TREMORS AND HOROCYCLE DYNAMICS ON THE MODULI SPACE OF TRANSLATION SURFACES

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Abstract. We introduce a 'tremor' deformation on strata of translation surfaces. Using it, we give new examples of behaviors of horocycle flow orbits Uq in strata of translation surfaces. In the genus 2 stratum $\mathcal{H}(1,1)$ we find orbits Uq which are generic for a measure whose support is strictly contained in $\overline{Uq}$ and find orbits which are not generic for any measure. We also describe a horocycle orbit-closure whose Hausdorff dimension is not an integer.

1. Introduction

A surprisingly fruitful technique for studying certain mathematical objects is to study dynamics on their moduli spaces. Examples of this phenomenon occur in the study of integral values of indefinite quadratic forms (motivating the study of dynamics of Lie group actions on homogeneous spaces) and billiard flows on polygonal tables (motivating the study of the $\mathrm{SL}_2(\mathbb{R})$-action on the moduli space of translation surfaces). In both cases, far-reaching results regarding the actions on the moduli spaces have been used to shed light on a wide range of problems in number theory, geometry, and ergodic theory. See [Zo, Wr2, KSS] for surveys of these developments.

Let $B \subset \mathrm{SL}_2(\mathbb{R})$ be the subgroup of upper triangular matrices, and let

$$U \stackrel{\mathrm{def}}{=} \{u_s : s \in \mathbb{R}\}, \quad \text{where} \quad u_s \stackrel{\mathrm{def}}{=} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}. \tag{1.1}$$

The U-action is an example of a unipotent flow and, in the case of strata of translation surfaces, is also known as the horocycle flow. The actions of these groups on moduli spaces are fundamental in both dynamical settings. For homogeneous spaces of Lie groups, actions of subgroups such as $\mathrm{SL}_2(\mathbb{R})$, B and U are strongly constrained and much is known about invariant measures and orbit-closures. For the action on a stratum $\mathcal{H}$ of translation surfaces, fundamental papers of McMullen, Eskin, Mirzakhani and Mohammadi [McM1, EM, EMM] have shown that the invariant measures and orbit closures for the $\mathrm{SL}_2(\mathbb{R})$-action and B-action on $\mathcal{H}$ are severely restricted and have remarkable geometric features; in particular an orbit-closure is the image of a manifold under an immersion. This behavior is very much like the behavior observed in the homogeneous setting.

In this paper we examine the degree to which such regular behavior might hold for the U-action or horocycle flow on strata. We give examples showing that, with respect to orbit-closures and the asymptotic behavior of individual orbits, the U-action on $\mathcal{H}$ has features which are absent in homogeneous dynamics.

In order to set the stage for this comparison we first recall some results about the dynamics of unipotent flows on homogeneous spaces. Special cases of these were proved by several authors and the results were proved in complete generality in celebrated work of Ratner (see [M] for a survey, and for the definitions used in the statement below).

Theorem 1.1 (Ratner). *Let G be a connected Lie group, Γ a lattice in G, $X = G/\Gamma$, and $U = \{u_s : s \in \mathbb{R}\}$ a one-parameter Ad-unipotent subgroup of G.*

(1) *For any $x \in X$, $\overline{Ux} = Hx$ is the orbit of a group H satisfying $U \subset H \subset G$, and Hx is the support of an H-invariant probability measure μ_x.*
(2) *Any $x \in X$ is* generic for μ_x, *i.e.*

$$\forall f \in C_c(X), \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_s x)ds = \int_X f d\mu_x.$$

Statement (1) is known as the orbit-closure theorem, and statement (2) is known as the genericity theorem.

1.1. **Main results.** We will introduce a method for constructing U-orbits with unexpected properties, and apply it in the genus two stratum $\mathcal{H}(1,1)$.

In the homogeneous setting, orbit-closures of unipotent flows are manifolds. It was known (see [SW2]) that horocycle orbit-closures could be manifolds with boundary in the setting of translation surfaces. We show here that they can be considerably wilder.

Theorem 1.2. *There is $q \in \mathcal{H}(1,1)$ for which the orbit-closure $\overline{Uq}$ has non-integer Hausdorff dimension. In fact, by appropriately varying the initial point, q, we can construct an uncountable nested chain of distinct horocycle orbit-closures of fractional Hausdorff dimension.*

We will give additional information about these orbit-closures in Theorems 1.8 and 1.9 below.

Let $\mathcal{E}_4 \subset \mathcal{H}_1(1,1)$ denote the set of unit-area surfaces which can be presented as two identical tori glued along a slit (in the notation and terminology of McMullen [McM1], $\mathcal{E}_4$ is the subset of area-one surfaces in the eigenform locus of discriminant $D = 4$).

From now on we write $G \stackrel{\mathrm{def}}{=} \mathrm{SL}_2(\mathbb{R})$ and $\mathcal{E} \stackrel{\mathrm{def}}{=} \mathcal{E}_4$. The locus $\mathcal{E}$ is 5-dimensional, is G-invariant, and is the support of a G-invariant ergodic probability measure $\mu_{\mathcal{E}}$.

Theorem 1.3. *There is $q \in \mathcal{H}(1,1)$ which is not contained in $\mathcal{E}$ but which is generic for the measure $\mu_{\mathcal{E}}$ supported on $\mathcal{E}$.*

Since $\mathcal{E} = \operatorname{supp} \mu_{\mathcal{E}}$ is strictly contained in $\overline{Uq}$, this orbit does not satisfy the analogue of Theorem 1.1(2). The next result shows that the analogue of Ratner's genericity theorem fails dramatically in $\mathcal{H}(1,1)$:

Theorem 1.4. *There is a dense G_δ subset of $q \in \mathcal{H}(1,1)$ and $f \in C_c(\mathcal{H}(1,1))$ so that*

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T f(u_s q)ds < \limsup_{T\to\infty} \frac{1}{T}\int_0^T f(u_s q)ds. \tag{1.2}$$

In particular such points are not generic for any measure on $\mathcal{H}(1,1)$, and there are such points whose forward and backward geodesic trajectories (i.e., in the notation (2.4), the sets $\{g_t q : t > 0\}$ and $\{g_t q : t < 0\}$) are both dense.

One property of unipotent flows on homogeneous spaces which played a crucial role in Ratner's work is 'controlled divergence of nearby trajectories'. The proof of Theorem 1.3 shows that in strata, divergence of nearby trajectories can be erratic. We make this precise in §8.3, see Theorem 8.6.

The proofs of Theorems 1.2, 1.3, and 1.4 rely on the tremor paths which we now introduce (the geological nomenclature is inspired by Thurston's earthquake paths, see [T2]).

1.2. **Tremors.** We can describe the action of the horocycle flow on a translation surface geometrically as giving us a family of surfaces obtained by changing the flat structure on the original surface by shearing it horizontally. An interesting modification of this procedure was studied by Alex Wright [Wr1]. Let $q \in \mathcal{H}$, let M_q be the corresponding surface, and suppose M_q contains a horizontal cylinder C. Then one can deform M_q by horizontally shearing the flat structure on C and leaving $M_q \smallsetminus C$ unchanged. This *cylinder shear* operation defines a flow on the subset of the stratum corresponding to surfaces containing a horizontal cylinder. This subset of $\mathcal{H}$ is invariant under the horocycle flow and on it, the flow defined by the cylinder shear commutes with the horocycle flow. The tremors we study in this paper are partially defined flows which commute with the horocycle flow on their domains of definition and are a common generalization of both cylinder shears and the horocycle flow. While Wright's analysis of cylinder shears focused on shears that keep points inside a G-invariant locus we will study tremors that move points in a G-invariant locus away from that locus and we will use these tremors to exhibit new behaviors of the horocycle flow.

We can think of both the cylinder shear and the horocycle flow as arising from transverse invariant measures to the horizontal foliation $\mathcal{F}_q$ on the surface M_q, where the amount and location of shearing is determined by the transverse measure. If the cylinder shear flow takes q to q' then the relationship between their period coordinates (see §2.1 and §2.2, where we will explain the notation and make our discussion more precise) is given by

$$\mathrm{hol}^{(x)}_{q'}(\gamma) = \mathrm{hol}^{(x)}_{q}(\gamma) + t \cdot \tau(\gamma), \quad \mathrm{hol}^{(y)}_{q'}(\gamma) = \mathrm{hol}^{(y)}_{q}(\gamma). \tag{1.3}$$

Here $\mathrm{hol}^{(x)}_{q}$ and $\mathrm{hol}^{(y)}_{q}$ denote the cohomology classes corresponding to the transverse measures dx and dy on M_q respectively, γ is an oriented closed curve or path joining singularities on M_q, t is the parameter for the cylinder shear flow, and τ is the cohomology class corresponding to the transverse measure which is the restriction of dy to the cylinder. The horocycle flow is given in period coordinates as

$$\mathrm{hol}^{(x)}_{u_s q}(\gamma) = \mathrm{hol}^{(x)}_{q}(\gamma) + s \cdot \mathrm{hol}^{(y)}_{q}(\gamma), \quad \mathrm{hol}^{(y)}_{u_s q}(\gamma) = \mathrm{hol}^{(y)}_{q}(\gamma). \tag{1.4}$$

Recalling that $\mathrm{hol}^{(y)}_{q}(\gamma)$ is the cohomology class corresponding to the transverse measure dy, and recalling that some surfaces may have additional transverse measures to the horizontal foliation $\mathcal{F}_q$, we will define a surface q' via the formula

$$\mathrm{hol}^{(x)}_{q'}(\gamma) = \mathrm{hol}^{(x)}_{q}(\gamma) + t \cdot \beta(\gamma), \quad \mathrm{hol}^{(y)}_{q'}(\gamma) = \mathrm{hol}^{(y)}_{q}(\gamma), \tag{1.5}$$

where β is the cohomology class associated with a transverse measure on M_q. In a sense that we will make precise in §5, this means that M_q is deformed by shearing nearby horizontal lines relative to each other, where the amount of shearing is specified by β and t (see Figure 1). We write $\mathrm{trem}_{t,\beta}(q)$ for q' and $\mathrm{trem}_{\beta}(q)$ for $\mathrm{trem}_{1,\beta}(q)$. We refer to a surface of the form $\mathrm{trem}_{t,\beta}(q)$ as a *tremor of* q. As we will show in §4.1.4 and §4.3, q' is uniquely determined by q, t and β.

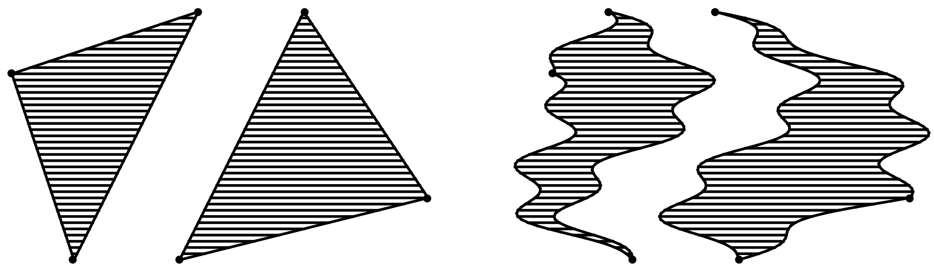

FIGURE 1. The effect of a tremor on a subset of a translation surface.

We now give some additional definitions needed for stating our results. If the transverse measure corresponding to β is absolutely continuous with respect to dy (see §4.1.3) we will say that both β and the tremor $\mathrm{trem}_\beta(q)$ are *absolutely continuous.* If q has no horizontal saddle connections and the transverse measure is not a scalar multiple of dy, we will say β and $\mathrm{trem}_\beta(q)$ are *essential*. We will denote the subspace of cohomology corresponding to signed transverse measures at q by $\mathcal{T}_q$. This can be related to the tangent space to the stratum, see §2.3 and §4.1.1. If the transverse measure is non-atomic, i.e. assigns zero measure to all horizontal saddle connections or closed leaves, then the tremor path can be continued for all time, see Proposition 4.13. The case of atomic transverse measures presents some technical difficulties which will be discussed in §13.

1.3. **More detailed results.** The importance of tremor maps for the study of the horocycle flow is that, where they are defined, they commute with the horocycle flow, i.e., $u_s\mathrm{trem}_\beta(q) = \mathrm{trem}_\beta(u_s q)$ (this makes sense because q and $u_s q$ have the same horizontal foliation, thus we can consider the same cohomology class β as an element of both $\mathcal{T}_q$ and $\mathcal{T}_{u_s q}$, see §4.3 for more details). In particular we will see that for many tremors, the surfaces $u_s q$ and $u_s\mathrm{trem}_\beta(q)$ stay close to each other, and this leads to the following:

Theorem 1.5. *Let $\mathcal{H}$ be any stratum, let $\mathcal{H}_1$ be its subset of area-one surfaces, and let $\mathcal{L} \subset \mathcal{H}_1$ be a closed U-invariant set which is the support of a U-invariant ergodic measure μ. Let $q \in \mathcal{L}$, $\beta \in \mathcal{T}_q$ and $q_1 = \mathrm{trem}_\beta(q)$. Then:*

(i) *If β is absolutely continuous then for the sup-norm distance* dist *on $\mathcal{H}$ (see §2.6), we have*

$$\sup_{s \in \mathbb{R}} \mathrm{dist}(u_s q, u_s q_1) < \infty. \tag{1.6}$$

(ii) *If β is absolutely continuous then for any q in $\overline{Uq_1} \smallsetminus \mathcal{L}$, the surface M_q has a non-uniquely ergodic horizontal foliation. In particular, if $\mathcal{L} \neq \mathcal{H}_1$ then Uq_1 is not dense in $\mathcal{H}_1$.*

(iii) *If μ-a.e. surface in $\mathcal{L}$ has no horizontal saddle connection and if q is generic for μ, then q_1 is also generic for μ.*

We will give examples of loci $\mathcal{L}$ and points q for which the hypotheses of Theorem 1.5 are satisfied, namely we will find $\mathcal{L}$ and q for which:

(I) The locus $\mathcal{L}$ is G-invariant and is the support of a G-invariant ergodic measure μ and the orbit Uq is generic for μ.

(II) The surface M_q has no horizontal saddle connections and the transverse measure corresponding to dy on M_q is not ergodic (and hence q admits essential absolutely continuous tremors).

(III) There is an essential absolutely continuous tremor q_1 of q which is not in $\mathcal{L}$.

There are many examples of strata $\mathcal{H}$ and loci $\mathcal{L}$ for which these properties hold. One particular example which we will study in detail is $\mathcal{L} = \mathcal{E} \subsetneq \mathcal{H}_1(1,1)$ (see §3.1 for more information on $\mathcal{E}$). Namely we will prove the following result which, in conjunction with Theorem 1.5, immediately implies Theorem 1.3.

Theorem 1.6. *There are points $q \in \mathcal{E}$ satisfying (I), (II) and (III) above. Moreover, for any $q \in \mathcal{E}$ which admits an essential tremor $\beta \in \mathcal{T}_q$, the points*

$$q_r \stackrel{\mathrm{def}}{=} \mathrm{trem}_{r,\beta}(q) \in \mathcal{H}(1,1) \ (\textit{where } r > 0)$$

satisfy

$$0 < r_1 < r_2 \implies \overline{Uq_{r_1}} \neq \overline{Uq_{r_2}}. \tag{1.7}$$

Remark 1.7. *Theorem 1.6 is also true if $\mathcal{E}$ is replaced with any of the other eigenform loci $\mathcal{E}_D \subset \mathcal{H}(1,1)$. See §8.2 for more details.*

For certain $q \in \mathcal{E}$ and $\beta \in \mathcal{T}_q$, we can give a complete description of the closure of Uq_1 where $q_1 = \mathrm{trem}_\beta(q)$. To state this result we will need a measurement of the size of a tremor and to do this we introduce the *total variation* $|L|_q(\beta)$ of $\beta \in \mathcal{T}_q$, see §4.1.2 for the definition. Also we say that $q \in \mathcal{E}$ is *aperiodic* if the horizontal foliation of M_q is not periodic, i.e. it is either minimal or contains a horizontal slit separating the surface into two tori so that the restriction of the horizontal foliation to each torus is minimal.

Theorem 1.8. *For any $a > 0$ there is $q_0 \in \mathcal{E}$ and an essential tremor $q_1 = \mathrm{trem}_{\beta_0}(q_0) \in \mathcal{H}(1,1)$ of q_0 such that*

$$\begin{aligned} \overline{Uq_1} &= \overline{\{\mathrm{trem}_\beta(q) : q \in \mathcal{E} \text{ is aperiodic},\ \beta \in \mathcal{T}_q,\ |L|_q(\beta) \leqslant a\}} \\ &\subset \{\mathrm{trem}_\beta(q) : q \in \mathcal{E},\ \beta \in \mathcal{T}_q,\ |L|_q(\beta) \leqslant a\}. \end{aligned} \tag{1.8}$$

Moreover, setting $q_r \stackrel{\mathrm{def}}{=} \mathrm{trem}_{r,\beta_0}(q_0)$, we have that the orbit-closure $\overline{Uq_r}$ admits the description in (1.8) with the constant a replaced by ra, and the points q_r satisfy the following strengthening of (1.7):

$$0 < r_1 < r_2 \implies \overline{Uq_{r_1}} \subsetneq \overline{Uq_{r_2}}. \tag{1.9}$$

The following more explicit result implies Theorem 1.2. Its proof relies on [CHM].

Theorem 1.9. *Let $q_1 \in \mathcal{H}(1,1)$ be the point described in Theorem 1.8. Then the Hausdorf dimension of the horocycle orbit closure of q_1 satisfies*

$$5.5 \leqslant \dim \overline{Uq_1} < 6.$$

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2. Basics

In this section we review basic concepts and set up notation. Some readers will find it useful to skip this section on a first reading, and refer back to it as needed.

2.1. **Strata and period coordinates.** There are several possible approaches for defining the topology and geometric structure on strata, see [FM, MaTa, Wr2, Y, Zo]. For the most part we follow the approach of [BSW], where the reader can find additional details.

Let M be a compact oriented surface of genus g and let $\Sigma \subset M$ be a non-empty finite set with k elements. We make the convention that the points of Σ are labeled as $p_1, \ldots, p_k$. Let $\mathbf{r}$ be a list of k non-negative integers satisfying $\sum r_j = 2g - 2$. A *translation surface of type* $\mathbf{r}$ is given by an atlas on M of orientation preserving charts $\mathcal{A} = (\psi_\alpha, U_\alpha)_{\alpha \in \mathcal{A}}$, where the $U_\alpha \subset M \smallsetminus \Sigma$ are open and cover $M \smallsetminus \Sigma$, the transition maps $\psi_\alpha \circ \psi_\beta^{-1}$ are restrictions of translations to the appropriate domains, and such that the planar structure in a neighborhood of each $p_j \in \Sigma$

completes to a cone angle *singularity* of total cone angle $2\pi(r_j + 1)$. A *translation equivalence* between translation surfaces is a homeomorphism h which preserves the labels and the translation structure.

These charts determine a metric on M and a measure which we denote by Leb. These charts also allow us to define natural 'coordinate' vector fields ∂_x and ∂_y and 1-forms dx and dy on M. The (partially defined) flow corresponding to ∂_x will be called the *horizontal straight-line flow*, and we will denote the trajectory parallel to ∂_x starting at $p \in M_q$ by $t \mapsto \Upsilon^{(p)}(t)$. The corresponding foliation of $M \smallsetminus \Sigma$, which we denote by $\mathcal{F}$, will be called the *horizontal foliation.* If we remove from M the horizontal trajectories that hit singular points, then the straightline flow becomes an actual flow defined on a dense G_δ subset of full Lebesgue measure. If this flow is *minimal*, i.e. all infinite horizontal straightline flow trajectories are dense, we will say that $\mathcal{F}$ *is minimal* or that M is *horizontally minimal.*

Fix $\mathbf{r}$ of length k, and g satisfying the relation $\sum r_j = 2g-2$. Choose a surface S of genus g and a set $\Sigma \subset S$ of cardinality k, whose elements are labelled by $1, \ldots, k$ (note that we use the same symbol Σ to denote finite subsets of S and of M, this should cause no confusion). We refer to (S, Σ) as the *model surface.* A *marking map* of a translation surface M is a homeomorphism $\varphi : (S, \Sigma) \to (M, \Sigma)$ which preserves labels on Σ. We say that two markings maps $\varphi : (S, \Sigma) \to (M, \Sigma)$ and $\varphi' : (S, \Sigma) \to (M', \Sigma)$ are equivalent if there is a translation equivalence $h : M \to M'$ so that $h \circ \varphi$ is isotopic to φ' via an isotopy which fixes Σ. An equivalence class of translation surfaces with marking maps is a *marked translation surface.* There is a forgetful map which takes a marked translation surface, which is the equivalence class of $\varphi : S \to M$, to the translation equivalence class of M. We will denote this map by π and usually denote an element of $\pi^{-1}(q)$ by $\widetilde{q}$.

The set of translation self-equivalences of M is a finite group which we denote by Γ_M. In particular we get a left action, by postcomposition, of Γ_M on the set of marking maps of M. Note that a marking map determines a marked translation surface, but the marked translation surface need not uniquely determine the marking map φ. Indeed, if $h \in \Gamma_M$ is nontrivial, and $\widetilde{q}$ is the equivalence class of a marking map φ, then φ and $h \circ \varphi$ are different (in fact, non-isotopic) representatives of the same marked translation surface.

As we have seen a flat surface structure on M determines two natural 1-forms dx and dy and these 1-forms determine cohomology classes in $H^1(M, \Sigma; \mathbb{R})$ which we denote by $\mathrm{hol}^{(x)}$ and $\mathrm{hol}^{(y)}$. Specifically for an oriented curve γ we have $\mathrm{hol}^{(x)}(\gamma) = \int_\gamma dx$ and $\mathrm{hol}^{(y)}(\gamma) = \int_\gamma dy$.

We can combine these classes to create an $\mathbb{R}^2$-valued cohomology class $\mathrm{hol}_M = (\mathrm{hol}^{(x)}, \mathrm{hol}^{(y)})$ in $H^1(M, \Sigma; \mathbb{R}^2)$. Conversely, any $\mathbb{R}^2$-valued cohomology class gives rise to two $\mathbb{R}$-valued cohomology classes via the identification $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$. We denote the corresponding direct sum decomposition by

$$H^1(M, \Sigma; \mathbb{R}^2) = H^1(M, \Sigma; \mathbb{R}_x) \oplus H^1(M, \Sigma; \mathbb{R}_y). \tag{2.1}$$

Now consider a marked translation surface $\widetilde{q}$ with choice of marking map $\varphi : (S, \Sigma) \to (M, \Sigma)$, where $M = M_{\widetilde{q}}$ is the underlying translation surface. In this situation we have a distinguished element $\mathrm{hol}_{\widetilde{q}} = \varphi^*(\mathrm{hol}_M) \in H^1(S, \Sigma; \mathbb{R}^2)$ given by using the map φ to pull back the cohomology class hol_M from $H^1(M, \Sigma; \mathbb{R}^2)$ to $H^1(S, \Sigma; \mathbb{R}^2)$. More concretely if γ is an oriented curve in S with endpoints in Σ then $\mathrm{hol}_{\widetilde{q}}(\gamma) = \mathrm{hol}_M(\varphi(\gamma))$. The cohomology class $\mathrm{hol}_{\widetilde{q}}$ is independent of the choice of the marking map, and depends only on the equivalence class $\widetilde{q}$. We write $\mathrm{dev}(\widetilde{q})$ for the cohomology class $\mathrm{hol}_{\widetilde{q}} \in H^1(S, \Sigma; \mathbb{R}^2)$.

2.2. **An atlas of charts on $\mathcal{H}_{\mathrm{m}}$.** Let $\mathcal{H}_{\mathrm{m}} = \mathcal{H}_{\mathrm{m}}(\mathbf{r})$ denote the collection of equivalence classes of marked translation surfaces of a fixed type $\mathbf{r}$. Let $\mathcal{H} = \mathcal{H}(\mathbf{r})$ denote the collection of translation equivalence classes of translation surfaces. We will use the developing map defined above to equip these sets with a topology, via a local coordinate system which is referred to as *period coordinates*.

We caution the reader that different variants of these definitions can be found in the literature, and they might not be equivalent to our definitions, specifically as regards the question of whether or not points of Σ are labelled. Our terminology and notation follows [BSW], but we introduce some additional notation related to comparison maps, which will be useful in §4.2 and §5. Readers who are familiar with these notions may choose to skip this subsection.

