundary. We define $M$ to be the result of taking two copies of the surface with boundary and gluing along the boundaries. The surface $M$ has a branched covering map to $T$ and a deck transformation which is an involution interchanging the two copies of $T$. A slit on a translation surface is a union of homologous saddle connections which disconnect the surface. Thus in this example, the preimage $\sigma$ of $\delta$ under the map $M \to T$ is a slit. We say that $M$ is built from the slit construction applied to $\sigma$.

**Proposition 3.2.** Every surface in $\mathcal{E}$ can be built from the slit construction in infinitely many ways. Two slits on the same torus determine the same surface in $\mathcal{E}$ if the corresponding homology classes $[\delta_1]$ and $[\delta_2]$ are equal as elements of $H_1(T; \mathbb{Z}/2\mathbb{Z})$.

**Proof.** As in the proof of Proposition 3.1, a surface in $\mathcal{E}$ corresponds to a class $\theta \in H^1(T \setminus \Sigma; \mathbb{Z}/2\mathbb{Z})$ for which the $\theta(\ell_\xi)$ are nonzero. If $\delta$ is any path from $\xi_0$ to $\xi_1$, it defines a class $[\delta] \in H_1(T; \mathbb{Z}/2\mathbb{Z})$, and we will say $\theta$ is represented by $\delta$ if $\theta$ is the class in $H^1(T \setminus \Sigma; \mathbb{Z}/2\mathbb{Z})$ which is Poincaré dual to $[\delta]$. Clearly, if $\theta$ is represented by some $\delta$ then $\theta$ satisfies the requirement $\theta(\ell_\xi) \neq 0$, and by a dimension count, any such $\theta$ is represented by some path $\delta$. It remains to show that each $\theta$ is represented by infinitely many paths $\delta$ which are homotopic to a saddle connection from $\xi_0$ to $\xi_1$. To see this, let $\delta_0$ be some path representing $\theta$, let $v_0 \overset{\text{def}}{=} \text{hol}_T(\delta_0)$, let $\Lambda \overset{\text{def}}{=} \text{hol}_T(H_1(T; \mathbb{Z}))$, and let $\Lambda' \overset{\text{def}}{=} \Lambda \cup (v_0 + \Lambda)$. Since $\mathbb{R}^2$ is the universal cover of $T$, $\Lambda$ is a lattice in $\mathbb{R}^2$, $v_0 \notin \Lambda$, and the required paths $\delta$ are those for which $\text{hol}_T(\delta) \in v_0 + 2 \cdot \Lambda$ and for which the straight segment in $\mathbb{R}^2$ from the origin to $\text{hol}_T(\delta)$ does not intersect $\Lambda'$ in its interior. It follows from this description that the set of such $\delta$ is infinite. □

For use in the sequel, we record the conclusion of Proposition 2.2 in the special case of the orbifold substratum $\mathcal{E}$:
Proposition 3.3. We can identify the tangent space $T(p)$ with the $+1$ eigenspace of the action of $\iota$ on $H^1(S, \Sigma; \mathbb{R}^2)$ and the normal bundle $N(p)$ with the $-1$ eigenspace. The bundle $N(p)$ has a splitting into flat sub-bundles

$$N(p) = N_x(p) \oplus N_y(p),$$

and each of these sub-bundles has a flat monodromy invariant volume form.

3.2. Dynamics on $E$. Here we state some important features of the straightline flow on surfaces in $E$.

Proposition 3.4. Let $q \in E$, let $M = M_q$ be the underlying surface, let $\iota : M \to M$ be the involution as described in Proposition 3.1, let $F$ be the horizontal foliation on $M$, and let $(dy)_q$ be the canonical transverse measure. Suppose that the foliation $F$ is not periodic. Then for any transverse measure $\nu$ to $F$, $\iota \ast \nu$ is also a transverse measure and there is $c > 0$ such that $\nu + \iota \ast \nu = c(dy)_q$. Moreover, if $F$ is not uniquely ergodic, then (up to multiplication by constants) it supports exactly two ergodic transverse measures which are images of each other under $\iota \ast$, and Leb is not ergodic for the horizontal straightline flow.

We omit the proof which uses the facts that $\iota$ commutes with the flow and that, under our aperiodicity assumption, the projection of $F$ to the torus is uniquely ergodic.

The following proposition is the main result of this section. Let $\theta \in [-\pi, \pi]$. In the following proposition we will write $F_{\theta}$ for the foliation in direction $\theta$ where $\theta = 0$ corresponds to the horizontal direction.

Proposition 3.5. Suppose $q \in E$ has the property that the horizontal foliation on $M_q$ is minimal but not ergodic and let $\mu$ be an invariant ergodic probability measure on $M_q$, for the horizontal straightline flow. Then there are directions $\theta_j$, such that the foliations $F_j$ in direction $\theta_j$ contain saddle connections $\delta_j$ satisfying the following:

(i) The union $\sigma_j = \delta_j \cup \iota(\delta_j)$ is a slit in $F_j$ separating $M_q$ into two isometric tori.
(ii) The holonomy $\operatorname{hol}_q(\delta_j) = (x_j, y_j)$ satisfies

$$|x_j| \to \infty, \ 0 \neq y_j \to 0 \text{ as } j \to \infty. $$

In particular the direction $\theta_j$ is not horizontal but tends to horizontal, and the length of $\delta_j$ tends to $\infty$. Moreover there are no saddle connections $\delta$ on $M_q$ with holonomy vector satisfying $|\operatorname{hol}_q^{(x)}(\delta)| < |x_j|$ and $|\operatorname{hol}_q^{(y)}(\delta)| < y_j$. 
(iii) For each $j$ we can choose one of the tori $A_j$ in $M_q \setminus \sigma_j$, such that the normalized restriction $\mu_j$ of $\text{Leb}$ to $A_j$ converges to $\mu$ as $j \to \infty$, w.r.t. the weak-$*$ topology on probability measures on $M_q$. Thus, letting $\nu$ and $\nu_j$ be the transverse measures corresponding to $\mu$ and $\mu_j$ (via Proposition 2.3), and letting $\beta_\nu$ and $\beta_j = \beta_{\nu_j}$ be the corresponding foliation cocycles in $H^1(M_q; \Sigma_q; \mathbb{R})$, we have $\beta_j \to \beta_\nu$.

Proof. We first show how to find the $\theta_j$ and $\delta_j$. We consider the projection map $\bar{\pi} : E \to \mathcal{H}(0)$ given by the composition of the map $P$ from Proposition 3.1 with the forgetful map forgetting the second marked point. In other words, $\bar{\pi} : M_q \mapsto M_q/\langle \iota \rangle$. Because $M_q$ has a minimal horizontal foliation, there exists a compact subset $K \subset \mathcal{H}(0)$, and a subsequence $j_1, j_2, \ldots$ so that $\tilde{g}_{t_j} \bar{\pi}(q) \in K$ for all $j$ (recall from (2.4) that $\tilde{g}_t = g_{-t}$). Indeed, $\mathcal{H}(0)$ can be identified with $\text{SL}_2(\mathbb{R})/\text{SL}_2(\mathbb{Z})$, the space of unimodular lattices. The horizontal foliation is minimal if and only if the corresponding lattice does not contain a nonzero horizontal vector, or equivalently, there is a compact set $K$ and a sequence $t_j \to \infty$ for that $\tilde{g}_{t_j} \bar{\pi}(q) \in K$.

Denote by $\mathcal{M}_g$ the moduli space of Riemann surfaces of genus $g$ and let $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification (see [B, §5] for a concise introduction). Passing to a further subsequence (which we will continue to denote by $t_j$ to simplify notation) we have that $\tilde{g}_{t_j}q$ converges to a stable curve in $\overline{\mathcal{M}}_2$. This curve projects to some torus in $\mathcal{M}_1$ (and not in its boundary $\overline{\mathcal{M}}_1 \setminus \mathcal{M}_1$) because the projection of $K$ to $\mathcal{M}_1$ is compact. So the limiting stable curve has area $1$. By [McM2, Theorem 1.4] the limit of $\tilde{g}_{t_j}q$ is not connected and so, considering the projection to $\mathcal{M}_1$ again, it is two tori with a node between them. Thus for all large $j$, the surfaces $M_{\tilde{g}_{t_j}q}$ are two copies of a torus $T_j \in \mathcal{K}$ glued along slits whose lengths are going to zero. These slits must be the union of two saddle connections that connect the two different singularities of $M_{\tilde{g}_{t_j}q}$. Indeed, the slit cannot project to a short curve on $T_j$ and it must be trivial in homology. Let $\delta_j$ denote one of the saddle connections that make up this slit, so that the other is $\iota(\delta_j)$. We have proven (i).

Now because the horizontal flow on $M_q$ is minimal, the $\delta_j$ are not horizontal, so we may assume they are all different. By the discreteness of holonomies of saddle connections this implies that $|\text{hol}(\delta_j)| \to \infty$. Because we have that $|\tilde{g}_{t_j} \delta_j| = |(e^{-t_j}x_j, e^{t_j}y_j)| \to 0$ we have that $y_j \to 0$ and so $x_j \to \infty$. Because the torus $T_j$ is in the compact set $\mathcal{K}$, the only short saddle connections of $M_{\tilde{g}_{t_j}q}$ are $\delta_j$ and $\iota(\delta_j)$ which implies the second assertion in (ii). This establishes (ii).
For the proof of (iii), let $\Upsilon_{t}^{(p)}$ denote the horizontal straightline flow. By the Birkhoff ergodic theorem, there is an increasing sequence $S_k \to \infty$ and an increasing sequence of subsets $E_k \subset M_q$ such that $\lim_{k \to \infty} \mu(M_q \setminus E_k) = 0$, $\Upsilon_{t}^{(p)}$ is defined for all $t \in \mathbb{R}$ and all $p \in E_k$, and for any $p = p_k \in E_k$, and any interval $I \subset \mathbb{R}$ around 0 of length $|I| \geq S_k$, the empirical measures $\eta_k = \eta(p, I)$ on $M_q$ defined by
\[
\int f d\eta_k = \frac{1}{|I|} \int_I f (\Upsilon_{t}^{(p)}) \, dt \quad (f \in C_c(M_q))
\]
satisfy
\[
\eta_k \to_{k \to \infty} \mu, \text{ with respect to the weak-\ast topology.} \quad (3.1)
\]

For each $j \in \mathbb{N}$, let
\[
M^{(0)} \overset{\text{def}}{=} M_q, \quad M^{(j)} \overset{\text{def}}{=} M_{\bar{g}, \bar{q}}, \quad \text{and let } \phi_j : M^{(0)} \to M^{(j)}
\]
be the affine comparison map corresponding to $\bar{g}_j$. Let $\sigma_j$ be the slit on $M^{(0)}$ as before, and $A_j, A'_j$ be the two tori comprising $M^{(0)} \setminus \sigma_j$ as in (i). Denote by $t$ the involution of Proposition 3.1, on both $M^{(0)}$ and $M^{(j)}$, so that $\phi_j$ commutes with $t$. Then $\phi_j(\sigma_j)$ is a slit on $M^{(j)}$ and its length $|\phi_j(\sigma_j)|$ satisfies $|\phi_j(\sigma_j)| \to 0$. Also let $C \overset{\text{def}}{=} \sup_j \text{diam}(M^{(j)})$, which is a finite number since $\pi(M^{(j)}) \in \mathcal{K}$ for all $j$. Let $\ell \subset \phi_j(A_j)$ be a vertical segment of length less than $C$, and let $\ell' \overset{\text{def}}{=} t(\ell)$. For each $x \in \ell \cup \ell'$ let $I(x)$ be the interval starting at 0, such that $H_x \overset{\text{def}}{=} \{ \Upsilon_{t}^{(x)} : t \in I(x) \}$ is the horizontal segment on $M^{(j)}$ starting at $x$ and ending at the first return to $\ell \cup \ell'$. Then the length of $I(x)$ is bounded above and below by positive constants independent of $j$ and $x$, and by adjusting $\ell$ and $C$ we can assume
\[
1 \leq |I(x)| \leq C.
\]
Let
\[
D_j \overset{\text{def}}{=} \{ x \in \ell \cup \ell' : \phi_j(\sigma_j) \cap H_x = \emptyset \}
\]
and
\[
B_j \overset{\text{def}}{=} \bigcup_{x \in D_j} H_x \quad \text{and} \quad \bar{B}_j \overset{\text{def}}{=} \bigcup_{x \in D_j \cap \ell} H_x.
\]
Then clearly $\iota(B_j) = B_j$ and moreover, since $|\phi_j(\sigma_j)| \to 0$, $\text{Leb}(B_j) \to 1$. Similarly $\bar{B}_j \subset \phi_j(A_j)$ and $\text{Leb}(\bar{B}_j) \to \frac{1}{2}$.

Let $k_j$ be the largest $k$ for which $e^{\ell_j} \geq S_k$. Then $k_j \to \infty$. By Proposition 3.4 we have $\text{Leb} = \mu + \iota_* \mu$ and so for large enough $j$, $\phi_j(E_{k_j} \cup \iota(E_{k_j})) \cap B_j \neq \emptyset$. Since $\iota(B_j) = B_j$ this implies $\phi_j(E_{k_j}) \cap B_j \neq \emptyset$.
\[ \phi_j(A_j \cap E_{k_j}) \cap B_j \neq \emptyset. \]

Let \( \mu_j \) be the normalized restriction of Leb to \( A_j \). Our goal is to show that for all \( \varepsilon > 0 \) and \( f \in C_c \left( M^{(0)} \right) \), for all \( j \) large enough we have

\[ \left| \int_{M^{(0)}} f d\mu_j - \int_{M^{(0)}} f d\mu \right| < \varepsilon, \tag{3.2} \]

and we assume with no loss of generality that \( \|f\|_\infty = 1 \).

Fix \( x_1 \in \phi_j(A_j \cap E_{k_j}) \cap B_j \) and let \( y_1 \overset{\text{def}}{=} \phi_j^{-1}(x_1) \). There is \( x \in \ell \cap D_j \) such that \( x_1 \in H_x \), and we let \( y \overset{\text{def}}{=} \phi_j^{-1}(x) \). Recall that \( \phi_j^{-1} \) maps horizontal and vertical straightline segments on \( M^{(j)} \) to horizontal and vertical straightline segments on \( M^{(0)} \), multiplying their lengths respectively by \( e^{\pm \ell_j} \). In particular \( \phi_j^{-1}(H_x) \) is a horizontal line segment on \( M^{(0)} \) of length at least \( e^{\ell_j} \) and containing \( y_1 \), and since \( y_1 \in E_{k_j} \), this implies that for \( j \) sufficiently large,

\[ \left| \frac{1}{e^{\ell_j}|I(x)|} \int_0^{e^{\ell_j}|I(x)|} f \left( \Upsilon_i^{(y)} \right) dt - \int_{M^{(0)}} f d\mu \right| < \frac{\varepsilon}{3}. \tag{3.3} \]

Let \( x' \in D_j \cap \ell \). Then there is a vertical segment from \( x \) to \( x' \) along \( \ell \), of length at most \( C \). This segment lies completely inside \( \phi_j(A_j) \). Furthermore, by considering the projection map \( M^{(j)} \to T_j \), we see that whenever the vertical straightline segment of length \( C \) starting at \( \Upsilon_i^{(x)} \) misses \( \sigma_j \), there is also a vertical segment from \( \Upsilon_i^{(x)} \) to \( \Upsilon_i^{(x')} \) of length at most \( C \), which lies completely inside \( \phi_j(A_j) \). Since \( |\phi_j(\sigma_j)| \to 0 \), this implies that there is a finite union of subintervals \( J = J(x') \subset I \), such that \( |J| = O(|\phi_j(\sigma_j)|) \to 0 \) and such that for all \( t \in I(x') \setminus J(x') \) there is a vertical line segment of length at most \( C \) from \( \Upsilon_i^{(x)} \) to \( \Upsilon_i^{(x')} \), and this segment stays completely in \( A_j \).

Thus for any \( x' \in D_j \cap \ell \), if we set \( y' \overset{\text{def}}{=} \phi_j^{-1}(x') \), then for all large enough \( j \) we have

\[ \frac{1}{e^{\ell_j}|I(x')|} \int_0^{e^{\ell_j}|I(x')|} \left| f \left( \Upsilon_i^{(y')} \right) - f \left( \Upsilon_i^{(y)} \right) \right| dt < \frac{\varepsilon}{3}. \tag{3.4} \]

Let \( \bar{\mu}_j \) be the normalized restriction of Leb to \( \phi_j^{-1}(B_j) \). Then using Fubini’s theorem to express \( \bar{\mu}_j \) as an integral of integrals along the
Since °B °p0°qă°φ °p0°q and °φ °1°p preserves Lebesgue measure, we have °φ °1°p°B °p0°qă°A °p0°q, Leb °φ °1°p°B °p0°qă°1 2 and hence for all large °j,,

\[ \left| \int_{M^{(0)}} f d\bar{\mu}_j - \int_{M^{(0)}} f d\mu \right| < \frac{\varepsilon}{3}. \tag{3.5} \]

Combining this with (3.5) gives (3.2). \hfill \Box

Similar ideas can be used to prove the following statement.

**Theorem 3.6.** Suppose °q ∈ °E is a surface for which the horizontal measured foliation is minimal but not ergodic. Then there is a sequence of decompositions of °M °q into pairs of tori °A °j, °B °j glued along a slit, and such that the set

\[ A_\infty = \bigcup_{i \geq i} A_j \]

is invariant under the horizontal flow, and has Lebesgue measure 1/2.

The statement will not be used in this paper and its proof is left to the reader.

4. Tremors

In this section we give a more detailed treatment of tremors and their properties.

4.1. Definitions and basic properties.

4.1.1. Semi-continuity of foliation cocycles. Let °q ∈ °H represent a surface °M °q with horizontal foliation °F °q. Recall from §2.5 that the transverse measures (respectively, signed transverse measures) define a cone °C °q of foliation cocycles (resp., a space °T °q of signed foliation cocycles) and these are subsets of \( H^1(M_q; \Sigma; \mathbb{R}_x) \). For a marking map °φ : °S → °M °q representing a marked translation surface \( \tilde{q} \in \pi^{-1}(q) \), the pullbacks \( \varphi^*(C_q^+), \varphi^*(T_q) \) are subsets of \( H^1(S, \Sigma; \mathbb{R}_x) \) and will be denoted by \( C_\tilde{q}^+, T_\tilde{q} \). Note that these notions are well-defined even at orbifold points (i.e. do not depend on the choice of the marking map) because translation equivalences map transverse measures to transverse measures. Recall that \( \beta \in C_\tilde{q}^+ \) is called non-atomic if \( \beta = \beta_\nu \) for a non-atomic transverse measure \( \nu \). We will mostly work with non-atomic
transverse measures as described in §2.5, and for completeness explain
the atomic case in §13.

Recall from §2.2 that for any \(q\), the tangent space \(T_q(\mathcal{H})\) at \(q\) is
identified with \(H^1(M_q, \Sigma_q; \mathbb{R}^2)\) (or with \(H^1(M_q, \Sigma_q; \mathbb{R}^2)/\Gamma_q\) if \(q\) is an
orbifold point), and that a marking map identifies the tangent space
\(T_{\tilde{q}}(\mathcal{H}_m)\), for \(\tilde{q}\) close to \(q\), with \(H^1(S, \Sigma; \mathbb{R}^2)\). The following proposition
expresses an important semi-continuity property for the cone of
foliation cocycles.

**Proposition 4.1.** The set
\[
C^+_\mathcal{H} \overset{\text{def}}{=} \left\{ (\tilde{q}, \beta) \in \mathcal{H}_m \times H^1(S, \Sigma; \mathbb{R}_x) : \beta \in C^+_\tilde{q} \right\}
\]
is closed. That is, suppose \(\tilde{q}_n \to \tilde{q}\) is a convergent sequence in \(\mathcal{H}_m\),
and let \(C^+_{\tilde{q}_n}, C^+_{\tilde{q}} \subset H^1(S, \Sigma; \mathbb{R}_x)\) be the corresponding cones. Suppose
that \(\beta_n \in H^1(S, \Sigma; \mathbb{R}_x)\) is a convergent sequence such that \(\beta_n \in C^+_{\tilde{q}_n}\) for
every \(n\). Then \(\lim_{n \to \infty} \beta_n \in C^+_{\tilde{q}}\).

Proposition 4.1 will be proved in §4.2 under an additional assumption
and in §13 in general. Note that care is required in formulating an
analogous property for \(\mathcal{T}_q\) because \(\dim \mathcal{T}_q\) can decrease when taking
limits. See Corollary 4.4.

### 4.1.2. Signed mass, total variation, and balanced tremors

We now define the **signed mass** and **total variation** of a signed foliation cocycle. Recall from §2 that \(dx = (dx)_q\) and \(\text{hol}^{(x)}_q\) denote respectively the
canonical transverse measure for the vertical foliation on a translation
surface \(q\), and the corresponding element of \(H^1(M_q, \Sigma_q; \mathbb{R})\). For \(q \in \mathcal{H}\)
and \(\beta \in H^1(M_q, \Sigma_q; \mathbb{R}),\) denote by \(L_q(\beta)\) the evaluation of the cup prod-
uct \(\text{hol}^{(x)}_q \cup \beta\) on the fundamental class of \(M_q\). In particular, if \(\beta = \beta_\nu\) for
a non-atomic signed transverse measure \(\nu\), \(L_q(\beta) = \int_{M_q} dx \wedge \nu;\) or
equivalently, if \(\mu = \mu_\nu\) is the horizontally invariant signed measure associated
with \(\nu\) by Proposition 2.3, then \(L_q(\beta) = \mu(M_q).\) We will refer to
\(L_q(\beta)\) as the **signed mass** of \(\beta\). Our sign conventions imply \(L_q(\beta) > 0\)
for any nonzero \(\beta \in C^+_q\).

Note that if \(h : M_q \to M_q\) is a translation equivalence then
\(L_q(\beta) = L_q(h^*(\beta))\). Thus, if \(\tilde{q} \in \pi^{-1}(q)\) is a marked translation surface repre-
sented by a marking map \(\varphi\), we can define \(L_{\tilde{q}}(\beta) \overset{\text{def}}{=} L_q(\varphi_\ast \beta)\) (where
\(\beta \in H^1(S, \Sigma; \mathbb{R}))\), and this definition does not depend on the choice of
the marking map \(\varphi\). In particular the mapping \((q, \beta) \mapsto L_q(\beta)\) defines
a map on \(T(\mathcal{H})\), even if \(q\) lies in an orbifold stratum.

Recall that every signed measure and every signed transverse measure
has a canonical Hahn decomposition \(\nu = \nu^+ - \nu^-\) as a difference
of measures. Thus any \( \beta \in \mathcal{T}_q \) can be written as \( \beta = \beta^+ - \beta^- \) where \( \beta^\pm \in \mathcal{C}^+_q \). As in the total variation of a measure we now define

\[
|L|_q(\beta) = L_q(\beta^+) + L_q(\beta^-),
\]

and call this the total variation of \( \beta \). Note that the signed mass is defined for every \( \beta \in \mathcal{H}^1(M_q; \Sigma; \mathbb{R}) \) but the total variation is only defined for \( \beta \in \mathcal{T}_q \). By linearity of the cup product, the maps

\[
T(\mathcal{H}) \to \mathbb{R}, \ (q, \beta) \mapsto L_q(\beta) \quad \text{and} \quad T(\mathcal{H}_m) \to \mathbb{R}, \ (\tilde{q}, \beta) \mapsto L_q(\beta)
\]

are both continuous. In combination with Proposition 4.1, this implies:

**Corollary 4.2.** The sets

\[
C^+_q \overset{\text{def}}{=} \{(q, \beta) : \beta \in \mathcal{C}^+_q, \ L_q(\beta) = 1\}
\]

and

\[
C^+_q \overset{\text{def}}{=} \{(\tilde{q}, \beta) : \beta \in \mathcal{C}^+_\tilde{q}, \ L_\tilde{q}(\beta) = 1\}
\]

are closed, and thus define closed subsets of \( T(\mathcal{H}_m) \) and \( T(\mathcal{H}) \).

The following special case will be important in the proofs of Theorem 1.2 and Theorem 1.4.

**Corollary 4.3.** Suppose \( q \in \mathcal{H} \) is a translation surface of area one whose horizontal foliation is uniquely ergodic, and denote its canonical foliation cocycle by \( \text{hol}^y_q \). Then for any sequence \( q_n \in \mathcal{H} \) such that \( q_n \to q \), and any \( \beta_n \in \mathcal{C}^+_q \), with \( L_q(\beta_n) = 1 \), we have \( \beta_n \to \text{hol}^y_q \).

The total variation of a tremor also has a continuity property:

**Corollary 4.4.** Suppose \( \tilde{q}_n \to \tilde{q} \) in \( \mathcal{H}_m \) and \( \beta_n \in \mathcal{T}_{\tilde{q}_n} \subset \mathcal{H}^1(S, \Sigma; \mathbb{R}) \) is a sequence of non-atomic signed foliation cocycles for which the limit \( \beta = \lim_{n \to \infty} \beta_n \) exists and \( \sup_n |L|_{\tilde{q}_n}(\beta_n) < \infty \). Then \( \beta \in \mathcal{T}_\tilde{q} \) and

\[
|L|_{\tilde{q}}(\beta) = \lim_{n \to \infty} |L|_{\tilde{q}_n}(\beta_n)
\]

(in particular, the limit exists).

Corollary 4.4 will also be proved in §4.2.

We say that \( \beta \in \mathcal{T}_q \) is balanced if \( L(\beta) = 0 \), and let \( \mathcal{T}_q^{(0)} \) denote the set of balanced signed foliation cocycles. Combining Propositions 3.4 and 3.3, for surfaces in \( \mathcal{E} \) we have the following description of balanced tremors:

**Corollary 4.5.** Let \( \mathcal{O} \) be an orbifold substratum of \( \mathcal{H} \) and \( q \in \mathcal{O} \). Then

\( \mathcal{T}_q \cap \mathcal{N}_q(\mathcal{O}) = \mathcal{T}_q^{(0)} \), with equality in the case \( \mathcal{O} = \mathcal{E} \); namely, if \( q \in \mathcal{E} \) is aperiodic then \( \mathcal{T}_q^{(0)} = \mathcal{N}_q(\mathcal{E}) \).
Proof. Let \( q \in \mathcal{O} \), let \( \Gamma_q \) be the group of translation equivalences of \( M_q \), let \( \mathcal{G} \overset{\text{def}}{=} \mathcal{G}_q \) be the local group as in \( \S 2.3 \) and let \( \gamma \in \mathcal{G} \). Recall that \( \Gamma_q \) and \( \mathcal{G} \) are isomorphic and by fixing a marking map, we can think of \( \gamma \) simultaneously as acting on \( M_q \) by translation automorphisms, and on \( H^1(S, \Sigma; \mathbb{R}^2) \) by the natural map induced by a homeomorphism. Since translation automorphisms of \( M_q \) preserve the canonical transverse measure \( (dx)_q \), we have \( \gamma^{\ast} \text{hol}_{q}^{(x)} = \text{hol}_{q}^{(x)} \), and thus for any \( \beta \),

\[
L_{q}(\gamma^{\ast}\beta) = (\text{hol}_{q}^{(x)} \cup \gamma^{\ast}\beta)(M_q) = (\text{hol}_{q}^{(x)} \cup \beta)(\gamma(M_q)) \\
= (\text{hol}_{q}^{(x)} \cup \beta)(M_q) = L_{q}(\beta).
\]

Hence, if \( \beta \in \mathcal{T}_q \cap \mathcal{M}_q(\mathcal{O}) \) then \( P^{+}(\beta) = 0 \), where \( P^{+} \) is as in (2.3), and we have

\[
L_{q}(\beta) = \frac{1}{|\mathcal{G}|} \sum_{\gamma \in \mathcal{G}} L_{q}(\gamma^{\ast}(\beta)) = L_{q} \left( \frac{1}{|\mathcal{G}|} \sum_{\gamma \in \mathcal{G}} \gamma^{\ast}(\beta) \right) = L_{q}(P^{+}(\beta)) = 0.
\]

Therefore \( \beta \in \mathcal{T}_q^{(0)} \).

Now if \( q \in \mathcal{E} \) is aperiodic and \( \beta \in \mathcal{T}_q^{(0)} \), then we can write \( \beta = \beta_{\nu} \) for a signed transverse measure \( \nu \), and let \( \mu = \mu_{\nu} \) be the associated horizontally invariant signed measure (see Proposition 2.3). Since \( \beta \in \mathcal{T}_q^{(0)} \) we have \( \mu(M_q) = 0 \). Recall from Proposition 3.4 that aperiodic surfaces in \( \mathcal{E} \) are either uniquely ergodic, or have two ergodic measures which are exchanged by the involution \( \iota = \iota_{q} \). By ergodic decomposition (applied to each summand in \( \mu = \mu^{+} - \mu^{-} \)) we can write \( \mu \) as a linear combination of ergodic measures (where the coefficients may be negative). If \( M_q \) is uniquely ergodic then this gives \( \mu = c \cdot \text{Leb} \) and since \( \mu(M_q) = 0 \) we have \( \mu = 0 \). If \( M_q \) has two ergodic probability measures \( \mu_1 \) and \( \mu_2 = \iota_{\ast}\mu_1 \) then \( \mu = c_1\mu_1 + c_2\iota_{\ast}\mu_1 \) and

\[
0 = \mu(M_q) = c_1\mu_1(M_q) + c_2\mu_1(\iota(M_q)) = c_1 + c_2,
\]

so \( c_1 = -c_2 \). In both cases we obtain \( \iota_{\ast}\mu = -\mu \), which implies \( \iota_{\ast}\beta = -\beta \). Thus, using Proposition 3.3, we see that \( \beta \in \mathcal{M}_q(\mathcal{E}) \). \( \square \)

4.1.3. Absolutely continuous foliation cycles. Let \( \nu_1 \) and \( \nu_2 \) be two signed transverse measures for \( \mathcal{F}_q \). We say that \( \nu_1 \) is absolutely continuous with respect to \( \nu_2 \) if the corresponding signed measures \( \mu_{\nu_1}, \mu_{\nu_2} \) given by Proposition 2.3 satisfy \( \mu_{\nu_1} \ll \mu_{\nu_2} \). We say that \( \nu \) is absolutely continuous if it is absolutely continuous with respect to the canonical transverse measure \( (dy)_q \). Since \( (dy)_q \) is non-atomic, so is any absolutely continuous signed transverse measure. For \( c > 0 \), we say \( \nu \) is
c-absolutely continuous if

\[ \int_{\gamma} d\nu \leq c \int_{\gamma} dy . \] (4.3)

We call a signed foliation cocycle \( \beta = \beta_\nu \) absolutely continuous (respectively, c-absolutely continuous) if it corresponds to a signed transverse measure \( \nu \) which is absolutely continuous (resp., c-absolutely continuous). Let \( \| \nu \|_{RN} \) denote the minimal \( c \) such that the above equation holds for all transverse arcs \( \gamma \) (our notation stems from the fact that \( \| \nu \|_{RN} \) is the sup-norm of the Radon-Nikodym derivative \( \frac{d\mu}{d\text{Leb}} \), although we will not be using this in the sequel). Given \( q \in H \) and \( c > 0 \), denote by \( C^+_q, RN(c) \) (respectively, by \( T^+_q, RN(c) \)) the set of absolutely continuous (signed) foliation cocycles \( \beta_\nu \) with \( \| \nu \|_{RN} \leq c \).

It is easy to see that

\[ C^+_q, RN(c) \subset \{ \beta \in C^+_q : L_q(\beta) \leq c \} \] (4.4)

and

\[ T^+_q, RN(c) \subset \{ \beta \in T_q : |L_q(\beta) | \leq c \} . \] (4.5)

As we will see in Lemma 8.3, for some surfaces we will also have a reverse inclusion.

We now observe that for aperiodic surfaces, the assumption of absolute continuity implies a uniform bound on the Radon-Nikodym derivative:

**Lemma 4.6.** Suppose \( M_q \) is a horizontally aperiodic surface, \( \nu \) is an absolutely continuous transverse measure, and \( \mu = \mu_\nu \) is the corresponding measure on \( M_q \), so that \( \mu \ll \text{Leb} \). Then there is \( c > 0 \) such that \( \| \nu \|_{RN} \leq c \). Moreover the constant \( c \) depends only on the coefficients appearing in the ergodic decompositions of \( \mu \) and \( \text{Leb} \), and if \( \mu \) is a probability measure and \( \text{Leb} = \sum a_i \nu_i \), where \( \{ \nu_i \} \) are the horizontally invariant ergodic probability measures and each \( a_i \) is positive, then \( \| \nu \|_{RN} \leq \max_i \frac{1}{a_i} \). The same conclusions hold if instead of assuming \( M_q \) is aperiodic, we assume the measure \( \nu \) is aperiodic, that is \( \mu \) assigns zero measure to any horizontal cylinder on \( M_q \).

**Proof.** Let \( \{ \mu_1, \ldots, \mu_d \} \) be the invariant ergodic probability measures for the horizontal straightline flow on \( M_q \). Since \( M_q \) is horizontally aperiodic, this is a finite collection, see e.g. [K]. Thus there only finitely many ergodic measures which are absolutely continuous with respect to \( \mu \), and we denote them by \( \{ \mu_1, \ldots, \mu_k \} \). The measures \( \mu_i \) are mutually singular. Write \( \text{Leb} = \sum_i a_i \mu_i \) and \( \mu = \sum_i b_i \mu_i \), where all \( a_i, b_j \) are non-negative and not all are zero. Since \( \mu \ll \text{Leb} \), we have

\[ b_i > 0 \implies a_i > 0 . \]
Set
\[ c = \max \left\{ \frac{b_i}{a_i} : b_i \neq 0 \right\}. \tag{4.6} \]

For any Borel set \( A \subseteq M_q \) we have
\[ \mu(A) = \sum_i b_i \mu_i(A) \leq c \sum_i a_i \mu_i(A) = c \text{Leb}(A). \]

This implies that the Radon Nikodym derivative satisfies \( \frac{d\mu}{d\text{Leb}} \leq c \) a.e. The horizontal invariance of \( \mu \) and \( \text{Leb} \) shows that the Radon-Nikodym derivative \( \frac{d\mu}{d\text{Leb}} \) is defined on almost every point of every transverse arc \( \gamma \), and the relation (2.6) shows that it coincides with the Radon-Nikodym derivative \( \frac{d\nu}{dy} \). Thus we get (4.3).

The second assertion follows from (4.6), and the last assertion follows by letting \( \mu_i \) denote the horizontally invariant measures on complement of the horizontal cylinders in \( M_q \), and repeating the argument given above. \( \square \)

4.1.4. Tremors as affine geodesics, and their domain of definition. Recall from §2.2 that we identify \( T(\mathcal{H}_m) \) with \( \mathcal{H}_m \times H^1(S, \Sigma; \mathbb{R}^2) \). Our particular interest is in affine geodesics tangent to signed foliation cocycles. That is, we take \( \beta \in \mathcal{T}_q \subseteq H^1(S, \Sigma; \mathbb{R}_x) \) (where the last inclusion uses a marking map \( \varphi : S \to M_q \) representing \( \tilde{q} \)). We write \( v = (\beta, 0) \in H^1(S, \Sigma; \mathbb{R}^2) \) and consider the parameterized line \( \theta(t) \) in \( \mathcal{H}_m \) satisfying \( \theta(0) = \tilde{q} \) and \( \frac{d}{dt} \theta(t) = v \) (where we have again used the marking to identify the tangent space \( T_{\theta(t)} \mathcal{H}_m \) with \( H^1(S, \Sigma; \mathbb{R}^2) \)). Unraveling definitions, for an element \( \gamma \in H_1(S, \Sigma) \) we find
\[ \text{hol}_{\theta(t)}^{(x)}(\gamma) = \text{hol}_{\tilde{q}}^{(x)}(\gamma) + t \cdot \beta(\gamma), \quad \text{hol}_{\theta(t)}^{(y)}(\gamma) = \text{hol}_{\tilde{q}}^{(y)}(\gamma). \tag{4.7} \]

By the uniqueness of solutions of differential equations, these equations uniquely define the affine geodesic \( \theta(t) \) for \( t \) in the maximal domain of definition \( \text{Dom}(\tilde{q}, v) \). We will denote \( \text{trem}_{\tilde{q}, \beta}(t) \) \( \equiv \theta(t) \). The mapping class group \( \text{Mod}(S, \Sigma) \) acts on each coordinate of \( T(\mathcal{H}_m) = \mathcal{H}_m \times H^1(S, \Sigma; \mathbb{R}^2) \), and by equivariance we find that \( \text{trem}_{\beta, q}(t) = \pi(\text{trem}_{\beta, \tilde{q}}(t)) \) and \( \text{Dom}(q, \beta) \equiv \text{Dom}(\tilde{q}, \beta) \) are well-defined, independently of the choice of \( \tilde{q} \in \pi^{-1}(q) \). As in the introduction, we denote \( \text{trem}_{\tilde{q}, \beta} \equiv \text{trem}_{\tilde{q}, \beta}(1) \) when \( 1 \in \text{Dom}(\tilde{q}, \beta) \). Comparing (4.7) with the defining equation (1.5), we see that we have given a formal definition of the tremor maps introduced in §1.

Basic properties of ordinary differential equations now give us:

**Proposition 4.7.** The set
\[ D = \{(\tilde{q}, v, s) \in T(\mathcal{H}_m) \times \mathbb{R} : s \in \text{Dom}(\tilde{q}, v)\} \]
is open in $T(\mathcal{H}_m) \times \mathbb{R}$, and the map
$$D \ni (\tilde{q}, v, s) \mapsto \tilde{q}_v(s)$$
is continuous. In particular the tremor map
$$\{(\tilde{q}, \beta) \in T\mathcal{H}_m : \beta \in \mathcal{T}_q \} \rightarrow \mathcal{H}_m ; \ (\tilde{q}, \beta) \mapsto \text{trem}_{\tilde{q}, \beta}$$is continuous where defined.

It is of interest to identify the set $\text{Dom}(q, \beta)$. In this regard we have:

**Proposition 4.8.** If $\beta \in \mathcal{T}_q$ is non-atomic then $\text{Dom}(q, \beta) = \mathbb{R}$.

The assumption in Proposition 4.8 that $\beta$ is non-atomic is important here, see §13. Proposition 4.8 follows from Proposition 5.1 which will be stated and proved below. It can also be deduced from [MW2, Thm. 1.2], which gives a general criterion for lifting a straightline path in $H^1(S, \Sigma; \mathbb{R}_x)$ to $\mathcal{H}_m$. Comparing (4.7) to the definition of the horocycle flow in period coordinates, we immediately see that for the canonical foliation cycle $dy = \text{hol}_{\tilde{q}}^{(q)}$, we have
$$\text{trem}_{\tilde{q}, s}dy = u_s\tilde{q}.$$ (4.8)

### 4.2. Tremors and polygonal presentations of surfaces

In this section we prove Proposition 4.1, under an additional hypothesis. This special case is easier to prove and suffices for proving our main results. We will prove the general case of Proposition 4.1 in §13. At the end of this section we deduce Corollary 4.4 from Proposition 4.1.

**Proposition 4.9.** Let $\tilde{q}_n \rightarrow \tilde{q}$ in $\mathcal{H}_m$, $\beta_n \rightarrow \beta$ in $H^1(S, \Sigma; \mathbb{R}_x)$ be as in the statement of Proposition 4.1. Write $q_n = \pi(\tilde{q}_n), q = \pi(\tilde{q})$ and suppose also that
there is $c > 0$ such that for all $n$, $\beta_n \in C^{+, R\mathbb{N}}_{q_n}(c)$. (4.9)
Then $\beta \in C^{+, R\mathbb{N}}_q(c)$.

Clearly Proposition 4.9 implies Proposition 4.1 in the case that (4.9) holds.

**Proof.** We will write $\beta_n = \beta_{\nu_n}$ for a sequence of $c$-absolutely continuous transverse measures $\nu_n$ on $M_{q_n}$ (in particular the $\nu_n$ are non-atomic). Our goal is to prove that there is a transverse measure $\nu$ on $M_q$ such that $\beta = \beta_\nu$. It suffices to consider the restriction of the transverse measure to a particular finite collection of transverse arcs, which we now describe.
Recall that any translation surface has a polygon decomposition. In such a decomposition, some edges might be horizontal, and corresponding edges on nearby surfaces may intersect the horizontal foliation with different orientations. This will cause complications and in order to avoid them, we introduce a special kind of polygon decomposition, which we will call an adapted polygon decomposition (APD). An APD is a polygon decomposition in which all polygons are either triangles with no horizontal edges, or quadrilaterals with one horizontal diagonal. Any surface has an APD, as can be seen by taking a triangle decomposition and merging adjacent triangles sharing a horizontal edge into quadrilaterals. We fix an APD of \( M_{q_n} \), with a finite collection of edges \( \{J_i\} \), all of which are transverse to the horizontal foliation on \( M_q \). Since we are considering marked surfaces, we can use a marking map and the comparison maps of §2.2 and think of the arcs \( J_i \) as arcs on \( S \), as well as on \( M_{q_n} \), for all large enough \( n \). Moreover for all large \( n \), the edges \( \{J_i\} \) are also a subset of the edges of an APD on \( M_{q_n} \) and they are also transverse to the horizontal foliation on \( M_{q_n} \). Note that on \( M_{q_n} \) the APD may contain additional edges that are not edges on \( M_q \), namely some of the horizontal diagonals on \( M_q \) might not be horizontal on \( M_{q_n} \) and in this case we add them to the \( \{J_i\} \) to obtain an APD on \( M_{q_n} \).

Since the polygons of a polygon decomposition are simply connected, a 1-cochain representing an element of \( H^1(S, \Sigma; \mathbb{R}) \) is determined by its values on the edges of the polygons. For each \( i \), each polygon \( P \) of the APD with \( J = J_i \subset \partial P \), and each \( x \in J \), there is a horizontal segment in \( P \) with endpoints in \( \partial P \) one of which is \( x \). The other endpoint of this segment is called the opposite point (in \( P \)) to \( x \) and is denoted by \( \text{opp}_P(x) \). The image of \( J \) under \( \text{opp}_P \) is a union of one or two sub-arcs contained in the other boundary edges of \( P \). The transverse measures \( \nu_n \) assign a measure to each \( J \). We will denote this either by \( \nu_n \), or by \( \nu_n|_J \) when confusion may arise. By definition of a transverse measure,

\[
(\text{opp}_P)_* \nu_n|_J = \nu_n|_{\text{opp}_P(J)},
\]

and this holds for any \( n, P \) and \( J \). We call (4.10) the invariance property. Conversely, given an APD for a translation surface \( M_q \), suppose we are given a collection of finite non-atomic measures \( \nu_J \) on the edges \( J \) as above, satisfying the invariance property. We can reconstruct from the \( \nu_J \) a transverse measure, by homotoping any transverse arc to subintervals of edges of the APD (this will be well-defined in view of the invariance property). Note that in this section, all measures under consideration are non-atomic, and we will not have to worry about whether intervals are open or closed (but in §13 this will be a concern).
Let $\tau$ be the triangulation of $M_q$ obtained by adding the horizontal diagonals to quadrilaterals in the APD. As discussed in §2.2, using $\tau$ and marking maps, we obtain maps $\varphi_n : S \to M_{q_n}$, $\varphi : S \to M_q$, such that for each $n$, the comparison map $\varphi_n \circ \varphi^{-1} : M_q \to M_{q_n}$ is piecewise affine, with derivative (in planar charts) tending to the identity map as $n \to \infty$. Let $P$ be one of the polygons of the APD and $K \subset \partial P$ a subinterval of the form $J$ or opp$_P(J)$ as above. For all large enough $n$, none of the sides $\varphi_n \circ \varphi^{-1}(K)$ are horizontal and all have the same orientation as on $M_q$. Let $\nu_{K}^{(n)}$ be the measure on $\varphi_n \circ \varphi^{-1}(K)$ corresponding to $\nu_n$. Using the marking $\varphi^{-1}_n$ we will also think of $\nu_{K}^{(n)}$ as a measure on $\varphi^{-1}_n(K)$.

Passing to subsequences and using the compactness of the space of measures of bounded mass on a bounded interval, we can assume that for each $K$, the sequence $(\nu_{K}^{(n)})_n$ converges to a measure $\nu_K$ on $\varphi^{-1}(K)$. It follows from (4.9) that $\nu_K$ is non-atomic, indeed it is $c$-absolutely continuous since all the $\nu_{K}^{(n)}$ are. Each of the measures $\nu_{K}^{(n)}$ satisfies the invariance property for the horizontal foliation on $M_{q_n}$, and we claim:

**Claim 4.10.** The measures $\nu_K$ satisfy the invariance property for the horizontal foliation on $M_q$.

To see this, suppose $K = J$ in the above notation, the case $K = \text{opp}_P(J)$ being similar. For each $n$ let $\text{opp}_P^{(n)}$ be the map corresponding to the horizontal foliation on $M_{q_n}$; it maps $J$ to a subset of an edge or two edges of the APD. Let $I$ be a compact interval contained in the interior of $J$. Then for all sufficiently large $n$, $\text{opp}_P^{(n)}(I) \subset \text{opp}_P(J)$, and the maps $\text{opp}_P^{(n)}|I$ converge uniformly to $\text{opp}_P|I$. By our assumption that the measure is non-atomic, the endpoints of $I$ have zero $\nu_J$-measure. Therefore, since $\nu_{J}^{(n)} \to \nu_J$, by the Portmanteau theorem we have $\nu_j(I) = \nu_{\text{opp}_P(j)}(\text{opp}_P(I))$. Such intervals $I$ generate the Borel $\sigma$-algebra on $J$, and so we have established the invariance property. This proves Claim 4.10.

Since the $\nu_K$ satisfy the invariance property, they define a transverse measure $\nu$, and we let $\beta' = \beta_{\nu}$. Recall that we have assumed $\beta_n \to \beta$ as cohomology classes in $H^1(S, \Sigma; \mathbb{R})$. For each edge $J$ of the APD,

$$\beta(J) \leftarrow \beta_n(J) = m_{J}^{(n)} \to m_J = \beta'(J),$$

and so $\beta' = \beta$. 

*Proof of Corollary 4.4 assuming Proposition 4.1.* For each $n$ write $\beta_n = \beta_{\nu_n}$ where $\nu_n$ is a transverse measure on $M_{q_n}$, let $\nu_n = \nu_n^+ - \nu_n^-$ be the
Hahn decomposition and let \( \mu_n^\pm \) be the horizontally invariant measure corresponding to \( \nu_n^\pm \) via Proposition 2.3. By assumption,
\[
\mu_n^\pm(M_{q_n}) = L_{\tilde{q}_n}(\beta_n^\pm) \leq |L_{\tilde{q}_n}(\beta_n)|
\]
is a bounded sequence. Using the comparison maps \( \varphi^{-1} \circ \varphi_n : M_{q_n} \to M_q \) used in the preceding proof, we can think of the \( \mu_n^\pm \) as measures on \( M_q \) with a uniform bound on their total mass, and we can pass to a subsequence to obtain \( \mu_n^\pm \to \mu^\pm \), and hence, using Proposition 4.1, \( \nu_n^\pm \to \nu^\pm \) for transverse measures \( \nu^\pm \) on \( M_q \). More precisely, the fact that the \( \nu_n \) converge to transverse measures on \( M_q \) follows from the proof of Proposition 4.1 under assumption (4.9), and the general case follows from the proof of Proposition 4.1 given in §13.

Let \( \nu = \nu^+ - \nu^- \) and let \( \beta' = \beta_\nu \). Using Proposition 4.1 we have \( \beta_n \to \beta' \). But since we have assumed \( \beta = \lim_n \beta_n \), we have \( \beta = \beta' \in \mathcal{T}_q \). Finally, since the \( \nu_n^\pm \) are (non-negative) transverse measures, \( \nu = \nu^+ - \nu^- \) is the Hahn decomposition of \( \nu \), and we have
\[
|L_{\tilde{q}}(\beta)| = |L_{\tilde{q}}(\beta') = L_{\tilde{q}}(\beta^-_\nu) + L_{\tilde{q}}(\beta^+_\nu) = \lim_{j \to \infty} \left( L_{\tilde{q}_{n_j}}(\beta^+_n) + L_{\tilde{q}_{n_j}}(\beta^-_n) \right) = \lim_{j \to \infty} |L_{\tilde{q}_{n_j}}(\beta_n)|.
\]
In particular the limit is independent of the choice of the subsequence. This proves (4.2). \( \square \)

5. The tremor comparison homeomorphism, and tremors on different surfaces

In the sequel we will analyze the interaction of tremors with certain other maps on \( \mathcal{H} \). Loosely speaking, for surfaces \( M_q \) and \( M_{q'} \) which ‘share the same horizontal foliation’, we will need to consider a signed foliation cocycle \( \beta \) simultaneously as an element of \( \mathcal{T}_q \) and \( \mathcal{T}_{q'} \), and compare trem\(_q,\beta\) and trem\(_{q'},\beta\). The following proposition will make it possible to make this idea precise.

**Proposition 5.1.** Let \( q_0 \in \mathcal{H} \), let \( M_0 = M_{q_0} \) be the underlying surface, let \( \varphi_0 : S \to M_0 \) be a marking map and let \( \tilde{q}_0 \in \pi^{-1}(q_0) \) be the corresponding marked translation surface. Let \( \nu \) be a non-atomic signed transverse measure on the horizontal foliation of \( M_0 \), and let \( \beta = \beta_\nu \). Let \( \kappa_t = \text{trem}_{q_0,t,\beta} \), \( \tilde{\kappa}_t = \text{trem}_{\tilde{q}_0,t,\beta} \), let \( M_t = M_{\kappa_t} \) be the underlying surface, and let \( \varphi_t : S \to M_t \) be a marking map representing \( \tilde{\kappa}_t \). Denote \( \text{hol}_{M_t} = \left( \text{hol}_t^{(x)}, \text{hol}_t^{(y)} \right) \). Then there is a unique homeomorphism
\( \psi_t : M_0 \to M_t \) which is isotopic to \( \varphi_t \circ \varphi_0^{-1} \) and satisfies

\[
\text{hol}_{t}^{(x)}(\psi_t(\gamma)) = \text{hol}_{0}^{(x)}(\gamma) + t \int_{\gamma} \nu \quad \text{and} \quad \text{hol}_{t}^{(y)}(\psi_t(\gamma)) = \text{hol}_{0}^{(y)}(\gamma) \quad (5.1)
\]

for any piecewise smooth path \( \gamma \) in \( M_0 \) between any two points.

**Definition 5.2.** We call \( \psi_t : M_0 \to M_t \) the tremor comparison homeomorphism (TCH).

The uniqueness of a tremor comparison homeomorphism implies the following important naturality property:

**Proposition 5.3.** With the notation of Proposition 5.1, suppose \( \varphi_0 \) and \( \varphi_0' \) are two different marking maps \( S \to M_0 \) and \( \psi_t, \psi'_t \) are the corresponding TCH’s. Then \( \psi_t = \psi'_t \circ \varphi_0' \circ \varphi_0^{-1} \).

**Remark 5.4.** When \( q \) belongs to an orbifold substratum (e.g. for \( q \in \mathcal{E} \)), two different choices of the marking map \( \varphi_0 \) give TCH’s that differ by the action of the local group, that is by translation equivalences. The reader may want to ignore this complication on a first reading, as it will complicate our notation in the sequel, but cause no essential difficulties.

**Proof of Proposition 5.1.** We begin by proving the existence of \( \psi_t \). Let \( \tau \) be a triangulation of \( S \) obtained as the pullback via \( \varphi_0 \) of a geodesic triangulation on \( M_0 \), and let \( U_\tau \subset H^1(S, \Sigma; \mathbb{R}^2) \) and \( V_\tau \subset \mathcal{H}_m \) be the open sets as in §2.2. For a sufficiently small interval \( I \) around 0 we have

\[
\{ \text{trem}_{q,t,\beta} : t \in I \} \subset V_\tau, \quad (5.2)
\]

and we will first prove the existence of \( \psi_t \) for \( t \in I \) where \( I \) satisfies (5.2). The existence for all \( t \) then follows by constructing the maps on small intervals \( I_1, \ldots, I_N \), covering the interval from 0 to \( t \), where each \( I_j \) satisfies (5.2) for some triangulation \( \tau_j \), and composing the corresponding maps.

Let

\[
\beta_t \overset{\text{def}}{=} \text{dev}(\overline{q}) + t\beta = \text{dev}(\text{trem}_{q,t,\beta}).
\]

In the notations of §2.2, we have \( \beta_t \in U_\tau \) for all \( t \in I \) and we can identify \( M_t \) with \( M_{\tau,\beta} \) and take \( \varphi_t \) to be the piecewise affine marking map \( \varphi_{\tau,\beta} \). Combining (4.7) with the definition of \( \beta_\nu \) in §2.5, we see that the maps \( \varphi_t \circ \varphi_0^{-1} \) satisfy (5.1) for all \( \gamma \in H_1(S, \Sigma) \). However, we would like (5.1) to hold for any piecewise smooth path on \( M_0 \) (not necessarily closed or joining singular points). In order to achieve this, we will modify the maps \( \varphi_t \) by an isotopy which moves points in \( M_0 \) along their horizontal leaf. The isotopy will be parameterized by \( s \in [0, 1] \). Instead of attempting to write down an explicit formula for the isotopy,
we will define it indirectly, using singular cohomology and working with the universal cover of $S$.

As discussed in §2.5, the 1-form $dx$ and the transverse measure $\nu$ on $M_0$ can both be thought of as cohomology classes. They can also be thought of as 1-cocycles in singular cohomology; i.e., they assign a real number $\int_{\gamma} dx$ to any piecewise smooth path $\gamma$ in $M_0$. Note that this relies on our assumption that $\nu$ is non-atomic. Denote by $dx_t$ and $dy_t$ the pullbacks to $S$ of the cocycles $dx$ and $dy$ on $M_t$ by $\varphi_t$, let $F$ be the pullback of the horizontal foliation on $M_0$ and let $\nu_0$ denote the pullback of the transverse measure. Define two families of singular $\mathbb{R}^2$-valued cocycles $a_t$ and $b_t$ on $S$ as follows:

$$a_t : \gamma \mapsto \left( \int_{\gamma} dx_t, \int_{\gamma} dy_t \right)$$

and

$$b_t : \gamma \mapsto \left( \int_{\gamma} td\nu_0 + dx_0, \int_{\gamma} dy_0 \right).$$

That is, $a_t$ is the cocycle corresponding to the flat surface structure on $M_t$ and the explicit marking map $\varphi_t$, and $b_t$ is the cocycle which would correspond to a marking map $\psi_t$ for which the desired formula (5.1) holds. Also define a two-parameter family of singular $\mathbb{R}^2$-valued cocycles, interpolating between $a_t$ and $b_t$, by

$$\xi_{s,t} = (1 - s)a_t + sb_t \quad (\text{where } s \in [0, 1], \ t \in I).$$

Let $\tilde{S}$ denote the universal cover of $S$, let $\tilde{\Sigma} \subset \tilde{S}$ denote the pre-image of $\Sigma$ under the covering map, and let $p_0 \in \tilde{\Sigma}$. Given a singular $\mathbb{R}^2$-valued cocycle $\alpha$ on $S$ we can pull it back to a class $\tilde{\alpha}$ on $\tilde{S}$ by the covering map. Since $\tilde{S}$ is contractible its first cohomology vanishes, thus the cocycle $\tilde{\alpha}$ is a coboundary and we can find a function $f : \tilde{S} \to \mathbb{R}^2$ that solves the equation $\delta(f) = \tilde{\alpha}$, where $\delta$ is the coboundary operator in singular cohomology. Since $\tilde{S}$ is connected any two solutions to this equation differ by a constant, and we can fix $f$ uniquely by requiring $f(p_0) = 0$. We call $f$ the candidate developing map corresponding to $\alpha$. Our terminology is motivated by the fact that the candidate developing map corresponding to the singular cohomology class associated with the marking map $\varphi : S \to \hat{M}$, is a special case of the ‘developing map’ defined by Thurston, see [T1, §3.5]; however not all candidate covering maps are developing maps. Note that $f$ is continuous, and precomposition by a deck transformation changes $f$ by a constant.

We let $f_{s,t}$ denote the candidate developing map corresponding to $\xi_{s,t}$. It follows from the definition of the cocycles $\xi_{s,t}$ that $f_{s,t}$ is a local
homeomorphism near nonsingular points, that is for every $x \in \tilde{S} \setminus \tilde{\Sigma}$ there is a neighborhood $U$ of $x$ such that $f_{s,t}|_U$ is a homeomorphism onto an open subset of $\mathbb{R}^2$. Furthermore, the family of maps $\{f_{s,t} : s \in [0,1], t \in I\}$ is uniformly equicontinuous: for $x_1, x_2 \in \tilde{S}$, the difference $\|f_{s,t}(x_1) - f_{s,t}(x_2)\|$ can be bounded from above by an expression involving the length of a path $\gamma$ connecting $x_1$ and $x_2$, and the transverse measures of $\nu$ along $\gamma$ and along the edges of the triangulation $\tau$.

Pulling back $F$ by the covering map we get a foliation of $\tilde{S} \setminus \tilde{\Sigma}$, which we continue to call the horizontal foliation. Each map $f_{s,t}$ inherits the following properties from $a_t$ and $b_t$:

- It maps horizontal leaves in $\tilde{S} \setminus \tilde{\Sigma}$ to horizontal lines on $\mathbb{R}^2$.
- It is monotone increasing on each horizontal leaf.
- It is proper, that is, sends intervals that are bounded (resp. unbounded) on the left (resp. right) to intervals that are bounded (resp. unbounded) on the left (resp. right).

**Claim:** There is a continuous family of homeomorphisms $\tilde{g}_{s,t}$ of $\tilde{S}$ (where $s \in [0,1], t \in I$), such that $\tilde{g}_{s,t}|_\tilde{\Sigma} = \text{Id}$, and such that the functions $f_{s,t}$ and $f_{0,t}$ satisfy $f_{s,t} = f_{0,t} \circ \tilde{g}_{s,t}$.

**Proof of claim:** By a closed interval in $\mathbb{R}$ we mean a closed connected subset containing more than one point (i.e. it may be all of $\mathbb{R}$, or a compact interval with nonempty interior, or a closed ray). We note that two monotone increasing proper functions $f_0$ and $f_1$ on a closed interval in $\mathbb{R}$ satisfy $f_1 = f_0 \circ g$ for some homeomorphism $g$, if and only if they have the same range, and in this case we can recover $g$ as

$$g(x) = f_0^{-1}(f_1(x)).$$  \hfill (5.3)

Motivated by this observation, we can construct the isotopy $\tilde{g}_{s,t}$ on the closure $\tilde{L}$ of a single horizontal leaf $\mathcal{L}$. Since we are working in the universal cover $\tilde{S}$, a leaf-closure $\tilde{L}$ has the structure of a closed interval in $\mathbb{R}$ where its endpoints (if there are any) are in $\tilde{\Sigma}$. Clearly connected closed subsets of horizontal lines in $\mathbb{R}^2$ also have the structure of a closed interval in $\mathbb{R}$.

We now check that the functions $f_{s,t}|_\partial$ and $f_{0,t}|_\partial$ have the same range. The cocycles $\delta(f_{s,t}) = \xi_{s,t}$ and $\delta(f_{0,t}) = \xi_{0,t}$ both represent the cohomology class $\beta_t$ in $H^1(\mathcal{S}; R^2)$. This implies that the difference of the values of $f_{s,t}$ and $f_{0,t}$ at a singular point $p \in \tilde{S}$ are independent of $p$. Since these functions agree at $p_0$ they agree at every singular point, and hence at the endpoints of $\mathcal{L}$. Furthermore, if $\mathcal{L}$ has no left or right endpoint, then by properness, neither does its image under the two maps $f_{s,t}, f_{0,t}$. Thus both maps have the same image. This implies
that we can define $\tilde{g}_{s,t}|_{\tilde{\mathcal{L}}}$ (separately on each $\tilde{\mathcal{L}}$), by using (5.3). It is easy to check that the definition makes sense (does not depend on the identifications of the range and domain of the maps $f_{s,t}, f_{0,t}$ with closed intervals in $\mathbb{R}$). Since $f_{s,t}(p) = f_{0,t}(p) = p$ for every $p \in \tilde{\Sigma}$, we have $\tilde{g}_{s,t}|_{\tilde{\Sigma}} = \text{Id}$. In particular, since distinct leaf-closures $\mathcal{L}_1, \mathcal{L}_2$ can intersect only in $\tilde{\Sigma}$, the map $\tilde{g}_{s,t} : \tilde{\mathcal{L}} \to \tilde{\mathcal{L}}$ is well-defined, and its restriction to each $\mathcal{L}$ is continuous.