A *geodesic triangulation* of a translation surface is a decomposition of the surface into triangles whose sides are saddle connections, and whose vertices are singular points, which need not be distinct. The existence of a geodesic triangulation of any translation surface is proved in [MS, §4]. Let $\varphi : (S, \Sigma) \to (M, \Sigma)$ be a marking map, let $\widetilde{q}$ be the corresponding marked translation surface, and let τ denote the pullback of a geodesic triangulation with vertices in Σ, from (M, Σ) to (S, Σ). The cohomology class $\mathrm{hol}_{\widetilde{q}}$ assigns coordinates in $\mathbb{R}^2$ to edges of the triangulation and thus can be thought of as giving a map from the triangles of τ to triangles in $\mathbb{R}^2$ (well-defined up to translation), and so each triangle in τ has a Euclidean structure coming from M. Let

U_τ be the collection of all cohomology classes which map each triangle of τ into a positively oriented non-degenerate triangle in $\mathbb{R}^2$. Each $\beta \in U_\tau$ gives a translation surface $M_{\tau,\beta}$ built by gluing together the corresponding triangles in $\mathbb{R}^2$ along parallel edges, as well as a distinguished marking map, which we denote by $\varphi_{\tau,\beta} : (S, \Sigma) \to (M_{\tau,\beta}, \Sigma)$, which is the unique map taking each triangle of the triangulation τ of S to the corresponding triangle of the triangulation of $M_{\tau,\beta}$ and which is affine on each triangle (with respect to the Euclidean structure coming from M). Let $\widetilde{q}_{\tau,\beta}$ denote the marked translation surface corresponding to the marking map $\varphi_{\tau,\beta}$. Let

$$V_\tau \stackrel{\text{def}}{=} \{\widetilde{q}_{\tau,\beta} : \beta \in U_\tau\} \quad \text{and} \quad \Psi_\tau : U_\tau \to V_\tau, \quad \Psi_\tau(\beta) = \widetilde{q}_{\tau,\beta}.$$

By construction, β agrees with $\mathrm{dev}(\widetilde{q}_{\tau,\beta})$ on edges of τ, and these edges generate $H_1(S, \Sigma)$. Thus the map

$$\Phi_\tau : V_\tau \to U_\tau, \quad \Phi_\tau(\widetilde{q}) = \mathrm{dev}(\widetilde{q})$$

is an inverse to Ψ_τ (and in particular Ψ_τ is injective). The collection of maps $\{\Phi_\tau\}$ gives an atlas of charts for $\mathcal{H}_{\mathrm{m}}$ and the collection of maps $\{\Psi_\tau\}$ gives an inverse atlas for $\mathcal{H}_{\mathrm{m}}$. These charts give $\mathcal{H}_{\mathrm{m}}$ a manifold structure for which the map dev is a local diffeomorphism. In fact this atlas determines an affine structure on $\mathcal{H}_{\mathrm{m}}$ so that dev is an affine map.

We denote the tangent space of $\mathcal{H}_{\mathrm{m}}$ at $\widetilde{q} \in \mathcal{H}_{\mathrm{m}}$ by $T_{\widetilde{q}}(\mathcal{H}_{\mathrm{m}})$ and by $T(\mathcal{H}_{\mathrm{m}})$ the tangent bundle of $\mathcal{H}_{\mathrm{m}}$. Using the fact that the developing map is a local diffeomorphism we can identify the tangent space at each point of $\mathcal{H}_{\mathrm{m}}$ with $H^1(S, \Sigma; \mathbb{R}^2)$ so $T(\mathcal{H}_{\mathrm{m}}) = \mathcal{H}_{\mathrm{m}} \times H^1(S, \Sigma; \mathbb{R}^2)$. We say that two tangent vectors $v_i \in T_{\widetilde{q}_i}(\mathcal{H}_{\mathrm{m}})$ $(i = 1, 2)$, or two subspaces $V_i \subset T_{\widetilde{q}_i}(\mathcal{H}_{\mathrm{m}})$ are *parallel* if they map to the same element or subspace of $H^1(S, \Sigma; \mathbb{R}^2)$. We say that a sub-bundle of $T(\mathcal{H}_{\mathrm{m}})$ is *flat* if the fibers over different points are parallel, and that a sub-bundle of $T(\mathcal{H})$ is *flat* if each of the connected components of its pullback to $T(\mathcal{H}_{\mathrm{m}})$ is flat.

Let

$$H^1(S, \Sigma; \mathbb{R}^2) = H^1(S, \Sigma; \mathbb{R}_x) \oplus H^1(S, \Sigma; \mathbb{R}_y) \tag{2.2}$$

be the analogue of (2.1) for the model surface S. This decomposition determines a foliation of $H^1(S, \Sigma; \mathbb{R}^2)$, whose leaves are pre-images of points under the projection $H^1(S, \Sigma; \mathbb{R}^2) \to H^1(S, \Sigma; \mathbb{R}_y)$. The pullback of this foliation to $\mathcal{H}_{\mathrm{m}}$ is the *horospherical foliation* (or 'strong unstable foliation', see [MW2, SSWY] for more information). We denote the horospherical leaf of a point $\widetilde{q} \in \mathcal{H}_{\mathrm{m}}$ by $W^{uu}(\widetilde{q})$.

Using the explicit marking maps $\varphi_{\tau,\beta} : (S, \Sigma) \to (M_{\tau,\beta}, \Sigma)$, we get explicit *comparison maps* between surfaces $M_{\tau,\beta}$ and $M_{\tau,\beta'} \in U_\tau$, taking

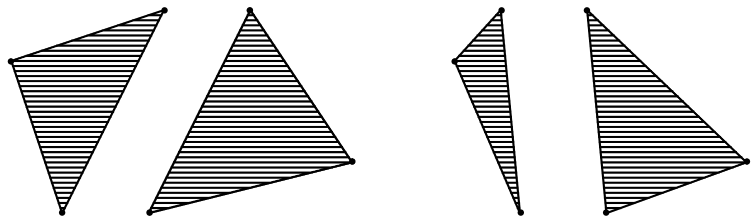

FIGURE 2. Subsurfaces of two surfaces in the same horospherical leaf.

triangles affinely to triangles, and having the form

$$\varphi_{\tau,\beta,\beta'} \stackrel{\text{def}}{=} \varphi_{\tau,\beta} \circ \varphi_{\tau,\beta'}^{-1} : M_{\tau,\beta'} \to M_{\tau,\beta}.$$

The maps $\varphi_{\tau,\beta,\beta'}$ are continuous and piecewise affine but are not in general affine mappings since they may have different derivatives on different triangles. If $M_{\tau,\beta}$ and $M_{\tau,\beta'}$ are in the same horospherical leaf, then the comparison map $\varphi_{\tau,\beta,\beta'}$ sends horizontal straightline leaves on $M_{\tau,\beta'}$ to horizontal straightline leaves on $M_{\tau,\beta}$, preserving the vertical distance between plaques (but the length measure on the leaves may be distorted). See Figure 2.

Let $\mathrm{Mod}(S,\Sigma)$ be the group of isotopy classes of homeomorphisms of S which fix Σ pointwise. This is the *pure mapping class group*. It acts on the right on marking maps by pre-composition, and this induces a well-defined action on $\mathcal{H}_{\mathrm{m}}$ (note that Γ_M acts on the left). It also acts on $T(\mathcal{H}_{\mathrm{m}}) = \mathcal{H}_{\mathrm{m}} \times H^1(S,\Sigma;\mathbb{R}^2)$ by $\gamma : (\varphi,\beta) \mapsto (\varphi\circ\gamma, \gamma^*(\beta))$. The developing map is $\mathrm{Mod}(S,\Sigma)$-equivariant with respect to these two right actions and thus the action of an element of $\mathrm{Mod}(S,\Sigma)$ on $\mathcal{H}_{\mathrm{m}}$, when expressed in charts, is linear. This implies that the $\mathrm{Mod}(S,\Sigma)$-action preserves the affine structure on $\mathcal{H}_{\mathrm{m}}$. This action is properly discontinuous, but not free. Elements with nontrivial stabilizer groups correspond to surfaces with nontrivial translation equivalences.

The group $\mathrm{Mod}(S,\Sigma)$ acts transitively on isotopy classes of marking maps hence each fiber of the forgetful map $\pi : \mathcal{H}_{\mathrm{m}} \to \mathcal{H}$ is a $\mathrm{Mod}(S,\Sigma)$-orbit. We can thus view $\mathcal{H}$ as the quotient $\mathcal{H}_{\mathrm{m}}/\mathrm{Mod}(S,\Sigma)$, and equip it with the quotient topology. The horospherical foliation on $\mathcal{H}_{\mathrm{m}}$ descends to a well-defined equivalence relation on $\mathcal{H}$, and we denote the equivalence class of $q \in \mathcal{H}$ by $W^{uu}(q)$. Loosely speaking, $W^{uu}(q)$ is the set of translation surfaces whose horizontal measured foliation is the same as that of M_q.

Viewed as a map between topological spaces the forgetful map is typically *not* a covering map due to to the presence of translation surfaces in $\mathcal{H}$ with non-trivial translation equivalences. To make this map behave more like a covering map we work in the category of orbifolds.

2.3. **The affine orbifold structure of a stratum.** An *orbifold* structure on a space X is given by an atlas of inverse charts. This consists of a collection of open sets W_j that cover X, a collection of maps $\phi_j : U_j \to W_j$ where U_j are open sets in a vector space V, and a collection of finite groups $\mathcal{G}_j$ acting linearly on the sets U_j so that each ϕ_j induces a homeomorphism from $U_j/\mathcal{G}_j$ to W_j. Furthermore we require that the transition maps on overlaps respect the group actions. The local groups $\mathcal{G}_j$ give rise to a local group $\mathcal{G}_x$, depending only on $x \in X$, and well-defined up to a conjugation. More information is contained in [AK, Definitions 2.1 & 2.2]. If we require that the overlap functions and finite group actions respect the affine structure then we get an *affine orbifold*.

We now modify our construction of the atlas for $\mathcal{H}_{\mathrm{m}}$ to give an affine orbifold atlas for $\mathcal{H}$. Let $q \in \mathcal{H}$, let $M = M_q$ be the underlying translation surface, and let $\Gamma_q = \Gamma_M$ be the group of translation equivalences of M_q. In order to construct an inverse chart in a neighborhood of q we choose a marking map $\varphi : (S, \Sigma) \to (M, \Sigma)$. By pulling back a triangulation from the quotient of M by Γ_q, we can find a geodesic triangulation τ' of M which is Γ_M-invariant, and we let $\tau = \varphi^{-1}(\tau')$ be the pullback of this triangulation to S. As before, let U_τ be the set of cohomology classes compatible with τ. Let $\mathcal{G}_q$ be the (conjugacy class of the) subgroup of $\mathrm{Mod}(S, \Sigma)$ corresponding to the isotopy classes of the elements $\{\varphi^{-1} \circ h \circ \varphi : h \in \Gamma_q\}$. Since τ' is Γ_q-invariant, the group $\mathcal{G}_q$ acts on U_τ, and the maps $\pi \circ \Psi_\tau : U_\tau \to \mathcal{H}$ induce maps from $U_\tau/\mathcal{G}_q$ to $\mathcal{H}$. By possibly replacing U_τ by a smaller neighborhood $U'_q \subset U_\tau$ on which this induced map is injective, we get a collection of inverse charts for an orbifold atlas for $\mathcal{H}$.

An orbifold structure on a space X determines a *local group* at a point $x \in X$. For $q \in \mathcal{H}$ this local group can be identified with $\mathcal{G}_q$. The *singular set* of an orbifold is the set of points where the local group is not the identity. The singular set has a stratification into submanifolds which we will call *orbifold substrata*, defined as the connected components of the subsets of the stratum on which the local group is constant. We will denote the orbifold substratum corresponding to $\mathcal{G}_q$ by $\mathcal{O}_q$.

The *tangent bundle* of an orbifold is defined in [AK, Prop. 4.1]. It is itself an orbifold, and is equipped with a projection map $T(X) \to X$, such that the fiber over x can be identified with the quotient of a vector space by a linear action of $\mathcal{G}_x$. The projection map $T(X) \to X$ is a bundle map in the category of orbifolds. Note that its fibers can vary from point to point.

We denote the orbifold tangent space of $\mathcal{H}$ at q by $T_q(\mathcal{H})$, and the tangent bundle of $\mathcal{H}$ by $T(\mathcal{H})$. We can identify $T(\mathcal{H})$ with the quotient of the tangent bundle of $\mathcal{H}_{\mathrm{m}}$ under the action of the pure mapping class group. The bundle $T(\mathcal{H})$ has a canonical $\mathrm{Mod}(S, \Sigma)$-invariant splitting coming from the decomposition (2.2) and we refer to the summands as the *horizontal and vertical sub-bundles*. Thus the horizontal sub-bundle is given by the tangent spaces to horospherical leaves in $\mathcal{H}_{\mathrm{m}}$.

Since $\mathcal{H}$ is the quotient of an affine manifold $\mathcal{H}_{\mathrm{m}}$ by a group acting affinely and properly discontinuously it inherits the structure of an *affine* orbifold. A map between affine orbifolds is *affine* if it can be expressed by affine maps in local charts.

With the above description of the orbifold tangent bundle of $\mathcal{H}$, we obtain a description of the sub-bundle corresponding to the orbifold substrata.

Proposition 2.1. *Let $q \in \mathcal{H}$ be a surface with a nontrivial local group $\mathcal{G}_q$ and let $\mathcal{O}_q$ be the corresponding orbifold substratum. A choice of $\tilde{q} \in \pi^{-1}(q)$ gives a component $\widetilde{\mathcal{O}}_q$ of $\pi^{-1}(\mathcal{O}_q)$ and a subgroup $\mathcal{G} \subset \mathrm{Mod}(S, \Sigma)$ in the conjugacy class $\mathcal{G}_q$, such that $\widetilde{\mathcal{O}}_q$ is an affine submanifold of $\mathcal{H}_{\mathrm{m}}$, and its tangent space $T_{\tilde{q}}(\widetilde{\mathcal{O}}_q)$ at $\tilde{q}$ is identified via the developing map with the set of vectors in $H^1(S, \Sigma; \mathbb{R}^2)$ fixed by $\mathcal{G}$.*

The proof is left to the reader.

We will need explicit formulas for the projections onto the tangent space to an orbifold substratum, and onto a normal sub-bundle. Let M_q be a surface with a non-trivial group of translation equivalences. Choose a marking map of M_q and let $\mathcal{G}_q$ be the corresponding local group acting on this chart. Define $P^+ : H^1(S, \Sigma; \mathbb{R}^2) \to H^1(S, \Sigma; \mathbb{R}^2)$ by

$$P^+(\beta) \stackrel{\mathrm{def}}{=} \frac{1}{|\mathcal{G}_q|} \sum_{\gamma \in \mathcal{G}_q} \gamma^*(\beta). \tag{2.3}$$

By Proposition 2.1, P^+ is a projection of $H^1(S, \Sigma; \mathbb{R}^2)$ onto the tangent space to the substratum. The kernel of P^+, which we denote by $\mathscr{N}(\mathcal{O}_q)$, is a natural choice for a normal bundle to $\mathcal{O}_q$. We denote by $P^- \stackrel{\mathrm{def}}{=} \mathrm{Id} - P^+$ the projection onto the normal space to the orbifold substratum. Note that $P^\pm$ depend on the orbifold substratum $\mathcal{O}_q$ (via $\mathcal{G}_q$) but this will be suppressed in the notation. It will also be useful to further decompose the normal bundle into its intersections with the horizontal and vertical sub-bundles, and we denote these sub-bundles by $\mathscr{N}_x(\mathcal{O}_q)$ and $\mathscr{N}_y(\mathcal{O}_q)$.

Proposition 2.2. *Given an orbifold sub-locus $\mathcal{O}$, the bundles $T(\mathcal{O})$, $\mathscr{N}(\mathcal{O})$, $\mathscr{N}_x(\mathcal{O})$ and $\mathscr{N}_y(\mathcal{O})$ are flat, and each has a volume form which is well-defined (independent of a marking).*

Proof. To see that the bundles in the statement are flat, note that $\mathrm{Mod}(S, \Sigma)$ acts on $H^1(S, \Sigma; \mathbb{R})$ and $H^1(S, \Sigma; \mathbb{R}^2)$ by linear transformations, and thus the set of vectors fixed by a subgroup $\mathcal{G}$ is a linear subspace. Now flatness follows using Proposition 2.1.

The map P^+ respects the splitting of cohomology into horizontal and vertical factors, i.e., it commutes with the two projections onto the summands in (2.2). Moreover, since the $\mathrm{Mod}(S, \Sigma)$-action on $H^1(S, \Sigma; \mathbb{R}^2)$ preserves $H^1(S, \Sigma; \mathbb{Z}^2)$, it takes integral classes to rational classes, i.e., is defined over $\mathbb{Q}$. It thus induces a map

$$H^1(S, \Sigma; \mathbb{R}_x) \supset H^1(S, \Sigma; \mathbb{Z}_x) \xrightarrow{P^+} H^1(S, \Sigma; \mathbb{Q}_x) \subset H^1(S, \Sigma; \mathbb{R}_x)$$

(with the obvious notations $\mathbb{Z}_x, \mathbb{Q}_x$ for the corresponding summands), and a corresponding map for the second summand $\mathbb{Z}_y, \mathbb{Q}_y, \mathbb{R}_y$. The kernels of these maps are lattices in $\mathscr{N}_x(\mathcal{O})$ and $\mathscr{N}_y(\mathcal{O})$ which are parallel. This means that the Lebesgue measure on $\mathscr{N}_x(\mathcal{O})$, coming from the affine structure of Proposition 2.1, has a natural normalization which does not depend on the choice of a particular lift $\widetilde{\mathcal{O}} \to \mathcal{O}$. $\square$

Affine structures do not give a metric geometry but some familiar notions from the theory of Riemannian manifolds have analogues for affine manifolds. Thus an *affine geodesic* is a path in an affine manifold N parametrized by an open interval in the real line which has the property that in any affine chart the parametrization is linear. We can also describe affine geodesics by saying that the tangent vector to the curve is invariant under parallel translation. Affine geodesics are projections of orbits of a partially defined flow on the tangent bundle which we call the *affine geodesic flow*. An affine geodesic has a maximal domain of definition which is a connected open subset of $\mathbb{R}$, which may or may not coincide with $\mathbb{R}$. We denote by $\mathrm{Dom}(\widetilde{q}, v) \subset \mathbb{R}$ the maximal domain of definition of the affine geodesic which is tangent at time $t = 0$ to $v \in T_{\widetilde{q}}(\mathcal{H}_{\mathrm{m}})$.

The space of marked translation surfaces with area one is a submanifold $\mathcal{H}_{\mathrm{m},1}$ of $\mathcal{H}_{\mathrm{m}}$, which is invariant under $\mathrm{Mod}(S, \Sigma)$. We refer to the quotient orbifold as the *normalized stratum* and denote it by $\mathcal{H}_1$. The normalized stratum is a codimension one sub-orbifold of $\mathcal{H}$ but it is not an affine sub-orbifold. The developing map dev maps $\mathcal{H}_{\mathrm{m},1}$ into a quadric in $H^1(S,\Sigma;\mathbb{R}^2)$, and the tangent space $T_{\tilde{q}}(\mathcal{H}_{\mathrm{m},1})$ is a linear subspace of $H^1(S,\Sigma;\mathbb{R}^2)$ on which area is constant to first order. This subspace varies with $\tilde{q}$. Nevertheless it is often quite useful to use the ambient affine coordinates to discuss it.

The intersection of horospherical leaves in $\mathcal{H}_m$ with $\mathcal{H}_{\mathrm{m},1}$ give the *horospherial foliation* of $\mathcal{H}_{\mathrm{m},1}$. Its leaves are of codimension one in the horospherical leaves of $\mathcal{H}_{\mathrm{m}}$. In general if we consider a vector tangent to $\mathcal{H}_1$ then the affine geodesic determined by this vector need not lie in $\mathcal{H}_1$ but in the particular case of vectors tangent to the horospherical foliation (e.g., horocycles and tremors) it will be the case that these paths lie in $\mathcal{H}_1$.

2.4. **The action of $G = \mathrm{SL}_2(\mathbb{R})$ on strata.** The group G acts linearly on $\mathbb{R}^2$ and this induces an action on translation surfaces by postcomposition. Thus if M is represented by an atlas $(\phi_\alpha, U_\alpha)_{\alpha\in\mathbb{A}}$, and $g \in G$, then gM is represented by $(g \circ \phi_\alpha U_\alpha)_{\alpha\in\mathbb{A}}$. This gives rise to an action of G on $\mathcal{H}_{\mathrm{m}}$, which commutes with the action of $\mathrm{Mod}(S,\Sigma)$ and preserves the normalized stratum $\mathcal{H}_{\mathrm{m},1}$. Thus we have an induced G-action on $\mathcal{H}$ and on $\mathcal{H}_1$, and the forgetful map $\pi : \mathcal{H}_{\mathrm{m}} \to \mathcal{H}$ is G-equivariant.

We now check that the action is affine in charts. There is a natural left action of G on $H^1(S,\Sigma;\mathbb{R}^2)$ which is given by the action of G on the coefficient system, i.e. by postcomposition of $\mathbb{R}^2$ valued 1-cochains. Let τ be a triangulation of S, and let $U_\tau \subset H^1(S,\Sigma;\mathbb{R}^2)$ be defined as in §2.2. For $\beta \in U_\tau$ and $g \in G$, we see that $g\beta \stackrel{\mathrm{def}}{=} g \circ \beta \in U_\tau$. Let $\varphi_{\tau,\beta,g\beta} : M_\beta \to M_{g\beta}$ be the comparison map. Notice that it has the same derivative on each triangle, namely its derivative is everywhere equal to the linear map g. In particular, the comparison map $\varphi_{\tau,\beta,g\beta}$ does not depend on τ. We will call it the *affine comparison map corresponding to g* and denote it by ψ_g. The action of g on $\mathcal{H}_{\mathrm{m}}$ can now be expressed as replacing a marking map $\varphi : S \to M$ by $\psi_g \circ \varphi : S \to gM$. Other affine maps $M_q \to M_{gq}$ with derivative g can be obtained by composing ψ_g with translation equivalences. Since the G-action commutes with the $\mathrm{Mod}(S,\Sigma)$-action, G preserves the orbifold stratification of $\mathcal{H}$. Additionally, the normal and tangent bundles of Propositions 2.1 and 2.2 are G-equivariant.

We introduce some notation for subgroups of G. Recall the group $U = \{u_s : s \in \mathbb{R}\}$ introduced in (1.1). We will also use the following notation for other subgroups:

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.4}$$

and

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}. \tag{2.5}$$

With this notation we note that the U-action is given in period coordinate charts by

$$\mathrm{hol}^{(x)}_{u_s \tilde{q}}(\gamma) = \mathrm{hol}^{(x)}_{\tilde{q}}(\gamma) + s \cdot \mathrm{hol}^{(y)}_{\tilde{q}}(\gamma), \quad \mathrm{hol}^{(y)}_{u_s \tilde{q}}(\gamma) = \mathrm{hol}^{(y)}_{\tilde{q}}(\gamma);$$

this now gives a precise meaning to equation (1.4). We see in particular that horocycle orbits are linearly parametrized affine geodesics.

Our next goal is to give a precise meaning to equation (1.5), by defining transverse measures and their associated cohomology class.

2.5. **Transverse (signed) measures and foliation cocycles.** In this section we define transverse measures and cocycles and cohomology classes associated with a non-atomic transverse measure. It will be useful to include signed transverse measures. In some settings it is useful to pass to limits of non-atomic transverse measures, and these limits may be certain atomic transverse measures. In §13 we will discuss the case of these atomic transverse measures.