We now show that $\tilde{g}_{s,t}$ is continuous. Let $x_i \to x$ be a convergent sequence in $\tilde{\mathcal{L}}$, and suppose first that $x \notin \tilde{\Sigma}$. Then $\tilde{g}_{s,t}(x_i)$ can be described as the unique $y_i$ on the horizontal leaf of $x_i$, for which $f_{0,t}(y_i) = f_{s,t}(x_i)$. Since $f_{s,t}$ is continuous, and since $f_{0,t}$ is a covering map in a neighborhood of $x$, we have $y_i \to \tilde{g}_{s,t}(x)$. This shows continuity at points of $\tilde{\mathcal{L}} \setminus \tilde{\Sigma}$, and moreover, since the maps $f_{s,t}, f_{0,t}$ are uniformly continuous, the same argument shows uniform continuity of $\tilde{g}_{s,t}$. Since $\tilde{\mathcal{L}}$ is the completion of $\tilde{\mathcal{L}} \setminus \tilde{\Sigma}$, continuity of $\tilde{g}_{s,t}$ follows.

A similar argument reversing the roles of $f_{0,t}$ and $f_{s,t}$ produces a continuous inverse to $\tilde{g}_{s,t}$. This shows that $\tilde{g}_{s,t}$ is a homeomorphism. The fact that the family $\tilde{g}_{s,t}$ depends continuously on the parameter $s$, follows from the equicontinuity of the collection of maps $\{f_{s,t} : s \in [0,1]\}$. This proves the claim.

We now claim that $\tilde{g}_{s,t}$ is invariant under the group of deck transformations, and hence induces an isotopy $g_{s,t}$ defined on $\mathcal{L}$. To see this, recall that applying a deck transformation changes both $f_{s,t}$ and $f_{0,t}$ by some constant. Since they are in the same cohomology class, the constant must be the same.

Taking coboundaries of both sides of the equation $f_{s,t} = f_{0,t} \circ \tilde{g}_{s,t}$ gives $\xi_{s,t} = \tilde{g}_{s,t}^*(\xi_{0,t})$. In particular $a_t = g_{0,t}^*(a_t)$ and $b_t = g_{1,t}^*(a_t)$. We define

$$\psi_{s,t} \overset{\text{def}}{=} \varphi_t \circ g_{s,t} \circ \varphi_0^{-1} \quad \text{and} \quad \psi_t \overset{\text{def}}{=} \psi_{1,t}.$$

Computing on the level of cocycles, we have:

$$\psi_t^*(\text{hol}(M_t)) = (\varphi_t \circ g_{1,t} \circ \varphi_0^{-1})^*(\text{hol}(M_t))$$

$$= \varphi_0^{-1*} \circ g_{1,t}^* \circ \varphi_t^*(\text{hol}(M_t))$$

$$= \varphi_0^{-1*} \circ g_{1,t}^*(a_t) = \varphi_0^{-1*}(b_t) = b_t. \quad (5.4)$$

Applying this equation to a path $\gamma$ in $M_0$ gives (5.1), and the family $\psi_{s,t}$ provides an isotopy between $\psi_{0,t} = \varphi_t \circ \varphi_0^{-1}$ and $\psi_{1,t} = \psi_t$.

We now prove uniqueness. If there were two isotopic maps $\psi_t$ and $\psi'_t$ satisfying the requirements then (5.1) implies that $\psi^{-1} \circ \psi'$ is a translation equivalence which is isotopic to the identity. The identity
map is the unique translation equivalence of $M_0$ isotopic to the identity so we have $\psi^{-1} \circ \psi' = I$ and $\psi = \psi'$.

Remark 5.5. It is instructive to compare our discussion of tremors, using Proposition 5.1, with the discussion of the Rel deformations in [BSW, §6]. Namely in [BSW, Pf. of Thm. 6.1], a map $\bar{f}_t : M_0 \to \text{Rel}(M_0)$ is constructed but the definition of this map involves some arbitrary choices. In particular it is not unique and is not naturally contained in a continuous one-parameter family of maps. Thus from this perspective, tremors are simpler to analyze than Rel deformations.

6. Properties of tremors

In this section we will apply the results of §5 in order to derive further properties of tremors.

6.1. Composing tremors. From the uniqueness in Proposition 5.1, we immediately deduce:

Proposition 6.1. With the same assumptions as in Proposition 5.1, suppose $\nu_1, \nu_2$ are two non-atomic signed transverse measures on the horizontal foliation of $M_0$, set $\nu_3 \overset{\text{def}}{=} \nu_1 + \nu_2$ and let $\beta_1, \beta_2, \beta_3 \in \mathcal{T}_{q_0}$ be the corresponding signed foliation cocycles. For $i = 1, 2, 3$ let $q^{(i)} \overset{\text{def}}{=} \text{trem}_{q_0, \beta_i}$ denote by $M^{(i)} = M_{q^{(i)}}$ the underlying surface, and by $\psi^{(i)} : M_0 \to M^{(i)}$ the corresponding TCH. Also write $\nu_{1,2} \overset{\text{def}}{=} \left(\psi^{(1)}\right)_* \nu_2$, let $\beta_{1,2}$ denote the corresponding foliation cocycle, and denote by $\psi^{(3)} : M^{(1)} \to M^{(3)}$ the TCH corresponding to the marking map $\psi^{(1)} \circ \varphi_0$ and the signed transverse measure $\nu_{1,2}$. Then:

(i) $\beta_{1,2} = \left(\psi^{(1)}\right)_* \beta_2$ and $\text{trem}_{q^{(1)}, \beta_{1,2}} = \text{trem}_{q_0, \beta_3}$.
(ii) We have $\psi^{(3)} = \psi^{(3,1)} \circ \psi^{(1)}$.

Proposition 6.1 gives an identification of $\mathcal{T}_q$ and $\mathcal{T}_{q'}$ when $q'$ is a tremor of $q$. The identification depends on the choice of marking maps, but several natural structures are carried over independently of marking maps.

Corollary 6.2. Suppose $q' = \text{trem}_{q, \beta}$ for some surfaces $q$ and $q'$ and some non-atomic signed foliation cocycle $\beta \in \mathcal{T}_q$. Let $M = M_q$ and $M' = M_{q'}$ be the underlying surfaces, let $\varphi_0 : S \to q$ be a marking map, and let $\psi : M \to M'$ be the corresponding TCH. Then $\psi$ and the induced map $\psi_* : H^1(M, \Sigma; \mathbb{R}^2) \to H^1(M', \Sigma; \mathbb{R}^2)$ on cohomology, and the induced map $\psi_*$ on transverse measures, satisfy the following:
\[ \psi^* \text{ maps } \mathcal{C}_q \text{ bijectively onto } \mathcal{C}'_{q'} \text{ and maps the cone } C_q^+ \text{ bijectively onto } C'_{q'}^+. \text{ It also maps the subsets of c-absolutely continuous and balanced signed foliation cycles in } \mathcal{T}_q \text{ bijectively onto the corresponding subsets in } \mathcal{T}'_{q'}. \]

\[ \psi_* \text{ maps the cone of transverse measures for } M_q \text{ bijectively to the cone of transverse measures for } M'_{q'}. \text{ For the canonical transverse measure } (dy)_q \text{ we have } \psi_*(dy)_q = (dy)'_{q'}. \text{ Also the image of a non-atomic transverse measure under } \psi_* \text{ is non-atomic.} \]

In the rest of this section we will simplify our notation by considering the identification \( \mathcal{T}_q = \mathcal{T}'_{q'} \) afforded by some TCH \( \psi \) as fixed, and using the same letter \( \beta \) to denote elements in these different spaces, if they are mapped to each other by \( \psi \). With this notation, Proposition 6.1(i) simplifies to

\[ \text{trem}_{q,\beta_1+\beta_2} = \text{trem}_{\text{trem}_{q,\beta_1},\beta_2}. \quad (6.1) \]

### 6.2. Tremors and G-action

We now discuss the interaction between the restriction of the \( G \)-action to the subgroups \( B \) and \( U \), and tremors.

**Proposition 6.3.** Let \( s \in \mathbb{R}, \) let \( u = u_s \in U, \) let \( q \in \mathcal{H} \) with underlying surface \( M_q \) and let \( \varphi_0 : S \to M_q \) be a marking map. Let \( \text{hol}^q_{(y)} \) be the foliation cocycle corresponding to the canonical transverse measure \( (dy)_q \). Then the affine comparison map \( \psi_u \) defined in §2.4 coincides with the TCH corresponding to \( s \cdot \text{hol}^q_{(y)} \), and in particular is independent of \( \varphi_0 \). Furthermore, for any \( \beta \in \mathcal{T}_q = \mathcal{T}_{u,s,q} \), we have

\[ \text{trem}_{q,\beta + s \cdot \text{hol}^q_{(y)}} = \text{trem}_{u,s,q,\beta}. \quad (6.2) \]

**Proof.** The assertion concerning \( \psi_u \) follows from the uniqueness in Proposition 5.1, and (6.2) follows from (4.8) and (6.1). \( \square \)

Under the identification above, for \( \beta \in \mathcal{T}_q = \mathcal{T}_{u,s,q} \), we have \( L_q(\beta) = L_{u,s,q}(\beta) \), and from (6.1) and (6.2) we deduce:

**Corollary 6.4.** Let \( \beta \in \mathcal{T}_q \) and \( s \overset{\text{def}}{=} L_q(\beta) \). Then \( \text{trem}_{q,\beta} = \text{trem}_{u,s,q,\beta - s(dy)_q} \) and \( \beta - s(dy)_q \in \mathcal{T}_{u,s,q} \) is balanced.

The interaction of tremors with the \( B \)-action is as follows.

**Proposition 6.5.** Let \( q \in \mathcal{H} \) and let

\[ b = \begin{pmatrix} a & z \\ 0 & a^{-1} \end{pmatrix} \in B, \text{ with } a = a(b) > 0. \quad (6.3) \]
Let \( M_q \) and \( M_{bq} \) be the underlying surfaces. Let \( \beta \in \mathcal{T}_q \), and using the obvious affine comparison maps, consider \( \beta \) also as an element of \( \mathcal{T}_{bq} \). Then

\[
\text{b trem}_{\beta}(q) = \text{trem}_{a,\beta}(bq), \quad \text{Dom}(bq, \beta) = a^{-1} \cdot \text{Dom}(q, \beta),
\]

(6.4)

where \( a = a(b) \) is as in (6.3). These bijections multiply the canonical transverse measure \( dy \) by \( a^{-1} \) and preserve the subsets of atomic and balanced foliation cocycles, and map c-approximately continuous foliation cocycles to \( ac \)-approximately continuous foliation cocycles. In particular, for \( u_s \in U \) we have

\[
u_{s} \text{trem}_{\beta}(q) = \text{trem}_{\beta}(u_{s}q), \quad \text{Dom}(u_{s}q, \beta) = \text{Dom}(q, \beta).
\]

(6.5)

Proof. Let \( q_1 = bq \) and denote the underlying surfaces by \( M = M_q, \ M_1 = M_{q_1} \). Let \( \psi = \psi_b : M \to M_1 \) be the affine comparison map. Since the linear action of \( b \) on \( \mathbb{R}^2 \) preserves horizontal lines, \( \psi \) sends the horizontal foliation on \( M \) to the horizontal foliation on \( M_1 \). As in Corollary 6.2, it sends transverse measures to transverse measures, with non-atomic transverse measures to non-atomic transverse measures, and the induced map \( \psi^* \) on cohomology sends \( \mathcal{T}_q \) to \( \mathcal{T}_{q_1} \) and \( C^+_q \) to \( C^+_{q_1} \).

Since \( \psi \) has derivative \( b \), the canonical transverse measure \( (dy)_q \) on \( q \) is sent to its scalar multiple \( a(b)^{-1} \cdot (dy)_{q_1} \) on \( q_1 \). Hence \( c \)-approximately continuous foliation cocycles are mapped to \( ac \)-approximately continuous foliation cocycles. To prove (6.4), let \( t \mapsto \tilde{\gamma}_t \) be the affine geodesic in \( \mathcal{H}_w \) with \( \tilde{\gamma}_0 = \tilde{q} \) and \( \frac{d}{dt}|_{t=0}\tilde{\gamma}_t = \beta \), so that \( \tilde{q}_1 = \text{trem}_{\beta}(\tilde{q}) \). The new path \( t \mapsto \tilde{\gamma}_t = b\tilde{\gamma}_t \) is also an affine geodesic and satisfies \( \tilde{\gamma}_0 = b\tilde{q} \). Now (6.4) follows from the fact that \( \frac{d}{dt}|_{t=0}\tilde{\gamma}_t = a(b) \cdot \beta \), since \( \mathcal{T}_q \) is embedded in the real space \( H^1(S, \Sigma; \mathbb{R}) \).

We now show that our affine comparison maps sends \( \mathcal{T}_q^{(0)} \) to \( \mathcal{T}_{q_1}^{(0)} \), that is, preserves balanced foliation cocycles. First suppose \( a(b) = 1 \), i.e. \( b = u \in U \). Since the horizontal transverse measure \( dx \) is the same on \( q \) and on \( q_1 \), we have

\[
L_{q_1}(\beta) = L_q(\beta) \quad \text{and} \quad |L|_q(\beta) = |L|_{q_1}(\beta),
\]

(6.6)

and thus \( \psi^*_u(\mathcal{T}_q^{(0)}) = \mathcal{T}_{q_1}^{(0)} \). Now for general \( b \in B \), by composing with an element of \( U \) we can assume \( b \) is diagonalizable, that is in (6.3) we have \( z = 0 \). The horizontal transverse measure \( dx \) on \( q_1 \) is obtained from the horizontal transverse measure on \( q \) by multiplication by the scalar \( a = a(b) \). Now suppose \( \beta \in \mathcal{T}_q^{(0)} \) so that \( \text{hol}_q^{(x)} \cup \beta = 0 \). By naturality of the cup product we get

\[
0 = a^{-1} \text{hol}_q^{(x)} \cup \beta = (\psi_b^{-1})^* \left( \text{hol}_{q_1}^{(x)} \cup \psi_b^* \beta \right) = \text{hol}_{q_1}^{(x)} \cup \psi_b^* \beta.
\]
6.3. Tremors and sup-norm distance. Let dist denote the sup-norm distance as in §2.6.

**Proposition 6.6.** If \( q \in \mathcal{H} \), \( \nu \) is a non-atomic absolutely continuous signed transverse measure on the horizontal foliation of \( M_q \), and \( \beta = \beta_{\nu} \), then

\[
\text{dist}(q, \text{trem}_{q,\beta}) \leq \|\nu\|_{RN} \tag{6.7}
\]

and there is \( q' \in U_q \) and \( \beta' \in T_{q'}^{(0)} \) with \( |L|_{q'}(\beta') \leq 2\|\nu\|_{RN} \) and

\[
\text{trem}_{q,\beta} = \text{trem}_{q',\beta'} \tag{6.8}
\]

**Proof.** Let \( q_1 = \text{trem}_{q,\beta} \) and let \( dy \) be the canonical transverse measure on \( q \). Let

\[
\{\gamma(t) : t \in [0,1]\}, \quad \text{where } \gamma_t := \text{trem}_{q,t\beta},
\]

be the affine geodesic from \( q \) to \( q_1 \). The tangent vector of \( \gamma \) is represented by the class \( \beta \), and by specifying a marking map \( \varphi_0 : S \to M_q \) we can lift the path to \( \mathcal{H}_m \), and find \( \tilde{q}, \tilde{q}_1 \), and \( \tilde{\gamma}(t), t \in [0,1] \) so that \( \pi(\tilde{q}_1) = q_1, \pi(\tilde{\gamma}(t)) = \gamma(t) \) with \( \tilde{\gamma}(0) = \tilde{q}, \tilde{\gamma}(1) = \tilde{q}_1 \), and \( \tilde{\gamma}(t) \) satisfies \( \text{dev}(\tilde{\gamma}(t)) = \text{dev}(\tilde{q}) + t\tilde{\beta} \), where \( \tilde{\beta} = (\varphi_0^{-1})^* \beta \in H^1(S, \Sigma; \mathbb{R}) \). We will use this path in (2.9) to give an upper bound on the distance from \( q \) and \( q_1 \). For each \( t \in [0,1] \), write \( q_t = \gamma(t) \) and denote the underlying surface by \( M_t \). Recall that we denote the collection of saddle connections on a surface \( q \) by \( \Lambda_q \). For any \( \sigma \in \Lambda_q \), we write \( \text{hol}_{q_t}(\sigma) \) component-wise as \( (x_{q_t}(\sigma), y_{q_t}(\sigma)) \), so that

\[
\ell_{q_t}(\sigma) = \|\text{hol}_{q_t}(\sigma)\| \geq |y_{q_t}(\sigma)|. \tag{6.9}
\]

By Corollary 6.2, applying the TCH between \( M_0 \) and \( M_t \) to \( \nu \) and \( dy \) we obtain transverse measures \( \nu_t \) and \( (dy)_t \) on each \( q_t \). Using this, for all \( t \in [0,1] \) we have

\[
\|\gamma'(t)\|_{\gamma(t)} = \|\tilde{\beta}\|_{\tilde{q}_t} = \sup_{\sigma \in \Lambda_{q_t}} \|\tilde{\beta}(\sigma)\|_{\ell_{\tilde{q}_t}(\sigma)} \leq (6.9) \sup_{\sigma \in \Lambda_{q_t}} \|\tilde{\beta}(\sigma)\| \frac{|\int_{q_t} d\nu|}{|y_{q_t}(\sigma)|} = \sup_{\sigma \in \Lambda_{q_t}} \frac{1}{|y_{q_t}(\sigma)|} \|\int_{\sigma} (dy)_t\|_{\ell_{\tilde{q}_t}(\sigma)} \leq \|\nu\|_{RN} \tag{4.3}.
\]

Integrating w.r.t. \( t \in [0,1] \) in (2.9) we obtain the bound (6.7). The second assertion follows from the first one, Corollary 6.4, and the triangle inequality. \( \square \)
6.4. Relations between tremors and other maps. We will now prove commutation and normalization relations between tremors and other maps, which extend those in Proposition 6.5. We will discuss the interaction of tremors with the action of $B$, all possible tremors for a fixed surface, real-Rel deformations, and the $\mathbb{R}^\ast$-action on the space of tremors.

We will use the notation and results of [BSW] in order to discuss real-Rel deformations. Let $Z$ be the subspace of $H^1(S, \Sigma; \mathbb{R}_x)$ which vanish on closed loops. Thus $Z$ represents the subspace of real rel deformations of surfaces in $H$ (see [BSW, §3] for more information).

Let $q \in H, M_q$ the underlying surface, $\varphi : S \to M_q$ a marking map and $\tilde{q} \in H_m$ the corresponding element in $\pi^{-1}(q)$. We define semi-direct products

$$ S_1^{(\varphi)} \overset{\text{def}}{=} B \ltimes \mathcal{T}_{\tilde{q}}, \quad S_2^{(\varphi)} \overset{\text{def}}{=} B \ltimes (\mathcal{T}_{\tilde{q}} \oplus Z), $$

where the group structure on $S_2^{(\varphi)}$ is defined by

$$ (b_1, v_1, z_1), (b_2, v_2, z_2) = (b_1 b_2, a^{-2}(b_2)v_1 + v_2, a^{-1}(b_2)z_1 + z_2), $$

where

$$ b_i \in B, \ v_i \in \mathcal{T}_{\tilde{q}}, \ z_i \in Z, $$

$a(b)$ is defined in (6.3). Also define the group structure on $S_1^{(\varphi)}$ by thinking of it as a subgroup of $S_2^{(\varphi)}$. Define the quotient semidirect products

$$ \tilde{S}_1^{(\varphi)} \overset{\text{def}}{=} S_1^{(\varphi)} / \sim, \quad \tilde{S}_2^{(\varphi)} \overset{\text{def}}{=} S_2^{(\varphi)} / \sim, $$

where $\sim$ denotes the equivalence relation $B \ni u_s \sim s \cdot \text{hol}^{(y)}_{\tilde{q}} \in \mathcal{T}_{\tilde{q}}$.

With these notations we have the following:

**Proposition 6.7.** Let $q, M_q, \varphi, \tilde{q}$ be as above, and suppose $M_q$ has no horizontal saddle connections so that tremors and real-Rel deformations have the maximal domain of definition. Define

$$ \Upsilon^{(\varphi)}_1 : S_1^{(\varphi)} \to H_m, \quad (b, \beta) \mapsto b \text{ trem}_\beta(\tilde{q}) $$

and

$$ \Upsilon^{(\varphi)}_2 : S_2^{(\varphi)} \to H_m, \quad (b, \beta, z) \mapsto b \text{ Rel}_z \text{ trem}_\beta(\tilde{q}). $$

Then the maps $\Upsilon^{(\varphi)}_i$ obey the natural group action law

$$ \Upsilon^{(\varphi)}_i (g_1 g_2) = \Upsilon^{(\varphi)}_i (g_1) - \Upsilon^{(\varphi)}_i (g_2) \quad (i = 1, 2). $$

Moreover these maps are continuous, and descend to well-defined immersions $\tilde{\Upsilon}^{(\varphi)}_i : \tilde{S}_i^{(\varphi)} \to H_m$. 

Proof. We will only prove the statement corresponding to \( i = 1 \). The case \( i = 2 \) will not be needed in the sequel and we will leave it to the reader. The fact that the map \( \Upsilon_1^{(ϕ)} \) satisfies the group action law (6.10) with respect to the group structure on \( S_1^{(ϕ)} \) is immediate from (6.1) and Proposition 6.5. The fact that \( \tilde{\Upsilon}_1^{(ϕ)} \) is well-defined on \( \tilde{S}_1^{(ϕ)} \) is a restatement of (6.2). The maps \( \Upsilon_1^{(ϕ)}, \tilde{\Upsilon}_1^{(ϕ)} \) are continuous because they are given as affine geodesics, and because of general facts on ordinary differential equations. The fact that \( \tilde{\Upsilon}_1^{(ϕ)} \) is an immersion can be proved by showing that when \( g_1, g_2 \) are two elements of \( S_1^{(ϕ)} \) that project to distinct elements of \( \tilde{S}_1^{(ϕ)} \), then \( \text{dev} \left( \tilde{\Upsilon}_1^{(ϕ)}(g_i) \right) \) are distinct, i.e. the operations have a different effect in period coordinates. □

There is also a natural action of the multiplicative group \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \) on \( T_q \) given by \((ρ, β) \mapsto ρβ\), where \( ρ \in \mathbb{R}^* \) and \( β \in T_q \). This action preserves the set of balanced tremors \( T_q^{(0)} \), and by Corollary 6.2 and Proposition 6.5, \( T_q^{(0)} \) is also invariant under the maps \( \Upsilon_1^{(ϕ)} \). We define a semidirect product \( S_3^{(ϕ)} = (\mathbb{R}^* × B) \rtimes T_q^{(0)} \), where \( \mathbb{R}^* \) acts on \( T_q^{(0)} \) by scalar multiplication and \( B \) acts on \( T_q^{(0)} \) as above. Arguing as in the Proposition 6.7 we obtain:

**Proposition 6.8.** Let \( q, M_q, ϕ, \tilde{q} \) be as in Proposition 6.7. Then the map

\[
S_3^{(ϕ)} \to \mathcal{H}_m, \ (ρ, b, β) \mapsto b \trem_{ρβ}(\tilde{q}),
\]

obeys the group action law is a continuous immersion.

**Remark 6.9.** Note that (as reflected by the notation) the objects \( S_i^{(ϕ)} \) and \( \Upsilon_i^{(ϕ)} \) discussed above depend on the choice of a marking map. This is needed because a TCH was used to identify \( T_q \) for different surfaces \( q \). On the other hand Proposition 6.3 makes sense irrespective of a choice of a marking map.

**Remark 6.10.** In addition to the deformations listed above, in the spirit of [Ve2, §1] (see also [CMW, §2.1]), for each horizontally invariant fully supported probability measure on \( M_q \), there is a topological conjugacy sending it to Lebesgue measure (on a different surface \( M_{q'} \)). This topological conjugacy also induces a comparison map \( M_q \to M_{q'} \) and corresponding maps on foliation cocycles and on the resulting tremors, and it is possible to write down the resulting group-action law which it the maps obeys when combined with those of Propositions 6.7 and 6.8. This will not play a role in this paper and is left to the assiduous reader.
7. Proof of Theorem 1.5

Proof of Theorem 1.5(i). Let $\beta = \beta_\nu$ be the signed foliation cocycle corresponding to a signed transverse measure $\nu$. We first claim that there is no loss of generality in assuming that $\nu$ is $c$-absolutely continuous for some $c > 0$. To see this, write $\nu = \nu_1 + \nu_2$ where $\nu_1$ is aperiodic and $\nu_2$ is supported on horizontal cylinders. By Lemma 4.6, $\beta_{\nu_1}$ is $c_1$-absolutely continuous for some $c_1 > 0$. We now modify $\nu_2$ so that for any horizontal cylinder $C$ on $M_q$, the restriction of $\nu_2$ to $C$ is equal to $a_C dy|_C$ for some constant $a_C$. Indeed, such a modification has no effect on $\beta_{\nu_1}$, and will thus have no effect on $\beta = \beta_{\nu_1} + \beta_{\nu_2}$. Thus, if $c_2$ is the maximum of the scalars $a_C$, then (after the modification), $\|\nu\|_{R^N} \leq c_1 + c_2$. Now using Propositions 6.3 and 6.6, we see that the left-hand side of (1.6) is bounded by $c_1 - c_2$. □

Proposition 7.1. Let $F \subset H$ be a closed set, and fix $c > 0$. Then the sets

$$F' \overset{\text{def}}{=} \bigcup_{q \in F, \beta \in C_q^+ \cap R^N(c)} \text{trem}_q,\beta$$

and

$$F'' \overset{\text{def}}{=} \bigcup_{q \in F, \beta \in T_q^R \cap (c)} \text{trem}_q,\beta$$

are also closed.

Proof. We first prove that $F'$ is closed. Let $q'_n \in F'$ be a convergent sequence, $q' = \lim_n q'_n$. We need to show that $q' \in F'$. Let $q_n \in F$ and $\beta_n \in C_{q_n}^+ \cap R^N(c)$ such that $q'_n = \text{trem}_{q_n,\beta_n}$. According to Proposition 6.6, the sequence $(q_n)$ is bounded with respect to the metric dist. Also, by a similar computation to the one appearing in the proof of Proposition 6.6, we have $\|\beta_n\|_{q_n} \leq c$, where $\|\cdot\|_{q_n}$ is the norm defined in (2.7). By Proposition 2.5 the sup-norm distance is proper, and hence the sequence $(q_n)$ has a convergent subsequence. Thus passing to a subsequence and using that $F$ is closed, we can assume $q_n \to q \in F$. Let $M_n$ be the underlying surfaces of $q_n$ and choose marking maps $\varphi_n : S \to M_{q_n}$, $\varphi : S \to M_q$ so that the corresponding $\tilde{q}_n \in H_m$ satisfy $\tilde{q}_n \to \tilde{q}$, and using these marking maps, identify $\beta_n$ with elements of $H^1(S, \Sigma; \mathbb{R}^2)$. By the continuity property of the norms $\|\cdot\|_{q_n}$ (see §2.6), this sequence of cohomology classes is bounded, and so we can pass to a further subsequence to assume $\beta_n \to \beta \in H^1(S, \Sigma; \mathbb{R}^2)$. Applying Proposition 4.9 we get that $\beta = \lim_{n \to \infty} \beta_n \in C_q^+(c)$ and using Proposition 4.7 we see that $q' = \text{trem}_{q,\beta} \in F'$. The proof that $F''$ is closed is similar. □
Proof of Theorem 1.5(ii). Let \( q_1 = \text{trem}_{q, \beta} \) where \( \beta = \beta_\nu \) and \( \nu \) is absolutely continuous. As in the proof of part (i) of the theorem, we can assume that \( \nu \) is \( c \)-absolutely continuous for some \( c \), i.e. \( \beta \in C^+_q(RN)(c) \), and set \( F = \overline{Uq} \). For any \( s \in \mathbb{R} \), we have \( u_s q_1 = \text{trem}_{u_s q, \beta} \) by (6.5). By Corollary 6.2 and Proposition 6.3, \( \beta \in C^+_u q(c) \) for all \( s \), and so \( u_s q_1 \in F' \), where \( F' \) is defined via (7.1). By Proposition 7.1 we have that any \( q_2 \in \overline{Uq_1} \setminus \mathcal{L} \) also belongs to \( F' \), so is a tremor of a surface in \( \mathcal{L} \). So we write \( q_2 = \text{trem}_{q_3, \beta'} \) for \( q_3 \in \mathcal{L} \). Let \( M_2, M_3 \) be the underlying surfaces. Since \( \mathcal{L} \) is \( U \)-invariant and \( q_2 \notin \mathcal{L} \), \( \beta' \) is not a multiple of the canonical foliation cocycle \( \text{hol}_{[q]}^{[q]} \), i.e. the horizontal foliation on \( M_3 \) is not uniquely ergodic. By Corollary 6.2, neither is the horizontal foliation on \( M_2 \).

Proof of Theorem 1.5(iii). We first claim that \( \mu \)-a.e. surface \( q \) has a uniquely ergodic horizontal foliation (see [LM] for a similar argument). Using [MW1], let \( K \) be a compact subset of \( \mathcal{H} \) such that any surface \( q \) with no horizontal saddle connections satisfies

\[
\liminf_{T \to \infty} \frac{1}{T} | \{ s \in [0, T] : u_s q \in K \} | > \frac{1}{2}.
\]

Then by the Birkhoff ergodic theorem, any \( U \)-invariant ergodic measure \( \nu \) on \( \mathcal{H} \), which gives zero measure to surfaces with horizontal saddle connections, satisfies \( \nu(K) > 1/2 \). If the claim is false, then by ergodicity \( \mu \)-a.e. surface has a minimal but non-uniquely ergodic horizontal foliation, and so by Masur’s criterion (applied to the horizontal foliation), for \( \mu \)-a.e. \( q \) the trajectory \( \{ \tilde{g}_t q : t > 0 \} \) is divergent (where \( \tilde{g}_t \) as in (2.4) is the time-reversed geodesic flow). Thus for \( \mu \)-a.e. \( q \) there is \( t_0 = t_0(q) \) such that for all \( t \geq t_0 \), \( \tilde{g}_t q \notin K \), and we can take \( t_1 \) large enough so that \( \mu(\{ q : t_0(q) < t_1 \}) > 1/2 \) and hence \( \nu = \tilde{g}_{t_1} \cdot \mu \) satisfies \( \nu(K) < 1/2 \). Since \( \nu \) is also \( U \)-ergodic, and also gives zero measure to surfaces with horizontal saddle connections, this gives a contradiction. The claim is proved.

Let \( \mu \) be the measure on \( \mathcal{L} \), let \( q \in \mathcal{L} \) be generic for \( \mu \), and let \( q_1 = \text{trem}_{q, \beta} \) for some \( \beta \). We need to show that \( q_1 \) is generic. Let \( f \) be a compactly supported continuous test function and let \( \varepsilon > 0 \). Let \( s_0 = L_q(\beta) > 0 \) and let \( q_2 = u_{s_0} q \). Since \( q_2 \) and \( q \) are in the same \( U \)-orbit, \( q_2 \) is also generic. We claim that for every \( \varepsilon > 0 \) and every \( \delta > 0 \), for all large enough \( T \), there is a subset \( A \subset [0, T] \) with \( |A| \geq (1 - \frac{\varepsilon}{4})T \) and for all \( s \in A \), \( \text{dist}(u_s q_1, u_s q_2) < \delta \). We will prove this claim below, but first we show how to use it to conclude the proof of the Theorem.

By the claim and by the uniform continuity of \( f \), there is \( \delta \) so that whenever \( \text{dist}(x, y) < \delta \) we have \( |f(x) - f(y)| < \frac{\varepsilon}{4} \). Apply the
claim with $\frac{\varepsilon}{8|f|_x}$ in place of $\varepsilon$. Then, choosing $T$ large enough so that
\[ \left| \frac{1}{T} \int_0^T f(u_sq_2) \, ds - \int f \, d\mu \right| < \frac{\varepsilon}{2} \] and using the triangle inequality, we see that for all large enough $T$:
\[
\left| \frac{1}{T} \int_0^T f(u_sq_1) \, ds - \int f \, d\mu \right|
\leq \left| \frac{1}{T} \int_0^T f(u_sq_1) \, ds - \frac{1}{T} \int_0^T f(u_sq_2) \, ds \right| + \left| \frac{1}{T} \int_0^T f(u_sq_2) \, ds - \int f \, d\mu \right|
\leq \frac{1}{T} \int_A |f(u_sq_1) - f(u_sq_2)| \, ds + \frac{1}{T} \int_{[0,T] \setminus A} 2\|f\|_\infty \, ds + \frac{\varepsilon}{2}
\leq \varepsilon + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.
\]

It remains to prove the claim. For this we use [MW1] again. Let $Q \subset \mathcal{H}$ be a compact set such that for all large enough $T$,
\[ \frac{|A_1|}{T} \geq 1 - \frac{\varepsilon}{2}, \text{ where } A_1 = \{s \in [0,T] : u_sq \in Q\}. \]

Let $\tilde{Q} \subset \mathcal{H}_m$ be compact such that $\pi(\tilde{Q}) = Q$. Fix some norm on $H^1(S, \Sigma, \mathbb{R})$. Since $\tilde{Q}$ is compact, and by the continuity in Proposition 4.7, there is $\delta'$ such that for any $\tilde{q}' \in \tilde{Q}$, and $\beta_1, \beta_2 \in C^+_\tilde{q}'$ for which $L_{\tilde{q}'}(\beta_1) = L_{\tilde{q}'}(\beta_2) = s_0$, we have
\[ \|\beta_1 - \beta_2\| < \delta' \implies \text{dist}(\text{trem}_{\tilde{q}', \beta_1}, \text{trem}_{\tilde{q}', \beta_2}) < \delta. \quad (7.3) \]

Let $\mathcal{L}'$ denote the collection of surfaces in $\mathcal{L}$ with no horizontal saddle connections and for which the horizontal foliation is uniquely ergodic. By assumption $\mu(\mathcal{L}') = \mu(\mathcal{L}) = 1$, and by Corollary 4.3 there is a neighborhood $\mathcal{U}$ of $\pi^{-1}(\mathcal{L}')$ such that
\[ \tilde{q}' \in \mathcal{U}, \beta \in C^+_\tilde{q}', L_{\tilde{q}'}(\beta) = s_0 \implies \|\beta - s_0(dy)_{\tilde{q}'}\| < \delta'. \quad (7.4) \]

Clearly $\pi(\mathcal{U})$ is an open subset of $\mathcal{L}$ of full $\mu$-measure. Since $q$ is generic, for all sufficiently large $T$ we there is a subset $A_2 \subset [0,T]$ with
\[ \frac{|A_2|}{T} > 1 - \frac{\varepsilon}{2} \text{ and } s \in A_2 \implies u_sq \in \pi(\mathcal{U}). \]

Now set $A = A_1 \cap A_2$, so that $|A| > (1 - \varepsilon)T$. Suppose $s \in A$. Then there is $\tilde{q}' \in \mathcal{U} \cap \tilde{Q}$ with $\pi(\tilde{q}') = u_sq$. We can view $\beta$ as an element of $C^+_{u_sq}$ and with respect to the marked surface $\tilde{q}'$ this corresponds to $\beta' \in C^+_{\tilde{q}'}$, and we have
\[ u_sq_1 = \text{trem}_{u_sq, \beta} = \pi(\text{trem}_{\tilde{q}', \beta'}) \text{ and } u_sq_2 = u_s q' = \pi(\text{trem}_{\tilde{q}', s_0(dy)}). \]
By (7.3) and (7.4), we find \( \text{dist}(u_1 q_1, u_2 q_2) < \delta \), and the claim is proved. \( \square \)

8. Points outside a locus \( \mathcal{L} \) which are generic for \( \mu_{\mathcal{L}} \)

In this section, after some preparations, we prove Theorem 1.6. At the end of the section we also discuss how tremored surfaces behave with respect to the divergence of nearby trajectories under the horocycle flow.

8.1. Tremors and rank-one loci. We now recall the notions of Rel deformations and of a rank-one locus. Define \( W \) to be the kernel of the restriction map \( \text{Res} : H^1(S, \Sigma; \mathbb{R}^2) \to H^1(S, \Sigma; \mathbb{R}^2) \) which takes a cochain to its restriction to absolute periods. For any \( q \in \mathcal{H} \), and any lift \( \tilde{q} \in \pi^{-1}(q) \), as in §2.2 we have an identification \( T_{\tilde{q}}(\mathcal{H}_m) \cong H^1(S, \Sigma; \mathbb{R}^2) \), and the subspace of \( T_q(\mathcal{H}) \) corresponding to \( W \) is called the Rel subspace and is independent of the marking (see [BSW, §3] for more details). Let \( g = g_q \) denote the tangent space to the \( G \)-orbit of \( q \) (we consider this as a subspace of \( T_q(\mathcal{H}) \) for any \( q \)). A \( G \)-orbit-closure \( \mathcal{L} \) is said to be a rank-one locus if there is a subspace \( V \subset W \) such that for any \( q \in \mathcal{L} \), the tangent space \( T_q(\mathcal{L}) \) is everywhere equal to \( g_q \oplus V \). The eigenform loci \( \mathcal{E}_D \) in \( \mathcal{H}(1, 1) \) are examples of rank-one loci. Rank-one loci were introduced and analyzed by Wright in [Wr1]. We have:

**Proposition 8.1.** Suppose \( \mathcal{L} \) is a rank-one locus. Then for any compact set \( K \subset \mathcal{L} \) there is \( \varepsilon > 0 \) such that if \( q \in K \) is horizontally minimal, and \( \beta \in T_q \) is an essential tremor satisfying \( |L|q(\beta) < \varepsilon \), then \( \text{trem}_\beta(q) \notin \mathcal{L} \). If \( q \) is horizontally minimal and \( \bar{U}q = \mathcal{L} \), then any essential tremor of \( q \) does not belong to \( \mathcal{L} \).

**Proof.** For the first assertion, since \( \mathcal{L} \) is closed and \( K \) is compact, it suffices to show that for any surface \( q \in \mathcal{L} \), any foliation cycle tangent to \( g \oplus W \supset T_q(\mathcal{L}) \) must be a multiple of the canonical foliation cycle \( dy \). To this end, let \( \beta = x + w \in (g \oplus W) \cap T_q \), where \( x \in g \) and \( w \in W \). We want to show that \( \beta \) is a multiple of \( (dy)_q \), and can assume that \( w \) and \( x \) are sufficiently small so that \( q_1 = \text{trem}_\beta(q) = g \text{Rel}_w q \), where \( g = \text{exp}(x) \in G \) and \( \text{Rel}_w q \) is the Rel deformation tangent to \( w \) (see [BSW]). Let \( \tilde{q}_1 \in \pi^{-1}(q_1) \) be a marked translation surface corresponding to a marking map \( \varphi : S \to M_q \), let \( \tilde{\gamma} \) be a closed loop on \( S \), and let \( \gamma = \varphi(\tilde{\gamma}) \). Since Rel deformations do not affect absolute periods, \( \text{dev}(\tilde{q}_1)(\gamma) = g \text{dev}(\tilde{q})(\gamma) \). Write \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). By
(4.7),
\[ c \text{hol}_q^{(x)}(\gamma) + d \text{hol}_q^{(y)}(\gamma) = \text{hol}_q^{(y)}(\gamma) = \text{hol}_{\text{trem}_\beta}(\gamma) = \text{hol}_q^{(y)}(\gamma), \]
and since this holds for every closed loop \( \gamma \), we must have \( c = 0, d = 1 \), i.e. \( q \in U \). Then by (4.8), \( x = s(dy)_q \) for some \( s \in \mathbb{R} \). Since \( w = \beta - x \) is now a tremor on a surface with a minimal horizontal foliation, which evaluates to zero against any element of absolute homology, by Proposition 2.4 we have \( w = 0 \), and \( \beta = s(dy)_q \).

For the second assertion, suppose by contradiction that \( \text{trem}_\beta(q) \in \mathcal{L} \) for some \( q \in \mathcal{L} \) with \( \mathcal{L} = \overline{Uq} \) and \( \beta \in \mathcal{T}_q \) an essential tremor. Let \( K \) be a bounded open subset of \( \mathcal{L} \) and let \( \varepsilon > 0 \) be as in the first assertion. As before, let \( \tilde{y}_t \) be the time-reversed geodesic flow. The translated set \( \tilde{y}_t Uq \) is also dense in \( \mathcal{L} \), and \( \tilde{y}_t u \text{trem}_\beta(q) \in \mathcal{L} \) for any \( u \in U \). By Proposition 6.5, \( \tilde{y}_t u \text{trem}_\beta(q) = \text{trem}_{e^{-t}\beta}(\tilde{y}_t u q) \). Taking \( t \) large enough so that \( |L| q(e^{-t}\beta) < \varepsilon \), and choosing \( u \) so that \( \tilde{y}_t u q \in K \), we get a contradiction to the choice of \( \varepsilon \).

**Corollary 8.2.** Suppose \( \mathcal{L} \) is a rank-one locus, \( q_1, q_2 \in \mathcal{L} \) are horizontally minimal and have dense \( U \)-orbits, and for \( i = 1, 2 \) there are \( \beta_i \in \mathcal{T}_q \) such that \( \text{trem}_{\beta_1}(q_1) = \text{trem}_{\beta_2}(q_2) \). Then there is \( u \in U \) such that \( u q_1 = q_2 \). Furthermore, if \( \beta_1, \beta_2 \) are balanced then \( q_1 = q_2 \) and \( \beta_1 \) is obtained from \( \beta_2 \) by applying a translation equivalence.

**Proof.** Let \( q_3 = \text{trem}_{\beta_1}(q_1) \), let \( M_3 \) be the underlying surface, and let \( \varphi : S \to M_3 \) be a marking map representing \( \tilde{q}_3 \in \pi^{-1}(q_3) \). For \( i = 1, 2 \), let \( \tilde{\beta}_i = \varphi^*(\beta_i) \in H^1(S, \Sigma; \mathbb{R}_x) \) be the cohomology classes for which \( \text{trem}_{\beta_i}(\tilde{q}_i) \equiv \tilde{q}_3 \) and \( \tilde{q}_i \in \pi^{-1}(q_i) \). By Proposition 6.7 we have \( \text{trem}_{\beta_1 - \beta_2}(\tilde{q}_1) \equiv \tilde{q}_2 \). It follows from Proposition 8.1 that \( \tilde{\beta}_1 - \tilde{\beta}_2 = s_0(dy)_{\tilde{q}_1} \), i.e. \( \text{trem}_{\tilde{\beta}_1 - \tilde{\beta}_2}(\tilde{q}_1) = u_{s_0} \tilde{q}_1 \) and \( u_{s_0} q_1 = q_2 \). If \( \beta_1, \beta_2 \) are balanced then
\[
 s_0 = \int_{M_{q_1}} dx \wedge s_0 dy = \int_{M_{q_1}} dx \wedge (\beta_1 - \beta_2) = L_{q_1}(\beta_1) - L_{q_1}(\beta_2) = 0,
\]
and this implies that \( q_1 = q_2 \). Now considering the expression (4.7) giving \( \text{dev}(\text{trem}_\beta(\tilde{q})) \), we see that the only possible ambiguity in the choice of \( \tilde{\beta}_i \) for which \( \text{trem}_{\tilde{\beta}_i}(\tilde{q}) = \text{trem}_{\tilde{\beta}_2}(\tilde{q}) \) is if \( \tilde{\beta}_1, \tilde{\beta}_2 \in H^1(S, \Sigma; \mathbb{R}_x) \) are exchanged by the action of \( \varphi^{-1} \circ h \circ \varphi \), where \( h \) is a translation equivalence of the underlying surface \( M_q \). This gives the last assertion.

We can use Proposition 8.1 to construct examples fulfilling property (III) in the discussion preceding the formulation of Theorem 1.6; namely we will use the rank-one locus \( \mathcal{L} = \mathcal{E} \). We remark that in
the introduction we explicitly required that \( q \) admit a tremor which is both essential and absolutely continuous. In fact this assumption is redundant, that is for surfaces in \( \mathcal{E} \), foliation cocycles are absolutely continuous. More precisely we have:

**Lemma 8.3.** For each aperiodic \( q \in \mathcal{E} \), and any \( \beta \in \mathcal{T}_q \),

\[
|L|_q(\beta) \leq 1 \implies \beta \text{ is 2-absolutely continuous.} \tag{8.1}
\]

Moreover, if \( \beta = \beta_\nu \) where \( \nu \) is 2-absolutely continuous signed transverse measure, then \( |L|_q(\beta) \leq 4 \).

**Proof.** First suppose \( \beta = \beta_\nu \in C^+_q \) with \( L_q(\beta) = 1 \). By Proposition 3.4 there is \( c_1 \) such that \( \nu + \iota_x \nu = c_1(dy)_q \). Since \( 1 = \int_{M_q} dx \wedge dy \) and \( 1 = L_q(\beta) = \int_{M_q} dx \wedge \nu = \int_{M_q} dx \wedge dt_\ast \nu \) we must have \( c_1 = 2 \), i.e.

\[
(dy)_q = \frac{1}{2}d\nu + \frac{1}{2}d\iota_x \nu.
\]

This implies that \( \beta \in C^+_{q,RN}(c) \) for \( c = 2 \). For a general \( \beta \in \mathcal{T}_q \), with \( |L|_q(\beta) \leq 1 \), write \( \beta = \beta_{\nu^+} - \beta_{\nu^-} \), with \( \beta_{\nu^\pm} \in C^+_q \) and repeat the argument. For any transverse positive arc \( \gamma \) we have \( \int_\gamma d\nu^\pm \in \left[0, 2\int_\gamma dy\right] \), which implies (4.3) with \( c = 2 \).

For the second assertion, assume first that \( \nu \) is a positive transverse measure which is 2-absolutely continuous and ergodic, and let \( \nu' = \iota_x \nu \) and let \( c \) so that \( (dy)_q = c(d\nu + d\nu') \). The measures \( \nu, \nu' \) are mutually singular so for any \( \varepsilon > 0 \) we can find a short arc \( \gamma \) such that \( \int_\gamma d\nu' \leq \varepsilon \int_\gamma d\nu \) and \( \int_\gamma d\nu > 0 \). Since \( \nu \) is 2-absolutely continuous, this gives

\[
\int_\gamma d\nu \leq 2 \int_\gamma dy = 2c \int_\gamma (d\nu + d\nu') \leq 2c(1 + \varepsilon) \int_\gamma d\nu.
\]

Taking \( \varepsilon \to 0 \) we see that \( c \geq \frac{1}{2} \). Since \( dx = d_x dx \) we have \( \int_{M_q} dx \wedge d\nu = \int_{M_q} dx \wedge d\nu' \), and thus

\[
1 = \int_{M_q} dx \wedge dy = c \int_{M_q} dx \wedge (d\nu + d\nu') = 2c \int_{M_q} dx \wedge d\nu = 2cL_q(\beta_\nu),
\]

and \( L_q(\beta_\nu) = \frac{1}{2c} \leq 1 \).

If \( \nu \) is a positive transverse measure which is not necessarily ergodic, and is 2-absolutely continuous, then we can write \( \nu = \nu_1 + \nu_2 \) where each \( \nu_i \) is ergodic and 2-absolutely continuous, and by the previous paragraph, \( L_q(\beta) = L_q(\beta_{\nu_1}) + L_q(\beta_{\nu_2}) \leq 2 \). If \( \nu \) is a signed transverse measure, then the conclusion follows by considering its Hahn decomposition. \( \Box \)
8.2. Nested orbit closures. Theorems 1.6 and 1.8 both exhibit one-parameter families of distinct orbit-closures for the $U$-action (see (1.7) and (1.9)). This property is proved using the following general statement.

Proposition 8.4. Let $F = \mathcal{E}$, let $c > 0$, and let $F''$ be the set defined by (7.2). Let $\mathfrak{F}_1$ be a subset of $F''$ containing an essential tremor of a surface $q_0$ in $\mathcal{E}$ with a dense $U$-orbit. For each $\rho > 0$ define

$$\mathfrak{F}_\rho \overset{\text{def}}{=} \{ \text{trem}_{q,\rho} : q \in \mathcal{E}, \; \beta \in \mathcal{T}_q^{(0)}, \; \text{trem}_{q,\beta} \in \mathfrak{F}_1 \}.$$ 

Then for $0 < \rho_1 < \rho_2$ we have $\mathfrak{F}_{\rho_1} \neq \mathfrak{F}_{\rho_2}$.

Proof. By Corollary 6.4, replacing $q_0$ with an element in its $U$-orbit, there is no loss of generality in assuming that $\mathfrak{F}_1$ contains an essential balanced tremor of $q_0$. Thus if we define

$$\mathcal{T}_q^{(0)}(\rho) \overset{\text{def}}{=} \{ \beta \in \mathcal{T}_q^{(0)} : \text{trem}_{q,\beta} \in \mathfrak{F}_\rho \},$$

then $\mathcal{T}_q^{(0)}(1)$ contains a nonzero vector. Clearly for all $\rho > 0$ we have $\mathcal{T}_q^{(0)}(\rho) = \rho \mathcal{T}_q^{(0)}(1)$, so each of the sets $\mathcal{T}_q^{(0)}(\rho)$ contain nonzero vectors as well. By (7.2) and Corollary 8.2, the sets $\mathcal{T}_q^{(0)}(\rho)$ are bounded for each $\rho$. Now suppose by contradiction that for $\rho_1 < \rho_2$ we have $\mathfrak{F}_{\rho_1} = \mathfrak{F}_{\rho_2}$. Then

$$\mathcal{T}_q^{(0)}(\rho_1) = \mathcal{T}_q^{(0)}(\rho_2) = \frac{\rho_2}{\rho_1} \mathcal{T}_q^{(0)}(\rho_1).$$

But $\frac{\rho_2}{\rho_1} > 1$ and a bounded subset of $\mathcal{T}_q^{(0)}$ cannot be invariant under a nontrivial dilation if it contains nonzero points. This is a contradiction. □

Proof of Theorem 1.6. We will find a surface satisfying conditions (I), (II) and (III) of the theorem. It was shown by Katok and Stepin [KS] that there is a surface $q \in \mathcal{E}$ with a horizontal foliation which is not uniquely ergodic and has no horizontal saddle connection (Veech [Ve1] proved an equivalent result on $\mathbb{Z}_2$-skew products of rotations, see [MaTa]). Thus the underlying surface $M_q$ satisfies condition (II). To see that $q$ satisfies condition (III) we apply Proposition 8.1 to the rank-one locus $\mathcal{E}$. To see that $q$ satisfies condition (I), we use [BSW, Thm. 10.1], which states that the $U$-orbit of every point in $\mathcal{E}$ is generic for some measure; furthermore, the result identifies the measure. In the terminology, of [BSW] the $G$-invariant ‘flat’ measure on $\mathcal{E}$ is the measure of type 7. The last bullet point of the theorem states that a surface is equidistributed with respect to flat measure if it has no horizontal saddle connection and is not the result of applying a real-Rel flow to a lattice surface. However lattice surfaces without horizontal
saddle connections have a uniquely ergodic horizontal foliation ([Ve4]) and the horizontal foliation is preserved under real-Rel deformations. This implies that $q$ cannot be a real-Rel deformation of a lattice surface.

For the proof of the second assertion, equation (1.7), we combine Propositions 6.8 and 8.4. Namely, we let $q_r = \text{trem}_{q,r\beta}'$ be as in the statement of the Theorem and define

$$\hat{\mathcal{F}}_\rho \overset{\text{def}}{=} Uq_\rho.$$

Since $q_r$ is obtained from $q_1$ using the $\mathbb{R}^*$-action, by naturality of the $\mathbb{R}^*$-action (see Proposition 6.8) we obtain that if we define $\hat{\mathcal{F}}_1 = \hat{\mathcal{F}}_1'$ and define $\hat{\mathcal{F}}_\rho$ by (8.2), then we have $\hat{\mathcal{F}}_\rho = \hat{\mathcal{F}}_\rho$. So (1.7) follows by Proposition 6.8. □

Remark 8.5. As we remarked in the introduction (see Remark 1.7), Theorem 1.6 remains valid for other eigenform loci $\mathcal{E}_D$ in place of $\mathcal{E} = \mathcal{E}_4$. Indeed, the results of [BSW] used above are valid for all eigenform loci, and for the existence of surfaces in $\mathcal{E}_D$ whose horizontal foliations are minimal but not ergodic, one can use [CM] in place of [KS]. Thus the proof given above goes through with obvious modifications. Finally we note that Lemma 8.3 is also true for other eigenform loci, provided the constant 2 on the right hand side of (8.1) is replaced with an appropriate constant depending on the discriminant $D$. We leave the details to the reader.

8.3. Erratic divergence of nearby horocycle orbits. A crucial ingredient in Ratner’s measure classification theorem is the polynomial divergence of nearby trajectories for unipotent flows. As we have seen in Corollary 2.7 there is a quadratic upper bound on the distance between two nearby horocycle trajectories in a stratum $\mathcal{H}$, with respect to the sup-norm distance. Such upper bounds can also be obtained in the homogeneous space setting, but in that setting they are accompanied by complementary lower bounds. Namely, Ratner used the fact that if \{u_s\} is a unipotent flow on a homogeneous space $X$, for some metric $d$ on $X$ we have (see e.g. [M, Cor. 1.5.18]):

\[(\ast) \text{ for any } \varepsilon > 0 \text{ and every } K \subset X \text{ compact, there is } \delta > 0 \text{ such that if } x_1, x_2 \in X \text{ and for some } T > 0 \text{ we have}
\]

$$|\{s \in [0, T] : d(u_s x_1, u_s x_2) < \delta, u_s x_1 \in K\}| \geq \frac{T}{2},$$

then for all $s \in [0, T]$ for which $u_s x_1 \in K$ we have $d(u_s x_1, u_s x_2) < \varepsilon$.

Our proof of Theorem 1.6 shows that $(\ast)$ fails for strata and in fact we have:
Theorem 8.6. There is a stratum $\mathcal{H}$, a compact set $K \subset \mathcal{H}$, $\epsilon > 0$, and $q_1, q_2 \in \mathcal{H}$, so that for any $\delta > 0$,
\[
\liminf_{T \to \infty} \frac{1}{T} \left\{ s \in [0, T] : \text{dist}(u_s q_1, u_s q_2) < \delta, \; u_s q_1 \in K \right\} > \frac{1}{2}, \tag{8.3}
\]
but the set
\[
\{ s \geq 0 : u_s q_1 \in K \text{ and } \text{dist}(u_s q_1, u_s q_2) \geq \epsilon \} \tag{8.4}
\]
is nonempty.

Proof. Take $q_1 \in \mathcal{L}$ for some $\mathcal{L}$ as in the proof of Theorem 1.6, where $q_1$ admits an essential tremor, and is generic for the $G$-invariant measure on $\mathcal{L}$, and let $q_2$ be a balanced essential tremor of $q_1$. Let $0 < \epsilon < \text{dist}(q_1, q_2)$, so that (8.4) holds. The main claim in the proof of Theorem 1.5(iii) implies (8.3). \(\square\)

Remark 8.7. The construction in \S 10 exhibits a stronger contrast to ($\ast$): it gives examples in which (8.3) holds while (8.4) is unbounded.

9. Existence of non-generic surfaces

In this section we will prove Theorem 1.4. Let $B$ be the upper-triangular group. We will need the following useful consequence of the interaction of tremors with the $B$-action.

Theorem 9.1. Let $\mathcal{H}$ be a stratum of translation surfaces and let $\mathcal{L} \subseteq \mathcal{H}$ be a $G$-invariant locus such that there is $q \in \mathcal{L}$ with $Gq = \mathcal{L}$ and such that $q$ admits an essential absolutely continuous tremor which does not belong to $\mathcal{L}$. Then the closure of the set
\[
\bigcup_{q' \in Bq} \{ \text{trem}_{q', \beta} : \beta \in C^+_q \text{ is an essential absolutely continuous tremor} \} \tag{9.1}
\]
is $G$-invariant and contains a $G$-invariant locus $\mathcal{L}'$ with $\dim \mathcal{L}' > \dim \mathcal{L}$.

In particular, if $\mathcal{L} = \mathcal{E} \subset \mathcal{H}(1, 1)$, then the set in (9.1) is dense in $\mathcal{H}(1, 1)$.

Proof. Let $\Omega$ be the set in (9.1), and let $F$ be the closure of $\Omega$. By assumption there is $q \in \mathcal{L}$ and an absolutely continuous $\beta \in C^+_q \setminus T_q(\mathcal{L})$, and hence for $\epsilon > 0$ sufficiently small, the curve
\[
t \mapsto q(t) \overset{\text{def}}{=} \text{trem}_{q, t\beta}, \quad t \in (-\epsilon, \epsilon)
\]
satisfies $q(t) \in \Omega \setminus \mathcal{L}$ for $t \neq 0$ and $q = \lim_{t \to 0} q(t)$; i.e., $q \in \overline{\Omega \setminus \mathcal{L}}$. By Proposition 6.5, $\Omega$ is $B$-invariant, and hence so is $F$. According to [EMM, Thm. 2.1], any $B$-invariant closed set is $G$-invariant, and is a
finite disjoint union of $G$-invariant loci. This implies that $\mathcal{L} = \overline{Bq} \subset F$, and also that we can write $F = F_1 \sqcup \cdots \sqcup F_k$ where each $F_i$ is a closed $G$-invariant locus supporting an ergodic $G$-invariant measure, and for $i \neq j$ we have $F_i \notin F_j$. There is an $i$ so that $\mathcal{L} \subset F_i$, and we claim $\mathcal{L} \subseteq F_i$. Suppose $\mathcal{L} = F_i$ and let $q(t)$ as above. Then for sufficiently small $t > 0$ we have $q(t) \notin F_i$. So there is some $j$ such that $F_j$ contains a sequence $q(t_n)$ with $t_n > 0$ and $t_n \to 0$. Since $F_j$ is closed we find that $q \in F_j$. But since $F_i = \overline{Gq}$ and $F_j$ is $G$-invariant and closed, we obtain that $F_i \subset F_j$, a contradiction proving the claim.

Thus if we set $L_1 \overset{\text{def}}{=} F_i$ we have $L \subseteq L_1$, and since both $L$ and $L_1$ are manifolds and each is the support of a smooth ergodic measure, we must have $\dim L \leq \dim L_1$, as claimed. To prove the second assertion, that $L_1 \neq H_{p_{1}}$, we note that by McMullen’s classification [McM1], there are no $G$-invariant loci $L_1$ satisfying $E \subseteq L_1 \subseteq H_{p_{1}}$.

**Proof of Theorem 1.4.** First we claim that a dense set of surfaces in $H_{p_{1}}$ are generic for $\mu_1 = \mu_E$, the natural measure on $E$. By Theorem 1.5(iii) it suffices to show that tremors of surfaces in $E$ with no horizontal saddle connections are dense in $H_{p_{1}}$. By Theorem 9.1 it suffices to show that there exists a surface in $E$ with no horizontal saddle connections that admits an essential tremor. Theorem 1.6 establishes this, and the claim is proved.