Let M be a translation surface, let $\theta \in \mathbb{S}^1$ be a direction (i.e., a unit vector $(\cos\theta, \sin\theta) \in \mathbb{R}^2$), and let $\mathcal{F}_\theta$ denote the foliation of M obtained by pulling back the foliation of $\mathbb{R}^2$ by lines parallel to θ. A *transverse arc* to $\mathcal{F}_\theta$ is a piecewise smooth curve $\gamma : (a, b) \to M \smallsetminus \Sigma$ of finite length which is everywhere transverse to leaves of $\mathcal{F}_\theta$. A *transverse measure* on $\mathcal{F}_\theta$ is a family $\{\nu_\gamma\}$ where γ ranges over the transverse arcs, the ν_γ are finite regular Borel measures defined on γ which are invariant under isotopy through transverse arcs and so that if $\gamma' \subset \gamma$ then $\nu_{\gamma'}$ is the restriction of ν_γ to γ'. Since transverse measures are defined via measures, the usual notions of measure theory (absolute continuity, Radon-Nikodym theorem, etc.) make sense for transverse measures (or a pair of transverse measures). In particular it makes sense to speak of atoms of a transverse measure, and we will say that ν is *non-atomic* if none of the ν_γ have atoms. In this paper, if transverse measures have atoms we require that the atoms be supported on closed loops, each of which is a closed leaf, or a union of saddle connections that meet at angles $\pm\pi$ (see §13 for a formal definition). These are the atomic transverse measures that can arise as limits of non-atomic transverse measures. We remark that in the literature, there are several different conventions regarding atomic transverse measures.

A (finite) *signed measure* on X is a map from Borel subsets of X to $\mathbb{R}$ satisfying all the properties satisfied by a measure. Recall that every signed measure has a canonical Hahn decomposition, i.e. a unique representation $\nu = \nu^+ - \nu^-$ as a difference of finite measures. A *signed transverse measure* is a system $\{\nu_\gamma\}$ of signed measures, satisfying the same hypotheses as a signed measure; or equivalently, the difference of two transverse measures $\{\nu_\gamma^+\}$, $\{\nu_\gamma^-\}$. In what follows, the words 'measure' and 'transverse measure' always refer to non-negative measures (i.e. measures for which $\nu^- = 0$). When we want to allow general signed measures we will include the word 'signed'. We say that ν is *non-atomic* if $\nu^\pm$ are both non-atomic. The sum $\nu^+(X) + \nu^-(X)$ is called the *total variation* of ν.

If M is a translation surface, $\mathcal{F}_\theta$ is a directional foliation on M, and ν is a non-atomic signed transverse measure on $\mathcal{F}_\theta$, we have a map β_ν from transverse line segments to real numbers, defined as follows. If γ is a transverse oriented line segment and the (counterclockwise) angle between the direction θ and the direction of γ is in $(0, \pi)$, set $\beta_\nu(\gamma) = \nu(\gamma)$. If the angle is in $(-\pi, 0)$ set $\beta_\nu(\gamma) = -\nu(\gamma)$. We extend this to all straight line segments by stipulating that $\beta_\nu(\gamma) = 0$ for any line segment γ that is contained in a leaf of the foliation. By linearity we extend β_ν to finite concatenations of oriented straight line segments. Similarly we can define $\beta_\nu(\gamma)$ for an oriented piecewise smooth curve γ, where the sign of an intersection is measured using the derivative of γ.

By a *polygon decomposition* of a translation surface M, we mean a decomposition into simply connected polygons for which all the vertices are singular points. As we saw every M admits a geodesic triangulation which is a special case of a polygon decomposition. Let β_ν be as in the preceding paragraph. Any element $\alpha \in H_1(M, \Sigma)$ has a representative $\widetilde{\alpha}$ that is a concatenation of edges of a polygon decomposition. The invariance property of a transverse measure ensures that the value $\beta_\nu(\widetilde{\alpha})$ depends only on α and not on the representative $\widetilde{\alpha}$; in particular it does not depend on the cell decomposition used, and β_ν is a cochain and defines a cohomology class in $H^1(M, \Sigma; \mathbb{R})$. We have defined a mapping $\nu \mapsto \beta_\nu$ from non-atomic signed transverse measures to $H^1(M, \Sigma; \mathbb{R}^2)$, and in §13 we will explain how to extend this map to atomic transverse measures. We will be primarily interested in transverse measures to the horizontal foliation. An element of cohomology which corresponds to a transverse measure (resp., a signed transverse measure) to the horizontal foliation will be called a *foliation cocycle* (respectively, *signed foliation cocycle*), and β_ν will be called the *(signed) foliation cocycle corresponding to ν.*

Identifying $\mathbb{R}$ with $\mathbb{R}_x$ and $H^1(M, \Sigma; \mathbb{R})$ with the first summand in (2.1), we identify the collection of all signed foliation cocycles with a subspace $\mathcal{T}_q \subset H^1(M, \Sigma; \mathbb{R}_x)$, and the collection of all foliation cocycles with a cone $C_q^+ \subset \mathcal{T}_q$. We refer to these respectively as the *space of signed foliation cocycles* and the *cone of foliation cocycles*. The Hahn decomposition of transverse measures implies that every $\beta \in \mathcal{T}_q$ can be written uniquely as $\beta = \beta^+ - \beta^-$ for $\beta^\pm \in C_q^+$. For every q, the 1-form dy gives rise to a *canonical transverse measure* and to the corresponding cohomology class $\mathrm{hol}_q^{(y)}$. When we want to think of this class as a foliation cocycle, we will denote it by dy or $(dy)_q$, and refer to it as the *canonical foliation cocycle*.

As discussed above for the horizontal direction, we can define a (partially defined) straightline flow in direction θ by lifting the vector field on $\mathbb{R}^2$ in direction θ and following lines parallel to θ. We write $\mathcal{F}_\theta$ for the foliation by lines in direction θ and write $\mathcal{F}$ for $\mathcal{F}_0$. We say that a finite Borel measure μ on M is *$\mathcal{F}_\theta$-invariant* if it is invariant under the straightline flow in direction θ. We have the following well-known relationship between transverse measures and invariant measures.

Proposition 2.3. *For each non-atomic transverse measure ν on $\mathcal{F}_\theta$ there exists an $\mathcal{F}_\theta$-invariant measure μ_ν with*

$$\mu_\nu(A) = \nu(v) \cdot \ell(h) \tag{2.6}$$

for every isometrically embedded rectangle A with one side h parallel to θ, and another side v orthogonal to θ, where ℓ is the Euclidean length. The map $\nu \mapsto \mu_\nu$ is a bijection between non-atomic transverse measures and $\mathcal{F}_\theta$-invariant measures that assign zero measure to leaves. It extends to a bijection between non-atomic signed transverse measures and $\mathcal{F}_\theta$-invariant signed measures assigning zero measure to leaves.

It is clear from (2.6) that two different transverse measures give different measures to some rectangle, and so the assignment is injective. To see that each $\mathcal{F}_\theta$-invariant measure arises from a transverse measure, partition M into rectangles and use disintegration of measures to define a transverse measure on each rectangle. This transverse measure will be non-atomic if the invariant measure gives zero measure to every horizontal leaf.

The map $\nu \mapsto \beta_\nu$ is almost injective. More precisely, we have:

Proposition 2.4 (Katok)**.** *If M_q has no horizontal cylinders and $\nu_1 \neq \nu_2$ are distinct non-atomic signed transverse measures to the horizontal foliation, then $\beta_{\nu_1} \neq \beta_{\nu_2}$, and moreover the restrictions of β_{ν_i} to the absolute period space $H_1(S)$ are different.*

For a proof see [K]. Katok considered measures rather than signed measures, but the passage to signed measures follows from the uniqueness of the Hahn decomposition.

2.6. **The Sup-norm Finsler metric.** We now recall the sup-norm Finsler metric on $\mathcal{H}_{\mathrm{m}}$ studied by Avila, Gouëzel and Yoccoz in [AGY]. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$. For a translation surface q, denote by Λ_q the collection of saddle connections on M_q and let $\ell_q(\sigma) = \|\mathrm{hol}_q(\sigma)\|$ be the length of $\sigma \in \Lambda_q$. For $\beta \in H^1(M_q, \Sigma_q; \mathbb{R}^2)$ we set

$$\|\beta\|_q \stackrel{\text{def}}{=} \sup_{\sigma \in \Lambda_q} \frac{\|\beta(\sigma)\|}{\ell_q(\sigma)}. \tag{2.7}$$

We now define a Finsler metric for $\mathcal{H}_{\mathrm{m}}$. Let $\varphi : (S, \Sigma) \to (M_q, \Sigma)$ be a marking map, which represents $\widetilde{q} \in \mathcal{H}_{\mathrm{m}}$. Recall that we can identify $T_{\widetilde{q}}(\mathcal{H}_{\mathrm{m}})$ with $H^1(S, \Sigma; \mathbb{R}^2)$. Then $\|\varphi^*\beta\|_{\widetilde{q}} = \|\beta\|_q$ is a norm on $H^1(S, \Sigma; \mathbb{R}^2)$, or equivalently:

$$\|\beta\|_{\widetilde{q}} \stackrel{\text{def}}{=} \sup_{\tau \in \Lambda_{\widetilde{q}}} \frac{\|\beta(\varphi(\tau))\|}{\ell_q(\varphi(\tau))}. \tag{2.8}$$

Note that $\Lambda_{\widetilde{q}}$ varies as $\widetilde{q}$ changes, and that $\|\theta\|_{\widetilde{q}}$ is well-defined (i.e. depends on $\widetilde{q}$ and not on the actual marking map φ). Recall that using period coordinates, the tangent bundle $T(\mathcal{H}_{\mathrm{m}})$ is a product $\mathcal{H}_{\mathrm{m}} \times H^1(S, \Sigma; \mathbb{R}^2)$. As shown in [AGY, Prop. 2.11], the map

$$T(\mathcal{H}_{\mathrm{m}}) \to \mathbb{R}, \quad (\widetilde{q}, \beta) \mapsto \|\beta\|_{\widetilde{q}} \tag{2.9}$$

is continuous.

The Finsler metric defines a distance function on $\mathcal{H}_{\mathrm{m}}$ which we call the *sup-norm distance* and define as follows

$$\mathrm{dist}(\widetilde{q}_0, \widetilde{q}_1) \stackrel{\text{def}}{=} \inf_{\gamma} \int_0^1 \|\gamma'(\tau)\|_{\gamma(\tau)} d\tau. \tag{2.10}$$

Here γ ranges over smooth paths $\gamma : [0,1] \to \mathcal{H}$ with $\gamma(0) = \widetilde{q}_0$ and $\gamma(1) = \widetilde{q}_1$. This distance is symmetric since $\|\beta\|_{\widetilde{q}} = \|-\beta\|_{\widetilde{q}}$.

The following was shown in [AGY, §2.2.2]:

Proposition 2.5. *The metric* dist *is proper, complete, and induces the topology on $\mathcal{H}_{\mathrm{m}}$ given by period coordinates. It is invariant under the action of the pure mapping class group.*

By Proposition 2.5, in order to compute the length of a path ρ, one can lift the path to $\mathcal{H}_{\mathrm{m}}$ and measure its length there. Note that dist need not be invariant under parallel translation.

Proof. The fact that the sup-norm distance is a Finsler metric giving the topology on period coordinates is [AGY, proof of Proposition 2.11]. The fact that the metric is proper is [AGY, Lemma 2.12]. Completeness is [AGY, Corollary 2.13]. The metric is invariant under the action of the mapping class group because its definition depends only on the collection of saddle connections in M_q which is independent of the marking. $\square$

We will now compute the deviation of nearby G-orbits with respect to the sup-norm distance. Let $\|g\|_{\mathrm{op}}$, g^{t} and $\mathrm{tr}(g)$ denote respectively the operator norm, transpose, and trace of $g \in G$. The operator norm can be calculated in terms of the singular values of g. Specifically the operator norm is the square root of the the largest eigenvalue of $g^{\mathrm{t}}g$. For a 2 by 2 matrix this eigenvalue can be expressed in terms of the trace and determinant of $g^{\mathrm{t}}g$:

$$\|g\|_{\mathrm{op}} = \sqrt{\frac{\mathrm{tr}(g^{\mathrm{t}}g) + \sqrt{\mathrm{tr}^2(g^{\mathrm{t}}g) - 4}}{2}} \tag{2.11}$$

Recall the affine comparison map $\psi_g : M_q \to M_{gq}$ with derivative g, from §2.4. For this map we have $\mathrm{hol}(\psi(\sigma)) = g(\mathrm{hol}(\sigma))$ and hence $\|\sigma\|_{gq} = \|g(\mathrm{hol}(\sigma))\|_q$. From this it is not hard to deduce that

$$\|g\beta\|_{g\widetilde{q}} \leqslant \|g\|_{\mathrm{op}} \cdot \|g^{-1}\|_{\mathrm{op}} \cdot \|\beta\|_{\widetilde{q}}.$$

Corollary 2.6 (See [AGY], equation (2.13))**.** *For any $s, t \in \mathbb{R}$ and any $\beta \in H^1(S, \Sigma; \mathbb{R}^2)$, we have*

$$\|u_s(\beta)\|_{u_s\widetilde{q}} \leqslant \left(1 + \frac{s^2 + |s|\sqrt{s^2+4}}{2}\right) \|\beta\|_{\widetilde{q}}$$

and

$$\|g_t(\beta)\|_{g_t\widetilde{q}} \leqslant e^{2|t|} \|\beta\|_{\widetilde{q}}.$$

Integrating these pointwise bounds and using the definition of the sup-norm distance, we find that nearby horocycle trajectories diverge from each other at most quadratically and nearby geodesic orbits diverge at most exponentially. Namely:

Corollary 2.7. *For $\widetilde{q}_0$ and $\widetilde{q}_1 \in \mathcal{H}_{\mathrm{m}}$ and any $s, t \in \mathbb{R}$,*

$$\left(1 + \frac{s^2 + |s|\sqrt{s^2+4}}{2}\right)^{-1} \mathrm{dist}(\widetilde{q}_0, \widetilde{q}_1) \leqslant \mathrm{dist}(u_s\widetilde{q}_0, u_s\widetilde{q}_1)$$
$$\leqslant \left(1 + \frac{s^2 + |s|\sqrt{s^2+4}}{2}\right) \mathrm{dist}(\widetilde{q}_0, \widetilde{q}_1)$$

and

$$e^{-2|t|}\mathrm{dist}(\widetilde{q}_0, \widetilde{q}_1) \leqslant \mathrm{dist}(g_t\widetilde{q}_0, g_t\widetilde{q}) \leqslant e^{2|t|}\mathrm{dist}(\widetilde{q}_0, \widetilde{q}_1). \tag{2.12}$$

In the case of unipotent flows in homogeneous dynamics nearby orbits diverge at most polynomially with respect to an appropriate metric. Corollary 2.7 shows that on strata, nearby horocycles orbits diverge from each other *at most* quadratically. In §8.3 we will discuss the more delicate question of *lower* bounds for the rate of divergence of horocycles, and show that erratic divergence is possible.

3. The space of pairs of tori glued along slits

In this section we collect some information we will need regarding the structure of $\mathcal{E}$ and the dynamics of the straightline flow on surfaces in $\mathcal{E}$. We also prove Proposition 3.5, which plays an important role in §10. It shows that for surfaces in $\mathcal{E}$, the ergodic measures in directions which are not uniquely ergodic have good approximations by splittings of the surface into two tori. This may be considered as a converse to a construction of Masur and Smillie [MaTa, §3.1].

3.1. The locus $\mathcal{E}$.

McMullen studied the eigenform loci $\mathcal{E}_D$, which are affine G-invariant suborbifolds of $\mathcal{H}(1,1)$ (see [McM1] and references therein). The description of $\mathcal{E} = \mathcal{E}_4$ which will be convenient for us is the following. Recalling that $\mathcal{H}(0,0)$ is the stratum of tori with two marked points, we have that $\mathcal{E}$ is the collection of $q \in \mathcal{H}(1,1)$ for which there is a branched 2 to 1 translation cover from M_q onto a torus in $\mathcal{H}(0,0)$. To avoid confusion with different conventions used in the literature, we remind the reader that we take the marked points in $\mathcal{H}(0,0)$ and $\mathcal{H}(1,1)$ to be labelled. See [BSW, §7] for additional information.

Given a torus $T \in \mathcal{H}(0,0)$ and a saddle connection δ joining the two marked points we can build a surface $M \in \mathcal{H}(1,1)$ by cutting T along δ, viewing the resulting surface as a surface with boundary. We define M to be the result of taking two copies of the surface with boundary and gluing along the boundaries. The surface M has a branched covering map to T and a deck transformation which is an involution interchanging the two copies of T. A *slit* on a translation surface is a union of homologous saddle connections which disconnect the surface. Thus in this example, the preimage σ of δ under the map $M \to T$ is a slit. We say that M is built from the *slit construction* applied to σ. Clearly surfaces built from the slit construction belong to $\mathcal{E}$.

The following proposition shows that, with respect to the terminology of §2.3, $\mathcal{E}$ consists of points in $\mathcal{H}(1,1)$ where the local orbifold group is non-trivial; namely, it is the group of order two generated by an involution in $\mathrm{Mod}(S, \Sigma)$.

Proposition 3.1 ([EMS]). *The locus $\mathcal{E}$ is connected. It admits a four to one covering map $P : \mathcal{E} \to \mathcal{H}(0,0)$ which is characterized by the following property: for every $q \in \mathcal{E}$ there is an order 2 translation equivalence $\iota = \iota_q : M_q \to M_q$, such that the quotient surface $M_q/\langle\iota\rangle$ is a translation surface which is translation equivalent to the torus $T_{P(q)}$.*

Proof. Connectedness of $\mathcal{E}$ is proved in [EMS, Theorem 4.4]. It remains to show that P is four to one. By definition, if $q \in \mathcal{E}$ then M_q has a translation automorphism ι such that $M_q/\langle\iota\rangle$ is a torus in $\mathcal{H}(0,0)$.

We begin by determining the fixed points of ι. If a translation automorphism fixes a nonsingular point it fixes a neighborhood of that point. Thus the set of nonsingular fixed points is open and closed. We conclude that the only possible fixed points are singularities and singularities are indeed fixed since they are labelled. We conclude that ι induces a branched covering map which has non-trivial branching at the two singular points.

Let T be a torus with $\Sigma = \{p_1, p_2\}$ corresponding to a point in $\mathcal{H}(0,0)$. Any $q \in \mathcal{E}$ for which $P(q) = T$ gives an unbranched cover $M_q \smallsetminus P^{-1}(\Sigma) \to T \smallsetminus \Sigma$. Conversely any unbranched cover of $T \smallsetminus \Sigma$ can be completed to a branched cover of T. This cover is ramified at $p_j \in \Sigma$ precisely when a small loop ℓ_j around p_j in T does not lift as a closed loop in M_q. So the cardinality of $P^{-1}(T)$ is the number of topologically distinct degree 2 covers of $T \smallsetminus \Sigma$ for which the loops ℓ_j do not lift as closed loops. Equivalently, it is the number of conjugacy classes of homomorphisms $\pi_1(T \smallsetminus \Sigma) \to \mathbb{Z}/2\mathbb{Z}$ for which the image of the class of each ℓ_j is nontrivial. Since $\mathbb{Z}/2\mathbb{Z}$ is abelian, the covering spaces are determined uniquely by elements $\theta \in H^1(T \smallsetminus \Sigma; \mathbb{Z}/2\mathbb{Z})$ which has dimension 3 and we are counting those θ for which both $\theta(\ell_j) \neq 0$. Since the loops ℓ_1 and ℓ_2 are homologous, this condition gives a single inhomogeneous linear equation on a $\mathbb{Z}/2\mathbb{Z}$ vector space of dimension 3, so we have four solutions. $\square$

As we saw surfaces built from the slit construction belong to $\mathcal{E}$. The following is a strong converse to this statement (a similar result holds for all eigenform loci, see [McM1, §7]).

Proposition 3.2. *Two saddle connections δ_1 and δ_2 on the same torus in $\mathcal{H}(0,0)$, connecting the singularities, give rise to the same surface in $\mathcal{E}$ if and only if the corresponding homology classes $[\delta_1]$ and $[\delta_2]$ are equal as elements of $H_1(T, \Sigma; \mathbb{Z}/2\mathbb{Z})$. In particular every surface in $\mathcal{E}$ can be built from the slit construction in infinitely many ways (that is, using infinitely many different δ).*

Proof. As in the proof of Proposition 3.1, a surface in $\mathcal{E}$ corresponds to a class $\theta \in H^1(T \smallsetminus \Sigma; \mathbb{Z}/2\mathbb{Z})$ for which the $\theta(\ell_j)$ are nonzero, for $j = 1, 2$. If δ is any path from p_1 to p_2, it defines a class $[\delta] \in H_1(T, \Sigma; \mathbb{Z}/2\mathbb{Z})$, and we will say θ is *represented by* δ if θ is the class in $H^1(T \smallsetminus \Sigma; \mathbb{Z}/2\mathbb{Z})$ which is Poincaré dual to $[\delta]$. Clearly, if θ is represented by some δ then θ satisfies the requirement $\theta(\ell_j) \neq 0$, and by a dimension count, any such θ is represented by some path δ. It remains to show that each θ is represented by infinitely many saddle connections δ from p_1 to p_2. To see this, let δ_0 be some path representing θ, let $v_0 \stackrel{\mathrm{def}}{=} \mathrm{hol}_T(\delta_0)$, let $\Lambda \stackrel{\mathrm{def}}{=} \mathrm{hol}_T(H_1(T; \mathbb{Z}))$, and let $\Lambda' \stackrel{\mathrm{def}}{=} \Lambda \cup (v_0 + \Lambda)$. Since $\mathbb{R}^2$ is the universal cover of T, Λ is a lattice in $\mathbb{R}^2$, $v_0 \notin \Lambda$, and the required paths δ are those for which $\mathrm{hol}_T(\delta) \in v_0 + 2 \cdot \Lambda$ and for which the straight segment in $\mathbb{R}^2$ from the origin to $\mathrm{hol}_T(\delta)$ does not intersect Λ' in its interior. It follows from this description that the set of such δ is infinite. $\square$

For use in the sequel, we record the conclusion of Proposition 2.2 in the special case of the orbifold substratum $\mathcal{E}$:

Corollary 3.3. *We can identify the tangent space $T(\mathcal{E})$ with the $+1$ eigenspace of the action of ι on $H^1(S, \Sigma; \mathbb{R}^2)$ and the normal bundle $\mathscr{N}(\mathcal{E})$ with the -1 eigenspace. The bundle $\mathscr{N}(\mathcal{E})$ has a splitting into flat sub-bundles*

$$\mathscr{N}(\mathcal{E}) = \mathscr{N}_x(\mathcal{E}) \oplus \mathscr{N}_y(\mathcal{E}),$$

and each of these sub-bundles has a flat monodromy-invariant volume form.

3.2. **Dynamics on $\mathcal{E}$.** Here we state some important features of the straightline flow on surfaces in $\mathcal{E}$.

Proposition 3.4. *Let $q \in \mathcal{E}$, let $M = M_q$ be the underlying surface, let $\iota : M \to M$ be the involution as described in Proposition 3.1, let $\mathcal{F}$ be the horizontal foliation on M, and let $(dy)_q$ be the canonical transverse measure. Suppose that the foliation $\mathcal{F}$ is not periodic. Then for any transverse measure ν to $\mathcal{F}$, $\iota_*\nu$ is also a transverse measure and there is $c > 0$ such that $\nu + \iota_*\nu = c\,(dy)_q$. Moreover, if $\mathcal{F}$ is not uniquely ergodic, then (up to multiplication by constants) it supports exactly two ergodic transverse measures which are images of each other under ι_*, and* Leb *is not ergodic for the horizontal straightline flow.*

This follows from the facts that ι commutes with the flow and that, under our aperiodicity assumption, the projection of $\mathcal{F}$ to the torus is uniquely ergodic. We leave the details to the reader.

The following proposition is the main result of this section. Recall that $\mathcal{F}_\theta$ denotes the foliation in direction θ, where $\theta = 0$ corresponds to the horizontal direction.

Proposition 3.5. *Suppose $q \in \mathcal{E}$ has the property that the horizontal foliation on M_q is minimal but not ergodic and let μ be an invariant ergodic probability measure on M_q for the horizontal straightline flow. Then there are directions θ_j, such that the foliations $\mathcal{F}_j$ in direction θ_j contain saddle connections δ_j satisfying the following:*

(i) *The union $\sigma_j = \delta_j \cup \iota(\delta_j)$ is a slit in $\mathcal{F}_j$ separating M_q into two isometric tori.*

(ii) *The holonomy $\mathrm{hol}_q(\delta_j) = (x_j, y_j)$ satisfies*

$$|x_j| \to \infty, \; 0 \neq y_j \to 0 \text{ as } j \to \infty.$$

In particular the direction θ_j is not horizontal but tends to horizontal, and the length of δ_j tends to ∞. Moreover there are no saddle connections δ on M_q with holonomy vector satisfying $\left|\mathrm{hol}_q^{(x)}(\delta)\right| < |x_j|$ and $\left|\mathrm{hol}_q^{(y)}(\delta)\right| < |y_j|$.