We now use a Baire category argument. Let $\mu_2$ be the natural flat measure on the entire stratum $H(1, 1)$. Let $f$ be a compactly supported continuous function with $\int f d\mu_1 \neq \int f d\mu_2$, and let $\varepsilon > 0$ be small enough so that

$$2\varepsilon < \left| \int f d\mu_1 - \int f d\mu_2 \right|.$$ 

For $j = 1, 2$ and $T > 0$ let

$$C_{j,T} \overset{\text{def}}{=} \left\{ q \in H(1, 1) : \left| \frac{1}{T} \int_{0}^{T} f(u_s q) ds - \int f d\mu_j \right| < \varepsilon \right\}$$

(which is an open subset of $H(1, 1)$), and let

$$C_j \overset{\text{def}}{=} \bigcap_{n \in \mathbb{N}} \bigcup_{T \geq n} C_{j,T}.$$

If $q$ is generic for $\mu_j$ then $q \in C_{j,T}$ for all $T$ sufficiently large. Since generic surfaces for $\mu_j$ are dense in $H(1, 1)$, each $C_j$ is a dense $G_\delta$-subset of $H(1, 1)$. By definition, for $q \in C_j$ we have a subsequence $T_n \to \infty$ such that $\frac{1}{T_n} \int_{0}^{T_n} f(u_s q) ds$ converges to a number $L$ with $|L - \int f d\mu_j| \leq \varepsilon$. In particular, any $q \in C_1 \cap C_2$ satisfies (1.2). For the last assertion, note
that surfaces with a uniquely ergodic horizontal foliation also comprise a dense $G_δ$ subset, and so intersect $C_1 \cap C_2$ nontrivially. □

Remark 9.2. It is straightforward that the set of surfaces with dense (forward or backward) orbit under $g_t$ is a residual set. So, there are such surfaces and in particular surfaces that are recurrent under $g_t$ that do not equidistribute for any measure.

10. A NEW HOROCYCLE ORBIT CLOSURE

In this section we will prove Theorem 1.8. We first show the inclusion between the two lines in equation (1.8), namely that

$$\{\text{trem}_{q,β} : q ∈ E \text{ is aperiodic, } β ∈ T_q, |L|q(β) ≤ a\}$$

$$⊂\{\text{trem}_{q,β} : q ∈ E, β ∈ T_q, |L|q(β) ≤ a\}. \quad (10.1)$$

It suffices to check that the second line of (10.1) is closed. Proposition 7.1 and Lemma 8.3 imply that any accumulation point of $\{\text{trem}_{q,β} : q ∈ E, β ∈ T_q, |L|q(β) ≤ a\}$ is also a tremor with total variation at most $4a$.

In particular $|L|q(β)$ is bounded independently of $β$, and Corollary 4.4 implies that the total variation is at most $a$.

For the last assertion of the Theorem, note that the inclusion in (1.9) is obvious from the first line of (1.8), and the naturality of the $\mathbb{R}^*$-action (Proposition 6.8). The inclusion is proper by Theorem 1.6.

It remains to show the existence of $q_1$ for which we have the equality in equation (1.8), namely for which

$$Uq_1 = \{\text{trem}_{q,β} : q ∈ E \text{ is aperiodic, } β ∈ T_q, |L|q(β) ≤ a\}. \quad (10.2)$$

Before doing this, we set up some notation to be used throughout this section and describe our strategy. Throughout this section we partition $E$ into the following subsets:

$E^{(\text{per})} = \{q ∈ E : M_q \text{ is horizontally periodic}\}$,

$E^{(\text{tor})} = \{q ∈ E : M_q \text{ is two tori glued along a horizontal slit}\} \setminus E^{(\text{per})}$,

$E^{(\text{min})} = E \setminus (E^{(\text{per})} ∪ E^{(\text{tor})})$

$$= \{q ∈ E : \text{all infinite horizontal trajectories are dense}\}.$$

Note that the set of aperiodic surfaces in $E$ is precisely $E^{(\text{tor})} ∪ E^{(\text{min})}$, and that each of the sets $E^{(\text{per})}, E^{(\text{tor})}, E^{(\text{min})}$ is dense in $E$. We further partition $E^{(\text{tor})}$ according to the length of the slit:

$E^{(\text{tor},H)} = \{q ∈ E^{(\text{tor})} : M_q \text{ is two tori glued along a horizontal slit of length } H\}$.

Although the individual sets $E^{(\text{tor},H)}$ are not dense in $E$, for each $H_0 > 0$ their union $∪_{H > H_0} E^{(\text{tor},H)}$ is dense in $E$. 

Now for positive parameters $a, H$ we define
\[ SF_{(\leq a)}^{(\min)} = \{ \text{trem}_{q, \beta} : q \in E^{(\min)}, \beta \in T_q, |L_q(\beta) \leq a \} \]
\[ SF_{(\leq a)}^{(\tor)} = \{ \text{trem}_{q, \beta} : q \in E^{(\tor)}, \beta \in T_q, |L_q(\beta) \leq a \} \]
\[ SF_{(\leq a)} = SF_{(\leq a)}^{(\min)} \cup SF_{(\leq a)}^{(\tor)} \]
\[ SF_{(\leq a)}^{(\tor, H)} = \{ \text{trem}_{q, \beta} \in SF_{(\leq a)}^{(\tor)} : q \in E^{(\tor, H)} \} \].

The letters $SF$ stand for ‘spiky fish’, and one can think of $SF_{(\leq a)}^{(\min)} \setminus E$ as the spikes of the spiky fish. For $q \in E^{(\tor)} \cup E^{(\min)}$, denote by $C_q^{+}\text{erg}$ the extreme rays in the cone of foliation cocycles. Thus by Proposition 3.4, $C_q^{+}\text{erg}$ consists of two rays interchanged by the involution $\iota$, and these rays are distinct if the horizontal direction is not uniquely ergodic on $M_q$. Further denote
\[ SF_{(= a)}^{(\min)} = \{ \text{trem}_{q, \beta} : q \in E^{(\min)}, \beta \in C_q^{+}\text{erg}, L_q(\beta) = a \} \]
\[ SF_{(= a)}^{(\tor)} = \{ \text{trem}_{q, \beta} : q \in E^{(\tor)}, \beta \in C_q^{+}\text{erg}, L_q(\beta) = a \} \]
\[ SF_{(= a)} = SF_{(= a)}^{(\min)} \cup SF_{(= a)}^{(\tor)} \]
\[ SF_{(= a)}^{(\tor, H)} = \{ \text{trem}_{q, \beta} \in SF_{(= a)}^{(\tor)} : q \in E^{(\tor, H)} \} \].

With this terminology it is clear that (10.2) (and hence Theorem 1.8) follows from:

**Theorem 10.1.** For any $a > 0$ there is $q_1 \in SF_{(= a)}^{(\min)}$, such that $\overline{Uq_1} = SF_{(= a)}^{(\tor)} = SF_{(= a)}^{(\tor, H)}$.

The proof of Theorem 10.1 will use the following intermediate statements. Throughout this section, dist refers to the sup-norm distance discussed in §2.6. We will restrict dist to $SF$, in particular the balls which will appear in the proof are subsets of $SF$.

**Proposition 10.2.** For any $q \in SF_{(\leq a)}^{(\min)}$ and any $\varepsilon > 0$ there is $q' \in SF_{(\leq a)}^{(\tor)}$ such that $\text{dist}(q, q') < \varepsilon$.

**Proposition 10.3.** For any positive constants $a$, $H$, and any $q \in SF_{(= a)}^{(\tor, H)}, Uq$ contains all of $SF_{(= a)}^{(\tor, H)}$.

**Proposition 10.4.** For any $a > 0$, any $q \in SF_{(\leq a)}^{(\tor)}$ and any $\varepsilon > 0$ there is $H_0$ such that for any $H > H_0$ there is $q' \in SF_{(= a)}^{(\tor, H)}$ such that $\text{dist}(q, q') < \varepsilon$.

For the remainder of this section we will denote $SF_{(\leq a)}$ by $SF$.™
Proof of Theorem 10.1 assuming Propositions 10.2, 10.3, and 10.4. The equality $\mathcal{SF}_{(\leq a)} = \mathcal{SF}_{(\leq a)}^{(\text{tor})}$ is clear from Proposition 10.2. We will prove:

(i) There is $q_1 \in \mathcal{SF}$ for which orbit $Uq_1$ is dense in $\mathcal{SF}$.
(ii) Any $q_1$ as in (i) satisfies $q_1 = \text{trem}_{q_1,\beta}$ for some $q \in \mathcal{E}^{(\text{min})}$ and $\beta \in C_{q,\text{erg}}^+$ with $L_q(\beta) = a$.

To prove (i), we will use the strategy of proof of the Baire category theorem. Given $\varepsilon > 0$ and a compact set $K \subset \mathcal{SF}$, let $\mathcal{V}_{K,\varepsilon}$ denote the set of points in $\mathcal{SF}$ whose $U$-orbit is $\varepsilon$-dense in $K$. By continuity of the horocycle flow and compactness of $K$, one sees that $\mathcal{V}_{K,\varepsilon}$ is open. We will show that $\mathcal{V}_{K,\varepsilon}$ is not empty. To see this, note that by Proposition 10.2, given a compact $K \subset \mathcal{SF}$ and $\varepsilon > 0$ there is a finite set $F \subset \mathcal{SF}_{(\leq a)}^{(\text{tor})}$ which is $\varepsilon$-dense in $K$. For $p \in F$, let $H_0 = H_0(p)$ be as in Proposition 10.4, for $q = p$ and $\varepsilon/2$ instead of $\varepsilon$. Let $H > \max_{p \in F} H_0(p)$. Then for each $p$ there is $q'_p \in \mathcal{SF}_{(\leq a)}^{(\text{tor},H)}$ such that $\text{dist}(p, q'_p) < \varepsilon/2$. Finally by Proposition 10.3, for any $q \in \mathcal{SF}_{(\leq a)}^{(\text{tor},H)}$, the closure of $Uq$ contains all of the $q'_p$. Thus the orbit $Uq$ comes within distance $\varepsilon/2$ of each $p \in F$ and in particular is $\varepsilon$-dense in $K$.

Now let $K_1 \subset K_2 \subset \cdots$ be an exhaustion of $\mathcal{SF}$ by compact sets with nonempty interiors. Let $\varepsilon_1$ be larger than the diameter of $K_1$, and let $B_0$ be a closed ball in the interior of $\mathcal{V}_{K_1,\varepsilon_1}$. We will iteratively choose $\varepsilon_1, \varepsilon_2, \ldots$ so that $\varepsilon_n \searrow 0$ and for all $n$, $B_n = B_0 \cap \bigcap_{j=1}^n \mathcal{V}_{K_j,\varepsilon_j}$ has nonempty interior. Assuming we have chosen such $\varepsilon_1, \ldots, \varepsilon_n$, let $\varepsilon_{n+1} < \varepsilon_n/2$ be small enough so that $B_n$ contains a closed ball of radius $\varepsilon_{n+1}$. We have shown that $\mathcal{V}_{K_{n+1},\varepsilon_{n+1}} \neq \emptyset$, so by definition it intersects the interior of $B_n$. Since $\mathcal{V}_{K_{n+1},\varepsilon_{n+1}}$ is open, $B_{n+1} = \overline{\mathcal{V}_{K_{n+1},\varepsilon_{n+1}}} \cap B_n$ has nonempty interior. In particular the sets $B_n$ form a nested sequence of compact sets satisfying the finite intersection property, so have nontrivial intersection. Since $\varepsilon_n \searrow 0$ and $\bigcup K_n = \mathcal{SF}$, for any point $q_1 \in \bigcap B_n$, we will have $\overline{Uq_1} = \mathcal{SF}$, and (i) is proved.

To prove assertion (ii), note that by Corollary 4.4, Proposition 7.1 and Lemma 8.3, the set

$$\mathcal{S}_{(\leq a)} \overset{\text{def}}{=} \{ \text{trem}_{q,\beta} : q \in \mathcal{E}, \beta \in \mathcal{T}_q, |L|_q(\beta) \leq a \}$$

is closed and contains $\mathcal{SF}_{(\leq a)}$. Thus $q_1$ is of the form $\text{trem}_{q_1,\beta}$ for some $q \in \mathcal{E}$ and $\beta \in \mathcal{T}_q$ with $|L|_q(\beta) \leq a$. We cannot have $q \in \mathcal{E}^{(\text{tor})} \cup \mathcal{E}^{(\text{per})}$ since in both of these cases $q$ would have a horizontal saddle connection of some length $H$, hence so would $q_1$, and hence any surface in $Uq_1$ would have a horizontal saddle connection of length at most $H$. This would contradict the fact that $Uq_1$ is dense in $\mathcal{SF}_{(\leq a)}$. So we must have
\( q \in \mathcal{E}^{(\text{min})}, \) and moreover \( q \) has no horizontal saddle connection. Let \( \nu_1 \) and \( \nu_2 = \iota_\ast \nu_1 \) be the ergodic transverse measures for the straightline flow on \( M_q \), normalized by \( L_q(\beta_i) = 1 \), where \( \beta_i \equiv \beta_{\nu_i}, i = 1, 2, \) and write \( \beta = a_1 \beta_1 + a_2 \beta_2 \) where \( |a_1| + |a_2| \leq a \). Assume with no loss of generality that \( a_2 \geq a_1 \). Since \( \beta \) is not a multiple of \( (dy)_q = \frac{1}{2} \nu_1 + \frac{1}{2} \nu_2 \), we have \( a_2 > a_1 \). Defining \( s = 2a_1 \) and using (6.2) we get
\[
\text{trem}_{q,\beta} = \text{trem}_{q,a_1 \beta_1 + a_2 \beta_2} = \text{trem}_{q,a_1(2\mathfrak{hol}(q_\beta) - \beta_2) + a_2 \beta_2} = \text{trem}_{u_s q,(a_2-a_1) \beta_2},
\]
and this shows that we may replace \( q \) with \( u_s q \) and \( \beta \) with \( (a_2 - a_1) \beta_2 \), which is an element of \( C^+_{q,\beta} \). So we assume that \( \beta \in C^+_{q,\beta} \) and \( L_q(\beta) \leq a \). Suppose \( L_q(\beta) = a' < a \), then writing \( \rho = \frac{a}{a'} > 1 \) and letting \( q_1 = \text{trem}_{q,\beta}, q_2 = \text{trem}_{q,\rho \beta} \in \mathcal{SF}(\leq a) = \overline{U_{q_1}}, \) we have using Proposition 6.8 that
\[
\mathcal{SF}(\leq \rho a) = \overline{U_{q_2}} \subset \overline{U_{q_1}} = \mathcal{SF}(\leq a) \subset \mathcal{SF}(\leq \rho a),
\]
and thus \( \mathcal{SF}(\leq \rho a) = \mathcal{SF}(\leq a) \). This contradicts Proposition 8.4. \( \square \)

We proceed with the proof of Propositions 10.2, 10.3 and 10.4. As we will see now, the main ingredient for proving Proposition 10.2 is Proposition 3.5.

**Proof of Proposition 10.2.** By Proposition 4.7, it is enough to show that for any \( q \) in \( \mathcal{E}^{(\text{min})} \), any \( \beta \in \mathcal{T}_q \) and any \( \varepsilon' > 0 \), there is \( q_1 \in \mathcal{E}^{(\text{tor})} \) and \( \beta_1 \in C^+_{q_1} \) such that \( \text{dist}(q, q_1) < \varepsilon' \) and \( \| \beta - \beta_1 \| < \varepsilon' \). Here \( \| \cdot \| \) is some norm on \( H^1(S, \Sigma; \mathbb{R}_2) \), and we identify the cones \( C^+_{q_1} \) with subsets of this vector space by using period coordinates; namely by choosing a marking map \( \varphi : S \to M_q \), using \( \varphi \) to pull back a triangulation \( \tau \) of \( M_q \) to \( S \), and equipping all \( q' \) in a small neighborhood of \( q \) with the marking map \( \varphi_{\tau,q_1} \circ \varphi \), where \( \varphi_{\tau,q,q'} : M_q \to M_{q'} \) is the comparison map of \( \S) \). We would like to use Proposition 3.5 (iii) and take \( q_1 = r_{-\theta_j} q \), where \( r_{-\theta_j} \) is the rotation of \( M_q \) which makes the slit \( \delta_j \) horizontal, and for \( \beta_1 \) take the cohomology class corresponding to restriction of Lebesgue measure to a torus on \( M_{q_1} \) which a connected component of the complement of the horizontal slit; i.e. the rotation of \( A_j \). It is clear that for large \( j \) this definition will fulfill all our requirements, except perhaps the requirement that \( q_1 \in \mathcal{E}^{(\text{tor})} \). Namely it could be that the two translation equivalent slit tori which appear in Proposition 3.5 are periodic in direction \( \theta_j \). If this happens, we recall that \( q_1 \) is presented as two tori glued along a horizontal slit, but the tori are horizontally periodic, so a small perturbation of these tori (in the space of tori \( \mathcal{H}(0) \)) will make them horizontally aperiodic. Pulling back to \( \mathcal{E} \), i.e. regluing the aperiodic tori along the same slit, we get a new surface \( q'_1 \) which is not horizontally periodic and can be made
arbitrarily close to \( q_1 \). The cohomology class \( \beta_1 \) corresponding to the restriction of Lebesgue measure to one of the two aperiodic tori can be made arbitrarily close to \( \beta_1 \), completing the proof. \( \square \)

Proposition 10.3 follows from a classical result of Hedlund asserting that any horizontally aperiodic surface has a dense \( U \)-orbit in the space of tori \( \mathcal{H}(0) \cong \text{SL}_2(\mathbb{R})/\text{SL}_2(\mathbb{Z}) \).

**Proof of Proposition 10.3.** Note that each surface \( q \) in \( \mathcal{E}^{(\text{tor}, H)} \) has a splitting into two parallel isometric tori \( A_1, A_2 \) glued along a horizontal slit of length \( H \), and swapped by the map \( \iota \) of Proposition 3.1. The two rays \( C_q^{+,\text{erg}} \) correspond, up to multiplication by scalars, to the restriction of the transverse measure \( (dy)_q \) to each of these tori. Thus if we set \( s = 2a \), then each \( q' \in \mathcal{S}F^{(\text{tor}, H)}_{(=a)} \) is obtained by a ‘subsurface shear’ of a surface in \( \mathcal{E}^{(\text{tor}, H)} \), namely by applying \( u_s \) to one of the tori \( A_i \) and not changing the other torus. The reason for taking \( s = 2a \) is that the area of each of the \( A_i \) is exactly \( 1/2 \). This description implies in particular that \( \mathcal{S}F^{(\text{tor}, H)}_{(=a)} \) is the image of \( \mathcal{E}^{(\text{tor}, H)} \) under a continuous map commuting with the \( U \)-action. So it suffices to show that the \( U \)-orbit of any \( q \in \mathcal{E}^{(\text{tor}, H)} \) is dense in \( \mathcal{E}^{(\text{tor}, H)} \).

For this, let \( \mathcal{H}(0)^{\text{tor}} \) denote the tori in \( \mathcal{H}(0) \) which are horizontally aperiodic. Note that any surface in \( \mathcal{E}^{(\text{tor}, H)} \) is obtained from a surface \( q' \in \mathcal{H}(0)^{\text{tor}} \) by forming two copies of \( q' \) and gluing them along a slit of length \( H \) starting at the marked point (the fact that the surface is aperiodic ensures that the slit exists). This defines a \( U \)-equivariant map \( \mathcal{H}(0)^{\text{tor}} \rightarrow \mathcal{E}^{(\text{tor}, H)} \), which is continuous when \( \mathcal{H}(0)^{\text{tor}} \) is equipped with its topology as a subset of \( \mathcal{H}(0) \). Thus to complete the proof it suffices to show that any surface in \( \mathcal{H}(0)^{\text{tor}} \) has a \( U \)-orbit which is dense in \( \mathcal{H}(0) \); this in turn is a well-known result of Hedlund [H]. \( \square \)

10.1. **Controlling tremors using checkerboards.** In order to prove Proposition 10.4 we will (among other things) have to deal with the following situation. Given \( q \in \mathcal{E} \) and \( \beta \in C_{q}^{+,\text{erg}} \), with \( L_q(\beta) < a \), we would like to find a surface \( q' \) and \( \beta' \in C_{q'}^{+,\text{erg}} \), such that \( L_{q'}(\beta') = a \) and \( \text{trem}_{q,\beta} \) is close to \( \text{trem}_{q',\beta'} \). Since \( L_q(\beta) < L_{q'}(\beta') \), it is clear from Corollary 4.2 that we cannot achieve this with \( q' \) close to \( q \), so we will choose \( s \) so that \( q_0 = u_{-s}q \) and \( \beta_0 = \beta + s \text{hol}_{q}^{(y)} \) satisfy \( \text{trem}_{q,\beta} = \text{trem}_{q_0,\beta_0} \) and \( L_{q_0}(\beta_0) = a \), and take \( q' \) close to \( q_0 \). This transforms our problem into finding \( \beta' \in C_{q'}^{+,\text{erg}} \) which closely approximates \( \beta_0 \in C_{q_0}^{+} \), where \( \beta_0 \) is not ergodic but rather is a nontrivial convex combination of \( \text{hol}_{q_0}^{(y)} \) and an ergodic foliation cocycle.
Controlling such convex combinations (see point (IV) below) is achieved using what we will refer to informally as a ‘checkerboard pattern’. A checkerboard on a torus $T$ is a pair of non-parallel line segments $\sigma_1, \sigma_2$ on $T$ which form the boundary of a finite collection of identical parallelograms which can be colored in two colors so that no two adjacent parallelograms have the same color (see Figures 1 and 2). If we equip two identical tori $T_1, T_2$ with checkerboard patterns defined by the same lines $\sigma_1, \sigma_2$, and in which the colors in the coloring are swapped, we can form a surface $M$ in $E$ by gluing $T_1$ to $T_2$ in two different ways, namely along each of the $\sigma_i$. Both of these gluings give the same surface $M$, but it is decomposed as a union of two tori glued along a slit in two different ways. One decomposition is into the original tori $T_1$ and $T_2$, and the other is into the unions $T'_1, T'_2$ of parallelograms of a fixed color. Our interest will be in the ‘area imbalance’ of the checkerboard, which is the difference between the areas of $T'_1 \times T'_1$ and $T'_2 \times T'_1$.

In our application the lines $\sigma_1, \sigma_2$ will both be nearly horizontal. Taking the normalized restriction $\text{Leb}|_{T'_1}$ to one of the tori in the decomposition $M = T'_1 \cup T'_2$ gives the ergodic foliation cocycle, and the checkerboard picture shows that it closely approximates a nontrivial convex combination of the two ergodic components of the other foliation cocycle, namely the one coming from the normalized restrictions $\text{Leb}|_{T_1}, \text{Leb}|_{T_2}$. Controlling the coefficients in this convex combination amounts to controlling the area imbalance parameter.

Checkerboards were originally introduced by Masur and Smillie in order to provide a geometric way to understand Veech’s examples of surfaces with a minimal and non-ergodic horizontal foliation, see [MaTa, p. 1039 & Fig. 7]. We now proceed to a more precise discussion.

Let $p \in \mathcal{H}(0,0)$ be a torus with two marked points $\xi_1$ and $\xi_2$. Let $T = T_p$ be the underlying surface. Let $\sigma_1, \sigma_2$ be two non-parallel saddle connections on $p$ from $\xi_1$ to $\xi_2$. Let $\bar{\sigma}_2$ be the segment obtained by reversing the orientation on $\sigma_2$, and let $\sigma$ be the concatenation of $\sigma_1$ and $\bar{\sigma}_2$ so that $\sigma$ is a closed loop on $T$. We have:

**Lemma 10.5.** The following are equivalent:

(i) The loop $\sigma$ is homologous to zero in $H_1(T; \mathbb{Z}/2\mathbb{Z})$.

(ii) It is possible to color the connected components of $T \setminus \sigma$ with two colors so that components which are adjacent along a segment forming part of $\sigma$ have different colors.

(iii) For $i = 1, 2$ let $M_i$ be the surface obtained from the slit construction applied to $\sigma_i$ (as in §3.1). Then $M_1$ and $M_2$ are translation equivalent.
Figure 1. A checkerboard: when the $\sigma_i$ (drawn in black) are long and orthogonal, the torus will be partitioned into small rectangles of alternating colors. The difference between the areas occupied by the colors is the area imbalance.

Figure 2. A key feature of this checkerboard is that the non-horizontal black segment crosses the horizontal segment immediately adjacent to its previous crossing, leading to strips of equal width and length.
Proof. The equivalence of (i) and (iii) follows from Proposition 3.2. We now show that (ii) is equivalent to the triviality of the class represented by $\sigma$. Consider the $\mathbb{Z}/2\mathbb{Z}$ valued 1-cochain Poincaré dual to $\sigma$. This cochain represents a trivial cocycle if and only if it is the coboundary of a $\mathbb{Z}/2\mathbb{Z}$-valued function. Associating colors to the values of such a function as in Figure 1 we have the checkerboard picture. Specifically being a coboundary with $\mathbb{Z}/2\mathbb{Z}$ coefficients means that two regions have the same color iff a generic path crosses $\sigma$ an even number of times to get from one to the other. \qed

Let $\sigma_1, \sigma_2$ cross each other an odd number of times and satisfy the conditions of Lemma 10.5, let $A$ be the area of $T$ and let $A_1, A_2$ be the areas of the two colors in the coloring in (iii) above, so that $A_1 + A_2 = A$. We will refer to the quantity $\left| \frac{A_1 - A_2}{A} \right|$ as the area imbalance of the subdivision given by $\sigma_1, \sigma_2$ (note that when $T_p$ has area one this is the same as $|A_1 - A_2|$).

Proof of Proposition 10.4. Let $q$ be as in the statement of Proposition 10.4, that is $q$ is obtained from $p \in \mathcal{H}(0)$ with minimal horizontal foliation, and from parameters $H_1 > 0$ and $s_1, s_2 \in \mathbb{R}$ satisfying $|s_1| + |s_2| \leq 2a$, as follows. First put a horizontal segment $\sigma_1$ of length $H_1$ on the underlying torus $T = T_p$, giving rise to a surface in $\mathcal{H}(0,0)$. Then apply the slit construction of §3.1 to obtain a surface $q_0 \in \mathcal{E}^{\text{tor}}$ which is a union of two tori $T_1, T_2$, with minimal horizontal foliation, glued along a horizontal slit of length $H_1$. Rescale so that this surface has area one, i.e. each $T_i$ has area 1/2. Then for $i = 1, 2$, apply the horocycle shear map $u_{s_i}$ to $T_i$, and glue the resulting aperiodic tori to each other to obtain $M_q$. By swapping the roles of $T_1, T_2$, replacing $p$ with $u_{-s}p$ and $s_i$ with $s_i + s$ for some $s \in \mathbb{R}$, we can assume that

$$0 \leq s_1 \leq s_2 \text{ and } s_1 + s_2 = 2a. \quad (10.3)$$

Let $c = \frac{s_2 - s_1}{2a}$. We will take $\sigma_1$ to be one of the segments comprising a checkerboard on $T$, and we will show that for any $\eta > 0$ there is $H_0$ such that for any $H > H_0$, there is a second segment $\sigma_2$ on $T$ joining the two endpoints of $\sigma_1$ for which the following hold:

(I) The segments $\sigma_1, \sigma_2$ on $T$ intersect an odd number of times and satisfy the conditions of Lemma 10.5;

(II) Let $\theta \in (-\pi, \pi)$ be the direction of $\sigma_2$. Then $|\theta| < \eta$ and the flow in direction $\theta$ is aperiodic on $T$;

(III) the length of $\sigma_2$ is in the interval $(H, (1 + \eta)H)$;

(IV) the area imbalance of $\sigma_1, \sigma_2$ is in the interval $(c - \eta, c + \eta)$.

Sublemma. The conclusion of Proposition 10.4 follows from (I)-(IV).
Proof of Sublemma. Let \( q_0 \in \mathcal{E}^{(\text{tor})} \) be the surface as in the above discussion, so that \( q = \text{trem}_{u_0, s_1 \beta_1 + s_2 \beta_2} \), where \( \beta_i = \beta \nu_i \) is the cohomology class corresponding to the transverse measure \( \nu_i \) obtained by restricting the canonical transverse measure \( (dy)_{q_0} \) to each of the tori \( T_i \), and the tori are glued along a slit of length \( H_1 \). Let \( \mathcal{U} \) denote the \( \varepsilon \)-ball around \( q \). Our goal is to show that \( \mathcal{U} \) contains some \( q' \) which is also tremor of a surface \( q'_0 \in \mathcal{E}^{(\text{tor})} \), but for which the parameters \( s_1, s_2 \) are prescribed, and so is the slit length \( H_0 \). More precisely \( M_{q'_0} \) is made of two minimal tori \( T', T'' \) glued a horizontal slit of length \( H, M_q' \) is obtained by applying the horocycle flow \( u_{2a} \) to \( T' \) (since \( T' \) has area \( \frac{1}{2} \) this will give a tremor of total variation exactly \( a \)), and we need to carry the construction out for all \( H > H_0 \) where \( H_0 \) is allowed to depend on \( \mathcal{U} \).

We obtain \( q'_0 \) as follows. We find \( \sigma_2 \) satisfying items (I–IV) above, for \( \eta \) sufficiently small (to be determined below). Define \( q'_0 \define g q_0 \) where \( g \in \text{SL}_2(\mathbb{R}) \) is the composition of a small rotation and small diagonal matrix, moving the slit \( \sigma_2 \) projecting to \( \sigma_2 \) to a horizontal slit of the required length \( H \). Note that in light of (II) and (III), \( g \) is close to the identity in the sense that we can bound the norm \( \| g - \text{Id} \| \) with a bound which goes to zero as \( \eta \to 0 \), so that by choosing \( \eta \) small we can make \( \text{dist}(q_0, q'_0) \) as small as we wish. Thus \( q \) is obtained from \( q_0 \) by shearing the two tori \( T_i \) (for \( i = 1, 2 \)) by \( u_{s_i} \), and \( q' \) is obtained from \( q'_0 \) by shearing the torus \( T' \) by \( u_{2a} \).