(iii) *For each j we can choose one of the tori A_j in $M_q \smallsetminus \sigma_j$, such that the normalized restriction μ_j of* Leb *to A_j converges to μ as $j \to \infty$, w.r.t. the weak-$*$ topology on probability measures on M_q. Thus, letting ν and ν_j be the transverse measures corresponding to μ and μ_j (via Proposition 2.3), and letting β_ν and $\beta_j = \beta_{\nu_j}$ be the corresponding foliation cocycles in $H^1(M_q, \Sigma_q; \mathbb{R})$, we have $\beta_j \to \beta_\nu$.*

We divide the argument below into steps.

Proof. **Step 1. Finding slits satisfying (i) and (ii): divergence in $\mathcal{E}$ versus convergence in $\mathcal{H}(0)$.**

We consider the projection map $\bar{\pi} : \mathcal{E} \to \mathcal{H}(0)$ given by the composition of the map $P : \mathcal{E} \to \mathcal{H}(0,0)$ from Proposition 3.1 with the forgetful map forgetting the second marked point. In other words, $\bar{\pi} : M_q \mapsto M_q/\langle \iota \rangle$. Since M_q has a minimal horizontal foliation, so does $M_{\bar{\pi}(q)}$. We normalize the area of M_q to be 2, so that $\bar{\pi}(q)$ has unit area, and thus belongs to the normalized stratum $\mathcal{H}_1(0)$, which can be identified with the space of unimodular lattices $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. The horizontal foliation is minimal if and only if the corresponding lattice does not contain a nonzero horizontal vector, and this implies that there is a compact set $\mathcal{K} \subset \mathcal{H}(0)$ for which

$$\text{there is } t_j \to \infty \text{ such that } g_{-t_j}\bar{\pi}(q) \in \mathcal{K}. \tag{3.1}$$

Denote by $\mathcal{M}_g$ the moduli space of Riemann surfaces of genus g and let $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification (see [B, §5] for a concise introduction). Passing to a further subsequence (which we will continue to denote by t_j to simplify notation) we have that $g_{-t_j}q$ converges to a stable curve in $\overline{\mathcal{M}}_2$. This curve projects to some torus in $\mathcal{M}_1$ (and not in its boundary $\overline{\mathcal{M}}_1 \smallsetminus \mathcal{M}_1$) because the projection of $\mathcal{K}$ to $\mathcal{M}_1$ is compact. So the limiting stable curve has area 2. By [McM2, Theorem 1.4], the limit of $g_{-t_j}q$ is not connected and so, considering the projection to $\mathcal{M}_1$ again, it is built from two tori connected at a node. Thus for all large j, the surfaces

$$M^{(j)} \stackrel{\mathrm{def}}{=} M_{g_{-t_j}q} \tag{3.2}$$

are built from two copies of a torus $T_j \in \mathcal{K}$ glued along slits whose lengths go to zero. These slits must be the union of two saddle connections that connect the two different singularities of $M^{(j)}$. Indeed, the slit cannot project to a short curve on T_j and it must be trivial in homology. Write $M^{(0)} = M_q$, let $\phi_j : M^{(0)} \to M^{(j)}$ be the affine comparison map corresponding to g_{-t_j}, and let $\delta_j \subset M^{(0)}$ denote the pullback under ϕ_j of one of the saddle connections that make up this slit, so that the other is $\iota(\delta_j)$. Letting θ_j be the direction of δ_j, we obtain (i).

Because the horizontal flow on $M^{(0)}$ is minimal, the δ_j are not horizontal. For any fixed non-horizontal segment δ on $M^{(0)}$, the length of $\phi_j(\delta)$ on $M^{(j)}$ goes to infinity as $j \to \infty$. Therefore we may assume that the δ_j are all different. By the discreteness of holonomies of saddle connections (see [MaTa]), this implies that $\|(x_j, y_j)\| \to \infty$, where $(x_j, y_j) = \mathrm{hol}_{M^{(0)}}(\delta_j)$. Since

$$\|\mathrm{hol}_{M^{(j)}}(\phi_j(\delta_j))\| = \|(e^{-t_j}x_j, e^{t_j}y_j)\| \to 0,$$

we have that $y_j \to 0$ and so $x_j \to \infty$. Because the torus T_j is in the compact set $\mathcal{K}$, the only short saddle connections of $M^{(j)}$ are δ_j and $\iota(\delta_j)$ which implies the second assertion in (ii). This establishes (ii).

Step 2. Sets of uniform convergence for the straightline flow on one side of the slit.

For the proof of (iii), recall that $\Upsilon^{(p)}(t)$ denote the horizontal straightline flow (starting at p, with time parameter t). By the Birkhoff ergodic theorem, there is an increasing sequence $S_k \to \infty$ and an increasing sequence of subsets $E_k \subset M^{(0)}$ such that $\lim_{k\to\infty} \mu(M^{(0)} \smallsetminus E_k) = 0$, $\Upsilon^{(p)}(t)$ is defined for all $t \in \mathbb{R}$ and all $p \in E_k$, and for any choice of $p_k \in E_k$, and an interval $I_k \subset \mathbb{R}$ around 0 of length $|I_k| \geqslant S_k$, the empirical measures η_k on $M^{(0)}$ defined by

$$\int f d\eta_k = \frac{1}{|I_k|}\int_{I_k} f\left(\Upsilon^{(p_k)}(t)\right) dt \quad (f \in C_c(M_q))$$

satisfy

$$\eta_k \to_{k\to\infty} \mu, \text{ with respect to the weak-}* \text{ topology.} \tag{3.3}$$

Step 3. Notation for $\Upsilon(t)$-orbit segments on one side of the slit.

Let σ_j be the slit on $M^{(0)}$ as before, and A_j, A'_j be the two tori comprising $M^{(0)} \smallsetminus \sigma_j$ as in (i). We will define certain segments in $M^{(j)}$ and use (3.1) in order to obtain bounds on their length.

Denote by ι the involution of Proposition 3.1, on both $M^{(0)}$ and $M^{(j)}$, so that ϕ_j commutes with ι. Then $\phi_j(\sigma_j)$ is a slit on $M^{(j)}$ and as we saw, its length $|\phi_j(\sigma_j)|$ satisfies $|\phi_j(\sigma_j)| \to 0$. Since, by (3.1), $\bar{\pi}\left(M^{(j)}\right) \in \mathcal{K}$ for all j, the diameter of $M^{(j)}$ is bounded above independently of j. Since $\phi_j(A_j)$ is one of the connected components of $M^{(j)} \smallsetminus \phi_j(\sigma_j)$, $\phi_j(A_j)$ contains a vertical segment whose length is a fixed number independent of j. We denote this segment by ℓ, and let $\ell' \stackrel{\text{def}}{=} \iota(\ell)$ (the segments ℓ and ℓ' depend on j but we omit this from the notation). For each $x \in \ell \cup \ell'$ let $I(x)$ be the interval starting at 0, such that $H_x \stackrel{\text{def}}{=} \left\{\Upsilon^{(x)}(t) : t \in I(x)\right\}$ is the horizontal segment on $M^{(j)}$ starting at x and ending at the first return to $\ell \cup \ell'$. Then, by considering the projection to $\mathcal{K}$, we see that the length of $I(x)$ is bounded above and below by positive constants independent of j and x, and by adjusting ℓ there is a constant C such that

$$\forall j,\ \forall x \in \ell \cup \ell', \quad \text{we have} \ \ 1 \leqslant |I(x)| \leqslant C.$$

Let

$$D_j \stackrel{\text{def}}{=} \{x \in \ell \cup \ell' : \phi_j(\sigma_j) \cap H_x = \varnothing\}$$

and

$$B_j \stackrel{\text{def}}{=} \bigcup_{x \in D_j} H_x \quad \text{and} \quad \bar{B}_j \stackrel{\text{def}}{=} \bigcup_{x \in D_j \cap \ell} H_x.$$

Thus B_j is the union of trajectories in $M^{(j)}$ starting and ending in $\ell \cup \ell'$ that do not pass through the slit $\phi_j(\sigma_j)$, and $\bar{B}_j$ is the set of such trajectories that stay in $\phi_j(A_j)$. Then clearly $\iota(B_j) = B_j$ and moreover, since $|\phi_j(\sigma_j)| \to 0$, $\mathrm{Leb}(B_j) \to \mathrm{Leb}(M^{(j)}) = 2$. Similarly we have $\mathrm{Leb}(\bar{B}_j) \to 1$.

Let k_j be the largest k for which $e^{t_j} \geqslant S_k$. Then $k_j \to \infty$. By Proposition 3.4 awe have $\mathrm{Leb} = \mu + \iota_*\mu$, and so for large enough j, $\phi_j(E_{k_j} \cup \iota(E_{k_j})) \cap B_j \neq \varnothing$. Since $\iota(B_j) = B_j$ this implies $\phi_j(E_{k_j}) \cap B_j \neq$

$\varnothing$. Since the two tori A_j, A'_j cover $M^{(0)}$, by replacing A_j with A'_j if necessary, we may assume that for all large enough j,

$$\phi_j(A_j \cap E_{k_j}) \cap B_j \neq \varnothing.$$

Step 4: Comparing $\Upsilon(t)$-orbit segments on one side of the slit, and Lebesgue measure restricted to that component.

Let μ_j be the restriction of Leb to A_j, so that μ_j is a probability measure. Our goal is to show that for all $\varepsilon > 0$ and $f \in C_c\left(M^{(0)}\right)$, for all j large enough we have

$$\left| \int_{M^{(0)}} f d\,\mu_j - \int_{M^{(0)}} f d\mu \right| < \varepsilon. \tag{3.4}$$

We will do this by showing that orbit segments of points in E_k, which are almost generic for μ, track orbit segments of other points, which approximate Leb. We assume with no loss of generality that $\|f\|_\infty = 1$.

Fix $x_1 \in \phi_j(A_j \cap E_{k_j}) \cap B_j$ and let $y_1 \stackrel{\mathrm{def}}{=} \phi_j^{-1}(x_1)$. There is $x \in \ell \cap D_j$ such that $x_1 \in H_x$, and we let $y \stackrel{\mathrm{def}}{=} \phi_j^{-1}(x)$. Recall that ϕ_j^{-1} maps horizontal and vertical straightline segments on $M^{(j)}$ to horizontal and vertical straightline segments on $M^{(0)}$, multiplying their lengths respectively by $e^{\pm t_j}$. In particular $J(y) \stackrel{\mathrm{def}}{=} \phi_j^{-1}(H_x)$ is a horizontal line segment on $M^{(0)}$ of length at least e^{t_j} and containing y_1, and since $y_1 \in E_{k_j}$, this implies via (3.3) that for j sufficiently large,

$$\left| \frac{1}{e^{t_j}|J(y)|} \int_0^{e^{t_j}|J(y)|} f\left(\Upsilon^{(y)}(t)\right) dt - \int_{M^{(0)}} f d\mu \right| < \frac{\varepsilon}{3}. \tag{3.5}$$

Let $x' \in D_j \cap \ell$. Then there is a vertical segment from x to x' along ℓ, of length at most C. Since $\ell \subset \phi_j(A_j)$, this segment lies completely inside $\phi_j(A_j)$. Arcs starting in $\phi_j(A_j)$ can only leave $\phi_j(A_j)$ by passing through the slit $\phi_j(\sigma_j)$. Thus, if $\Upsilon^{(x)}(t)$ is in $\phi_j(A_j)$ and the vertical straightline segment of length C starting at $\Upsilon^{(x)}(t)$ misses $\phi_j(\sigma_j)$, there is also a vertical segment from $\Upsilon^{(x)}(t)$ to $\Upsilon^{(x')}(t)$ of length at most C, which lies completely inside $\phi_j(A_j)$. Since $|\phi_j(\sigma_j)| \to 0$, this implies that there is a finite union of subintervals $J_1 = J_1(y') \subset J(y')$, such that $|J_1| = O(|\phi_j(\sigma_j)|) \to 0$ and such that for all $t \in J(y') \smallsetminus J_1(y')$ there is a vertical line segment of length at most C from $\Upsilon^{(x)}(t)$ to $\Upsilon^{(x')}(t)$, and this segment stays completely in $\phi_j(A_j)$.

Thus for any $x' \in D_j \cap \ell$, if we set $y' \stackrel{\text{def}}{=} \phi_j^{-1}(x')$, then for all large enough j we have

$$\frac{1}{e^{t_j}|J(y')|}\int_0^{e^{t_j}|J(y')|}\left|f\left(\Upsilon^{(y')}(t)\right) - f\left(\Upsilon^{(y)}(t)\right)\right| dt < \frac{\varepsilon}{3}. \tag{3.6}$$

Let $\bar{\mu}_j$ be the restriction of Leb to $\phi_j^{-1}(\bar{B}_j)$. Then using Fubini's theorem to express $\bar{\mu}_j$ as an integral of integrals along the lines $\phi_j^{-1}(H_{x'})$, for $x' \in D_j \cap \ell$, we find from (3.5) and (3.6) that

$$\left|\int_{M^{(0)}} f d\bar{\mu}_j - \int_{M^{(0)}} f d\mu\right| < \frac{2\varepsilon}{3}. \tag{3.7}$$

Since $\bar{B}_j \subset \phi_j(A_j)$ and ϕ_j^{-1} preserves Lebesgue measure, we have

$$\phi_j^{-1}(\bar{B}_j) \subset A_j, \quad \mathrm{Leb}(\phi_j^{-1}(\bar{B}_j)) \to 1 = \mathrm{Leb}(A_j)$$

and hence for all large j,

$$\left|\int_{M^{(0)}} f\, d\bar{\mu}_j - \int_{M^{(0)}} f\, d\mu_j\right| < \frac{\varepsilon}{3}.$$

Combining this with (3.7) gives (3.4). □

Similar ideas can be used to prove the following statement.

Theorem 3.6. *Suppose $q \in \mathcal{E}$, and the horizontal measured foliation of the underlying surface M_q is minimal but not ergodic. Then there is a sequence of decompositions of M_q into pairs of tori A_j and B_j glued along slits, and such that the set*

$$A_\infty = \bigcup_i \bigcap_{j \geqslant i} A_j$$

is invariant under the horizontal flow, and has Lebesgue measure $1/2$.

The statement will not be used in this paper and its proof is left to the reader.

4. Tremors

In this section we give a more detailed treatment of tremors and their properties.

4.1. **Definitions and basic properties.**

4.1.1. *Semi-continuity of foliation cocycles.* Let $q \in \mathcal{H}$ represent a surface M_q with horizontal foliation $\mathcal{F}_q$. Recall from §2.5 that the transverse measures (respectively, signed transverse measures) define a cone C_q^+ of foliation cocycles (resp., a space $\mathcal{T}_q$ of signed foliation cocycles) and these are subsets of $H^1(M_q, \Sigma; \mathbb{R}_x)$. For a marking map $\varphi : S \to M_q$ representing a marked translation surface $\widetilde{q} \in \pi^{-1}(q)$, the pullbacks $\varphi^*(C_q^+)$ and $\varphi^*(\mathcal{T}_q)$ are subsets of $H^1(S, \Sigma; \mathbb{R}_x)$ and will be denoted by $C_{\widetilde{q}}^+$ and $\mathcal{T}_{\widetilde{q}}$. Note that these notions are well-defined even at orbifold points (i.e. do not depend on the choice of the marking map) because translation equivalences map transverse measures to transverse measures. Recall that $\beta \in C_q^+$ is called non-atomic if $\beta = \beta_\nu$ for a non-atomic transverse measure ν. We will mostly work with non-atomic transverse measures as described in §2.5, and for completeness explain the atomic case in §13.

Recall from §2.2 that for any q, the tangent space $T_q(\mathcal{H})$ at q is identified with $H^1(M_q, \Sigma_q; \mathbb{R}^2)$ (or with $H^1(M_q, \Sigma_q; \mathbb{R}^2)/\Gamma_q$ if q is an orbifold point), and that a marking map identifies the tangent space $T_{\widetilde{q}_1}(\mathcal{H}_{\mathrm{m}})$, for $\widetilde{q}_1$ close to $\widetilde{q}$, with $H^1(S, \Sigma; \mathbb{R}^2)$. The following proposition expresses an important semi-continuity property for the cone of foliation cocycles.

Proposition 4.1. *The set*

$$C_{\mathcal{H}}^+ \stackrel{\mathrm{def}}{=} \left\{(\widetilde{q}, \beta) \in \mathcal{H}_{\mathrm{m}} \times H^1(S, \Sigma; \mathbb{R}_x) : \beta \in C_{\widetilde{q}}^+\right\}$$

is closed. That is, suppose $\widetilde{q}_n \to \widetilde{q}$ is a convergent sequence in $\mathcal{H}_{\mathrm{m}}$, and let $C_{\widetilde{q}_n}^+, C_{\widetilde{q}}^+ \subset H^1(S, \Sigma; \mathbb{R}_x)$ be the corresponding cones. Suppose that $\beta_n \in H^1(S, \Sigma; \mathbb{R}_x)$ is a convergent sequence such that $\beta_n \in C_{\widetilde{q}_n}^+$ for every n. Then $\lim_{n\to\infty} \beta_n \in C_{\widetilde{q}}^+$.

Proposition 4.1 will be proved in §4.2 under an additional assumption and in §13 in general. Note that care is required in formulating an analogous property for $\mathcal{T}_q$ because $\dim \mathcal{T}_q$ can decrease when taking limits. See Corollary 4.4.

4.1.2. *Signed mass, total variation, and balanced tremors.* We now define the *signed mass* and *total variation* of a signed foliation cocycle. Recall from §2 that $dx = (dx)_q$ denotes the canonical transverse measure for the vertical foliation on a translation surface q and $\mathrm{hol}_q^{(x)}$ denotes the corresponding element of $H^1(M_q, \Sigma_q; \mathbb{R})$. Given $q \in \mathcal{H}$ and $\beta \in H^1(M_q, \Sigma_q; \mathbb{R})$, denote by $L_q(\beta)$ the evaluation of the cup product $\mathrm{hol}_q^{(x)} \cup \beta$ on the fundamental class of M_q. In particular, if $\beta = \beta_\nu$ for a non-atomic signed transverse measure ν then

$$L_q(\beta) = \int_{M_q} dx \wedge \nu;$$

or equivalently, if $\mu = \mu_\nu$ is the horizontally invariant signed measure associated to ν by Proposition 2.3, then $L_q(\beta) = \mu(M_q)$. We will refer to $L_q(\beta)$ as the *signed mass* of β. Our sign conventions imply that $L_q(\beta) > 0$ for any nonzero $\beta \in C_q^+$.

Note that if $h : M_q \to M_q$ is a translation equivalence then $L_q(\beta) = L_q(h^*(\beta))$. Thus, if $\widetilde{q} \in \pi^{-1}(q)$ is a marked translation surface represented by a marking map φ, and $\beta' \in H^1(S, \Sigma; \mathbb{R})$ satisfies $\beta = \varphi_* \beta'$, then we can define $L_{\widetilde{q}}(\beta') \stackrel{\mathrm{def}}{=} L_q(\beta)$, and this definition does not depend on the choice of the marking map φ representing $\widetilde{q}$. In particular the mapping $(q, \beta) \mapsto L_q(\beta)$ defines a map on $T(\mathcal{H})$, even if q lies in an orbifold substratum.

Recall that every signed measure and every signed transverse measure has a canonical Hahn decomposition $\nu = \nu^+ - \nu^-$ as a difference of measures. Thus any $\beta \in \mathcal{T}_q$ can be written as $\beta = \beta^+ - \beta^-$ where $\beta^\pm \in C_q^+$. In analogy with the total variation of a measure we now define

$$|L|_q(\beta) = L_q(\beta^+) + L_q(\beta^-), \tag{4.1}$$

and call this the *total variation* of β. Note that the signed mass is defined for every $\beta \in H^1(M_q, \Sigma; \mathbb{R})$ but the total variation is only defined for $\beta \in \mathcal{T}_q$. The linearity of the cup product implies that the maps

$$T(\mathcal{H}) \to \mathbb{R},\ (q, \beta) \mapsto L_q(\beta) \quad \text{and} \quad T(\mathcal{H}_{\mathrm{m}}) \to \mathbb{R},\ (\widetilde{q}, \beta) \mapsto L_{\widetilde{q}}(\beta)$$

are both continuous. In combination with Proposition 4.1, this implies:

Corollary 4.2. *The sets*

$$C^+_{\mathcal{H}_{\mathrm{m}},1} \stackrel{\mathrm{def}}{=} \{(\widetilde{q}, \beta) : \beta \in C^+_{\widetilde{q}},\ L_{\widetilde{q}}(\beta) = 1\}$$

and

$$C^+_{\mathcal{H},1} \stackrel{\mathrm{def}}{=} \{(q, \beta) : \beta \in C^+_q,\ L_q(\beta) = 1\}$$

are closed, and thus define closed subsets of $T(\mathcal{H}_{\mathrm{m}})$ and $T(\mathcal{H})$.

The following special case will be important in the proofs of Theorem 1.3 and Theorem 1.4.

Corollary 4.3. *Let $q \in \mathcal{H}$, and denote its canonical foliation cocycle by $\mathrm{hol}^{(y)}_q$. Suppose the underlying translation surface M_q has area one and is horizontally uniquely ergodic. Then for any sequence $q_n \in \mathcal{H}$ such that $q_n \to q$, and any $\beta_n \in C^+_{q_n}$ with $L_{q_n}(\beta_n) = 1$, we have $\beta_n \to \mathrm{hol}^{(y)}_q$.*

The total variation of a foliation cocycle also has a semicontinuity property:

Corollary 4.4. *Suppose $\widetilde{q}_n \to \widetilde{q}$ in $\mathcal{H}_{\mathrm{m}}$ and $\beta_n \in \mathcal{T}_{\widetilde{q}_n} \subset H^1(S, \Sigma; \mathbb{R})$ is a sequence of non-atomic signed foliation cocycles for which the limit $\beta = \lim_{n\to\infty} \beta_n$ exists and $\sup_n |L|_{\widetilde{q}_n}(\beta_n) < \infty$. Then $\beta \in \mathcal{T}_{\widetilde{q}}$ and*

$$|L|_{\widetilde{q}}(\beta) \leqslant \liminf_{n\to\infty} |L|_{\widetilde{q}_n}(\beta_n). \tag{4.2}$$

Corollary 4.4 will also be proved in §4.2.

We say that $\beta \in \mathcal{T}_q$ is *balanced* if $L(\beta) = 0$, and we let $\mathcal{T}_q^{(0)}$ denote the set of balanced signed foliation cocycles. Combining Corollary 3.3 and Proposition 3.4, for surfaces in $\mathcal{E}$ we see that balanced foliation cocycles are those that are 'normal' to $\mathcal{E}$:

Corollary 4.5. *Let $\mathcal{O}$ be an orbifold substratum of $\mathcal{H}$ and $q \in \mathcal{O}$. Then $\mathcal{T}_q \cap \mathscr{N}_x(\mathcal{O}) \subset \mathcal{T}_q^{(0)}$, with equality in the case $\mathcal{O} = \mathcal{E}$; namely, if $q \in \mathcal{E}$ is aperiodic then $\mathcal{T}_q^{(0)} = \mathscr{N}_x(\mathcal{E})$.*

Proof. Let $q \in \mathcal{O}$, let Γ_q be the group of translation equivalences of M_q, let $\mathcal{G} \stackrel{\text{def}}{=} \mathcal{G}_q$ be the local group as in §2.3 and let $\gamma \in \mathcal{G}$. Recall that Γ_q and $\mathcal{G}$ are isomorphic and by fixing a marking map, we can think of γ simultaneously as acting on M_q by translation automorphisms, and on $H^1(S, \Sigma; \mathbb{R}^2)$ by the natural map induced by a homeomorphism. Since translation automorphisms of M_q preserve the canonical transverse measure $(dx)_q$, we have $\gamma^*\mathrm{hol}_q^{(x)} = \mathrm{hol}_q^{(x)}$, and thus for any β,

$$\begin{aligned} L_q(\gamma^*\beta) &= (\mathrm{hol}_q^{(x)} \cup \gamma^*\beta)(M_q) = (\mathrm{hol}_q^{(x)} \cup \beta)(\gamma(M_q)) \\ &= (\mathrm{hol}_q^{(x)} \cup \beta)(M_q) = L_q(\beta). \end{aligned}$$

Hence, if $\beta \in \mathcal{T}_q \cap \mathscr{N}_x(\mathcal{O})$ then $P^+(\beta) = 0$, where P^+ is the projection onto the tangent space of $\mathcal{O}$ given in (2.3), and we have

$$L_q(\beta) = \frac{1}{|\mathcal{G}|} \sum_{\gamma\in\mathcal{G}} L_q(\gamma^*(\beta)) = L_q\left(\frac{1}{|\mathcal{G}|} \sum_{\gamma\in\mathcal{G}} \gamma^*(\beta)\right) = L_q(P^+(\beta)) = 0.$$

Therefore $\beta \in \mathcal{T}_q^{(0)}$.