We now wish to show using (II) and (IV) that by making \( \eta \) small and \( H \) large we can ensure that \( q' \in \mathcal{U} \). To see this, we will work in period coordinates. We will choose a marking map \( \varphi : S \to M_{g_0} \) and use it to define an explicit basis for \( H_1(S, \Sigma) \), by pulling back a basis of \( H_1(M_{g_0}, \Sigma_{g_0}) \). Then we will show that for all \( \eta \) small enough and \( H \) large enough, when evaluating \( \text{hol}_q \) and \( \text{hol}_{q'} \) on the elements \( \alpha \) of this basis, the differences \( \| \text{hol}_q(\alpha) - \text{hol}_{q'}(\alpha) \| \) can be made as small as we wish. The basis is described as follows. For \( i = 1, 2 \), let \( \alpha_1^{(i)}, \alpha_2^{(i)} \) be straight segments in \( T_i \) generating the homology, so that \( \{ \alpha_j^{(i)} : i, j = 1, 2 \} \cup \{ \sigma_1 \} \) form a basis for \( H_1(M_{g_0}, \Sigma_{g_0}; \mathbb{Z}) \). We now compute the holonomy vectors of these elements, corresponding to \( q \) and \( q' \).

By the description of \( q \) from the preceding paragraph,

\[
\text{hol}_q \left( \alpha_j^{(i)} \right) = u_{s_i} \text{hol}_{g_0} \left( \alpha_j^{(i)} \right) = \text{hol}_{g_0} \left( \alpha_j^{(i)} \right) + s_i \left( \text{hol}_{g_0}^{(i)}(\alpha_j^{(i)}) \right) \tag{10.4}
\]

and

\[
\text{hol}_q \left( \sigma_1 \right) = \text{hol}_{g_0} \left( \sigma_1 \right). \tag{10.5}
\]
Now let \( \nu' \) be the transverse measure given by restricting the canonical transverse measure \((dy)_{q_0'}\) to \(T'\). Then by the description of \( q' \) from the preceding paragraph we also have that

\[
\text{hol}_{q'}(\alpha_{j}^{(i)}) = \text{hol}_{q_0'}(\alpha_{j}^{(i)}) + 2a \left( \nu'(\alpha_{j}^{(i)}) \right) \tag{10.6}
\]

and

\[
\text{hol}_{q'}(\sigma_1) = \text{hol}_{q_0'}(\sigma_1) + 2a \left( \nu'(\sigma_1) \right) \tag{10.7}
\]

Let \( \mu' \) be the restriction of Lebesgue measure to \( T' \), so that in the notation of Proposition 2.3 we have \( \mu' = \mu_{\nu'} \). The choice of \( c \) and (10.3) along with (IV) ensure that the numbers \( \mu'(T' \cap T_i) \) come closer to \( s_i \) the smaller \( \eta \) is. By (II), choosing \( \eta \) small forces \( \theta \) to be close to 0, and this forces each connected component of \( T' \cap T_i \) to wrap around \( T_i \) many times; i.e. the restriction of the transverse measure \( \nu' \) to \( T_i \) approaches the restriction of \( dy \) to \( T_i \), weighted by the scalar \( \mu'(T' \cap T_i) \).

Furthermore, for \( \eta \) small, the differences \( \|\text{hol}_{q_0'}(\alpha_{j}^{(i)}) - \text{hol}_{q_0'}(\alpha_{j}^{(i)})\| \) and \( \|\text{hol}_{q_0'}(\sigma_1) - \text{hol}_{q_0'}(\sigma_1)\| \) can be made as small as we wish. Thus for \( \eta \) small enough we can make the difference between (10.4) and (10.6) as small as we like. This completes the proof of the Sublemma.

It remains to show that we can choose \( \sigma_2 \) so that (I)–(IV) hold. We make a change of variables which maps \( T_p \) to the standard torus \( \mathbb{R}^2/\mathbb{Z}^2 \). Since the horizontal direction on \( T_p \) is aperiodic, this change of variables maps \( \sigma_1 \) to a segment with holonomy \((x, \alpha x)\) for some \( \alpha \notin \mathbb{Q} \) and \( x > 0 \). Let \( \xi_1, \xi_2 \) be the endpoints of \( \sigma_1 \). We will choose \( k \) an even positive integer, and a simple closed curve \( \ell \) from \( \xi_1 \) to \( \xi_1 \), and let \( \sigma_2 \) be the shortest curve homotopic to the concatenation of \( k \) copies of \( \ell \) and \( \sigma_1 \). Since \( k \) is even, the curve \( \sigma \) of Lemma 10.5 is homologous to an even multiple of \( \ell \) and thus (I) holds. The choice of the curve \( \ell \) corresponds to the choice of \((m, n) \in \mathbb{Z}^2 \) with \( \gcd(m, n) = 1 \). Since \( \alpha \) is irrational, the linear form \((m, n) \mapsto m\alpha - n \) assumes a dense set of values on pairs \((m, n) \in \mathbb{Z}^2 \) with \( \gcd(m, n) = 1 \) (see [CE] for a stronger
We choose \( m, n \) so that
\[
\left| x(m\alpha - n) - (1 - c) \right| < \eta. \tag{10.8}
\]
We can make this choice with \( m, n \) large enough, so that the direction of \( \ell \) approaches the direction of slope \( \alpha \). Note that for all \( k \), the direction of \( \sigma_2 \) is closer to the direction of slope \( \alpha \). Hence for such \( (m, n) \) and all large \( k, \theta \) is small. Because \( \alpha \notin \mathbb{Q} \) the slope of \( \sigma_2 \) is irrational and so we have (II). As we vary \( k \), the length of \( \sigma_2 \) increases by approximately twice the length of \( \ell \). So if \( H > H_0 \) where \( H_0 \eta \) is larger than twice the length of \( \ell \), (III) will hold for some choice of \( k \).

We now verify (IV), which requires describing the region and coloring given by \( \sigma_1 \) and \( \sigma_2 \) as in Lemma 10.5. The holonomy of \( \sigma_2 \) is \( k(m, n) + (x, x\alpha) \). The curves \( \sigma_1 \) and \( \sigma_2 \) intersect in \( k + 1 \) points (including \( \xi_1, \xi_2 \)) and these intersection points divide each \( \sigma_i \) into \( k \) equal length pieces. Consecutive pieces of the division of \( \sigma_2 \) bound strips of the coloring given by Lemma 10.5. So we obtain a region \( R \) composed of \( k - 1 \) strips of alternating color where each strip is a flat parallelogram with sides \( \frac{1}{k}(k(m, n) + (x, x\alpha)) \) and \( \frac{1}{k}(x, x\alpha) \). As \( k - 1 \) is odd all but one of these strips cancel out. This gives that the contribution of \( R \) to the area imbalance of \( R \) is equal to the area \( A \) of one strip. We have
\[
A = \left| \det \left( \frac{m + \frac{x}{k}}{n + \frac{x\alpha}{k}} \frac{x}{k} \right) \right|.
\]
The complement of \( R \) has one color and area \( 1 - (k - 1)A \). This implies that the total area imbalance is
\[
1 - (k - 1)A - A = 1 - kA = 1 - x(m\alpha - n).
\]
So (IV) follows from (10.8), and the proof is complete. \( \square \)

11. Non-integer Hausdorff dimension

The purpose of this section is to prove Theorem 1.9. Throughout this section we use the notation of §10. We denote the Hausdorff dimension of a subset \( A \) of a metric space \( X \) by \( \dim A \). We will use the following well-known facts about Hausdorff dimension (see e.g. [Mat]):

**Proposition 11.1.** Let \( X \) and \( X' \) be metric spaces.

1. If \( f : X \to X' \) is a Lipschitz map then \( \dim X \geq \dim f(X) \). In particular, Hausdorff dimension is invariant under bi-Lipschitz homeomorphisms.
2. For a countable collection \( X_1, X_2, \ldots \) of subsets of \( X \) we have \( \dim \bigcup X_i = \sup_i \dim X_i \).
(3) Let $A, B$ be Borel subsets of Euclidean space and let $X \subset A \times B$ be such that for all $a \in A$, $\dim \{ b \in B : (a, b) \in X \} \geq d$. Then

$$\dim X \geq \dim A + d.$$

In particular $\dim (A \times B) \geq \dim A + \dim B$.

Note that when stating Theorem 1.9 we did not specify a metric on $\mathcal{H}(1,1)$. For concreteness one can take the metric to be the metric dist defined in §2.6, but note that in view of items (1) and (2) of Proposition 11.1, the Hausdorff dimension of a set with respect to two different metrics on $\mathcal{H}(1,1)$ is equal, as long as they are mutually bi-Lipschitz on compact sets. We will use this fact repeatedly.

Let $U \subset \mathcal{H}$ be an open set and $\pi : \mathcal{H}_m \to \mathcal{H}$ be the forgetful map of §2.1. In this section, we say that $U$ is an adapted neighborhood if there is a triangulation of $S$ such that a connected component of $\pi^{-1}(U)$ is contained in $V_\tau$, where $V_\tau$ is described in §2.2. Additionally we will say that a relatively open $U \subset E$ is an adapted neighborhood (in $E$) if it is the intersection of an adapted neighborhood in $\mathcal{H}(1,1)$, with the locus $\mathcal{E}$.

11.1. **Proof of lower bound.** We use the notation introduced in §10, and begin with the proof of the easier half of the theorem.

**Proof of lower bound in Theorem 1.9.** For each $\delta > 0$, we define a Borel subset $X_0 \subset \mathcal{S}\mathcal{F}_{(\leq a)}$ and a surjective Lipschitz map $f : X_0 \to X_1 \times X_2$ where $\dim X_1 \geq 4.5 - \delta$ and $\dim X_2 = 1$. The statement will then follow via Proposition 11.1.

Let $U \subset \mathcal{H}(1,1)$ be an adapted neighborhood, so that we can identify $U$ with an open subset of $H^1(S, \Sigma; \mathbb{R}^2)$. Fix a norm $\| \cdot \|$ on $H^1(S, \Sigma; \mathbb{R})$ which is invariant under translation equivalence. According to Corollaries 6.4 and 8.2, for any $q' \in \mathcal{S}\mathcal{F}^{(\text{min})}_{(\leq a)}$ there is a unique $q = q(q') \in \mathcal{E}^{(\text{min})}$ and a unique $\beta = \beta(q') \in \mathcal{T}^{(0)}_{q'}$ (up to translation equivalence) such that $q' = \text{trem}_{q, \beta}$. Define

$$\bar{f} : \mathcal{S}\mathcal{F}^{(\text{min})}_{(\leq a)} \to \mathcal{E}^{(\text{min})} \times \mathbb{R}_{\geq 0} \text{ by } \bar{f}(q') \overset{\text{def}}{=} (q(q'), \| \beta(q') \|).$$

Note that because translation equivalences preserve $\| \cdot \|$ this is well-defined. By Corollary 4.5 and Proposition 3.3 we have that $\beta(q') \in \mathcal{N}_x(\mathcal{E})$ for all $q'$, where $\mathcal{N}_x(\mathcal{E})$ is a flat subbundle. So by making $U$ small enough, $\bar{f}$ is a Lipschitz map.
Fix $\eta > 0$ and set

$$X_1^{(n)} \overset{\text{def}}{=} \{ q \in \mathcal{E}^{(\min)} : \text{there is } \beta \in T_q^{(0)} \text{ with } |L|_q(\beta) \leq a \text{ and } \|\beta\| = \eta \},$$

$$X_0 \overset{\text{def}}{=} \left\{ q' \in \mathcal{S}\mathcal{F}_{(\leq a)}^{(\min)} : q(q') \in X_1^{(n)} , \|\beta(q')\| \leq \eta \right\},$$

$$X_2 \overset{\text{def}}{=} [0, \eta],$$

and define

$$f : X_0 \to X_1^{(n)}, \ f \overset{\text{def}}{=} \bar{f}|_{X_0}.$$ 

Then $f$ is Lipschitz on the intersection of $X_0$ with any compact set, and the definitions ensure that $f$ is surjective. So it remains to show that for $\eta > 0$ small enough we have

$$\dim X_1^{(n)} \geq 4.5 - \delta. \quad (11.1)$$

Let

$$X_1 = \{ q \in \mathcal{E}^{(\min)} : \text{horizontal flow on } M_q \text{ is not uniquely ergodic} \}.$$ 

Since $X_1 = \bigcup_{n>0} X_1^{(n)}$, by Proposition 11.1 (2) it suffices to show that $\dim X_1 \geq 4.5$. This is deduced from work of Cheung, Hubert and Masur as follows. By the general theory of local cross-sections (see e.g. [MSY]), the action of the group $\{ r_\theta : \theta \in S^1 \}$ on $\mathcal{E}$ admits a cross-section, that is, we can parameterize a small neighborhood in $\mathcal{E}$ by $(q,\theta) \mapsto r_\theta q$, where $q$ ranges over a 4-dimensional smooth manifold $U$, $\theta$ ranges over an open set in $S^1$, and the parameterizing map is Bi-Lipschitz. Thus these coordinates identify a neighborhood in $\mathcal{E}$ with a Cartesian product $U \times I$ where $I$ is an interval in $S^1$. It is shown in [CHM] that $U$ contains a Borel subset $A$ of full measure, such that for each $q \in A$ there is a subset $\Theta_q \subset S^1$ so that for $q \in A$, $\theta \in \Theta_q$ we have $r_\theta q \in X_1$, and $\dim \Theta_q = 0.5$. Items (1) and (3) of Proposition 11.1 now imply (11.1). \qed

11.2. **Proof of upper bound.** Establishing the upper bound occupies the rest of this section, as well as §12. We begin with a brief guide to its proof. We can think of a neighborhood of $\mathcal{E}$ as being modelled on a neighborhood of the zero section in the total space of the normal bundle $\mathcal{N}(\mathcal{E})$ (see Proposition 3.3). Thus we can think of $\mathcal{S}\mathcal{F}_{(\leq a)}$ as a subset of the total space of $\mathcal{N}(\mathcal{E})$. For all $q \in \mathcal{E}$, the intersection of $\mathcal{N}_q(\mathcal{E})$ with $\mathcal{S}\mathcal{F}_{(\leq a)}$ is either a point or a line segment. By [CHM] the set of $q \in \mathcal{E}$, for which this set is not a point has Hausdorff dimension 4.5. To cover $\mathcal{S}\mathcal{F}_{(\leq a)}$ efficiently, we find convex subsets $J_i$ of $\mathcal{E}$, where the fibers intersected with $\mathcal{S}\mathcal{F}_{(\leq a)}$ vary in a controlled way (Lemmas 11.7 and 11.8). Using a result of Athreya [At] (see Proposition 11.3), we can efficiently cover the set of $q \in \mathcal{E}$ whose horizontal foliations are
not uniquely ergodic. For a technical reason explained below in Remark 11.4, Proposition 11.3 is more convenient for our argument than [CHM]. To efficiently cover the fiber bundle intersected with \( SF_{(\leq a)} \), we cover (a super set of) the fiber bundle over each \( J_i \) intersected with \( SF_{(\leq a)} \) individually. Proposition 11.2 sets this up axiomatically. In turn, the proof of Proposition 11.2 uses Corollary 12.3, which is a technical result about covering neighborhoods of convex sets. Note that Proposition 11.2 and Lemma 11.8 involve constants that depend on \( q \), but \( SF_{(\leq a)} \) can be presented as a countable (nested) union over subsets on which where we may assume that the constants are fixed. Following Proposition 11.1(2), it suffices to bound the Hausdorff dimension of these sets uniformly, by some number smaller than 6. See the argument around (11.10).

We begin with our axiomatic result for exploiting efficient covers of convex sets. Let \( Y \subset \mathbb{R}^d \) and let \( |Y| \) denote the Lebesgue measure of \( Y \). Let \( \mathcal{N}(\varepsilon)(Y) \) denote the \( \varepsilon \)-neighborhood of \( Y \), that is \( \mathcal{N}(\varepsilon)(Y) = \bigcup_{y \in Y} B(y, \varepsilon) \). The inradius of \( Y \subset \mathbb{R}^d \) is defined to be the supremum of \( r \geq 0 \) such that \( Y \) contains a ball of radius \( r \).

**Proposition 11.2.** Let \( P_1 \subset \mathbb{R}^d, P_2 \subset \mathbb{R}^2 \) be balls. Let \( Z \subset P_1 \times P_2 \), and \( \{Z(t) : t \in \mathbb{N}\} \) be a collection of subsets of \( P_1 \times P_2 \), such that for any \( T > 0 \), \( Z \subset \bigcup_{t \leq T} Z(t) \). Assume furthermore that there are positive constants \( c_1, c_2 \), and \( \delta < 1 \) and that for each \( t \in \mathbb{N} \), \( Z(t) \) is a finite disjoint union of sets \( X_i(t) \times Y_i(t) \), with \( X_i(t) \subset P_1, Y_i(t) \subset P_2 \), for which the following hold:

(i) Each \( X_i(t) \) is contained in a convex set \( J_i(t) \) such that the \( J_i(t) \) are pairwise disjoint, and each has inradius at least \( c_1 e^{-2t} \).

(ii) Each \( Y_i(t) \) is a rectangle whose shorter side has length at most \( c_2 e^{-2t} \).

(iii) \( \left| \bigcup_i \mathcal{N}(e^{-2t})(X_i(t)) \right| \leq c_2 e^{-\delta t} \).

Then

\[
\dim Z \leq d + 1 - \frac{\delta}{5}.
\] (11.2)

The proof of Proposition 11.2 relies on additional statements about efficient covers of convex sets, and is deferred to §12. To obtain an upper bound on the Hausdorff dimension of \( SF_{(\leq a)} \), we will verify the assumptions of Proposition 11.2, with \( d = 5 \). In our setup, a small adapted neighborhood \( \mathcal{U} \subset \mathcal{E} \) (to be defined below) will play the role of a neighborhood in \( \mathbb{R}^5 \), and the 2-dimensional subspace \( \mathcal{N}_2(\mathcal{E}) \) will play the role of \( \mathbb{R}^2 \).
Recall that by Masur’s criterion, if the horizontal foliation on \( M_q \) is not uniquely ergodic then \( \hat{\gamma}_t q \to \infty \) as \( t \to \infty \), where \( \hat{\gamma}_t \) is the time-reversed geodesic flow as in (2.4); i.e., the trajectory eventually leaves every compact set. The following result gives (for a fixed surface) an upper bound for the measure of directions in which the orbit has escaped a large compact set by a fixed time.

**Proposition 11.3 (Athreya).** For any stratum \( \mathcal{H} \) there is \( \delta > 0 \), and a compact subset \( K \subset \mathcal{H} \) such that for any compact set \( Q \subset \mathcal{H} \) and any \( T_0 > 0 \) there is \( C > 0 \) so that for all \( q \in Q \) and all \( T > 0 \), we have

\[
|\{ \theta \in S^1 : \forall t \in [T_0, T_0 + T], \hat{\gamma}_t q \notin K \}| \leq C e^{-\delta T}.
\]

The formulation given above is stronger than the statement of [At, Thm. 2.2]. Namely, in [At], the constant \( C \) is allowed to depend on \( q \), while we claim that \( C \) can be chosen uniformly over the compact set \( Q \). One can check that the stronger Proposition 11.3 follows from the proof given in [At]. Alternatively, one can derive it from [AAEKMU, Prop. 3.7]. Indeed, in the notation of [AAEKMU], set \( \delta = \frac{2}{3}, a < 2 \cdot \frac{2}{3} C_1^{-\frac{2}{3}} \) and \( C = a^{-2T_0} C(x) \), and note that for \( N > \frac{2T_0}{t} \) we have

\[
Z \left( X_{\leq t}, N, 1, \frac{2}{3} \right) \geq \{ q : \alpha(g_t q) \leq M \text{ for all } T_0 \leq t \leq N \}.
\]

**Remark 11.4.** Proposition 11.3 is convenient for our covering arguments because if \( \hat{\gamma}_t q \notin Q \) for all \( t \in [T_0, T_0 + T] \) then when \( q' \) is in small neighborhood of \( q \) we have \( \hat{\gamma}_t q' \notin Q' \) for all \( t \in [T_0, T_0 + T_0] \), where \( Q' \) a slightly larger compact set. Applying Proposition 11.3 to \( Q' \) we have exponential decay (in \( T \)) of the measure of a neighborhood of the set we are covering.

In order to verify hypotheses (i) and (ii) of Proposition 11.2 we need to choose convex sets in \( \mathcal{E} \) so that the \( \mathcal{N}_z(\mathcal{E}) \) fibers intersected with \( \mathcal{S}_z(\mathcal{E}) \) vary in a controlled way. To do this, we now get good approximations for the cone of foliation cocycles which will be constant on our convex subsets of \( \mathcal{E} \).

Let \( \bar{q} \in \mathcal{H}_m \) and let \( M_q \) be the underlying translation surface. A transverse system on \( M_q \) is a finite collection of disjoint arcs of finite length which are transverse to the horizontal foliation on \( M_q \), do not contain points of \( \Sigma \), and intersect every horizontal leaf. The arcs may contain points of \( \Sigma \) in their closure. For example, if the horizontal foliation on \( M_q \) is minimal then \( \sigma \) could be any short vertical arc not passing through singularities, and if \( M_q \) is aperiodic and \( \varepsilon \) is an arbitrary positive number, \( \sigma \) could be the union of downward pointing vertical prongs of length \( \varepsilon \) starting at all singular points (and where
the singular points at their extremities are not considered a part of the prong.

We now define some structures associated with a transverse system. We mark one point on each connected component of $\sigma$. A $\sigma$-almost horizontal segment is a continuous oriented path $\ell$ from $\sigma$ to $\sigma$, which starts and ends at marked points, is a concatenation of an edge along $\sigma$, a piece of a horizontal leaf in $M_q \setminus \Sigma_q$ which does not intersect $\sigma$ in its interior, and another edge along $\sigma$. The orientation of a $\sigma$-almost horizontal segment is the one given by rightward motion along horizontal leaves. Two $\sigma$-almost horizontal segments are said to be isotopy equivalent if they are homotopic with fixed endpoints, and where the homotopy is through $\sigma$-almost horizontal segments. Up to isotopy equivalence there are only finitely many $\sigma$-almost horizontal segments. A $\sigma$-almost horizontal loop is a continuous oriented loop which is a concatenation of $\sigma$-almost horizontal segments, where the orientation of the loop is consistent with the orientation of each of the segments. We say that a $\sigma$-almost horizontal loop is reduced if it intersects each connected component of $\sigma$ at most once. With each $\sigma$-almost horizontal loop $\gamma$ we associate a cohomology class $\beta_\gamma \in H^1(M_q, \Sigma_q; \mathbb{R})$ via Poincaré duality.

We will need the following:

**Lemma 11.5.** For any transverse system $\sigma$, the cohomology classes corresponding to all $\sigma$-almost horizontal loops generate $H^1(M_q, \Sigma_q; \mathbb{Z})$.

**Proof.** The union of $\sigma$-almost horizontal segments in one isotopy equivalence class is the union of sub-arcs of $\sigma$ and a topological disc foliated by parallel horizontal segments. The union of these topological discs gives a presentation of $M_q \setminus \Sigma$ as a cell complex. We call it the cell complex associated with $\sigma$. This generalizes the well-known Veech zippered rectangles construction [Ve3]; namely the Veech construction is the one associated with a specific choice of $\sigma$ with one connected component. See [MW2, §2.2] for a related construction.

By Poincaré duality it suffices to show that the $\sigma$-almost horizontal loops generate $H_1(M_q \setminus \Sigma; \mathbb{Z})$. Since the cells of the cell complex are contractible, each element of $H_1(M_q \setminus \Sigma; \mathbb{Z})$ can be written as a concatenation of $\sigma$-almost horizontal segments. That is, for each $\alpha \in H_1(M_q \setminus \Sigma; \mathbb{Z})$ we can find $\sigma$-almost horizontal segments $\delta_1, \ldots, \delta_k$ and integers $a_1, \ldots, a_k$ such that $\alpha = \sum_i a_i \delta_i$ (and the $\delta_i$ are equipped with the rightward orientation). Let $\beta$ be a $\sigma$-almost horizontal loop of the form $\sum_i b_i \delta_i$, where $b_i \geq |a_i|$ for each $i$. Then

$$\beta_1 \overset{\text{def}}{=} \alpha + \beta = \sum_i c_i \delta_i$$
has $c_i \geq 0$ for all $i$, and it suffices to show that that $\beta_1$ is a finite sum of $\sigma$-almost horizontal loops.

We show this by induction on $\sum_i c_i$. If $\sum_i c_i = 0$ then all the $c_i$ are zero and there is nothing to prove. Otherwise, by omitting some of the $c_i$ we can assume that $c_i > 0$ for all $i$, and in particular $c_1 > 0$.

Since $\beta_1$ has no boundary, either $\delta_1$ is closed, or the terminal point of $\delta_1$ is on a connected component of $\sigma$ on which there is an initial point of another $\delta_j$ with $c_j > 0$. Since $\sigma$ has finitely many connected components, repeating this observation finitely many times we find a reduced $\sigma$-almost horizontal loop $\beta_2$ such that $\beta_1 - \beta_2 = \sum_i c'_i \delta_i$ where $c'_i \geq 0$ for all $i$, and we can apply the induction hypothesis to $\beta_1 - \beta_2$. □

Given a marking map $S \to M_q$ we can think of each $\beta_\gamma$ as an element of $H^1(S, \Sigma; \mathbb{R})$. We denote by $C_++(\sigma)$ the convex cone over all of the $\beta_\gamma$, that is

$$C_++(\sigma) = \text{conv}\{t\beta_\gamma : \gamma \text{ is a } \sigma\text{-almost horizontal loop on } M_q \text{ and } t > 0\}.$$ 

Note that $C_++(\sigma)$ is a finitely generated cone. Indeed, if we let $L = \mathcal{L}_{q,\sigma}$ denote the reduced $\sigma$-almost horizontal loops, then $C_++(\sigma)$ is the convex cone generated by $\beta_\gamma, \gamma \in \mathcal{L}$. Since $\beta_\gamma$ only depends on the homotopy class of $\gamma$, and there are only finitely many isotopy classes of $\sigma$-almost horizontal segments, this shows the finite generation of $C_++$.

Clearly, if $\sigma \subset \sigma'$ are transverse systems then $C_++(\sigma) \subset C_++(\sigma')$. We have the following standard fact.

**Proposition 11.6.** Suppose $M_q$ has no horizontal saddle connections and let $\sigma_1 \supset \sigma_2 \supset \cdots$ be a nested sequence of transverse systems for the horizontal foliation on $M_q$, with total length going to zero. Then

$$C_++ = \bigcap_{n=1}^{\infty} C_++(\sigma_n).$$ (11.3)

**Proof of Proposition 11.6.** To see that $C_++ \subset C_++(\sigma_n)$ for all $n$, we use the Birkhoff ergodic theorem. Take an ergodic invariant probability measure $\nu$ for the straightline flow on $M_q$ and take a horizontal leaf $\ell$ which lies on a generic horizontal straightline trajectory for $\nu$. Let $\sigma'_n$ be a connected component of $\sigma_n$ which intersects $\ell$ infinitely many times. Then we can find a sequence of intersections of $\ell$ and $\sigma'_n$ such that the horizontal lengths of subsegments of $\ell$ between consecutive intersections grow longer and longer. Closing up these segments along $\sigma'_n$ gives longer and longer $\sigma_n$-almost horizontal loops, and taking the Poincaré dual of a renormalized sum of a large number of them gives a sequence approaching $\nu$. This implies $\beta_\nu \in C_++(\sigma_n)$. 

In this paper we will not need the reverse inclusion, so we only sketch its proof. Let $\mathcal{L}_n \overset{\text{def}}{=} \mathcal{L}_{q,\sigma_n}$ denote the finite collection of reduced $\sigma_n$-almost horizontal loops as above. Then $C_q^+(\sigma_n)$ is the convex cone generated by $\{\beta_\gamma : \gamma \in \mathcal{L}_n\}$, and it is not hard to see that

$$
\min \left\{ \text{hol}_q^{(z)}(\gamma) : \gamma \in \mathcal{L}_n \right\} \rightarrow \infty \quad \text{and} \quad \max \left\{ \text{hol}_q^{(y)}(\gamma) : \gamma \in \mathcal{L}_n \right\} \rightarrow 0.
$$

Using this, a standard argument (see e.g. [MW2, Proof of Thm. 1.1]) shows that for any $\gamma_n \in \mathcal{L}_n$, any convergent subsequence of $\beta_\gamma$ converges to $\beta_\nu$ for some transverse measure $\beta_\nu$. This implies the reverse inclusion. □

We now specialize to $\mathcal{H}(1,1)$ and specify the collection of transverse systems $\{\sigma_n\}$ explicitly. Recall our convention that singularities for a surface in $\mathcal{H}(1,1)$ are labeled. Each $q \in \mathcal{H}(1,1)$ has two vertical prongs issuing from the first singular point in a downward direction, and we denote by $\bar{\sigma}_t$ the union of the corresponding vertical segments of length $e^{-t}$. On any compact subset of $\mathcal{H}(1,1)$ there is a lower bound on the length of a shortest saddle connection, and so for $t$ large enough the vertical prongs do not hit singular points and so $\bar{\sigma}_t$ is well-defined. If $\mathcal{M}_q$ is horizontally minimal then each horizontal leaf intersects $\bar{\sigma}_t$ and in particular each horizontal separatrix starting at a singularity has a first intersection with $\bar{\sigma}_t$. Say that $\varepsilon = \varepsilon(q,t)$ is the maximal length, along $\bar{\sigma}_t$, of a segment starting at a singularity and ending at the first intersection of a horizontal separatrix $\xi$ with $\bar{\sigma}_t$. Let $\hat{\sigma}_t \subset \bar{\sigma}_t$ be the union of the two vertical prongs taken of length $\varepsilon$. Note that $\hat{\sigma}_t$ is a transverse system on $\mathcal{M}_q$ if $\mathcal{M}_q$ is horizontally minimal, but some non-minimal surfaces have horizontal leaves that miss $\hat{\sigma}_t$.

Fix an adapted neighborhood $\mathcal{U}$, and recall that by choosing a connected component of $\pi^{-1}(\mathcal{U})$, we can equip all $q \in \mathcal{U}$ with a marking map (up to equivalence), and this identifies each $C_q^+$ with a cone in $H^1(S,\Sigma;\mathbb{R}_x)$. For those $q \in \mathcal{U}$ for which $\mathcal{M}_q$ has no horizontal saddle connections, the marking map also determines the cone $C_q^+(\hat{\sigma}_t)$ as a cone in $H^1(S,\Sigma;\mathbb{R})$. We denote it by $\hat{C}_q^+(t)$ in order to lighten the notation. Since $\hat{\sigma}_t$ is invariant under the map $\iota$, this identification does not depend on the choice of the marking map (within its equivalence class). As in Proposition 3.3 let $H^1(S,\Sigma;\mathbb{R}^2) = T(\mathcal{E}) \oplus \mathcal{N}(\mathcal{E})$ be the decomposition into $\iota$ invariant and anti-invariant classes. By Corollary 4.5, a balanced signed foliation cocycle belongs to $\mathcal{N}(\mathcal{E})$. As in the proof of Proposition 3.5, let $\bar{\pi} : \mathcal{E} \rightarrow \mathcal{H}(0)$ be the projection which map a surface $q \in \mathcal{E}$ to the torus $\mathcal{M}_q/\langle \iota \rangle$, and forgets the marked point (one
of the two endpoints of the slit) corresponding to the second singular point of $M_q$.

The area-one condition in the definition of $\mathcal{E}$ means that $\mathcal{E}$ is not a linear space. For our proof we will need to cover $\mathcal{E}$ by convex subsets, and in order to make the notion of convexity meaningful we work locally, as follows. Recall that $\mathcal{U} \subset \mathcal{E}$ is an adapted neighborhood (in $\mathcal{E}$) if it is the intersection of $\mathcal{E}$ with an adapted neighborhood in the stratum. In this case there is a triangulation $\tau$ of $S$ such that $\pi^{-1}(\mathcal{U})$ is contained in the intersection of the set $V_{\tau}$ (as in §2.2) with the locus of area-one surfaces. Let $q \in \mathcal{U}$ and fix a marking map of $\varphi : S \to q$ representing a surface $\tilde{q} \in V_{\tau}$. Let $\Phi = \Phi_q$ be the map which sends $x \in T_{\tilde{q}}(\mathcal{E})$ to the surface $q'$ satisfying $\text{hol}(q') = c(\text{hol}(\tilde{q}) + x)$, where $\tilde{q}$ is given by the marking map determined by $\varphi$ and $\tau$ (see §2.2) and the rescaling factor $c$ is chosen so that the surface $q'$ has area one. A convex adapted neighborhood of $q$ is $\Phi(\mathcal{W})$ where $\mathcal{W}$ is an open convex subset of $T_{\tilde{q}}(\mathcal{E})$ so that $\Phi|_{\mathcal{W}}$ is a homeomorphism onto its image, which is contained in $\mathcal{U}$. When discussing diameters, convex sets, etc., we will do this with respect to the linear structure on $\mathcal{W}$. When we say that a collection $\mathcal{J}$ of convex subsets of a convex adapted neighborhood is a convex partition up to boundary we mean that $\bigcup_{J \in \mathcal{J}} J$ covers all horizontally minimal surfaces in $\mathcal{U}$, and the elements of $\mathcal{J}$ are disjoint.

For $t \in \mathbb{R}$ and $\mathcal{U} \subset \mathcal{E}$, define

$$\Psi_t : \mathcal{U} \to \mathcal{H}, \quad \Psi_t(q) \overset{\text{def}}{=} \tilde{g}_t q.$$  \hspace{1cm} (11.4)

**Lemma 11.7.** Let $\mathcal{U} \subset \mathcal{E}$ be a convex adapted neighborhood. Then for any $t \in \mathbb{N}$, there is a convex partition up to boundary $\mathcal{J}_t$ of $\mathcal{U}$, such that $\tilde{C}_q^+(t)$ is constant on each $J \in \mathcal{J}_t$; that is, if $J \in \mathcal{J}_t$ and $q_1, q_2 \in J$ then $\tilde{C}_{q_1}(t) = \tilde{C}_{q_2}(t)$. The partition satisfies

$$q \in J \in \mathcal{J}_t \iff \tilde{g}_t q \in \Psi_t(J) \in \mathcal{J}_0.$$ \hspace{1cm} (11.5)

**Proof.** Let $\tau$ be a triangulation as in the definition of an adapted neighborhood. Since $\mathcal{U}$ is adapted, we can choose marking maps $\varphi_q : S \to M_q$ such that for $q, q' \in \mathcal{U}$, $\varphi_q \circ \varphi_q^{-1} : M_q \to M_{q'}$ is the comparison map defined via $\tau$ as in §2.2. Each of the two connected components of $\tilde{\sigma}_t$ has an endpoint in $\Sigma$, so, by pulling back via $\varphi_q$, we can think of $\tilde{\sigma}_t$-almost horizontal segments as curves on $S$ starting and ending at $\Sigma$. For fixed horizontally minimal $q \in \mathcal{U}$, let $\xi_1(q), \ldots, \xi_k(q)$ be all the horizontal segments from $\Sigma$ to $\tilde{\sigma}_t$, continued along $\tilde{\sigma}_t$ so they begin and end at points of $\Sigma$, and where the indices are chosen so that $\xi_1$ has the largest vertical component (see the definition of $\varepsilon(q, t)$ above).
Define $J = J(q)$ to be the set of $q' \in \mathcal{U}$ such that there is a bijection between the collections of $\hat{\sigma}_t$-almost horizontal segments for $q$ and $q'$, and another bijection between the arcs $\xi_i(q), \xi_i'(q')$, where both of these bijections preserve the homotopy class of segments (with endpoints fixed), and the second bijection preserves the order of the points at which the $\xi_i$ hit $\hat{\sigma}_t$. Clearly the collection $\mathcal{J}$ of $J$ defined in this way (as $q$ varies over all horizontally minimal elements of $\mathcal{U}$), is a disjoint collection and covers all horizontally minimal surfaces in $\mathcal{U}$. Since any two surfaces in $J$ have the same collection of $\hat{\sigma}_t$-almost horizontal segments, they have the same collection of $\hat{\sigma}_t$-almost horizontal loops.

We now show that each $J \in \mathcal{J}$ is convex. Fix a $\hat{\sigma}_t$-almost horizontal segment $\ell$ on $M_q$. Let $\eta_1, \ldots, \eta_m$ (respectively, $\zeta_1, \ldots, \zeta_n$) denote the vertical segments going down (respectively, up) from singular points to an intersection point with $\ell$, and have no additional intersection points with $\ell$. Let $\bar{\eta}_i, \bar{\zeta}_j$ denote the paths going from $\Sigma$ to $\Sigma$ which start along an initial segment of $\ell$ and end along $\eta_i$ (resp., $\zeta_j$). The fact that $\ell$ is $\hat{\sigma}_t$-almost horizontal can be expressed in terms of period coordinates by the conditions

$$\text{hol}_q^y(\bar{\eta}_i) > \text{hol}_q^y(\bar{\zeta}_j), \quad \text{hol}_q^x(\bar{\eta}_i) < \text{hol}_q^x(\ell), \quad \text{hol}_q^x(\bar{\zeta}_j) < \text{hol}_q^x(\ell)$$

for all $i, j$, and

$$0 < \text{hol}_q^y(\ell) \leq \text{hol}_q^y(\zeta_1).$$

Furthermore it is clear that the set of $q' \in \mathcal{U}$ for which (11.6) and (11.7) hold is a convex subset of $\mathcal{U}$.

Similarly we see that the condition on $q'$ involving $\xi_i$ can be expressed as requiring that (possibly up to permutation), for all $i, j$,

$$\text{hol}_q^y(\xi_i) > \text{hol}_q^y(\xi_j) \iff \text{hol}_q^y(\xi_i) > \text{hol}_q^y(\xi_j)$$

and for all $i$,

$$\text{hol}_q^y(\xi_i) \in (0, e^{-t}).$$

These also give convex conditions in period coordinates. The intersection of all these convex subsets, taken over all $\hat{\sigma}_t$-almost horizontal segments $\ell$ and all the $\xi_i$, is the set $J$. Therefore $J$ is convex.

Finally, the naturality property (11.5) follows from the fact that if we let $\psi_{\tilde{g}_t}$ denote the affine comparison map $M_q \to M_{\tilde{g}_t q}$, then $\psi_{\tilde{g}_t}$ maps the transverse system $\hat{\sigma}_t$ on $M_q$ to the transverse system $\hat{\sigma}_0$ on $M_{\tilde{g}_t q}$, and preserves all the arcs $\xi_i, \eta_i, \zeta_j$ appearing in the above discussion. □

We note that Lemma 11.7 remains true, with the same proof, if $\mathcal{E}$ is replaced by any $G$-invariant locus, and $\hat{\sigma}_t$ is replaced with any transverse system. We now use the additional structure of $\mathcal{E}$ in order
to state and prove bounds on the objects associated with a transverse system.

**Lemma 11.8.** Let $\mathcal{U} \subset \mathcal{E}$ be a convex adapted neighborhood, let $\mathcal{J}_t$ be the partitions as in Lemma 11.7, let $K_1 \subset \mathcal{H}(0)$ be compact, and let $a > 0$. If $q \in \mathcal{U} \cap \mathcal{E}^{(\min)}$ is horizontally minimal then there are positive constants $c_1$ and $c_2$ such that if $t > 0$ satisfies $\tilde{g}_t \pi(q) \in K_1$, then the following hold:

(a) The length of each $\hat{\sigma}_t$-almost horizontal loop is at least $c_1 e^t$, and the inradius of $J$ is at least $c_1 e^{-2t}$, where $J \in \mathcal{J}_t$ is the partition element containing $q$.

Suppose furthermore that $q$ is not horizontally uniquely ergodic, and let $P^- \beta$ be as in §2.3. Then

(b) $P^- \left\{ \beta \in \tilde{C}_q^+ : L_q(\beta) \leq a \right\}$ (11.8)

is contained in a rectangle with diameter in the interval $[c_1, c_2]$.

(c) One side of the rectangle in (b) has length bounded above by $c_2 e^{-2t}$.

**Proof.** In order to obtain the bounds in (a), note that the transverse system $\hat{\sigma}_t$ is the preimage under $\pi$ of a transverse system $\hat{\sigma}_0$ on the torus $\pi(M_q)$. Using the affine comparison map $\psi_{\tilde{g}_t}$ corresponding to $\tilde{g}_t$ as in §2.4, we can consider the image of this transverse system on $\tilde{g}_t \pi(q)$. If $\tilde{g}_t \pi(q) \in K_1$ there exists $c_1'$ depending only on $K_1$ so that any almost-horizontal loop, with respect to a transverse system of bounded length, has length at least $c_1'$. Considering the effect of the map $\psi^{-1}_{\tilde{g}_t}$, we obtain the required lower bound on the length of a $\hat{\sigma}_t$-almost horizontal segment on $M_q$. Now take some lower bound $c_1''$ for the inradius of an element $J$ in the partition $\mathcal{J}_0$, intersecting $K_1$. Such a lower bound exists because $K_1$ is compact and the collection $\mathcal{J}_0$ is locally finite. By (11.5), we can pull back to $\mathcal{J}_t$ using the $\Psi_t^{-1}$ (see (11.4)), and use (2.11) to obtain the lower bound of $c_1'' e^{-2t}$ on the inradius of elements of $\mathcal{J}_t$. Taking $c_1 = \min(c_1', c_1'')$ we obtain (a).

We now prove assertion (b). The upper bound on the diameter of the set described in equation (11.8) is clear from compactness, the semi-continuity in Proposition 4.1, and the continuity of $(q, \beta) \mapsto L_q(\beta)$ and of $P^-$ (see §4.1.2). Since $q$ admits an essential tremor, there is $\beta_0 \in \tilde{C}_q^+$ for which $P^-(\beta_0) \neq 0$ and this implies the lower bound in (b).

In the proof of (c) we will write $A \ll B$ if $A$ and $B$ are two quantities depending on several parameters, and $A \leq CB$ for some constant $C$ (the implicit constant) independent of these parameters. If $A \ll B$ and
$B \ll A$ we will write $A = B$. In this proof the implicit constant is allowed to depend on $q$ but not on $t$.

It follows from Propositions 2.2, 3.3 and Corollary 4.5, that the projection $P^-$ is defined over $\mathbb{Q}$ and hence maps the lattice of $\mathbb{Z}$-points $H^1(S, \Sigma; \mathbb{Z})$ to a sublattice $\Lambda$ in $\mathcal{N}_2(\mathcal{E}, \mathbb{Z}) = \mathcal{N}_2(\mathcal{E}) \cap H^1(S, \Sigma; \mathbb{Z})$.

Let $M_t$ be the underlying surface of $\tilde{g}_t q$ denote by $\psi_t : M_q \to M_t$ the affine comparison map defined in §2.4. Let $\mathcal{L}(q)$ and $\mathcal{L}(\tilde{g}_t q)$ denote respectively the set of reduced $\tilde{\gamma}_t$ (resp., $\psi_t(\tilde{\gamma}_t)$) almost horizontal loops on $q$ (resp., on $\tilde{g}_t q$). By Lemma 11.5, for $\mathcal{L}$ equal to either of $\mathcal{L}(q)$ and $\mathcal{L}(\tilde{g}_t q)$, we have that $\{\beta_\gamma : \gamma \in \mathcal{L}\}$ contains a basis of $H^1(S, \Sigma; \mathbb{Z})$, and hence the projection $P^- (\{\beta_\gamma : \gamma \in \mathcal{L}\})$ generates $\Lambda$. Let $\Psi_t$ be as in (11.4). By choosing a marking map $\varphi : S \to S$, this gives $\beta$ as in Proposition 6.5, we have that for $\beta \in T_q$, we set for $\beta \in \mathcal{T}_q$, we have that $\beta$ as a marking map for $M_t$, this induces a map $\Psi_t : H^1(S, \Sigma; \mathbb{R}^2) \to H^1(S, \Sigma; \mathbb{R}^2)$. Since the map $\iota$ of Proposition 3.1 commutes with the map $\psi_t$, the map $P^-$ commutes with $\Psi_t$, and hence we have the following diagram:

$$
\begin{array}{ccc}
H^1(S, \Sigma; \mathbb{R}_x) & \xrightarrow{\Psi_t} & H^1(S, \Sigma; \mathbb{R}_x) \\
\downarrow P^- & & \downarrow P^- \\
\mathcal{N}_2(\mathcal{E}) & \xrightarrow{\Psi_t|_{\mathcal{N}_2(\mathcal{E})}} & \mathcal{N}_2(\mathcal{E})
\end{array}
$$

The preceding discussion shows that $|\det \Psi_t|_{\mathcal{N}_2(\mathcal{E})} = 1$. (11.9)

We have that $\mathcal{L}(q) = \Psi_t^{-1}(\mathcal{L}(\tilde{g}_t q))$ and $\tilde{\mathcal{C}}_q^+(0) = \Psi_t(\tilde{\mathcal{C}}_q^+(t))$. Also, as in Proposition 6.5, we have that for $\beta \in \mathcal{T}_q$, we set $\beta' \equiv \Psi_t(\beta)$ then $L_{\tilde{g}_t q}(\beta') = e^{-t} L_{\tilde{g}_t q}(\beta)$. This gives

$$
P^-(\{\beta \in \tilde{\mathcal{C}}_q^+(t) : L_{\tilde{g}_t q}(\beta) \leq a\})
$$

$$
= \Psi_t^{-1} \circ P^- \circ \Psi_t \left(\{\beta \in \tilde{\mathcal{C}}_q^+(t) : L_{\tilde{g}_t q}(\beta) \leq a\}\right)
$$

$$
= \Psi_t^{-1} \circ P^- \left(\{\beta' \in \tilde{\mathcal{C}}_q^+(0) : L_{\tilde{g}_t q}(\beta') \leq e^{-t} a\}\right)
$$

$$
\subseteq \Psi_t^{-1} \left(\{\beta'' \in \mathcal{N}_2(\mathcal{E}) : \|\beta''\| \leq e^{-t} a\}\right).
$$

Thus, using (11.9), the set in the left hand side of (11.8) is a convex subset of $\mathcal{N}_2(\mathcal{E})$ of area $\ll e^{-2t}$. On the other hand, by (b), it contains a vector of length $\gg 1$. This means that it is contained in a rectangle whose small sidelength is $\ll e^{-2t}$, as claimed. $\square$

We are now ready for the
Proof of the upper bound in Theorem 1.9. For each \( H_0 > 0 \), the subset \( \bigcup_{H \leq H_0} \mathcal{E}^{(\text{tor}, H)} \) is a proper submanifold of \( \mathcal{E} \) (with boundary) and the map \((q, s) \mapsto \text{trem}_{q, \beta}\), where \( \beta \in \mathcal{T}_q^{(0)} \) satisfies \(|L|_q(\beta) = s\), is Lipschitz. Thus by Proposition 11.1, the subset of \( \mathcal{SF}_{(\leq a)} \) consisting of tremors of surfaces in \( \mathcal{E}^{(\text{tor})} \) has Hausdorff dimension at most 5. A similar argument applies to the subset of \( \mathcal{SF}_{(\leq a)} \) consisting of tremors of surfaces in \( \mathcal{E}^{(\text{per})} \). So we need only bound the Hausdorff dimension of the set of tremors \( \text{trem}_{q, \beta} \) where \( q \) is horizontally minimal and non-uniquely ergodic, i.e. bound the dimension of the essential tremors in \( \mathcal{SF}_{(\leq a)} \).

Define

\[
\mathcal{E}' \overset{\text{def}}{=} \{ q \in \mathcal{E} : M_q \text{ admits an essential tremor} \},
\]

and write \( \mathcal{H} \) for \( \mathcal{H}(1, 1) \). Let \( \delta > 0 \) and \( K \subset \mathcal{H} \) be a compact set as in Proposition 11.3. We assume with no loss of generality that \( \delta < 1 \). Let \( \text{dist} \) be the metric of §2.6 and let

\[
K' \overset{\text{def}}{=} \{ q \in \mathcal{H}(1, 1) : \text{dist}(q, K) \leq 1 \}.
\]

By Proposition 2.5, \( K' \) is compact. Let \( K_1 \subset \mathcal{H}(0) \) be a compact set so that for each \( q \in \mathcal{H}(0) \) for which the horizontal foliation is aperiodic, the set of return times \( \{ t \in \mathbb{N} : g_t q \in K_1 \} \) is unbounded.

We will use Proposition 11.1(2) and partition \( \mathcal{SF}_{(\leq a)} \) into countably many subsets, and give a uniform upper bound on the Hausdorff dimension of each. We can cover \( \mathcal{E}' \) with countably many convex adapted neighborhoods with compact closures. Given such a convex adapted neighborhood \( \mathcal{U} \subset \mathcal{E} \), and given a parameter \( T_0 > 0 \), let \( C = C(\mathcal{U}, T_0) \) be as in Proposition 11.3 with \( Q \overset{\text{def}}{=} \overline{\mathcal{U}}^\circ \). If \( q \in \mathcal{U} \cap \mathcal{E}' \) and \( \beta \in \mathcal{T}_q^{(0)} \), there are \( c_1 = c_1(q), c_2 = c_2(q) \) so the conclusions of Lemma 11.8 are satisfied. Masur’s criterion [MaTa] implies that the trajectory \( \{ g_t q : t > 0 \} \) is divergent, and in particular, there is \( T_1(q) \) such that for all \( t \geq T_1(q) \), \( g_t q \notin K' \). For each \( \mathcal{U} \) in the above countable collection, each \( T_0 \in \mathbb{N} \), and each \( c \in \mathbb{N} \) with \( c \geq C(\mathcal{U}, T_0) e^{\delta T_0} \), let \( Z = Z(\mathcal{U}, T_0, c) \) denote the set of tremors \( \text{trem}_{q, \beta} \) where \( q \in \mathcal{U} \cap \mathcal{E}' \) and \( \beta \in \mathcal{T}_q^{(0)} \) satisfy the bounds

\[
|L|_q(\beta) \leq a, T_1(q) \leq T_0, c_2(q) \leq c, c_1(q) \geq 1/c.
\]

Then in light of Proposition 11.1(2) it suffices to show that

\[
\dim Z \leq 6 - \frac{\delta}{5}, \quad (11.10)
\]

which we will now do with the help of Proposition 11.2.

By (2.11) and the definition of \( K' \) we see that for any \( q_0 \in \mathcal{N}^{(e^{-2a})}(q) \) we must have \( \tilde{g}_t q_0 \notin K' \). Thus if \( \mu_\mathcal{E} \) denotes the flat measure on \( \mathcal{E} \),
Proposition 11.3 and a Fubini argument show that for each $t \in \mathbb{N}$,
\[ \mu_{\mathcal{U}} \left( \mathcal{N}(e^{-2t}) \left( \{ q \in \mathcal{U} \cap \mathcal{E}' : T_i(q) \leq T_0 \} \right) \right) \leq C e^{\delta T_0} e^{-\delta t}, \]
where $C = C(\mathcal{U}, T_0)$. The choice of $K_1$ ensures that for any $T > 0$,
\[ Z \subset \bigcup_{t \in \mathbb{N}, t \geq T} Z(t), \]
where
\[ Z(t) \overset{\text{def}}{=} \left\{ \text{trem}_{q, \beta} \in Z : q \in \mathcal{U} \cap \mathcal{E}', \beta \in \mathcal{T}_q^{(0)}, \tilde{y}_q\tilde{\pi}(q) \in K_1 \right\}. \]
Let
\[ X(t) \overset{\text{def}}{=} \left\{ q \in \mathcal{U} \cap \mathcal{E}' : \tilde{y}_q\tilde{\pi}(q) \in K_1 \right\}. \]
Using Lemmas 11.7 and 11.8, for each $t$ define finitely many convex sets $J_i(t)$ of inradius at least $c_1 e^{-2t}$ which cover $X(t)$ and for which the map $q \mapsto \tilde{C}_q^+(t)$ is constant on $J_i(t)$, and set
\[ X_i(t) \overset{\text{def}}{=} X(t) \cap J_i(t) \]
and
\[ Y_i(t) \overset{\text{def}}{=} \bigcup_{q \in X_i(t)} P^- \left( \left\{ \beta \in \tilde{C}_q^+(t) : L_q(\beta) \leq a \right\} \right). \]
With these definitions, it follows from Lemma 11.8 (with $c_2 = c = 1/c_1$) that all conditions of Proposition 11.2 are satisfied and the result follows.

12. Effective covers of convex sets

In this section we prove Proposition 11.2. For its proof, we need some useful results about convex sets in $\mathbb{R}^d$. In this section the notation $|A|$ may mean one of several different things: if $A \subset \mathbb{R}^d$ then $|A|$ denotes the Lebesgue measure of $A$. Let $S^{d-1}$ denote the $d-1$ dimensional unit sphere in $\mathbb{R}^d$, then for $A \subset S^{d-1}$, $|A|$ denotes the measure of $A$ with respect to the unique rotation invariant probability measure on $S^{d-1}$. If $A \subset \mathbb{R}^d \times S^{d-1}$, then $|A|$ denotes the measure of $A$ with respect to the product of these measures.

**Proposition 12.1.** For any $d \geq 2$ there are positive constants $c, C$ such that for any compact convex set $K \subset \mathbb{R}^d$ with inradius $R > 0$, and any $\varepsilon \in (0, 1)$, the set
\[ K^{(\varepsilon)} \overset{\text{def}}{=} \{ x \in K : |B(x, \varepsilon R) \cap K| \leq c(\varepsilon R)^d \} \]
(12.1)
satisfies
\[ |K^{(\varepsilon)}| \leq C \varepsilon^2 |K|. \]
For the proof of Proposition 12.1 we will need the following preliminary statement. Since the statement of Proposition 12.1 is invariant under homotheties, we can and will assume that $R = 1$. For $\psi \in S^{d-1}$, and $x \in \mathbb{R}^d$, let $\tau_\psi(x) \eqdef \{ x + s\psi \mid s \in \mathbb{R} \}$ be the line through $x$ in direction $\psi$, and let

$$K^{(e)}(\psi) \eqdef \{ x \in K^{(e)} : |\tau_\psi(x) \cap K| < \varepsilon \}.$$

**Lemma 12.2.** For any $d \geq 2$ there is a positive constant $c$ so that for the set $K^{(e)}$ defined in (12.1), there is $\psi \in S^{d-1}$ such that

$$\left| K^{(e)}(\psi) \right| \geq \frac{1}{2} |K^{(e)}|.$$ (12.2)

**Proof.** Let $c = \frac{1}{2d+\varepsilon^d}$, and suppose $x \in K^{(e)}$, so that $|B(x, \varepsilon) \cap K| \leq c\varepsilon^d$. For each $\theta \in S^{d-1}$, we write $T_\theta(x) = |\tau_\theta(x) \cap K|$ and $\rho(\theta) = \sup \{ s > 0 : x + s\theta \in K \}$. Then $\max(\rho(\theta), \rho(-\theta)) \geq \frac{T_\theta(x)}{2}$. Computing the volume of $B(x, \varepsilon) \cap K$ in polar coordinates, we have

$$c\varepsilon^d \geq |B(x, \varepsilon) \cap K| = \int_{S^{d-1}} \int_0^{\rho(\theta)} r^{d-1} dr \, d\theta \geq \frac{1}{2d+1} \int_{S^{d-1}} T_\theta(x)^d d\theta.$$

So by Markov’s inequality and the choice of $c$,

$$|\{ \theta \in S^{d-1} : T_\theta(x) < \varepsilon \}| \geq \frac{1}{2}.$$ (12.3)

Now consider the set

$$A \eqdef \{ (x, \theta) \in K^{(e)} \times S^{d-1} : T_\theta(x) < \varepsilon \}.$$

From (12.3) and Fubini we have

$$\frac{|K^{(e)}|}{2} \leq |A| = \int_{S^{d-1}} |K^{(e)}(\theta)| \, d\theta.$$}

Thus for some $\psi \in S^{d-1}$ we have (12.2). \qed

**Proof of Proposition 12.1.** Let $e_1, \ldots, e_d$ denote the standard basis of $\mathbb{R}^d$ and let $p_0$ be a point for which $B(p_0, 1) \subset K$. Applying a rotation and a translation, we may assume that $p_0 = 0$ and $\psi = e_d$, where $\psi$ is as in Lemma 12.2. We will make computations in cylindrical coordinates, i.e. we will consider the sphere $S^{d-2}$ as embedded in span $(e_1, \ldots, e_{d-1})$ and write vectors in $\mathbb{R}^d$ as $r\theta + z e_d$. In these coordinates, $d$-dimensional Lebesgue measure is given by $\alpha r^{d-2} dr \, d\theta \, dz$, where $d\theta$ is the rotation
invariant probability measure on $S^{d-2}$ and $\alpha = \alpha_{d-1}$ is a constant. For each $\theta \in S^{d-2}$, define
$$\rho_\theta = \sup\{r \in \mathbb{R} : r \theta \in K\}$$
i.e., $f_\theta(r)$ is the length of the intersection with $K$ of the vertical line through $r \theta$. Let
$$K' = K \cap \left\{ r \theta + ze_d : r \in \left[ \frac{\rho_\theta}{3}, \frac{2\rho_\theta}{3} \right] \right\}.$$ Since $K$ is convex, the function $f_\theta$ is concave, and since $B(0, 1) \subset K$, $f_\theta(0) \geq 1$. This implies that whenever $r \theta + ze_d \in K^{(\varepsilon)}(e_d)$, $r \geq (1 - \varepsilon)\rho_\theta$. Furthermore, whenever $r \theta + ze_d \in K'$ we have $f_\theta(r) \geq \frac{1}{3}$. Clearly $f_\theta(r) \leq \varepsilon$ whenever there is $z$ for which $r \theta + z \in K^{(\varepsilon)}$, and hence
$$|K^{(\varepsilon)}(e_d)| \leq \alpha \int_{S^{d-2}} \int_{(1-\varepsilon)\rho_\theta}^{\rho_\theta} \varepsilon r^{d-2} dr d\theta \leq \alpha \varepsilon \int_{S^{d-2}} \int_{(1-\varepsilon)\rho_\theta}^{\rho_\theta} \rho_\theta^{d-2} dr d\theta = \alpha \varepsilon^2 \rho_\theta^{d-1} = \alpha \varepsilon^2 \int_{S^{d-2}} \rho_\theta^{d-1} d\theta = C' \alpha \varepsilon^2 \int_{S^{d-2}} \frac{2\rho_\theta}{3} r^{d-2} dr d\theta \leq C' \alpha \varepsilon^2 3 \int_{S^{d-2}} \frac{2\rho_\theta}{3} f_\theta(r) r^{d-2} dr d\theta = 3C' \varepsilon^2 |K'|,$$
where
$$C' = \frac{3^{d-1}(d - 1)}{2^{d-1} - 1}.$$ Since $K' \subset K$, we have shown
$$|K^{(\varepsilon)}(e_d)| \leq 3C' \varepsilon^2 |K|.$$ Now taking $C = 6C'$, recalling that $\psi = e_d$, and combining Lemma 12.2 with (12.4) we obtain the desired result. \hfill $\Box$

Let $N(A, R)$ denote the minimal number of balls of radius $R$ needed to cover $A \subset \mathbb{R}^d$.