Now if $q \in \mathcal{E}$ is aperiodic and $\beta \in \mathcal{T}_q^{(0)}$, then we can write $\beta = \beta_\nu$ for a signed transverse measure ν, and let $\mu = \mu_\nu$ be the associated horizontally invariant signed measure (see Proposition 2.3). Since $\beta \in \mathcal{T}_q^{(0)}$ we have $\mu(M_q) = 0$. Recall from Proposition 3.4 that aperiodic surfaces in $\mathcal{E}$ are either uniquely ergodic, or have two ergodic measures which are exchanged by the involution $\iota = \iota_q$. By ergodic decomposition (applied to each summand in $\mu = \mu^+ - \mu^-$) we can write μ as a linear combination of ergodic measures (where the coefficients may be negative). If M_q is uniquely ergodic then this gives $\mu = c \cdot \mathrm{Leb}$ and since $\mu(M_q) = 0$ we have $\mu = 0$. If M_q has two ergodic probability measures μ_1 and $\mu_2 = \iota_* \mu_1$ then $\mu = c_1\mu_1 + c_2 \iota_* \mu_1$ and

$$0 = \mu(M_q) = c_1\mu_1(M_q) + c_2\mu_1(\iota(M_q)) = c_1 + c_2,$$

so $c_1 = -c_2$. In both cases we obtain $\iota_*\mu = -\mu$, which implies $\iota_*\beta = -\beta$. Thus, using Corollary 3.3, we see that $\beta \in \mathscr{N}_x(\mathcal{E})$. ☐

4.1.3. *Absolutely continuous foliation cocycles.* Let ν_1 and ν_2 be two signed transverse measures for $\mathcal{F}_q$. We say that ν_1 *is absolutely continuous with respect to* ν_2 if the corresponding signed measures μ_{ν_1}, μ_{ν_2} given by Proposition 2.3 satisfy $\mu_{\nu_1} \ll \mu_{\nu_2}$. We say that ν *is absolutely continuous* if it is absolutely continuous with respect to the canonical transverse measure $(dy)_q$. Since $(dy)_q$ is non-atomic, so is any absolutely continuous signed transverse measure. For $c > 0$, we say ν is *c-absolutely continuous* if

$$\text{for any transverse arc } \gamma \text{ on } M_q, \quad \left|\int_\gamma d\nu\right| \leqslant c \left|\int_\gamma dy\right|. \tag{4.3}$$

We call a signed foliation cocycle $\beta = \beta_\nu$ *absolutely continuous* (respectively, *c-absolutely continuous*) if it corresponds to a signed transverse measure ν which is absolutely continuous (resp., c-absolutely continuous). Let $\|\nu\|_{RN}$ denote the minimal c such that the above equation holds for all transverse arcs γ (our notation stems from the fact that $\|\nu\|_{RN}$ is the sup-norm of the Radon-Nikodym derivative $\frac{d\mu_\nu}{d\mathrm{Leb}}$, although we will not be using this in the sequel). Given $q \in \mathcal{H}$ and $c > 0$, denote by $C_q^{+,RN}(c)$ (respectively, by $\mathcal{T}_q^{RN}(c)$) the set of absolutely continuous (signed) foliation cocycles β_ν with $\|\nu\|_{RN} \leqslant c$.

Remark 4.6. *As the reader will note, we will use both $|L|_q(\beta)$ and $\|\nu\|_{RN}$ to measure the 'size' of a foliation cocycle $\beta = \beta_\nu$. For most purposes in this paper, $|L|_q$ is easier to work with. Additionally, it is more broadly defined, making sense when the tremor corresponds to a singular measure. However, $\|\cdot\|_{RN}$ is more suitable for estimates involving the distance function* dist *(see Proposition 6.7) and plays an essential role in the proof of Proposition 7.1.*

It is easy to see that

$$C_q^{+,RN}(c) \subset \{\beta \in C_q^+ : L_q(\beta) \leqslant c\} \tag{4.4}$$

and

$$\mathcal{T}_q^{RN}(c) \subset \{\beta \in \mathcal{T}_q : |L|_q(\beta) \leqslant c\}. \tag{4.5}$$

As we will see in Lemma 8.3, for some surfaces we will also have a reverse inclusion.

We now observe that for aperiodic surfaces, the assumption of absolute continuity implies a uniform bound on the Radon-Nikodym derivative:

Lemma 4.7. *Suppose M_q is a horizontally aperiodic surface, ν is an absolutely continuous transverse measure, and $\mu = \mu_\nu$ is the corresponding measure on M_q, so that $\mu \ll \mathrm{Leb}$. Then there is $c > 0$ such that $\|\nu\|_{RN} \leqslant c$. Moreover the constant c depends only on the coefficients appearing in the ergodic decompositions of μ and* Leb*, and if μ is a probability measure and $\mathrm{Leb} = \sum a_i \nu_i$, where $\{\nu_i\}$ are the horizontally invariant ergodic probability measures and each a_i is positive, then $\|\nu\|_{RN} \leqslant \max_i \frac{1}{a_i}$. The same conclusions hold if instead of assuming M_q is aperiodic, we assume the measure ν is aperiodic, that is μ assigns zero measure to any horizontal cylinder on M_q.*

Proof. Let $\{\mu_1, \dots, \mu_d\}$ be the invariant ergodic probability measures for the horizontal straightline flow on M_q. Since M_q is horizontally aperiodic, this is a finite collection, see e.g. [K]. Thus there only finitely many ergodic measures which are absolutely continuous with respect to μ, and we denote them by $\{\mu_1, \dots, \mu_k\}$. The measures μ_i are mutually singular. Write $\mathrm{Leb} = \sum_i a_i \mu_i$ and $\mu = \sum_i b_i \mu_i$, where all a_i, b_j are non-negative and not all are zero. Since $\mu \ll \mathrm{Leb}$, we have

$$b_i > 0 \implies a_i > 0.$$

Set

$$c \stackrel{\mathrm{def}}{=} \max \left\{ \frac{b_i}{a_i} : b_i \neq 0 \right\}. \tag{4.6}$$

For any Borel set $A \subset M_q$ we have

$$\mu(A) = \sum_i b_i \mu_i(A) \leqslant c \sum_i a_i \mu_i(A) = c\, \mathrm{Leb}(A).$$

This implies that the Radon Nikodym derivative satisfies $\frac{d\mu}{d\mathrm{Leb}} \leqslant c$ a.e. The horizontal invariance of μ and Leb shows that the Radon-Nikodym derivative $\frac{d\mu}{d\mathrm{Leb}}$ is defined on almost every point of every transverse arc γ, and the relation (2.6) shows that it coincides with the Radon-Nikodym derivative $\frac{d\nu}{(dy)_q}$. Thus we get (4.3).

The second assertion follows from (4.6), and the last assertion follows by letting μ_i denote the horizontally invariant measures on the complement of the union of the horizontal cylinders in M_q, and repeating the argument given above. ∎

4.1.4. *Tremors as affine geodesics, and their domain of definition.* Recall from §2.2 that we identify $T(\mathcal{H}_{\mathrm{m}})$ with $\mathcal{H}_{\mathrm{m}} \times H^1(S, \Sigma, \mathbb{R}^2)$. Our particular interest is in affine geodesics tangent to signed foliation cocycles. That is, we take

$$\beta \in \mathcal{T}_{\widetilde{q}} \subset H^1(S, \Sigma; \mathbb{R}_x)$$

(where the last inclusion uses a marking map $\varphi : S \to M_q$ representing $\widetilde{q}$). We write $v = (\beta, 0) \in H^1(S, \Sigma; \mathbb{R}^2)$ and consider the parameterized line $\theta(t) = \theta_{\widetilde{q},v}(t)$ in $\mathcal{H}_{\mathrm{m}}$ satisfying

$$\theta(0) = \widetilde{q} \quad \text{and} \quad \frac{d}{dt}\theta(t) = v \tag{4.7}$$

(where we have again used the marking to identify the tangent space $T_{\theta(t)}(\mathcal{H}_{\mathrm{m}})$ with $H^1(S, \Sigma; \mathbb{R}^2)$). By the uniqueness of solutions of differential equations, these equations uniquely define the affine geodesic $\theta(t)$ for t in the maximal domain of definition $\mathrm{Dom}(\widetilde{q}, v)$. As in the introduction we now have $\mathrm{trem}_{t,\beta}(\widetilde{q}) = \theta(t)$ and $\mathrm{trem}_{\beta}(\widetilde{q}) = \theta(1)$ when $1 \in \mathrm{Dom}(\widetilde{q}, \beta)$. Equation (4.7) and uniqueness of solutions imply that for $c > 0$ we have $\theta_{\widetilde{q},cv}(t) = \theta_{\widetilde{q},v}(ct)$ and $\mathrm{Dom}(\widetilde{q}, v) = c\,\mathrm{Dom}(\widetilde{q}, cv)$. In particular $\mathrm{trem}_{t,c\beta}(\widetilde{q}) = \mathrm{trem}_{ct,\beta}(\widetilde{q})$ and thus $\mathrm{trem}_{t,\beta}(\widetilde{q}) = \mathrm{trem}_{t\beta}(\widetilde{q})$.

Since the developing map is affine, we find

$$\mathrm{hol}^{(x)}_{\mathrm{trem}_\beta(\widetilde{q})}(\gamma) = \mathrm{hol}^{(x)}_{\widetilde{q}}(\gamma) + \beta(\gamma), \quad \mathrm{hol}^{(y)}_{\mathrm{trem}_\beta(\widetilde{q})}(\gamma) = \mathrm{hol}^{(y)}_{\widetilde{q}}(\gamma). \tag{4.8}$$

Comparing equations (4.8) and (1.5), we see that we have given a formal definition of the tremors introduced in §1.2.

The pure mapping class group $\mathrm{Mod}(S, \Sigma)$ acts on each coordinate of $T(\mathcal{H}_{\mathrm{m}}) = \mathcal{H}_{\mathrm{m}} \times H^1(S, \Sigma, \mathbb{R}^2)$, and by equivariance we find that

$$\mathrm{trem}_\beta(q) = \pi(\mathrm{trem}_\beta(\widetilde{q})) \quad \text{and} \quad \mathrm{Dom}(q, \beta) \stackrel{\mathrm{def}}{=} \mathrm{Dom}(\widetilde{q}, \beta)$$

are well-defined and independent of the choice of $\widetilde{q} \in \pi^{-1}(q)$.

Basic properties of ordinary differential equations now give us:

Proposition 4.8. *The set*

$$\mathcal{D} = \{(\widetilde{q}, v, s) \in T(\mathcal{H}_{\mathrm{m}}) \times \mathbb{R} : s \in \mathrm{Dom}(\widetilde{q}, v)\}$$

is open in $T(\mathcal{H}_{\mathrm{m}}) \times \mathbb{R}$, and the map

$$\mathcal{D} \ni (\widetilde{q}, v, s) \mapsto \theta_{\widetilde{q},v}(s)$$

is continuous. In particular the tremor map

$$\{(\widetilde{q}, \beta) \in T\mathcal{H}_{\mathrm{m}} : \beta \in \mathcal{T}_q\} \to \mathcal{H}_{\mathrm{m}}, \quad (\widetilde{q}, \beta) \mapsto \mathrm{trem}_{\widetilde{q},\beta}$$

is continuous where defined.

Comparing equation (4.8) to the definition of the horocycle flow in period coordinates, we immediately see that for the canonical foliation cocycle $dy = \mathrm{hol}^{(y)}_{\widetilde{q}}$, we have

$$\mathrm{trem}_{sdy}(\widetilde{q}) = u_s \widetilde{q}. \tag{4.9}$$

4.2. **Tremors and polygonal presentations of surfaces.** In this section we prove Proposition 4.1, under an additional hypothesis. This special case is easier to prove and suffices for proving our main results. We will prove the general case of Proposition 4.1 in §13. At the end of this section we deduce Corollary 4.4 from Proposition 4.1.

Proposition 4.9. *Let $\widetilde{q}_n \to \widetilde{q}$ in $\mathcal{H}_{\mathrm{m}}$, $\beta_n \to \beta$ in $H^1(S, \Sigma; \mathbb{R}_x)$ be as in the statement of Proposition 4.1. Write $q_n = \pi(\widetilde{q}_n)$, $q = \pi(\widetilde{q})$ and suppose also that*

$$\text{there is a } c > 0 \text{ such that for all } n,\ \beta_n \in C^{+,RN}_{q_n}(c). \tag{4.10}$$

Then $\beta \in C^{+,RN}_q(c)$.

Clearly Proposition 4.9 implies Proposition 4.1 in the case that (4.10) holds.

Recall that any translation surface has a polygon decomposition, and that fixing a polygon decomposition on a marked surface makes it possible to consider the same polygon decomposition on nearby marked surfaces. For the proof of Proposition 4.9, we introduce polygon decompositions which are useful for understanding transverse measures to the horizontal foliation.

In a general polygon decomposition of a surface, some edges might be horizontal, and corresponding edges on nearby surfaces may intersect the horizontal foliation with different orientations. This will cause complications and in order to avoid them, we introduce an *adapted polygon decomposition (APD)* of a surface. An APD is a polygon decomposition in which all polygons are either triangles with no horizontal edges, or quadrilaterals with one horizontal diagonal. Any surface has an APD, as can be seen by taking a triangle decomposition and merging adjacent triangles sharing a horizontal edge into quadrilaterals. We fix an APD of M_q, with a finite collection of edges $\{J_i\}$, all of which are transverse to the horizontal foliation on M_q. Since we are considering marked surfaces, we can use a marking map representing $\widetilde{q} \in \pi^{-1}(q)$ and the comparison maps of §2.2 and think of the arcs J_i as arcs on S, as well as on $M_{q'}$ for any marked translation surface $\widetilde{q'}$ sufficiently close to $\widetilde{q}$. Moreover, the edges $\{J_i\}$ are also a subset of the edges of an APD on $M_{q'}$ and they are also transverse to the horizontal foliation on $M_{q'}$. Note that on $M_{q'}$ the APD may contain additional edges that are not edges on M_q, namely some of the horizontal diagonals on M_q might not be horizontal on $M_{q'}$ and in this case we add them to the $\{J_i\}$ to obtain an APD on $M_{q'}$.

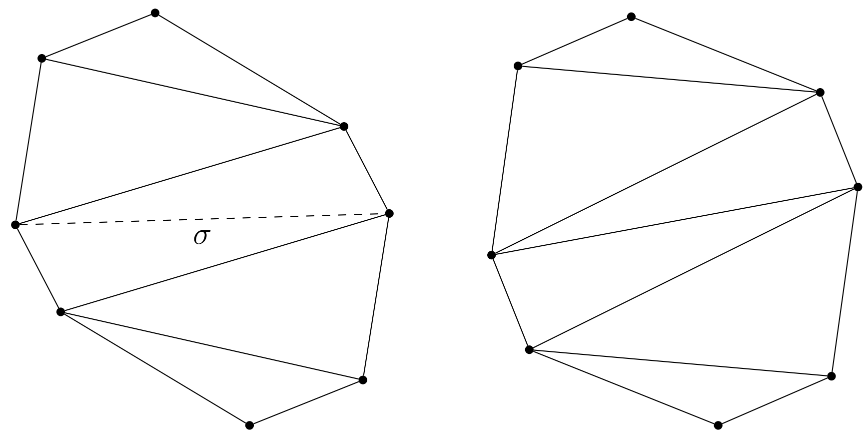

FIGURE 3. Two APD's on nearby surfaces in $\mathcal{H}(2)$. The dotted horizontal line represents a diagonal of a quadrilateral on the first surface and is an edge of a triangle on the second surface since it is no longer horizontal.

Since the polygons of a polygon decomposition are simply connected, a 1-cochain representing an element of $H^1(S, \Sigma; \mathbb{R})$ is determined by its values on the edges of the polygons. For each i, each polygon P of the APD with $J = J_i \subset \partial P$, and each $x \in J$, there is a horizontal segment in P with endpoints in ∂P one of which is x. The other endpoint of this segment is called the *opposite point (in P) to x* and is denoted by $\mathrm{opp}_P(x)$. The image of J under opp_P is a union of one or two sub-arcs contained in the other boundary edges of P.

A transverse measure ν for the horizontal foliation on M_q assigns a measure to each J. We will denote this either by ν, or by $\nu|_J$ when confusion may arise. By definition of a transverse measure,

$$(\mathrm{opp}_P)_* \nu|_J = \nu|_{\mathrm{opp}_P(J)}, \tag{4.11}$$

and this holds for any P and J. We call (4.11) the *invariance property*. Note that in this section, all measures under consideration are non-atomic, and we will not have to worry about whether intervals are open or closed (but in §13 this will be a concern).

Proposition 4.10. *Given an APD for a translation surface M_q, and a collection of finite non-atomic measures ν_J on the edges J as above, satisfying the invariance property, there is a transverse measure ν on M_q for which $\nu|_J = \nu_J$.*

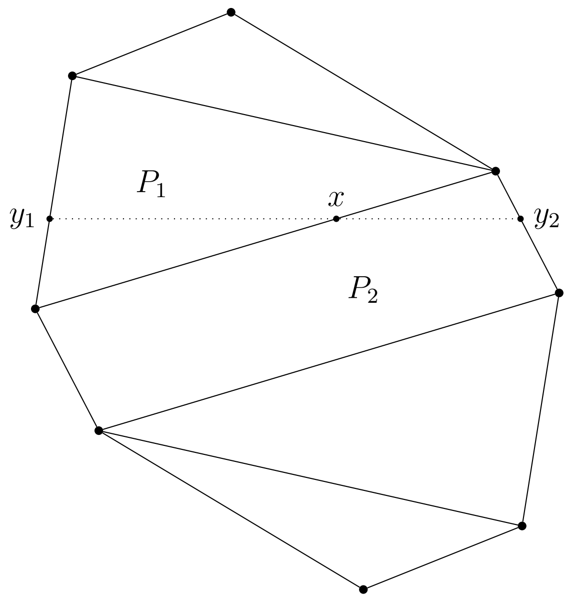

FIGURE 4. The opposite point map, with $y_1 = opp_{P_1}(x)$ and $y_2 = opp_{P_2}(x)$.

Proof. We can reconstruct ν from the ν_J, by homotoping any transverse arc to subintervals of edges of the APD along horizontal leaves (this is well-defined in view of the invariance property). □

Proposition 4.10 makes it possible to reduce questions about transverse measures on surfaces, to finitely many measures on some arcs. We use this idea in the following:

Proof of Proposition 4.9. We will write $\beta_n = \beta_{\nu_n}$ for a sequence of c-absolutely continuous transverse measures ν_n on M_{q_n} (in particular the ν_n are non-atomic). Our goal is to prove that there is a transverse measure ν on M_q such that $\beta = \beta_\nu$. The main idea of the proof is to use APD's to reduce the discussion to measures on finitely many transverse arcs. It suffices to consider the restriction of the transverse measure to a particular finite collection of transverse arcs, which we now describe.

Let τ be the triangulation of M_q obtained by adding the horizontal diagonals to quadrilaterals in an APD. As discussed in §2.2, using τ and marking maps, we obtain maps $\varphi_n : S \to M_{q_n}$, $\varphi : S \to M_q$, such that for each n, the comparison map $\varphi_n \circ \varphi^{-1} : M_q \to M_{q_n}$ is piecewise affine, with derivative (in planar charts) tending to the identity map as $n \to \infty$. Let P be one of the polygons of the APD and $K \subset \partial P$ a subinterval of the form J or $\mathrm{opp}_P(J)$ as above. For all large enough n, none of the sides $\varphi_n \circ \varphi^{-1}(K)$ are horizontal and all have the same orientation as on M_q. Let $\nu_K^{(n)}$ be the measure on $\varphi_n \circ \varphi^{-1}(K)$ corresponding to ν_n. Using the marking φ_n^{-1} we will also think of $\nu_K^{(n)}$ as a measure on $\widetilde{K} = \varphi^{-1}(K)$.

Passing to subsequences and using the compactness of the space of measures of bounded mass on a bounded interval, we can assume that for each K, the sequence $\left(\nu_K^{(n)}\right)_n$ converges to a measure ν_K on $\widetilde{K}$. It follows from (4.10) that ν_K is non-atomic, indeed it is c-absolutely continuous since all the $\nu_K^{(n)}$ are. Each of the measures $\nu_K^{(n)}$ satisfies the invariance property for the horizontal foliation on M_{q_n}, and we claim:

Claim 4.11. *The measures ν_K satisfy the invariance property for the horizontal foliation on M_q.*

To see this, suppose $K = J$ in the above notation, the case $K = \mathrm{opp}_P(J)$ being similar. For each n let $\mathrm{opp}_P^{(n)}$ be the map corresponding to the horizontal foliation on M_{q_n}; it maps J to a subset of an edge or two edges of the APD. Let I be a compact interval contained in the interior of J. Then for all sufficiently large n, $\mathrm{opp}_P^{(n)}(I) \subset \mathrm{opp}_P(J)$, and the maps $\mathrm{opp}_P^{(n)}|_I$ converge uniformly to $\mathrm{opp}_P|_I$. By our assumption that the measure is non-atomic, the endpoints of I have zero ν_J-measure. Therefore, since $\nu_J^{(n)} \to \nu_J$, by the Portmanteau theorem we have $\nu_J(I) = \nu_{\mathrm{opp}_P(J)}(\mathrm{opp}_P(I))$. Such intervals I generate the Borel σ-algebra on J, and so we have established the invariance property. This proves Claim 4.11. △

By Proposition 4.10, the ν_K define a transverse measure ν, and we let $\beta' = \beta_\nu$. Recall that we have assumed $\beta_n \to \beta$ as cohomology classes in $H^1(S, \Sigma; \mathbb{R})$. For each edge J of the APD,

$$\beta(J) \leftarrow \beta_n(J) = m_J^{(n)} \to m_J = \beta'(J),$$

and so $\beta' = \beta$. □

We now deduce Corollary 4.4. We will use the conclusion of Proposition 4.1. At this point we have only established the validity of this conclusion under assumption (4.10), and the general case will be established in §13.

Proof of Corollary 4.4 assuming Proposition 4.1. For each n write $\beta_n = \beta_{\nu_n}$ where ν_n is a transverse measure on M_{q_n}, let $\nu_n = \nu_n^+ - \nu_n^-$ be the Hahn decomposition and let $\mu_n^\pm$ be the horizontally invariant measure corresponding to $\nu_n^\pm$ via Proposition 2.3. By assumption,

$$\mu_n^\pm(M_{q_n}) = L_{\tilde{q}_n}(\beta_n^\pm) \leqslant |L|_{\tilde{q}_n}(\beta_n)$$

is a bounded sequence. Using the comparison maps $\varphi^{-1} \circ \varphi_n : M_{q_n} \to M_q$ used in the preceding proof, we can think of the $\mu_n^\pm$ as measures on M_q with a uniform bound on their total mass, and we can pass to a subsequence to obtain $\mu_{n_j}^\pm \to \mu_\infty^\pm$, and hence, using Proposition 4.1, $\nu_{n_j}^\pm \to \nu_\infty^\pm$ for transverse measures $\nu_\infty^\pm$ on M_q. More precisely, the fact that the ν_{n_j} converge to transverse measures on M_q follows from the proof of Proposition 4.1 under assumption (4.10), and the general case follows from the proof of Proposition 4.1 given in §13.

Let $\mu = \mu_\infty^+ - \mu_\infty^-$, $\nu = \nu_\infty^+ - \nu_\infty^-$ and let $\beta' = \beta_\nu$. Using Proposition 4.1 we have $\beta_{n_j} \to \beta'$. But since we have assumed $\beta = \lim_n \beta_n$, we have $\beta = \beta' \in \mathcal{T}_q$. The Hahn decomposition $\mu = \mu^+ - \mu^-$ minimizes the sum of the total masses of σ^+ and σ^-, over measures $\sigma^\pm$ with $\mu = \sigma^+ - \sigma^-$. Thus, even though $\mu = \mu_\infty^+ - \mu_\infty^-$ might not be the Hahn decomposition of μ, we have that the sum of the total masses of $\mu_\infty^\pm$ is bounded below by the sum of the total masses of $\mu^\pm$. From this we have

$$\begin{aligned} |L|_{\tilde{q}}(\beta) &= |L|_{\tilde{q}}(\beta') \leqslant L_{\tilde{q}}(\beta_{\nu^+}) + L_{\tilde{q}}(\beta_{\nu^-}) \\ &= \lim_{j\to\infty} \left(L_{\tilde{q}_{n_j}}(\beta_{n_j}^+) + L_{\tilde{q}_{n_j}}(\beta_{n_j}^-) \right) = \lim_{j\to\infty} |L|_{\tilde{q}_{n_j}}(\beta_{n_j}). \end{aligned}$$

Since this holds for any choice of the subsequence, we obtain (4.2). □

4.3. **The domain of definition of a tremor, and foliation cocycles in a fixed horospherical leaf.** In this subsection we will set up a canonical identification of $\mathcal{T}_q$ and $\mathcal{T}_{q'}$, when q and q' belong to the same horospherical leaf. For this, the notation of an APD, introduced in the §4.2, will turn out to be useful. As a consequence, and using results of [MW2], we will show that for a non-atomic tremor, the domain of definition $\mathrm{Dom}(q, \beta)$ is the entire real line, and we will obtain useful 'group action' properties of tremors on a fixed horospherical leaf.

Recall from §2.2 that via the identification of $T(\mathcal{H}_{\mathrm{m}})$ with the product $\mathcal{H}_{\mathrm{m}} \times H^1(S, \Sigma; \mathbb{R}^2)$, for any $\tilde{q}_1, \tilde{q}_2 \in \mathcal{H}_{\mathrm{m}}$, every $v_1 \in T_{\tilde{q}_1}(\mathcal{H}_{\mathrm{m}})$ has a unique parallel vector $v_2 \in T_{\tilde{q}_2}(\mathcal{H}_{\mathrm{m}})$. We say that v_2 is obtained from v_1 by *parallel transport.*

Proposition 4.12. *If $\tilde{q}_1$ and $\tilde{q}_2$ are elements of $\mathcal{H}_{\mathrm{m}}$ belonging to the same horospherical leaf W^{uu}, then parallel transport takes $\mathcal{T}_{\tilde{q}_1}$ to $\mathcal{T}_{\tilde{q}_2}$, takes $C_{q_1}^+$ to $C_{q_2}^+$, takes non-atomic tremors to non-atomic tremors, and takes $(dy)_{q_1} \in \mathcal{T}_{\tilde{q}_1}$ to $(dy)_{q_2} \in \mathcal{T}_{\tilde{q}_2}$.*

Proof. Since $\tilde{q}_1, \tilde{q}_2$ are both in W^{uu}, there is a path $\rho : [a, b] \to W^{uu}$ such that $\rho(a) = \tilde{q}_1$, $\rho(b) = \tilde{q}_2$. For each $t_0 \in [a, b]$, fix an APD on

$\rho(t_0)$, and let $\tau = \tau(t_0)$ be the triangulation obtained from this APD by adding diagonals to quadrilaterals, as in the proof of Proposition 4.9. Let V_τ be the open subset of $\mathcal{H}_{\mathrm{m}}$ associated with τ as in §2.2. We obtain a covering of $[a, b]$ by $\left\{\rho^{-1}\left(V_{\tau(t_0)}\right) : t_0 \in [a, b]\right\}$, and by compactness we can pass to a finite covering. Thus in proving the Proposition we may assume that the image of ρ is contained in one V_τ, where $\tau = \tau(a)$ is the triangulation obtained from an APD on M_{q_1}.

Let $\phi : M_{q_1} \to M_{q_2}$ be the comparison map which is affine on triangles of τ, as defined in §2.2. Since $\tilde{q}_1, \tilde{q}_2$ belong to the same horospherical leaf, a segment is horizontal on M_{q_1} if and only if its image under ϕ is horizontal on M_{q_2}. In particular the APD on M_{q_1} is sent to an APD on M_{q_2}, and the restriction of ϕ to edges of the APD commutes with the opposite point maps (this situation is illustrated in Figure 2). This implies via Proposition 4.10 that ϕ induces a bijection between signed transverse measures on M_{q_1} and M_{q_2}, and this bijection maps positive (respectively, atomic) transverse atomic transverse measures to positive (resp. atomic) transverse measures. Also, again using that $\tilde{q}_1, \tilde{q}_2$ are in the same horospherical leaf, the map ϕ sends $(dy)_{q_1}$ to $(dy)_{q_2}$. Thus the map $\phi^* : H^1(M_{q_2}, \Sigma_{q_1}; \mathbb{R}^2) \to H^1(M_{q_2}, \Sigma_{q_2}; \mathbb{R}^2)$ induced by ϕ sends $\mathcal{T}_{q_2}$ to $\mathcal{T}_{q_1}$ and sends $(dy)_{q_2}$ to $(dy)_{q_1}$. Finally, since $\tilde{q}_2$ is obtained from $\tilde{q}_1$ by pre-composing charts by ϕ, the definition of parallel vectors given in §2.2 shows that the map induced by ϕ^* is parallel transport. ☐

Proposition 4.13. *If* $\beta \in \mathcal{T}_q$ *is non-atomic then* $\mathrm{Dom}(q, \beta) = \mathbb{R}$.

The assumption that β is non-atomic is important here, see §13.

Proof. Let $\tilde{q} \in \pi^{-1}(q)$, let $\beta \in H^1(S, \Sigma; \mathbb{R}_x)$, let $v = (\beta, 0)$, let $\theta(t)$ be the parameterized line (4.7), and let $\mathrm{Dom}(q, \beta)$ denote its domain of definition. Let $\gamma_s = \mathrm{hol}^{(x)}(\tilde{q}) + s\beta$ be the corresponding line in $H^1(S, \Sigma; \mathbb{R}_x)$. We can define γ_s for all $s \in \mathbb{R}$, and for $s \in \mathrm{Dom}(q, \beta)$ we have $\gamma_s = \mathrm{dev}(\theta(s))$. Thus γ is a line in $H^1(S, \Sigma; \mathbb{R}_x)$, θ is its lift via dev, and our goal is to show that this lift is well-defined for all $s \in \mathbb{R}$.

We denote by $\mathcal{F}$ the foliation on $S \smallsetminus \Sigma$ obtained by pulling the horizontal foliation on M_q by φ. By Proposition 4.12, for all the surfaces $\tilde{q}'$ in any lift of γ, $\mathcal{F}$ is also the pullback of the horizontal foliation on $M_{q'}$. Let $\mathbb{B}(\mathcal{F})$ denote the set of cohomology classes $\gamma' \in H^1(S, \Sigma; \mathbb{R})$ satisfying the following conditions:

(i) For any oriented saddle connection δ on M_q with $\mathrm{hol}^{(x)}(\delta) > 0$, we have $\varphi^*\gamma'(\delta) > 0$.
(ii) For any non-atomic transverse measure ν to $\mathcal{F}$, γ' has a positive cup product with $\tau \stackrel{\mathrm{def}}{=} \beta_\nu$.

By [MW2, Thm. 1.1, see also Thm. 11.2] (but swapping the roles of horizontal and vertical foliations), in order to show that the path γ lifts, it suffices to show that $\gamma_s \in \mathbb{B}(\mathcal{F})$ for all s. Since β is non-atomic, it vanishes on horizontal saddle connections, and this implies that for any horizontal saddle connection δ, the function $s \mapsto \gamma_s(\delta)$ is constant. Therefore $\gamma_s(\delta) = \gamma_0(\delta) = \mathrm{hol}^{(x)}(\delta) > 0$, and this implies (i). In order to check (ii), let τ be the cohomology class corresponding to a non-atomic transverse measure. Then

$$\int \gamma_s \wedge \tau = \int ((dx)_{\tilde{q}} + s\beta) \wedge \tau = \int (dx)_{\tilde{q}} \wedge \tau + s\beta \wedge \tau = \int (dx)_{\tilde{q}} \wedge \tau > 0.$$

We have used here the fact that two cohomology classes arising from non-atomic transverse measures have cup product zero (see [K, Prop. 4]). ☐

It follows from Proposition 4.12 that for any horospherical leaf W^{uu} in $\mathcal{H}_{\mathrm{m}}$, there is a fixed subspace $\mathcal{T}^{(\mathrm{na})}_{W^{uu}} \subset H^1(S, \Sigma; \mathbb{R}_x)$, so that for each $\tilde{q} \in W^{uu}$, the collection of non-atomic foliation cocycles in $\mathcal{T}_{\tilde{q}}$ is canonically identified with $\mathcal{T}^{(\mathrm{na})}_{W^{uu}}$. Note that if $\tilde{q}$ has no horizontal saddle connections, then the same is true for any surface in the horospherical leaf of $\tilde{q}$, and in this case $\mathcal{T}_{\tilde{q}} = \mathcal{T}^{(\mathrm{na})}_{W^{uu}}$. We define a map

$$\mathcal{T}^{(\mathrm{na})}_{W^{uu}} \times W^{uu} \to W^{uu}, \quad (\beta, \tilde{q}) \mapsto \mathrm{trem}_\beta(\tilde{q}). \tag{4.12}$$

This map is well-defined in light of Proposition 4.13.

Proposition 4.14. *The map in* (4.12) *satisfies the 'group-action' law*

$$\mathrm{trem}_{\beta_1+\beta_2}(\tilde{q}) = \mathrm{trem}_{\beta_1}(\mathrm{trem}_{\beta_2}(\tilde{q}))$$

for all $\tilde{q} \in W^{uu}$ *and* $\beta_1, \beta_2 \in \mathcal{T}^{(\mathrm{na})}_{W^{uu}}$.

Proof. For any $s_1, s_2 \in \mathbb{R}$, the path

$$\gamma_{s_1,s_2} : \mathbb{R} \to H^1(S, \Sigma; \mathbb{R}_x), \quad \gamma_{s_1,s_2}(t) \stackrel{\mathrm{def}}{=} \mathrm{hol}_{\tilde{q}} + t(s_1\beta_1 + s_2\beta_2)$$

can be lifted to a path θ_{s_1,s_2} by Proposition 4.13. This implies that $\mathrm{trem}_{s_1\beta_1+s_2\beta_2}(q)$ is well-defined. Since dev is a local homeomorphism, it has a unique lifting property. That is, for any path $\gamma : [0,1] \to H^1(S, \Sigma; \mathbb{R}_x)$ and any $\tilde{q}_0$ with $\gamma(0) = \mathrm{dev}(q_0)$, there is at most one path $\theta : [0,1] \to \mathcal{H}_{\mathrm{m}}$ with $\theta(0) = \tilde{q}_0$ and $\gamma = \mathrm{dev} \circ \theta$. The two paths

$$s \mapsto \mathrm{trem}_{\beta_1}(\mathrm{trem}_{s\beta_2}(\tilde{q})), \quad s \mapsto \mathrm{trem}_{\beta_1+s\beta_2}(\tilde{q})$$

are continuous by Proposition 4.8, and commutativity of addition in $H^1(S, \Sigma; \mathbb{R}_x)$ shows that they are lifts of the same path in $H^1(S, \Sigma; \mathbb{R}_x)$. Thus they are the same, and setting $s = 1$ we get the required result. See [BSW, Prop. 4.5] for a similar argument. ☐

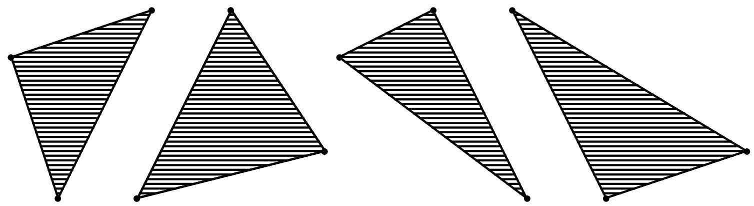

FIGURE 5. Along a horocycle, the affine comparison map also preserves horizontal lengths.

Corollary 4.15. *For any $u \in U$ and $\beta \in \mathcal{T}_q$, we have*

$$u\,\mathrm{trem}_\beta(q) = \mathrm{trem}_\beta(uq), \qquad \mathrm{Dom}(uq, \beta) = \mathrm{Dom}(q, \beta). \tag{4.13}$$

Proof. If β is non-atomic, this is immediate from (4.9) and Proposition 4.14. The proof when β is atomic is similar to the proof of Proposition 4.14. In this paper, we will not be using (4.13) when β is atomic, and we leave the details to the reader. □

5. The tremor comparison homeomorphism

Recall from §4.3 that two points $\tilde{q}_0$ and $\tilde{q}_1$ in the same horospherical leaf share the same space of foliation cocycles. This was proved in Proposition 4.12 by analyzing the effect of a composition of finitely many comparison maps $\varphi : M_{q_0} \to M_{q_1}$, each of which is affine on each triangle of a triangulation. The map φ respects horizontal foliations, that is maps the leaves of the horizontal foliation $\mathcal{F}$ on M_{q_0} to horizontal leaves on M_{q_1}, and preserves the canonical transverse measure dy measuring the 'height displacement' between leaves. In this section we will show that if $\tilde{q}_1$ is obtained from $\tilde{q}_0$ by a non-atomic tremor, then there is a comparison map $M_{q_0} \to M_{q_1}$ that *shears along horizontal leaves*; that is, respects the horizontal foliations $\mathcal{F}$ on M_{q_0} and M_{q_1}, preserves the transverse measure dy, and in addition, preserves the length parameter along horizontal leaves. In the language of flows, the comparison map from §4.3 commutes with the horizontal straightline flow up to a time change, and in this section we will produce a map commuting with straightline with no time change. This map need not be affine on triangles. The difference between these maps is illustrated in Figures 1 and 2. We note that for the horocycle flow, the affine comparison maps defined in §2.4 are both affine on triangles, and act by shearing horizontal leaves with respect to each other (see Figure 5).

As we will see in Proposition 5.9, the existence of a comparison homeomorphism that shears along horizontal leaves characterizes the property of lying on the same tremor path.

Proposition 5.1. *Let $q_0 \in \mathcal{H}$ and let $M_0 = M_{q_0}$ be the corresponding surface. Let $\varphi_0 : S \to M_0$ be a marking map and let $\widetilde{q}_0 \in \pi^{-1}(q_0)$ be the corresponding marked translation surface. Let ν be a non-atomic signed transverse measure on the horizontal foliation of M_0 and let $\beta = \beta_\nu$. Let $q_t = \mathrm{trem}_{t\beta}(q_0)$ and $\widetilde{q}_t = \mathrm{trem}_{t\beta}(\widetilde{q}_0)$, let $M_t = M_{q_t}$ be the underlying surface, and let $\varphi_t : S \to M_t$ be a marking map representing $\widetilde{q}_t$. Denote $\mathrm{hol}_{\widetilde{q}_t} = \left(\mathrm{hol}_t^{(x)}, \mathrm{hol}_t^{(y)}\right)$. Then there is a unique homeomorphism $\psi_t : M_0 \to M_t$ which is isotopic to $\varphi_t \circ \varphi_0^{-1}$, preserves horizontal foliations and satisfies*

$$\mathrm{hol}_t^{(x)}(\psi_t(\gamma)) = \mathrm{hol}_0^{(x)}(\gamma) + t\int_\gamma \nu \quad \text{and} \quad \mathrm{hol}_t^{(y)}(\psi_t(\gamma)) = \mathrm{hol}_0^{(y)}(\gamma) \tag{5.1}$$

for any piecewise smooth path γ in M_0 between any *two points.*

Definition 5.2. *We call $\psi_t : M_0 \to M_t$ the* tremor comparison homeomorphism *(TCH).*

The uniqueness of a tremor comparison homeomorphism implies the following important naturality property:

Corollary 5.3. *With the notation of Proposition 5.1, suppose φ_0 and φ_0' are two different marking maps $S \to M_0$ representing $\widetilde{q}_0$, so that $\varphi_0' \circ \varphi_0^{-1}$ is isotopic to a translation equivalence h of M_0. Then the TCH's ψ_t and ψ_t' satisfy $\psi_t = \psi_t' \circ h$.*

In order to construct ψ_t, we start with a comparison map φ which is only assumed to satisfy (5.1) in case γ is a saddle connection. We then modify φ by means of an isotopy which moves points along leaves of the horizontal foliation of the target surface M_t. The signed distance along horizontal leaves will be chosen so that (5.1) holds for all piecewise smooth curves γ connecting any two points. Since the horizontal straightline flow may not be defined for all times, one of the complications we will address is to ensure that we can move points horizontally by the required amount.

Proof of Proposition 5.1. We begin by proving the existence of ψ_t. Let τ be a triangulation of S obtained as the pullback via φ_0 of a geodesic triangulation on M_0. Since we will be using the opposite point map defined in §4.2, we will take τ to be given by adding horizontal diagonals to the polygons of an APD, as in the proof of Proposition 4.12. As in §2.2, let $U_\tau \subset H^1(S, \Sigma; \mathbb{R}^2)$ be the set of cohomology classes which assign to the edges of this triangulation vectors leading to positively oriented triangles, and let V_τ be the corresponding open subset of $\mathcal{H}_{\mathrm{m}}$

of marked translation surfaces compatible with this triangulation. For a sufficiently small $\varepsilon > 0$, in the interval $I = [0, \varepsilon]$ we have

$$\{\mathrm{trem}_{t\beta}(q) : t \in I\} \subset V_\tau, \tag{5.2}$$

and we will first prove the existence of ψ_t for $t \in I$ where I satisfies (5.2). The existence for all t then follows by composing maps defined on small intervals, as in the first paragraph of the proof of Proposition 4.12. With this in mind we can re-parameterize I, and replace β by its multiple by a positive constant, to assume that $t = 1$, $I = [0, 1]$ and $\widetilde{q}_0, \widetilde{q}_1 \in V_\tau$.

Let τ_0, τ_1 denote respectively the pushforward of the triangulation τ to M_0, M_1, and let $\varphi : M_0 \to M_1$ be the comparison map which is affine and orientation-preserving on triangles of τ as in §2.2. Thus φ sends τ_0 to τ_1. The definition of tremors gives us (5.1) with φ in place of ψ_t, and for any path γ on M_0 with endpoints in Σ. Recall from Proposition 4.12 that φ takes the horizontal foliation of M to the horizontal foliation of M' and takes $(dy)_{\widetilde{q}_0}$ to $(dy)_{\widetilde{q}_1}$. Also φ preserves the rightward orientation on horizontal lines. We will construct the homeomorphism ψ by composing φ with a map which moves a point in M_1 along its horizontal leaf. Recall that $\Upsilon^{(x)}(s)$ denotes the image of $x \in M_1$ under horizontal straightline flow, to (signed) distance s. With this notation, for a continuous function $\bar{s} : M_0 \to \mathbb{R}$, we write

$$\psi(p) \stackrel{\mathrm{def}}{=} \Upsilon^{(\varphi(p))}(\bar{s}(p)) \tag{5.3}$$

(where it is implicit, and will be guaranteed later by the choice of the function $\bar{s}$, that $\psi(p)$ is well-defined, i.e., the horizontal straightline motion starting at $\varphi(p)$ is defined to time $\bar{s}(p)$). That is, for $p \in M_0$, $\psi(p)$ is obtained by motion along the horizontal leaf of $\varphi(p)$ in M_1, by the signed distance $\bar{s}(p)$; see Figure 6. Clearly such a map will satisfy the second equation in (5.1), and $\bar{s}(p)$ will be chosen so that the first equation in (5.1) holds as well. The construction of $\bar{s}(p)$ and proof that it has the desired properties will be broken up into several lemmas.

We begin by specifying the values of the function $\bar{s}$, on each of the edges of the triangulation τ_0 of M_0. On the horizontal edges of the triangulation, we set $\bar{s}$ equal to zero. Let $\sigma : [0, 1] \to M_0$ denote an affine parameterization of a non-horizontal edge of τ_0. We define

$$\bar{s}(\sigma(t)) = \int_{\sigma(0)}^{\sigma(t)} d\nu - t \int_{\sigma(0)}^{\sigma(1)} d\nu, \tag{5.4}$$

where the integrals are taken along the path σ between the indicated limits.

Lemma 5.4. *The following hold for each edge σ:*

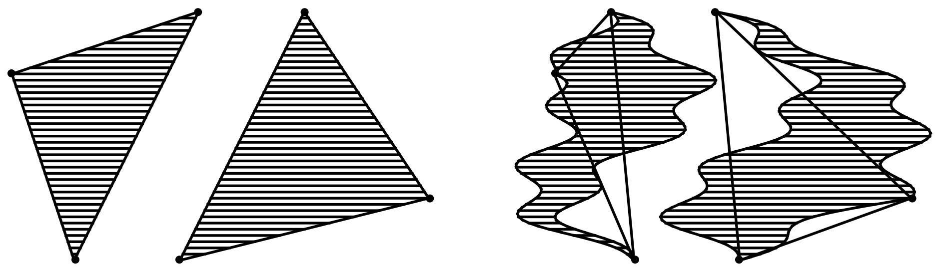

FIGURE 6. The images of two triangles in τ under ψ and φ.

(a) *The definition* (5.4) *does not depend on the choice of orientation for* σ*; that is, defines the same function on the edge, if one uses* $\bar{\sigma}(1-t)$ *instead of* $\sigma(t)$*.*
(b) *The map* $t \mapsto \bar{s}(\sigma(t))$ *is continuous.*
(c) $\bar{s}(\sigma(0)) = \bar{s}(\sigma(1)) = 0$.

Proof. Assertion (a) follows from a computation using (5.1) for the curve σ; we leave this to the reader. Assertion (b) follows from the fact that ν is non-atomic. Assertion (c) follows from (5.4). $\triangle$

We now check that when using (5.4), a map defined via (5.3) has the required property of preserving distances along horizontal lines, for two points on opposite sides of a polygon P of the APD. To this end, let opp_P be the opposite point map as in §4.2, and let σ, σ' denote two affine parameterizations of sides of P, so that

$$x = \sigma(t) \in \partial P, \quad y = \mathrm{opp}_P(x) = \sigma'(t')$$

for appropriate $t, t' \in [0,1]$. Let d_0, d_1 denote respectively the horizontal signed distance between x, y and $\varphi(x), \varphi(y)$ in M_0, M_1; that is,

$$y = \Upsilon^{(x)}(d_0), \qquad \varphi(y) = \Upsilon^{(\varphi(x))}(d_1).$$

Here we swap if necessary the roles of x and y to assume that $d_0 > 0$, for the definition of d_i we refer to straightline flow on M_i, and in case the horizontal trajectory of x is periodic we use the parameterization of paths through the interior of P and $\varphi(P)$. Note that the straightline flow from x to y is well-defined by definition of opp_P, and straightline flow from $\varphi(x)$ to $\varphi(y)$ is well-defined since φ maps horizontal segments to horizontal segments and preserves their orientation.

Lemma 5.5. *We have* $d_0 = d_1 - \bar{s}(x) + \bar{s}(y)$*.*

Note that Lemma 5.5 does not assume that ψ as in (5.3) is well-defined; but if one assumes that ψ is well-defined, one concludes from the Lemma that d_0, the signed horizontal distance between x and y, is the same as d_1, the signed horizontal distance between $\psi(x)$ and $\psi(y)$.

Proof. By decomposing P into triangles, we can assume with no loss of generality that P is a triangle. We can further assume, using Lemma 5.4(a), that P has one vertex at ξ, where σ and σ' are affine parameterizations of opposite edges of P with $\sigma(0) = \sigma'(0) = \xi$ and $\mathrm{hol}^{(y)}(\sigma) < \mathrm{hol}^{(y)}(\sigma')$, as shown in Figure 7. Let α denote a path from x to y along the horizontal segment through P. Then α is homotopic to the path from x to y along the edges of P, and hence

$$d_0 = \mathrm{hol}_0(\alpha) = t' \,\mathrm{hol}_0^{(x)}(\sigma') - t\,\mathrm{hol}_0^{(x)}(\sigma). \tag{5.5}$$

Similarly

$$d_1 = \mathrm{hol}_1(\varphi(\alpha)) = t' \,\mathrm{hol}_1^{(x)}(\varphi(\sigma')) - t\,\mathrm{hol}_1^{(x)}(\varphi(\sigma)). \tag{5.6}$$

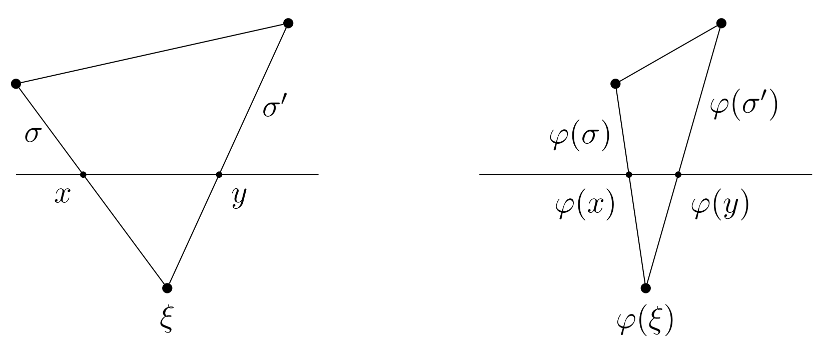

FIGURE 7. Paths used in the proof of Lemma 5.5.