**Corollary 12.3.** For any $d \geq 2$ there exist positive constants $\bar{c}, \bar{C}$ so that if $K \subset \mathbb{R}^d$ is a convex set with inradius $R$ then the set
$$K^{(\varepsilon, \bar{c})} \overset{\text{def}}{=} \{ x \in K : |B(x, \varepsilon R) \cap K| < \bar{c} |B(x, \varepsilon R)| \}$$
satisfies
$$N(K^{(\varepsilon, \bar{c})}, \varepsilon R) \leq \bar{C} |K| \varepsilon^{2-d} R^{-d}.$$ \hfill (12.5)

**Proof.** Let $K^{(\varepsilon)}$, $c$, $C$ be as in Proposition 12.1, and let $\bar{c}$ be small enough so that
$$\bar{c} |B(x, \varepsilon R)| < c \left( \frac{\varepsilon}{2} R \right)^d.$$
This choice ensures that if \( x \in K^{(e,\bar{c})} \) and \( y \in B \left( x, \frac{\varepsilon}{2} R \right) \) then \( y \in K^{(e/2)} \); i.e., \( B \left( x, \frac{\varepsilon}{2} R \right) \subset K^{(e/2)} \). Let \( B_1, \ldots, B_N \) be a minimal collection of balls of radius \( \varepsilon R \) which cover \( K^{(e,\bar{c})} \) and have centers \( x_1, \ldots, x_N \) in \( K^{(e,\bar{c})} \). Then for each \( i, \left| B_i \cap K^{(e/2)} \right| = \left| B \left( x_i, \frac{\varepsilon}{2} R \right) \right| = \kappa \varepsilon^d R^d \) for a constant \( \kappa \) depending on \( d \). By the Besicovitch covering theorem (see e.g. [Mat, Chap. 2]), each point in \( K^{(e,\bar{c})} \) is covered at most \( N_\varepsilon \) times, where \( N_\varepsilon \) is a number depending only on \( d \). Therefore, using Proposition 12.1 we have

\[
N \kappa \varepsilon^d R^d = \sum_{i=1}^{N} \left| B \left( x_i, \frac{\varepsilon}{2} R \right) \right| \leq \sum_{i=1}^{N} \left| B_i \cap K^{(e/2)} \right| \leq N \kappa \varepsilon^d R^d \leq N_\varepsilon C \varepsilon^2 R^d |K|.
\]

Setting \( \tilde{C} = \frac{N_\varepsilon C}{4\kappa} \) we obtain the required estimate. \( \Box \)

We are now ready for the

**Proof of Proposition 11.2.** For each \( t \in \mathbb{N} \) we will find an efficient cover of \( Z(t) \) by balls of radius \( e^{-(2+\frac{H}{2})t} \). We will lighten the notation by writing \( \hat{N}(P, t) \) for \( N \left( P, e^{-(2+\frac{H}{2})t} \right) \). We will continue with the notation \( A \ll B \) used in the proof of Lemma 11.8. In this proof the implicit constant is allowed to depend on \( d, c_1, c_2, \delta, P_1, P_2 \).

We claim that

\[
\hat{N}(Z(t), t) \ll e^{(2+\frac{H}{2}) (d+1) - \frac{\delta}{2}) t}. \tag{12.6}
\]

To prove (12.6), we will find an efficient cover for each set \( X_i(t) \) and each \( Y_i(t) \), and combine them. By assumption (ii), \( \hat{N}(Y_i(t), t) \ll e^{(2+\frac{H}{2})t} e^{\frac{\delta}{2}t} = e^{(2+\delta)t} \) for each \( i \). Indeed, the first term in this product comes from covering the long side, of length \( \ll 1 \), and the second term is needed for covering the short side of length \( \ll e^{-2t} \).

So it suffices to show

\[
\sum_i \hat{N}(X_i(t), t) \ll e^{((2+\frac{H}{2})d - \delta) t}. \tag{12.7}
\]

With the notation of (12.5) define

\[
J'_i(t) \overset{\text{def}}{=} J_i \left( e^{-\frac{H}{2}t}, \varepsilon \right).
\]

We will consider the sets \( \hat{X}_i(t) = X_i(t) \setminus J'_i(t) \) and \( X_i(t) \cap J'_i(t) \) separately, finding efficient covers for each. If \( x \in \hat{X}_i(t) \) then

\[
\left| B \left( x, e^{-(2+\frac{H}{2})t} \right) \cap J_i(t) \cap \mathcal{N}(e^{-2t})(X_i(t)) \right| \ll \left| B \left( x, e^{-(2+\frac{H}{2})t} \right) \right| \ll e^{-d(2+\frac{H}{2})t}. \tag{12.8}
\]
Let \( \{B_j^{(i)}\}_j \) be a minimal collection of balls of radius \( e^{-(2+\frac{\delta}{5})t} \) centered at points in \( X_i(t) \) needed to cover \( X_i(t) \). By the Besicovitch covering theorem, the collection \( \{B_j^{(i)}\} \) has bounded multiplicity, i.e. for each \( x \) and \( i \), \( \# \{ j : x \in B_j^{(i)} \} \ll 1 \). Since the \( J_i(t) \) are disjoint, the collection \( B_t = \{B_j^{(i)} \cap J_i(t)\}_{i,j} \) is also of bounded multiplicity. Taking into account \((12.8)\), we have

\[
\sum_i \hat{N}(\hat{X}_i(t), t) \ll \#B_t \ll e^{d(2+\frac{\delta}{5})t} \left| \bigcup_i \mathcal{N}(e^{-2t})(X_i(t)) \right| \ll e^{d(2+\frac{\delta}{5})}t.
\]

(12.9)

We also have from Corollary 12.3 (with \( R = e^{-2t} \) and \( \varepsilon = e^{-\frac{\delta}{5}}t \)) that

\[
\sum_i \hat{N}(J_i'(t), t) \ll \sum_i e^{\frac{\delta}{2}(d-2)t} e^{2dt\mid J_i(t)\mid} \ll e^{(2+\frac{\delta}{5})d-\delta)t} \left| \bigcup_i J_i(t) \right| \ll e^{((2+\frac{\delta}{5})d-\delta)t}.
\]

(12.10)

Combining the estimates \((12.9)\) and \((12.10)\), we obtain \((12.7)\), and thus \((12.6)\).

We now prove \((11.2)\). Let

\[
s > d + 1 - \frac{\delta}{5}
\]

and set

\[
s' \overset{\text{def}}{=} \frac{\delta}{2} - \left(2 + \frac{\delta}{2}\right) \cdot \frac{\delta}{5} > 0
\]

(12.11)

(where we have used \( \delta < 1 \)). We need to show that for any \( \eta > 0 \), we can cover \( Z \) by a collection of balls \( B \) of radius at most \( \eta \), so that \( \sum_{B \in B} \text{diam}(B)^s \ll 1 \). To this end, choose \( T \) so that \( e^{-(2+\frac{\delta}{5})T} < \eta \). For each \( t \geq T \) let \( B_t \) be a collection of \( N(t) \) balls of radius \( e^{-(2+\frac{\delta}{5})t} \) covering \( Z(t) \) and let \( B = \bigcup_t B_t \). Then by \((12.6)\) we have

\[
\sum_{B \in B} \text{diam}(B)^s \ll \sum_{t \geq T} N(t) e^{-(2+\frac{\delta}{5})st} \ll \sum_{t \geq T} e^{((2+\frac{\delta}{5})(d+1)-\frac{\delta}{5}-(2+\frac{\delta}{5})(d+1-\frac{\delta}{5}))t} \ll \sum_{t \geq T} e^{-s't} \to_{T \to \infty} 0.
\]

So for large enough \( T \) we have our required cover. \( \square \)
13. Atomic tremors

In this section we complete the proof of Proposition 4.1 (and thus of Corollary 4.4). We recall that in §4.2, these results were already proved in a special case (namely assuming (4.9)), and that this special case is sufficient for the proofs of Theorems 1.5, 1.8 and 1.9.

We note that in the literature there are several different conventions regarding atomic transverse measures. We now recall and explain in greater detail, the definition (given in the second paragraph §2.5) which we use in this paper. We say that a finite, cyclically ordered collection of horizontal saddle connections \( \delta_1, \ldots, \delta_t \) form a loop if the right endpoint of \( \delta_i \) is the left endpoint of \( \delta_{i+1} \) (addition mod \( t \)). Any singular point \( \xi \in \Sigma \) of degree \( a \), is contained in a neighborhood \( U_\xi \) naturally parameterized by polar coordinates \( (r \cos \theta, r \sin \theta) \), for \( 0 \leq r < r_0 \) and \( \theta \in \mathbb{R}/(2\pi(a+1)\mathbb{Z}) \), where \( r = 0 \) corresponds to \( \xi \) (see [BSW, §2.5]). If \( \xi \in \Sigma \) is a right endpoint of \( \delta_i \) and a left endpoint of \( \delta_{i+1} \), we can parameterize the intersections of \( \delta_i, \delta_{i+1} \) with \( U_\xi \) using polar coordinates, and the \( i \)-th turning angle is the difference in angle between \( \delta_i \) and \( \delta_{i+1} \). The turning angle is well-defined modulo \( 2\pi(a+1)\mathbb{Z} \) and is an odd multiple of \( \pi \). We say that the loop is continuously extendible if for each \( i \) the \( i \)-th turning angle is \( \pm \pi \), and we say it is primitive if whenever we have a repetition \( \delta_i = \delta_j, i \neq j \), we must have that the turning angle at both of the endpoints of \( \delta_i \) differs in sign from that of \( \delta_j \). Thus on each surface there are only finitely many primitive continuously extendible loops. In this paper we will always assume that the atomic part of a transverse measure is a finite linear combination of Dirac masses on periodic trajectories or on primitive continuously extendible loops. Furthermore, the mass of a saddle connection \( \delta \) (i.e., the measure assigned to it by the transverse measure) in such an atomic measure is obtained as follows. Writing \( \ell_k = (\delta_{k1}^{(k)}, \ldots, \delta_{ks_k}^{(k)}) \) for the primitive extendible loops in the support of the measure, there are numbers \( a_k \) such that the mass of \( \delta \) is \( \sum_k a_k \# \left\{ i : \delta_i^{(k)} = \delta \right\} \).

We now motivate this definition. The leaves of the horizontal foliation \( \mathcal{F} \) on a surface \( M_q \) have a natural metric inherited from the 1-form \( dx \) on the plane, and we say that a leaf is critical if it is incomplete with respect to this metric (and then its completion contains a singularity). A bounded critical leaf is a horizontal saddle connection, a bounded non-critical leaf is periodic, and an unbounded critical leaf is isometric to a ray and is called a separatrix. Since transverse measures are assumed to be a system of finite measures, and infinite leaves (critical or noncritical leaves) have nontrivial accumulation points, the atoms of a
transverse measure cannot be on infinite leaves. Thus the cases left to discuss are of transverse measures with atoms on periodic horizontal leaves and on horizontal saddle connections. We will be interested in transverse measures \( \nu \) which are naturally associated with cohomology classes \( \beta_\nu \in H^1(M_q, \Sigma_q; \mathbb{R}) \). As we explained in \( \S 2.5 \), if \( \nu \) is non-atomic, it determines a 1-cochain on \( H_1(M_q, \Sigma_q) \), and this gives the assignment \( \nu \mapsto \beta_\nu \). If \( \nu \) is atomic we will define \( \beta_\nu \) as a cohomology class rather than an explicit cochain. The continuous extension \( \delta \), \( \delta \) is a continuous closed curve homotopic to \( \delta \) with all its points in \( S \setminus \bigcup \xi U_\xi \), which is the same as \( \delta \) outside the neighborhoods \( U_\xi \), and such that for each \( i \), the intersection of \( \delta_i, \delta_{i+1} \) with \( U_\xi \) is replaced with a curve on \( \partial U_\xi \) corresponding to \( r = r_0 \) and \( \theta \) in an interval of length \( \pi \). Continuously extendible loops can be thought of as ‘ghosts of departed cylinders’; i.e. they can be seen as the outcome of collapsing the height of a horizontal cylinder to zero.

The assignment \( \nu \mapsto \beta_\nu \) is now defined as follows. By linearity, we only need to define this assignment in the case that \( \nu \) assigns unit mass to one primitive continuously extendible loop or to one closed horizontal leaf. In the former case we let \( \delta \) denote the continuous extension of the continuously extendible loop, and in the latter case we let \( \delta \), \( \delta \) denote the horizontal periodic loop supporting the measure. These are closed loops avoiding \( \Sigma_q \) so represent elements of \( H_1(M_q, \Sigma_q) \), and hence, by Poincaré duality, of \( H^1(M_q, \Sigma_q; \mathbb{R}) \).

The assignment \( \nu \mapsto \beta_\nu \) is not injective, indeed atomic transverse measures supported on distinct homotopic closed horizontal curves yield the same cohomology class. It follows from Proposition 2.4 that this is the only source of non-injectivity, that is, when \( q \) has no horizontal cylinders, the map \( \nu \mapsto \beta_\nu \) is injective. Furthermore, when \( q \) does have a horizontal cylinder \( C \) with area \( A \), the Dirac atomic measure on a core curve of \( C \) defines the same cohomology class as \( \frac{1}{A} dy \big|_C \).

With this definition of \( \beta_\nu \), for any cycle \( \alpha \) represented by a horizontal closed path, we have \( \beta_\nu(\alpha) = 0 \) (where \( \beta_\nu(\cdot) \) is the evaluation map), and the same is true if \( \alpha = \eta \) is represented by a continuously extendible loop. The same is true if \( \nu \) is non-atomic and \( \alpha \) is represented by a concatenation of horizontal saddle connections. However, if \( \nu \) has atoms and \( \alpha \) is represented by a concatenation of horizontal saddle connections, it may happen that \( \beta_\nu(\alpha) \neq 0 \). If this happens, the tremor to time \( s = -\beta_\nu(\alpha) \) will not be defined; indeed, if the surface \( q' = \text{trem}_{q,s;\beta_\nu} \) existed, then using (4.7), we would have \( \text{hol}_{q'}(\alpha) = (0,0) \), which is impossible. This shows why the requirement in Proposition 4.8 that the tremor is non-atomic, is essential. Finally we note that
13.1. **Refining an APD.** Our discussion of tremors for atomic transverse measures will rely on the construction in §4.2. Recall from the proof of Proposition 4.9 that an APD for $q$ is a polygon decomposition of the underlying surface $M_q$, into triangles and quadrilaterals, such that the quadrilaterals contain a horizontal diagonal. We consider all edges of an APD as open, i.e. they do not contain their endpoints. In order to pay attention to atomic measures, we further subdivide each edge of an APD into finitely many subintervals by removing the points that lie on horizontal saddle connections. We will denote by $J_i$ these open intervals lying on edges of an APD. We will refer to an APD whose edges have been additionally subdivided as above, as a **refined APD**. For each $i$, each polygon $P$ with $J_i \subset \partial P$, and each $x \in J_i$, we define the opposite point $\text{opp}_P(x)$ as in the proof of Proposition 4.9.

Let $J = J_{i_0}$ for some $i_0$, $J \subset \partial P$, and let $J' = \text{opp}_P(J)$. Then $J'$ is a union of either one or two of the intervals $J_i$, for $i \neq i_0$, depending on whether a point of $J$ has an opposite point in $\Sigma$. In the former case we set $J_0 = J$ and in the latter case we set $J_0 = J \setminus \text{opp}_P^{-1}(\Sigma)$. With these definitions $\text{opp}_P|_{J_0} : J_0 \rightarrow J'$ is a bijection. Note that each endpoint of $J$ lies on a horizontal saddle connection or in $\Sigma$, and each endpoint of $J_0$ is either an endpoint of some $J_i$ or lies on an infinite critical leaf.

Let $\nu$ be a transverse measure on $M_q$ whose atoms, if any, are on non-critical periodic trajectories. It assigns a measure to each of the intervals $J, J', J_0$, and by our condition that any atoms lie on periodic trajectories, the restriction to $J$ has the same mass as the restriction to $J_0$. The measures will be denoted by $\nu_J, \nu_{J'}, \nu_{J_0}$. They satisfy the invariance property as in the proof of Proposition 4.9.

Conversely, given a refined APD for a translation surface $q$, suppose we are given a collection of finite measures $\nu_J$ on the edges $J$ as above, satisfying the invariance property. Since an infinite leaf has an accumulation point in one of the $J$, by the invariance property, any atoms of the measures $\nu_J$ lie on finite leaves. The points of $M_q$ lying on horizontal saddle connections are not in any of the $J$’s, and thus we can reconstruct from the $\nu_J$ a transverse measure all of whose atoms (if any) are on periodic trajectories. The cohomology class $\beta$ corresponding to this transverse measure satisfies $\beta(E) = \sum_{J \in E} \nu_J(J)$ for any edge $E$ of the APD.

13.2. **Proof of Proposition 4.1.**
Proof of Proposition 4.1. Let \( \tilde{q}_n \to \tilde{q} \) and \( \beta_n \to \beta \) be as in the statement of the Proposition, let \( q_n = \pi(\tilde{q}_n), q = \pi(\tilde{q}) \) be the projections to \( \mathcal{H} \), and let \( M_{q_n}, M_q \) the underlying surfaces. As in §4.2, we can assume that \( \tilde{q}_n \) and \( \tilde{q} \) are represented by marking maps \( \varphi_n \to M_{q_n}, \varphi : S \to M_q \) such that \( \varphi_n \circ \varphi^{-1} \) is piecewise affine with derivative tending to \( \text{Id} \) as \( n \to \infty \). Let \( K \subset M_q \) range over each of the intervals \( J, J', J_0 \) in a refined APD for \( q \). Let us first assume that each \( \beta_n = \beta_{\nu_n} \) for some non-atomic transverse measure \( \nu_n \). The general case will be discussed further below.

Let \( \nu_K^{(n)} \) denote the measure on \( K \) given by the pushforward of \( \nu_n \) under \( \varphi \circ \varphi^{-1} \), and denote the total variation of \( \nu_K^{(n)} \) by \( m_K^{(n)} \). This number can be expressed as the evaluation of \( \beta_n \) on a path \( \sigma = \sigma_K \) from singular points to singular points that is a concatenation of \( K \) with parts of horizontal saddle connections. Since \( \beta_n \to \beta \), we have \( m_K^{(n)} \to_{n \to \infty} m_K = \beta(\sigma) \). Let \( \tilde{K} = \varphi^{-1}(K) \subset S \). Since \( K \) is open and not horizontal, \( \tilde{K} \) has a natural compactification \( \bar{K} \) in which we add bottom and top endpoints \( x^b_K, x^t_K \) to \( \bar{K} \). Note that we consider \( \bar{K} \) abstractly, and not as a subset of \( S \). Because the \( \nu_K^{(n)} \) are non-atomic, each measure \( \nu_K^{(n)} \) can be viewed as a measure on the compact interval \( \bar{K} \), assigning mass zero to endpoints. Passing to further subsequences, we can assume each sequence \( \left( \nu_K^{(n)} \right)_n \) converges to a measure \( \nu_K \) on \( \bar{K} \) such that \( \nu_K = \nu_K|_\bar{K} \). We have

\[
m_K = \nu_K(\bar{K}) = \nu_K(\bar{K}) + e^b_K + e^t_K, \tag{13.1}
\]

where the numbers \( e^b_K = \nu_K(x^b_K), e^t_K = \nu_K(x^t_K) \) record the escape of mass to endpoints. We can concretely express the \( e_K^{b,t} \) by subdividing \( K \) into two half-intervals \( K^b, K^t \) whose common endpoint is an interior point of \( K \) which has zero measure under \( \nu_K \). In these terms

\[
e^b_K = \lim_{n \to \infty} \nu_K^{(n)}(K^b) - \nu_K(K^b) \tag{13.2}
\]

(and this limit does not depend on the decomposition \( K = K^b \cup K^t \)).

Since the collection of measures \( \{ \nu_K \} \) satisfies the invariance property, it defines a transverse measure, and we let \( \beta' \) be the corresponding cohomology class. Suppose first that there is no escape of mass, i.e. all the \( e_K^{b,t} \) are 0. Using the fact that each \( \beta_n \) is non-atomic, for each edge \( E \) of the refined APD we have:

\[
\beta(E) = \beta_n(E) = \sum_K m_K^{(n)} \to \sum_K m_K = \sum_K \nu_K(\bar{K}) = \beta'(E),
\]
where the sum ranges over open intervals $K \subset E$ covering all but finitely many points of $E$. In this case we have shown that $\beta = \beta'$ corresponds to a transverse measure, and we are done. This establishes the statement when no mass is lost to the endpoints of the APD.

In order to treat the case that some of the $e_{K}^{b,t}$ are positive, we will need to record additional information about the invariance property satisfied by the measures $\nu_{K}$. It will be useful to use boundary-marked surfaces (see [BSW, §2.5]) for this. Let $\hat{S} \to \hat{q}$ be a blown up marked version of the marked surface $S \to q$. Let $\xi \in \Sigma$ and recall that $\hat{q}$ replaces $\xi$ with a circle parameterized by an angular variable $\theta$ taking values in $\mathbb{R}/(2(a + 1)\mathbb{Z})$, where $a$ is the order of $\xi$. Each $\theta$ will be called a prong at $\xi$ which can be thought of as the tangent direction of an infinitesimal line segment of angle $\theta \mod 2\pi$ ending at $\xi$. The infinitesimal line is horizontal if and only if $\theta \in \pi \mathbb{Z}$. In a similar way we can blow up nonsingular points of $S$, replacing them with a circle parameterized by $\mathbb{R}/(2\pi \mathbb{Z})$, and thus talk about the prongs at a regular point (this corresponds to a singularity of order $a = 0$). For each $k \in \mathbb{Z}/(2(a + 1)\mathbb{Z})$, and each $\xi$ two prongs at $\xi$ are called bottom-adjacent (resp. top-adjacent) if their angular parameter belongs to the same interval $[k\pi, (k + 1)\pi]$ with $k$ even (resp. odd), and adjacent if they are either bottom- or top-adjacent. By definition of an APD, at each $\xi$ and each $k$, there is at least one edge $E$ with an endpoint in $(k\pi, (k + 1)\pi)$.

We have compactified the line segments $K$ corresponding to $J, J_{0}, J'$ as above by abstract points $x_{K}^{b}, x_{K}^{t}$, and these points map to points in $\hat{S}$ by continuously continuing the embedding $\hat{K} \to \hat{S}$. We will denote these points in $\hat{S}$ by their angular parameters $\theta_{K}^{b,t}$ and call them prongs of the APD. Since the APD contains no horizontal segments, $\theta_{K}^{b,t} \notin \pi \mathbb{Z}$. Note that for $k$ even (resp. odd), all prongs of the APD with angular parameter in $(k\pi, (k + 1)\pi)$ are of form $\theta_{K}^{b,t}$ (resp. $\theta_{K}^{t}$). Via the $x_{K}^{b,t}, x_{K}^{t}$, we have associated to each of these prongs an ‘escape of mass’ quantity $e_{K}^{b,t}$.

**Claim:**

1. The weights of prongs of the APD only depend on their adjacency class. More precisely, if $K, K'$ are edges of the APD with bottom- (resp. top-) adjacent prongs $\theta_{K}^{b,t}, \theta_{K'}^{b,t}$ (resp. $\theta_{K}^{t}, \theta_{K'}^{t}$) then $e_{K}^{b} = e_{K'}^{b}$ (resp. $e_{K}^{t} = e_{K'}^{t}$).

2. For any horizontal saddle connection $\sigma$, let $\xi_{1}, \xi_{2}$ in $S$ be consecutive points of $\sigma$ lying on edges of the APD (the $\xi_{i}$ could either be singular points or interior points of edges of the APD which are endpoints of subintervals $K$). For $i = 1, 2$, let $\theta_{i}^{(\sigma)}$
represent the two prongs of $\sigma$ at $\xi_i$, and let $K_i$ (resp. $L_i$) be intervals with prongs at $\xi_i$ which are part of the APD, such that $\theta_{K_i}$ (resp. $\theta_{L_i}$) is bottom- (resp. top-) adjacent to $\theta_{i}^{(\sigma)}$. Then

$$e_{K_i}^b + e_{L_i}^t = e_{K_2}^b + e_{L_2}^t.$$  \((13.3)\)

(3) If a horizontal prong adjacent to $\theta_{K_i}^{b,t}$ is on an infinite critical leaf then $e_{K_i}^b = 0$.

Proof of Claim: Because adjacent prongs are in the same $(k\pi, (k+1)\pi)$ interval of direction, they are exchanged by opp$_P$ and so statement (1) follows from (13.2) and the invariance property of the measures $\nu_K$. To see (2), note that the assumption that $\xi_i$ are consecutive along $\sigma$ means that $K_1, L_1, K_2, L_2$ are both subintervals of edges of one polygon $P$ for the APD, with opp$_P(K_1) = K_2$ and opp$_P(L_1) = L_2$. By (13.2) we have

$$e_{K_i}^b + e_{L_i}^t = \lim_{n \to \infty} \left( \nu_{K_i}^{(n)}(K_i^b) + \nu_{L_i}^{(n)}(L_i^t) \right) - \left( \nu_{K_i}(K_i^b) + \nu_{L_i}(L_i^t) \right)$$

for each $i$, and (13.3) follows from the invariance property of each of the $\nu_K^{(n)}$ on $K_i^b, L_i^t, K_i^b \cup L_i^t$.

For (3), any critical leaf $\ell$ intersects some interval $J$ of the APD in its interior infinitely many times. If $e_{K_i}^{b,t} \neq 0$ for a prong $\theta_{K_i}^{b,t}$ adjacent to a prong defined by an endpoint of $\ell$, we obtain infinitely many atoms in the interior of $J$, and by the invariance property, they all have the $\nu_J$-mass. This contradicts the finiteness of the measure $\nu_J$. $\triangle$

We can now interpret extendible loops for boundary marked surfaces using our notion of adjacency: an extendible loop is a loop formed as a concatenation of saddle connections which are bottom- or top-adjacent at each of their endpoints. Thus each meeting of consecutive saddle connections represents an adjacency class and we say that $\delta$ represents each of the classes defined by these meeting points. By (1), the loss of mass parameters $e_{K_i}^{b,t}$ assign numbers $e_A$ to each bottom/top adjacency class $\mathcal{A}$. Now suppose that some of the $e_{K_i}^{b,t}$ are positive, and choose the adjacency class $\mathcal{A}_1$ for which

$$e_{\mathcal{A}_1} = \min\{e_A : e_A > 0\}.$$  

We claim that $M_q$ contains a primitive extendible loop, such that all the adjacency classes $\mathcal{A}$ represented by this loop satisfy $e_A \geq e_{\mathcal{A}_1}$. To see this, let $\delta_1$ be an outgoing prong in $\mathcal{A}_1$. According to (3), $\delta_1$ is part of a horizontal saddle connection. Let $\mathcal{A}_2^{b,t}$ be the two adjacency classes of the terminal point of $\delta_1$. Then according to (13.3), at least one of
\( e^{b,t}_{A_{ij}} \) is positive, and hence is bounded below by \( e_{A_1} \). We choose \( \delta_2 \) to lie on an outgoing prong of this adjacency class, and continue, finding \( \delta_1, \delta_2, \ldots \) which form an extendible loop, such that the adjacency class \( A_i \) represented by the meeting of \( \delta_i, \delta_{i+1} \) has \( e_{A_i} \geq e_{A_1} \). We have completed the proof of the claim.

By a straightforward induction, the claim implies that there is an integer \( s \geq 0 \), primitive extendible loops \( \delta_{p_{ij}} \), \( j = 1, \ldots, s \) and finitely many positive real numbers \( c_1, \ldots, c_s \) such that for each adjacency class \( A \),

\[
e_A = \sum_{\delta^{(j)} \text{ represents } A} c_j. \tag{13.4}
\]

We now show

\[
\beta - \beta' = \sum_j c_j \beta^{(j)}, \tag{13.5}
\]

where \( \beta^{(j)} \) is the class Poincaré dual to \( \delta^{(j)} \). Indeed, it is enough to check this identity by evaluating on the paths \( \alpha = \sigma_K \) introduced in the second paragraph of the proof, since such paths represent cycles which generate \( H_1(M_q, \Sigma_q) \). For such paths, (13.5) is immediate from (13.1) and (13.4).

Equation (13.5) completes the proof of Proposition 4.1, under the assumption that the \( \beta_n \) are non-atomic. For the general case, for each \( n \), write \( \beta_n = \beta_{n}^{\text{na}} + \beta_{n}^{\text{at}} \) as a sum of cohomology classes represented by non-atomic and atomic tremors respectively. Since \( L_{q_n}(\beta_{n}^{\text{na}}) \leq L_{q_n}(\beta_n) \rightarrow L_q(\beta) \), the sequence \( L_{q_n}(\beta_{n}^{\text{na}}) \) is bounded and hence the sequence \( (\beta_{n}^{\text{na}})_{n \in \mathbb{N}} \subset H^1(S, \Sigma; \mathbb{R}) \) is also bounded. Therefore we can pass to subsequences to assume \( \beta_{n}^{\text{na}} \rightarrow_{n \rightarrow \infty} u_1 \) and hence \( \beta_{n}^{\text{at}} \rightarrow_{n \rightarrow \infty} u_2 = \beta - u_1 \). By what we have already shown, \( u_1 \) is a signed foliation cocycle. Now write \( \beta_{n}^{\text{at}} \) as a sum \( \sum_{j=1}^{s_n} c_{j}^{(n)} I(\cdot, \delta_{n}^{(j)}) \) where \( \delta_{n}^{(j)} \) is an extendible loop. Since there is a bound on the number of horizontal saddle connections on a surface in a fixed stratum, we can pass to a subsequence to assume that \( s = s_n \) is fixed independently of \( n \) and for each \( j = 1, \ldots, s \), each \( \delta_{n}^{(j)} \) passes through the same prongs in the same order. Passing to further subsequences we can assume that for each \( j, c_j = \lim_{n \rightarrow \infty} c_{j}^{(n)} \) exists. Passing to further subsequences and re-indexing, there is \( r \leq s \) such that for \( j \leq r \), the total length of each \( \delta_{n}^{(j)} \) is bounded independently of \( n \), and for \( j > r \), the total length of \( \delta_{n}^{(j)} \) tends to infinity. Let \( j \leq r \). Since the lengths of the \( \delta_{n}^{(j)} \) are bounded we can pass to further subsequences to assume that the number of intersection points of each edge \( E \) of the APD with each curve \( \delta_{n}^{(j)} \) is fixed. This implies that the sequence of cohomology classes \( I(\cdot, \delta_{n}^{(j)}) \)
converges, and the limit is a signed foliation cocycle associated with an atomic transverse measure.

Now let \( j > r \), and fix a refined APD for \( q \). Then for all large enough \( n \), the curve \( \delta_n^{(j)} \) is longer than all horizontal cylinders or saddle connections on \( M_q \). Thus for all large \( n \), the \( \delta_n^{(j)} \) give rise to measures on the sides \( E \) of the APD that assign zero measure to endpoints of segments. Thus we can repeat the analysis in the first part of the proof to the atomic measures \( \delta_n^{(j)} \).

\[ \square \]

References


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