Applying the opposite point invariance property (4.11), we obtain

$$\int_{\sigma(0)}^{\sigma(t)} \nu = \int_{\sigma'(0)}^{\sigma'(t')} \nu. \tag{5.7}$$

By (5.1) (which holds for the saddle connections σ and σ'), along with (5.5) and (5.6), we get

$$\begin{aligned} d_1 - d_0 =& t' \left(\mathrm{hol}_1^{(x)}(\varphi(\sigma')) - \mathrm{hol}_0^{(x)}(\sigma')\right) - t\left(\mathrm{hol}_1^{(x)}(\varphi(\sigma)) - \mathrm{hol}_0^{(x)}(\sigma)\right) \\ =& t' \int_{\sigma'(0)}^{\sigma'(1)} \nu - t\int_{\sigma(0)}^{\sigma(1)} \nu, \end{aligned}$$

and by (5.4) and (5.7) we also get

$$\bar{s}(x) - \bar{s}(y) = \bar{s}(\sigma(t)) - \bar{s}(\sigma'(t')) = t' \int_{\sigma'(0)}^{\sigma'(1)} \nu - t\int_{\sigma(0)}^{\sigma(1)} \nu.$$

This gives the required identity. △

We now extend $\bar{s}$ by affine interpolation to the interiors of triangles. For any point $p \in M_0$, let x, y denote the two intersections of the horizontal leaf of p with ∂P, so that $y = \mathrm{opp}_P(x)$, and let d_0 be as above. Then there is $t \in (0,1)$ so that $p = \Upsilon^{(x)}(td_0) = \Upsilon^{(y)}((t-1)d_0)$. We define

$$\bar{s}(p) \stackrel{\mathrm{def}}{=} (1-t)\bar{s}(x) + t\bar{s}(y). \tag{5.8}$$

Since φ and (5.8) are both affine and orientation-preserving, the conclusion of Lemma 5.5 continues to hold for any two points x' and y' which are on a horizontal segment passing from side to side of a polygon of the APD, and with d_0, d_1 measuring signed distances defined using these points.

With this extended definition we claim:

Lemma 5.6. *For any $p \in M_0$, the horizontal straightline flow from $\varphi(p)$ to signed distance $\bar{s}(p)$ on M_1 is defined, and thus the map ψ defined via (5.3) is well-defined.*

Proof. Suppose by way of contradiction that for some $p \in M_0$, the straightline flow trajectory from $\varphi(p)$ to signed distance $\bar{s}(p)$ is not defined. We know from Lemma 5.4(c) that p is not a singular point. Assume with no loss of generality that $\bar{s}(p) > 0$; our assumption means that for some $0 < t_{\mathrm{crit}} \leqslant \bar{s}(p)$ we have $\{\Upsilon^{(\varphi(p))}(t) : 0 \leqslant t < t_{\mathrm{crit}}\}$ is well-defined but the one-sided limit

$$\xi \stackrel{\mathrm{def}}{=} \lim_{t \to t_{\mathrm{crit}}^-} \Upsilon^{(\varphi(p))}(t)$$

is a singular point on M_1. Let k be the number of times the trajectory $\left\{\Upsilon^{(\varphi(p))}(t) : 0 < t < t_{\mathrm{crit}}\right\}$ crosses edges of the APD. We can choose p with the above properties so that k is minimal. We will reach a contradiction in both cases $k = 0$ and $k > 0$.

If $k = 0$ then there is a polygon Δ of the APD on M_0 such that $\xi = \varphi(\xi)$ is a vertex of $\varphi(\Delta)$ and $p \in \Delta$. Let x be the point on $\partial\Delta$ which is opposite to ξ, so that p is on the segment from x to ξ. We define d_0, d_1 as above, with p and ξ playing respectively the roles of x and y. Since $t_{\mathrm{crit}} > 0$ we must have that $\varphi(p)$ is to the left of ξ in the polygon $\varphi(\Delta)$, and hence $d_1 > 0$. Since φ preserves the orientation of horizontal lines we must have $d_0 > 0$. By our contradiction assumption, $t_{\mathrm{crit}} \geqslant d_1$. Finally $\bar{s}(\xi) = 0$ by Lemma 5.4(c). Putting these together and using Lemma 5.5 we get the contradiction

$$\bar{s}(p) \geqslant t_{\mathrm{crit}} \geqslant d_1 = d_0 + \bar{s}(p) - \bar{s}(\xi) > \bar{s}(p).$$

Now suppose $k > 0$. Let Δ be a polygon of the APD containing p and let y' be the endpoint of the rightward oriented segment from p to $\partial\Delta$. Let d_0, d_1 be defined as above, using the points p and y' instead of x and y. We compute the numbers t'_{crit}, k' corresponding to y' instead of p. We have $t'_{\mathrm{crit}} = t_{\mathrm{crit}} - d_1$ and $k' = k - 1$. Using Lemma 5.5 we have

$$\bar{s}(y') = d_0 - d_1 + \bar{s}(p) \geqslant d_0 - d_1 + t_{\mathrm{crit}} = d_0 + t'_{\mathrm{crit}} \geqslant t'_{\mathrm{crit}}.$$

This implies that y' also satisfies that the straightline flow from $\varphi(y)$ to distance $\bar{s}(y)$ is not defined, and contradicts the minimality in the choice of p. $\triangle$

Lemma 5.7. *The map ψ is a homeomorphism which is isotopic rel Σ to φ and satisfies*

$$\Upsilon^{(\psi(p))}(t) = \psi(\Upsilon^{(p)}(t)) \tag{5.9}$$

for any $p \in M_0$ and any $t \in \mathbb{R}$ for which one (hence both) of these terms is defined.

Proof. The function $x \mapsto \bar{s}(x)$ is continuous by Lemma 5.4 and (5.8). This implies that ψ is continuous. Since M_0 is compact, in order to show that ψ is a homeomorphism, it is enough to show that it is bijective. To this end, we first note that (5.9) holds. Indeed by Lemma 5.5, (5.9) holds for any interval I for which the path $\{\Upsilon^{(p)}(t) : t \in I\}$ is contained in a polygon of the APD, and thus, by induction on the number of times a horizontal straightline segment from p to $\Upsilon^{(p)}(t)$ crosses edges of the APD, it holds for all t.

It follows from Lemma 5.4(c) that $\psi|_\Sigma = \varphi|_\Sigma$ and hence that ψ is a label-preserving bijection on Σ. It follows from (5.9) that the restriction of ψ to a horizontal straightline flow trajectory is an isometry (with respect to the metric induced by the 1-form dx). Since the restriction of ψ to a horizontal trajectory is an isometry mapping singular points to singular points, the restriction of ψ to any horizontal trajectory is a bijection. Moreover, by (5.3), the image of a horizontal trajectory under ψ is the same as its image under φ, and since φ is a bijection, we obtain that ψ is also a bijection.

Consider the one-parameter family of maps

$$g^{(r)}(x) \stackrel{\mathrm{def}}{=} \Upsilon^{(\varphi(x))}(r\bar{s}(x)) \quad (r \in [0,1]).$$

Clearly, this family gives a homotopy between φ and ψ fixing Σ pointwise. To see that each $g^{(r)}$ is a homeomorphism, arguing as before we see that it suffices to show that it is bijective on each horizontal straightline flow trajectory. For such a trajectory, it is indeed bijective as it is a linear homotopy between order-preserving homeomorphisms. This shows that φ and ψ are isotopic rel Σ. △

Lemma 5.8. *The map ψ satisfies* (5.1).

Proof. We first claim that it is enough to prove the claim for paths γ whose image is contained in edges of the APD. Indeed, if (5.1) holds for two paths it holds for their concatenation. Thus, in order to prove the result for an arbitrary path, it suffices to prove the result for a path γ contained in one polygon Δ of the APD. Let γ' is obtained by sliding every point in γ to an edge σ of Δ; that is, $\gamma'(t) = \Upsilon^{(\gamma(t))}(\rho(t))$, where $\rho(t)$ is the horizontal signed distance from $\gamma(t)$ to σ. Using formula (5.9), we see that $\psi(\gamma')$ is obtained from $\psi(\gamma)$ by sliding horizontally by the same amount $\rho(t)$. From this one easily sees that if (5.1) holds for γ', it also holds for γ.

It remains to check that (5.1) holds for paths whose image is contained in an edge $\sigma \subset \partial\Delta$. This follows easily from the definition (5.4) of $\bar{s}$ along edges of Δ; we leave the verification to the reader. △

Completing the proof of Proposition 5.1. Lemmas 5.7 – 5.8 establish the existence of ψ with the required properties. We complete the proof by proving uniqueness. Let ψ and ψ' be isotopic maps from M_0 to M_1 satisfying (5.1) for arbitrary paths. This equation implies that $\psi^{-1} \circ \psi'$ preserves the holonomy of paths and is thus a translation equivalence. Since the maps ψ and ψ' are isotopic the map $\psi^{-1} \circ \psi'$ is isotopic to the identity. The identity map is the unique translation equivalence of M_0 isotopic to the identity so we have $\psi^{-1} \circ \psi' = I$ and $\psi = \psi'$. □

Proposition 5.9. *Suppose that for $i = 0, 1$, $\widetilde{q}_i$ are marked translation surfaces represented by the marking maps $\varphi_i : S \to M_i$. Suppose $\widetilde{q}_0, \widetilde{q}_1$ belong to the same horospherical leaf, and there is a homeomorphism $\psi : M_0 \to M_1$ isotopic to $\varphi_1 \circ \varphi_0^{-1}$ for which the conclusion of Lemma 5.7 holds. Then $\widetilde{q}_1 = \mathrm{trem}_\beta(\widetilde{q}_0)$ for some non-atomic foliation cocycle $\beta \in \mathcal{T}_{q_0}$.*

Since we will not be using this result in this paper, we only outline the argument.

Sketch of proof. Since $\widetilde{q}_0, \widetilde{q}_1$ belong to the same horospherical leaf and ψ is isotopic to $\varphi_1 \circ \varphi_0^{-1}$, $\mathrm{hol}^{(y)}_{M_0} = \mathrm{hol}^{(y)}_{M_1}(\psi(\gamma))$ for any path γ joining singular points. We will define a non-atomic signed transverse measure ν satisfying

$$\mathrm{hol}^{(x)}_{M_1}(\psi(\gamma)) = \mathrm{hol}^{(x)}_{M_0}(\gamma) + \int_\gamma \nu. \tag{5.10}$$

This will show that (5.1) holds (with $t = 1, \psi = \psi_1$), for any path joining singular points, thus showing that $\widetilde{q}_1 = \mathrm{trem}_{\beta_\nu}(\widetilde{q}_0)$.

Let $\varepsilon > 0$ be such that horizontal straightline flow is defined on all points of both γ and $\psi(\gamma)$, to time s, for all $|s| < \varepsilon$. We define the *horizontal diameter* of a topological disc in a translation surface to be the supremum of horizontal holonomies of any curve contained in $\mathcal{U}$. We can cover the image of γ by topological discs $\mathcal{U}$ such that the horizontal diameter of both $\mathcal{U}$ and $\psi(\mathcal{U})$ is smaller than ε. The subarcs γ' of γ contained in such a topological disc $\mathcal{U}$ generate the Borel σ-algebra on γ. For each such γ' we define

$$\int_{\gamma'} \nu = \mathrm{hol}^{(x)}_{M_1}(\psi(\gamma')) - \mathrm{hol}^{(x)}_{M_0}(\gamma').$$

Using the Carathéodory extension theorem, one can show that this defines ν as a signed measure on γ. By linearity, ν satisfies (5.10), and one can check using (5.9) that ν defined in this way is invariant under holonomy along horizontal lines, and thus defines a transverse measure. $\square$

Remark 5.10. *It is instructive to compare our discussion of tremors, using Proposition 5.1, with the discussion of the Rel deformations in* [BSW, §6]*. Namely in* [BSW, Pf. of Thm. 6.1]*, a map* $\bar{f}_t : M_0 \to \mathrm{Rel}_t(M_0)$ *is constructed but the definition of this map involves some arbitrary choices. In particular it is not unique and is not naturally contained in a continuous one-parameter family of maps.*

6. Properties of tremors

In this section we will derive further properties of tremors.

6.1. Composing tremors and other maps.

Recall from Proposition 4.14 that we have

$$\mathrm{trem}_{\beta_1+\beta_2}(q) = \mathrm{trem}_{\beta_2}(\mathrm{trem}_{\beta_1}(q)). \tag{6.1}$$

Here, and in the rest of this section, we have in mind the identification of $\mathcal{T}_{\widetilde{q}_1}$ with $\mathcal{T}_{\widetilde{q}_2}$, for all $\widetilde{q}_1, \widetilde{q}_2$ in the same horospherical leaf; in particular, on the left-hand side of (6.1), β_2 belongs to $\mathcal{T}_q$, and on the right-hand side, to $\mathcal{T}_{q_1}$ for $q_1 = \mathrm{trem}_{\beta_1}(q)$, and these spaces are identified by choosing appropriate lifts $\widetilde{q}, \widetilde{q}_1$. With this convention recall also from (4.13) that $\mathrm{trem}_\beta(uq) = u\,\mathrm{trem}_\beta(q)$, for any $u \in U$.

Note that the identification of $\mathcal{T}_{\widetilde{q}_1}$ with $\mathcal{T}_{\widetilde{q}_2}$ in Proposition 4.12 need not send balanced tremors to balanced tremors. However, the horocycle flow commutes with horizontal straightline flows, and therefore for $u \in$

U and $\beta \in \mathcal{T}_q \cong \mathcal{T}_{uq}$, we have $L_q(\beta) = L_{uq}(\beta)$. From (6.1) and (4.13) we deduce:

Corollary 6.1. *Let $\beta \in \mathcal{T}_q$ and $s \stackrel{\text{def}}{=} L_q(\beta)$. Then*

- *$\beta - s(dy)_q \in \mathcal{T}_{u_s q}$ is balanced.*
- *If β is balanced in $\mathcal{T}_q$ then β is balanced in $\mathcal{T}_{uq}$, for any $u \in U$.*

Recall that $B \subset G$ denotes the upper triangular group. We now discuss the interaction between the B-action and tremors. Note that while an element $\mathbf{b} \in B$ maps horospherical leaves to horospherical leaves, it does not necessarily preserve individual horospherical leaves, so we cannot use Proposition 4.12 to identify $\mathcal{T}_q$ with $\mathcal{T}_{\mathbf{b}\tilde{q}}$. Instead, we use the derivative of the affine comparison map ψ_b defined in §2.4 to identify $\mathcal{T}_{\tilde{q}}$ with $\mathcal{T}_{\mathbf{b}q}$. Note that the subgroup of B preserving horospherical leaves is U, and for $u \in U$ the map ψ_u acts on $H^1(S, \Sigma; \mathbb{R}_x)$ trivially, and thus this identification coincides with the identfication via parallel transport that is used in Proposition 4.12.

The interaction of tremors with the B-action is as follows.

Proposition 6.2. *Let $q \in \mathcal{H}$ and let*

$$\mathbf{b} = \begin{pmatrix} a & z \\ 0 & a^{-1} \end{pmatrix} \in B, \text{ with } a = a(\mathbf{b}) > 0. \tag{6.2}$$

Let M_q and $M_{\mathbf{b}q}$ be the underlying surfaces, and let $\tilde{q} \in \pi^{-1}(q)$. The above identification $\mathcal{T}_{\tilde{q}} \to \mathcal{T}_{\mathbf{b}\tilde{q}}$ multiplies the canonical transverse measure dy by a^{-1} (where $a = a(\mathbf{b})$ is as in (6.2)), preserves the subsets of atomic and balanced foliation cocycles, and maps c-absolutely continuous foliation cocycles to ac-absolutely continuous foliation cocycles. Furthermore,

$$\mathbf{b}\,\mathrm{trem}_\beta(q) = \mathrm{trem}_{a\cdot\beta}(\mathbf{b}q), \qquad \mathrm{Dom}(\mathbf{b}q, \beta) = a^{-1} \cdot \mathrm{Dom}(q, \beta). \tag{6.3}$$

Proof. Let $q_1 = \mathbf{b}q$, denote the underlying surfaces by $M = M_q$, $M_1 = M_{q_1}$ and write $\psi = \psi_{\mathbf{b}} : M \to M_1$ for the affine comparison map. Since the linear action of $\mathbf{b}$ on $\mathbb{R}^2$ preserves horizontal lines, ψ sends the horizontal foliation on M to the horizontal foliation on M_1. As in Proposition 4.12, ψ sends transverse measures to transverse measures, non-atomic transverse measures to non-atomic transverse measures, and the induced map ψ^* on cohomology sends $\mathcal{T}_q$ to $\mathcal{T}_{q_1}$ and C_q^+ to $C_{q_1}^+$. Since ψ is an affine map with derivative $\mathbf{b}$, the canonical transverse measure $(dy)_q$ on M_q is sent to its scalar multiple $a(\mathbf{b})^{-1} \cdot (dy)_{q_1}$ on M_{q_1}. Hence c-absolutely continuous foliation cocycles are mapped to ac-absolutely continuous foliation cocycles. To prove equation (6.3), let $t \mapsto \tilde{q}_t$ be the affine geodesic in $\mathcal{H}_{\mathrm{m}}$ with $\tilde{q}_0 = \tilde{q}$ and $\frac{d}{dt}|_{t=0}\tilde{q}_t = \beta$, so that $\widetilde{q}_1 = \mathrm{trem}_\beta(\widetilde{q})$. The new path $t \mapsto \hat{q}_t = \mathbf{b}\widetilde{q}_t$ is also an affine geodesic and satisfies $\hat{q}_0 = \mathbf{b}\widetilde{q}$. Now (6.3) follows from the fact that $\frac{d}{dt}|_{t=0}\hat{q}_t = a(\mathbf{b}) \cdot \beta$, since $\mathcal{T}_q$ is embedded in the real space $H^1(S, \Sigma; \mathbb{R}_x)$.

We now show that our affine comparison map sends $\mathcal{T}_q^{(0)}$ to $\mathcal{T}_{q_1}^{(0)}$, that is, preserves balanced foliation cocycles. Since the horizontal direction is fixed by $\mathbf{b}$ and scaled by a factor of $a = a(\mathbf{b})$, $(dx)_{q_1}$ is obtained from $(dx)_q$ by multiplication by a. Now suppose $\beta \in \mathcal{T}_q^{(0)}$ so that $\mathrm{hol}_q^{(x)} \cup \beta = 0$. By naturality of the cup product we get

$$0 = a^{-1}\mathrm{hol}_q^{(x)} \cup \beta = \left(\psi^{-1}\right)^* \left(\mathrm{hol}_{q_1}^{(x)} \cup \psi^*\beta\right) = \mathrm{hol}_{q_1}^{(x)} \cup \psi^*\beta.$$

□

6.2. **Relations between tremors and other maps.** We will now prove commutation and normalization relations between tremors and other maps, which extend those in Proposition 6.2. We will simultaneously discuss the interaction of tremors with the action of B, all possible tremors for a fixed surface, real-Rel deformations, and the $\mathbb{R}^*$-action on the space of tremors.

We will use the notation and results of [BSW] in order to discuss real-Rel deformations. Let Z be the subspace of $H^1(S, \Sigma; \mathbb{R}_x)$ of cohomology classes which evaluate to zero on closed loops. Thus Z represents the subspace of real rel deformations of surfaces in $\mathcal{H}$ (see [BSW, §3] for more information).

Let $q \in \mathcal{H}$, M_q the underlying surface, $\varphi : S \to M_q$ a marking map and $\widetilde{q} \in \mathcal{H}_{\mathrm{m}}$ the corresponding element in $\pi^{-1}(q)$. We define semi-direct products

$$S_1^{(\varphi)} \stackrel{\mathrm{def}}{=} B \ltimes \mathcal{T}_{\widetilde{q}}, \qquad S_2^{(\varphi)} \stackrel{\mathrm{def}}{=} B \ltimes (\mathcal{T}_{\widetilde{q}} \oplus Z),$$

where the group structure on $S_2^{(\varphi)}$ is defined by

$$(b_1, v_1, z_1).(b_2, v_2, z_2) = \left(b_1 b_2, a^{-2}(b_2)v_1 + v_2, a^{-1}(b_2)z_1 + z_2\right),$$

where

$$b_i \in B, \ v_i \in \mathcal{T}_{\widetilde{q}}, \ z_i \in Z,$$

$a(b)$ is defined in (6.2). Also define the group structure on $S_1^{(\varphi)}$ by thinking of it as a subgroup of $S_2^{(\varphi)}$. Define the quotient semidirect products

$$\bar{S}_1^{(\varphi)} \stackrel{\mathrm{def}}{=} S_1^{(\varphi)}/\sim, \quad \bar{S}_2^{(\varphi)} \stackrel{\mathrm{def}}{=} S_2^{(\varphi)}/\sim,$$

where $\sim$ denotes the equivalence relation $B \ni u_s \sim s \cdot \mathrm{hol}_{\widetilde{q}}^{(y)} \in \mathcal{T}_{\widetilde{q}}$.

With this notation we have the following:

Proposition 6.3. *Let q, M_q, φ and $\widetilde{q}$ be as above, and suppose M_q has no horizontal saddle connections (so that tremors and real-Rel deformations have the maximal domain of definition). Define*

$$\Theta_1^{(\varphi)} : S_1^{(\varphi)} \to \mathcal{H}_{\mathrm{m}}, \quad (b, \beta) \mapsto b \, \mathrm{trem}_\beta(\widetilde{q})$$

and

$$\Theta_2^{(\varphi)} : S_2^{(\varphi)} \to \mathcal{H}_{\mathrm{m}}, \quad (b, \beta, z) \mapsto b \, \mathrm{Rel}_z \, \mathrm{trem}_\beta(\widetilde{q}).$$

Then the maps $\Theta_i^{(\varphi)}$ obey a 'group action' law

$$\Theta_i^{\left(\psi_{g_2} \circ \varphi\right)}(g_1) = \Theta_i^{(\varphi)}(g_1 g_2) \quad (i = 1, 2). \tag{6.4}$$

Moreover these maps are continuous, and descend to well-defined immersions $\bar{\Theta}_i^{(\varphi)} : \bar{S}_i^{(\varphi)} \to \mathcal{H}_{\mathrm{m}}$.

We will only prove the statement corresponding to $i = 1$. The case $i = 2$ will not be needed in the sequel and we will leave it to the reader. Specifically, in case $i = 2$, the comparison map ψ_{g_2} appearing in (6.4) is defined up to isotopy in [BSW], see the map $\bar{f}_t$ in Remark 5.10.

Proof. The fact that the map $\Theta_1^{(\varphi)}$ satisfies the group action law (6.4) with respect to the group structure on $S_1^{(\varphi)}$ is immediate from Propositions 4.14 and 6.2. The fact that $\bar{\Theta}_1^{(\varphi)}$ is well-defined on $\bar{S}_1^{(\varphi)}$ follows from (4.9) and (4.13). The maps $\Theta_1^{(\varphi)}$, $\bar{\Theta}_1^{(\varphi)}$ are continuous because they are given as affine geodesics, and because of general facts on ordinary differential equations. The fact that $\bar{\Theta}_1^{(\varphi)}$ is an immersion can be proved by showing that when g_1, g_2 are two elements of $S_1^{(\varphi)}$ that project to distinct elements of $\bar{S}_1^{(\varphi)}$, then $\mathrm{dev}\left(\bar{\Theta}_1^{(\varphi)}(q_i)\right)$ are distinct, i.e. the operations have a different effect in period coordinates. $\square$

There is also a natural action of the multiplicative group $\mathbb{R}^* = \mathbb{R} \smallsetminus \{0\}$ on $\mathcal{T}_{\widetilde{q}}$ given by $(\rho, \beta) \mapsto \rho\beta$, where $\rho \in \mathbb{R}^*$ and $\beta \in \mathcal{T}_{\widetilde{q}}$. This action preserves the set of balanced tremors $\mathcal{T}_{\widetilde{q}}^{(0)}$. By Proposition 4.12 and Proposition 6.2, $\mathcal{T}_{\widetilde{q}}^{(0)}$ is a normal subgroup of $\mathcal{T}_{\widetilde{q}}$ and $S_1^{(\varphi)}$. It is not hard to show using Corollary 6.1 that $B \ltimes \mathcal{T}_{\widetilde{q}}^{(0)}$ is a normal subgroup of $S_1^{(\varphi)}$ isomorphic to the group $\bar{S}_1^{(\varphi)}$. We define a third semidirect product $S_3^{(\varphi)} \stackrel{\mathrm{def}}{=} (\mathbb{R}^* \times B) \ltimes \mathcal{T}_{\widetilde{q}}^{(0)}$, where $\mathbb{R}^*$ acts on $\mathcal{T}_{\widetilde{q}}^{(0)}$ by scalar multiplication and B acts on $\mathcal{T}_{\widetilde{q}}^{(0)}$ as above. Arguing as in the proof of Proposition 6.3 we obtain:

Proposition 6.4. *Let q, M_q, φ, $\widetilde{q}$ be as in Proposition 6.3. Then the map*

$$S_3^{(\varphi)} \to \mathcal{H}_{\mathrm{m}}, \qquad (\rho, b, \beta) \mapsto b\,\mathrm{trem}_{\rho\beta}(\widetilde{q}),$$

obeys the group action law and is a continuous immersion.

Remark 6.5. *Note that (as reflected by the notation) the objects $S_i^{(\varphi)}$ and $\Theta_i^{(\varphi)}$ discussed above depend on the choice of a marking map. This is needed because the marking map was used to identify $\mathcal{T}_q$ for different surfaces q. On the other hand (4.13) makes sense irrespective of a choice of a marking map.*

Remark 6.6. *In addition to the deformations listed above there is another deformation that could be considered. In the spirit of* [Ve2, §1] *(see also* [CMW, §2.1]*), for each horizontally invariant fully supported probability measure ν on M_q, there is a topological conjugacy sending ν to Lebesgue measure (on a different surface $M_{q'}$). This topological conjugacy also induces a comparison map $M_q \to M_{q'}$ and corresponding maps on foliation cocycles and on the resulting tremors, and it is possible to write down the resulting group-action law which the map obeys when combined with those of Propositions 6.3 and 6.4. This will not play a role in this paper and is left to the assiduous reader.*

6.3. **Tremors and sup-norm distance.** Let dist denote the sup-norm distance as in §2.6.

Proposition 6.7. *If $q \in \mathcal{H}$, ν is a non-atomic absolutely continuous signed transverse measure on the horizontal foliation of M_q, and $\beta = \beta_\nu$ then*

$$\mathrm{dist}(q, \mathrm{trem}_\beta(q)) \leqslant \|\nu\|_{RN}. \tag{6.5}$$

Proof. Let $q_1 = \mathrm{trem}_\beta(q)$ and let dy be the canonical transverse measure on q. Let

$$\{\gamma(t) : t \in [0,1]\}, \quad \text{where } \gamma_t \stackrel{\mathrm{def}}{=} \mathrm{trem}_{t\beta}(q),$$

be the affine geodesic from q to q_1. The tangent vector of γ is represented by the class β, and by specifying a marking map $\varphi_0 : S \to M_q$ we can lift the path to $\mathcal{H}_{\mathrm{m}}$, and find $\widetilde{q}$, $\widetilde{q}_1$ and $\widetilde{\gamma}(t)$, $t \in [0,1]$ so that

$$\pi(\widetilde{q}_1) = q_1, \ \pi(\widetilde{\gamma}(t)) = \gamma(t) \ \text{ with } \ \widetilde{\gamma}(0) = \widetilde{q}, \ \ \widetilde{\gamma}(1) = \widetilde{q}_1,$$

and $\widetilde{\gamma}(t)$ satisfies

$$\mathrm{dev}(\widetilde{\gamma}(t)) = \mathrm{dev}(\widetilde{q}) + t\bar{\beta}, \ \text{ where } \bar{\beta} = \left(\varphi_0^{-1}\right)^* \beta \in H^1(S, \Sigma; \mathbb{R}).$$

We will use this path in (2.10) to give an upper bound on the distance from q and q_1. For each $t \in [0,1]$, write $q_t = \gamma(t)$ and denote the underlying surface by M_t. Recall that we denote the collection of saddle connections on a surface q by Λ_q. We let Λ'_q denote the saddle connections in Λ_q which are not horizontal on M_q; for horizontal saddle connections σ we have $\bar{\beta}(\sigma) = 0$. For any $\sigma \in \Lambda_q$, we write $\mathrm{hol}_{q_t}(\sigma)$ component-wise as $(x_{q_t}(\sigma), y_{q_t}(\sigma))$, so that

$$\ell_{q_t}(\sigma) = \|\mathrm{hol}_{q_t}(\sigma)\| \geqslant |y_{q_t}(\sigma)|. \tag{6.6}$$

By Proposition 4.12, we obtain transverse measures ν_t and $(dy)_t$ on each q_t. Using this, for all $t \in [0,1]$ we have

$$\begin{aligned}\|\gamma'(t)\|_{\gamma(t)} = \|\bar{\beta}\|_{\tilde{q}_t} &= \sup_{\sigma\in\Lambda_{\tilde{q}_t}} \frac{\|\bar{\beta}(\sigma)\|}{\ell_{\tilde{q}_t}(\sigma)} = \sup_{\sigma\in\Lambda'_{\tilde{q}_t}} \frac{\|\bar{\beta}(\sigma)\|}{\ell_{\tilde{q}_t}(\sigma)} \\ &\overset{(6.6)}{\leqslant} \sup_{\sigma\in\Lambda_{\tilde{q}_t}} \frac{\|\bar{\beta}(\sigma)\|}{|y_{\tilde{q}_t}(\sigma)|} = \sup_{\sigma\in\Lambda_{\tilde{q}_t}} \frac{|\int_\sigma d\nu_t|}{|\int_\sigma (dy)_t|} \overset{(4.3)}{\leqslant} \|\nu\|_{RN}.\end{aligned}$$

Integrating w.r.t. $t \in [0,1]$ in (2.10) we obtain the bound (6.5). ☐

By moving along a horocycle orbit, small absolutely continuous tremors can be realized by small *balanced* tremors. Namely:

Corollary 6.8. *With the notations and assumptions of Proposition 6.7, there is $q' \in Uq$ and $\beta' \in \mathcal{T}^{(0)}_{q'}$ with $|L|_{q'}(\beta') \leqslant 2\|\nu\|_{RN}$ and*

$$\mathrm{trem}_\beta(q) = \mathrm{trem}_{\beta'}(q'). \tag{6.7}$$

Proof. This follows from Corollary 6.1, (6.5), and the triangle inequality. ☐

7. Proof of Theorem 1.5

We will now deduce the three assertions of Theorem 1.5 from the results of the preceding sections. Throughout this section we write q_1 for $\mathrm{trem}_\beta(q)$ where $\beta \in \mathcal{T}(q)$. The first assertion of the Theorem is that, for β absolutely continuous, the distance between $u_s q$ and $u_s q_1$ remains bounded.

Proof of Theorem 1.5(i). Let $\beta = \beta_\nu$ be the signed foliation cocycle corresponding to a signed transverse measure ν. We first claim that there is no loss of generality in assuming that ν is c-absolutely continuous for some $c > 0$. To see this, write $\nu = \nu_1 + \nu_2$ where ν_1 is aperiodic and ν_2 is supported on horizontal cylinders. By Lemma 4.7, β_{ν_1} is c_1-absolutely continuous for some c_1. Now modify ν_2 so that for any horizontal cylinder C on M_q, the restriction of ν_2 to C is equal to $a_C\, dy|_C$ for some positive constant a_C. Such a modification has no effect on β_{ν_2}, and will thus have no effect on $\beta = \beta_{\nu_1} + \beta_{\nu_2}$. Thus, if

$c_2 = \max_C a_C$, then (after the modification), $\|\nu\|_{RN} \leqslant c_1 + c_2$. Now using (4.13) and Proposition 6.7, we see that the left-hand side of (1.6) is bounded by $c_1 + c_2$. $\square$

The second assertion of the Theorem is that if β is absolutely continuous and essential then the horizontal foliation of a surface in the closure of the orbit Uq_1 is not uniquely ergodic. For this we will need the following statement, which will also be useful in §10.

Proposition 7.1. *Let $F \subset \mathcal{H}$ be a closed set, and fix $c > 0$. Then the sets*

$$F' \stackrel{\mathrm{def}}{=} \bigcup_{q \in F} \bigcup_{\beta \in C_q^{+,RN}(c)} \mathrm{trem}_\beta(q) \tag{7.1}$$

and

$$F'' \stackrel{\mathrm{def}}{=} \bigcup_{q \in F} \bigcup_{\beta \in \mathcal{T}_q^{RN}(c)} \mathrm{trem}_\beta(q) \tag{7.2}$$

are also closed.

Recall from §4.1.3 that $C_q^{+,RN}(c)$ (respectively, $\mathcal{T}_q^{RN}(c)$) denotes the set of absolutely continuous (signed) foliation cocycles $\beta_\nu \in \mathcal{T}_q$ with $\|\nu\|_{RN} \leqslant c$.

Proof. We first prove that F' is closed. Let $q'_n \in F'$ be a convergent sequence with $q' = \lim_n q'_n$. We need to show that $q' \in F'$. Let $q_n \in F$ and $\beta_n \in C_{q_n}^{+,RN}(c)$ such that $q'_n = \mathrm{trem}_{\beta_n}(q_n)$. We will show that $q' = \mathrm{trem}_\beta(q)$ where q and β are accumulation points of the sequences (q_n) and (β_n). According to Proposition 6.7, the sequence (q_n) is bounded with respect to the metric dist. Also, a computation similar to the one appearing in the proof of Proposition 6.7, gives $\|\beta_n\|_{q_n} \leqslant c$, where $\|\cdot\|_{q_n}$ is the norm given by the Finsler structure defined in (2.7). By Proposition 2.5 the sup-norm distance is proper, and hence the sequence (q_n) has a convergent subsequence. Thus passing to a subsequence and using the fact that F is closed, we can assume $q_n \to q \in F$. Let M_n be the underlying surfaces of q_n. Choose marking maps $\varphi_n : S \to M_{q_n}$ and $\varphi : S \to M_q$ so that the corresponding points $\widetilde{q}_n \in \mathcal{H}_{\mathrm{m}}$ satisfy $\widetilde{q}_n \to \widetilde{q}$. Using these marking maps, identify β_n with elements of $H^1(S, \Sigma; \mathbb{R}^2)$. By the continuity property of the norms $\|\cdot\|_{q_n}$ (see §2.6), this sequence of cohomology classes is bounded, and so we can pass to a further subsequence to assume that β_n converges to $\beta \in H^1(S, \Sigma; \mathbb{R}^2)$. Applying Proposition 4.9 we get that $\beta = \lim_{n\to\infty} \beta_n \in C_{\widetilde{q}}^{+}(c)$ and using Proposition 4.8 we see that $q' = \mathrm{trem}_\beta(q) \in F'$. The proof that F'' is closed is similar. $\square$

Proof of Theorem 1.5(ii). Let $q_1 = \mathrm{trem}_\beta(q)$ where $\beta = \beta_\nu$ and ν is absolutely continuous. As in the proof of part (i) of the theorem, we can assume that ν is c-absolutely continuous for some c, i.e. $\beta \in C_q^{+,RN}(c)$, and set $F = \overline{Uq}$. By commutation of tremors and horocycles (see (4.13)), for any $s \in \mathbb{R}$, we have $u_s q_1 = \mathrm{trem}_\beta(u_s q)$. By Proposition 4.12, $\beta \in C^+_{u_s q}(c)$ for all s, and so $u_s q_1 \in F'$, where F' is defined via (7.1). By Proposition 7.1 we have that any $q_2 \in \overline{Uq_1} \smallsetminus \mathcal{L}$ also belongs to F', so is a tremor of a surface in $\mathcal{L}$.

So we write $q_2 = \mathrm{trem}_{\beta'}(q_3)$ for $q_3 \in \mathcal{L}$ and $\beta' \in \mathcal{T}_{q_3}$, and write M_2, M_3 for the underlying surfaces. Our goal is to show that the horizontal foliation on M_2 is not uniquely ergodic. Since $\mathcal{L}$ is U-invariant and $q_2 \notin \mathcal{L}$, β' is not a multiple of the canonical foliation cocycle $\mathrm{hol}^{(y)}_{q_3}$, i.e. the horizontal foliation on M_3 is not uniquely ergodic. By Proposition 4.12, neither is the horizontal foliation on M_2. □

The third assertion is that when q is generic for some U-invariant ergodic measure μ, assigning zero measure to surfaces with horizontal saddle connections, then q_1 is also generic for μ (but note that q_1 need not belong to $\mathrm{supp}\,\mu$). A heuristic explanation of this phenomenon is that for most values of s, the surface $u_s q$ is close to surfaces with a uniquely ergodic horizontal foliation, which means that $C^+_{u_s q}$ is a narrow cone centered around the canonical transverse measure tangent to the horocycle flow. By continuity of tremors, in this case $u_s q_1$ is very close to $u_{s+s_0} q$ for some s_0.

Proof of Theorem 1.5(iii). We first employ an argument of [LM], to prove the following:

Claim 1: For μ-a.e. surface q, the horizontal foliation on the underlying surface M_q is uniquely ergodic.

Indeed, from [MW1] we find that there is a compact subset $K \subset \mathcal{H}$ such that any surface q with no horizontal saddle connections satisfies

$$\liminf_{T\to\infty} \frac{1}{T} |\{s \in [0,T] : u_s q \in K\}| > \frac{1}{2}$$

(where $|A|$ denotes the Lebesgue measure of $A \subset \mathbb{R}$. Then by the Birkhoff ergodic theorem, any U-invariant ergodic measure ν on $\mathcal{H}$, which gives zero measure to surfaces with horizontal saddle connections, satisfies $\nu(K) > 1/2$. If the claim is false, then by ergodicity μ-a.e. surface has a minimal but non-uniquely ergodic horizontal foliation. Applying Masur's criterion (see e.g. [MaTa]) to the horizontal foliation, we find that for μ-a.e. q, the ray $\{g_t q : t < 0\}$ is divergent. Thus for μ-a.e. q there is $t_0 = t_0(q)$ such that for all $t \geqslant t_0$, $g_{-t} q \notin K$.

Moreover, we can take t_1 large enough so that $\mu(\{q : t_0(q) < t_1\}) > 1/2$ and hence $\nu = (g_{-t_1})_*\mu$ satisfies $\nu(K) < 1/2$. Since ν is also U-ergodic, and also gives zero measure to surfaces with horizontal saddle connections, this gives a contradiction. The claim is proved. $\triangle$

Let μ be the measure on $\mathcal{L}$, let $q \in \mathcal{L}$ be generic for μ, and let $q_1 = \mathrm{trem}_\beta(q)$ for some β. We need to show that q_1 is generic. Let f be a compactly supported continuous test function and let $\varepsilon > 0$. Let $s_0 = L_q(\beta)$ and let $q_2 = u_{s_0}q$. Since q_2 and q are in the same U-orbit, q_2 is also generic. We now claim:

Claim 2: For every $\varepsilon > 0$, every $\delta > 0$ and for all large enough T there is a subset $A \subset [0, T]$ with $|A| \geqslant (1-\varepsilon)T$ so that for all $s \in A$, $\mathrm{dist}(u_s q_1, u_s q_2) < \delta$.

We first use Claim 2 to conclude the proof of the Theorem.

By the uniform continuity of f, there is δ so that whenever $\mathrm{dist}(x, y) < \delta$ we have $|f(x) - f(y)| < \frac{\varepsilon}{4}$. Apply Claim 2 with $\frac{\varepsilon}{8\|f\|_\infty}$ in place of ε. Since q_2 is generic, for all large enough T we have

$$\left|\frac{1}{T}\int_0^T f(u_s q_2)\, ds - \int f d\mu\right| < \frac{\varepsilon}{2}.$$

Using the triangle inequality, we see that for all large enough T:

$$\begin{aligned}
&\left|\frac{1}{T}\int_0^T f(u_s q_1)\, ds - \int f d\mu\right| \\
\leqslant &\left|\frac{1}{T}\int_0^T f(u_s q_1)\, ds - \frac{1}{T}\int_0^T f(u_s q_2)\, ds\right| + \left|\frac{1}{T}\int_0^T f(u_s q_2)\, ds - \int f d\mu\right| \\
\leqslant &\frac{1}{T}\int_A |f(u_s q_1) - f(u_s q_2)|\, ds + \frac{1}{T}\int_{[0,T]\smallsetminus A} 2\|f\|_\infty\, ds + \frac{\varepsilon}{2} \\
\leqslant &\frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This shows that q_1 is generic.

It remains to prove Claim 2. For this we use [MW1] again. Let $Q \subset \mathcal{H}$ be a compact set such that for all large enough T,

$$\frac{|A_1|}{T} \geqslant 1 - \frac{\varepsilon}{2}, \quad \text{where } A_1 = \{s \in [0, T] : u_s q \in Q\}.$$

Let $\widetilde{Q} \subset \mathcal{H}_{\mathrm{m}}$ be compact such that $\pi(\widetilde{Q}) = Q$. Fix some norm on $H^1(S, \Sigma; \mathbb{R})$. Since $\widetilde{Q}$ is compact, and by the continuity in Proposition 4.8, there is δ' such that for any $\widetilde{q}' \in \widetilde{Q}$, and $\beta_1, \beta_2 \in C^+_{\widetilde{q}'}$ for which

$L_{\widetilde{q}'}(\beta_1) = L_{\widetilde{q}'}(\beta_2) = s_0$, we have

$$\|\beta_1 - \beta_2\| < \delta' \implies \mathrm{dist}\,(\mathrm{trem}_{\beta_1}(\widetilde{q}'), \mathrm{trem}_{\beta_2}(\widetilde{q}')) < \delta. \tag{7.3}$$

Let $\mathcal{L}'$ denote the collection of surfaces in $\mathcal{L}$ with no horizontal saddle connections and for which the horizontal foliation is uniquely ergodic. By assumption $\mu(\mathcal{L}') = \mu(\mathcal{L}) = 1$, and by Corollary 4.3 there is a neighborhood $\mathcal{U}$ of $\pi^{-1}(\mathcal{L}')$ such that

$$\widetilde{q}' \in \mathcal{U},\ \beta \in C^+_{\widetilde{q}'},\ L_{\widetilde{q}'}(\beta) = s_0 \implies \|\beta - s_0\,(dy)_{\widetilde{q}'}\| < \delta'. \tag{7.4}$$

Clearly $\pi(\mathcal{U})$ is an open set of full μ-measure. Since q is generic, for all sufficiently large T there is a subset $A_2 \subset [0, T]$ with

$$\frac{|A_2|}{T} > 1 - \frac{\varepsilon}{2} \text{ and } s \in A_2 \implies u_s q \in \pi(\mathcal{U}).$$

Now set $A = A_1 \cap A_2$, so that $|A| > (1-\varepsilon)T$. Suppose $s \in A$. Then there is $\widetilde{q}' \in \mathcal{U} \cap \widetilde{Q}$ with $\pi(\widetilde{q}') = u_s q$. We can view β as an element of $C^+_{u_s q}$ and with respect to the marked surface $\widetilde{q}'$ this corresponds to $\beta' \in C^+_{\widetilde{q}'}$, and we have

$$u_s q_1 = \mathrm{trem}_\beta(u_s q) = \pi(\mathrm{trem}_{\beta'}(\widetilde{q}')) \text{ and } u_s q_2 = u_{s_0} q' = \pi(\mathrm{trem}_{s_0 dy}(\widetilde{q}')).$$

By (7.3) and (7.4), we find $\mathrm{dist}(u_s q_1, u_s q_2) < \delta$, and the claim is proved. ∎

8. Points outside a locus $\mathcal{L}$ which are generic for $\mu_{\mathcal{L}}$

In this section, after some preparations, we prove Theorem 1.6. At the end of the section we also discuss how tremored surfaces behave with respect to the divergence of nearby trajectories under the horocycle flow.

8.1. **Tremors and rank-one loci.** We now recall the notions of Rel deformations and of a rank-one locus. Define $W \subset H^1(S, \Sigma; \mathbb{R}^2)$ to be the kernel of the restriction map $\mathrm{Res} : H^1(S, \Sigma) \to H^1(S)$ which takes a cochain to its restriction to absolute periods. For any $q \in \mathcal{H}$, and any lift $\widetilde{q} \in \pi^{-1}(q)$, as in §2.2 we have an identification $T_{\widetilde{q}}(\mathcal{H}_{\mathrm{m}}) \cong H^1(S, \Sigma; \mathbb{R}^2)$, and the subspace of $T_q(\mathcal{H})$ corresponding to W is called the Rel subspace and is independent of the marking (see [BSW, §3] for more details). Let $\mathfrak{g} = \mathfrak{g}_q$ denote the tangent space to the G-orbit of q (we consider this as a subspace of $T_q(\mathcal{H})$ for any q). A G-orbit-closure $\mathcal{L}$ is said to be a *rank-one locus* if there is a subspace $V \subset W$ such that for any $q \in \mathcal{L}$, the tangent space $T_q(\mathcal{L})$ is everywhere equal to $\mathfrak{g}_q \oplus V$. Rank-one loci were introduced and analyzed by Wright in [Wr1], and the eigenform loci $\mathcal{E}_D$ in $\mathcal{H}(1, 1)$ are examples of rank-one loci. The following result, which can be seen as a strengthening of an infinitesimal statement given in Corollary 4.5, is valid for all rank-one loci.

Proposition 8.1. *Suppose $\mathcal{L}$ is a rank-one locus. Then for any compact set $K \subset \mathcal{L}$ there is an $\varepsilon > 0$ such that if $q \in K$ is horizontally aperiodic, and $\beta \in \mathcal{T}_q$ is an essential tremor satisfying $|L|_q(\beta) < \varepsilon$, then $\mathrm{trem}_\beta(q) \notin \mathcal{L}$. If q is horizontally minimal and $\overline{Uq} = \mathcal{L}$, then no essential tremor of q belongs to $\mathcal{L}$.*

Proof. For the first assertion, since $\mathcal{L}$ is closed and K is compact, it suffices to show that for any aperiodic surface q in $\mathcal{L}$, any foliation cycle tangent to $\mathfrak{g} \oplus W \supset T_q(\mathcal{L})$ must be a multiple of the canonical foliation cycle $(dy)_q$. To this end, let

$$\beta = x + w \in (\mathfrak{g} \oplus W) \cap \mathcal{T}_q, \qquad \text{where } x \in \mathfrak{g} \;\text{ and } w \in W.$$

We want to show that β is a multiple of $(dy)_q$, and can assume that w and x are sufficiently small so that $q_1 = \mathrm{trem}_\beta(q) = g\,\mathrm{Rel}_w q$, where $g = \exp(x) \in G$ and $\mathrm{Rel}_w q$ is the Rel deformation tangent to w (see [BSW]). Let $\widetilde{q}, \widetilde{q}_1$ be marked surfaces with $\widetilde{q}_1 = g\mathrm{Rel}_w\widetilde{q}$, let $\varphi : S \to M_q$ be a marking map representing $\widetilde{q}$, let $\bar{\gamma}$ be a closed loop on S, and let $\gamma = \varphi(\bar{\gamma})$. Since Rel deformations do not change absolute periods, $\mathrm{dev}(\widetilde{q}_1)(\gamma) = g\,\mathrm{dev}(\widetilde{q})(\gamma)$. Write $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By (4.8),

$$c\,\mathrm{hol}^{(x)}_{\widetilde{q}}(\gamma) + d\,\mathrm{hol}^{(y)}_{\widetilde{q}}(\gamma) = \mathrm{hol}^{(y)}_{\widetilde{q}_1}(\gamma) = \mathrm{hol}^{(y)}_{\mathrm{trem}_{\widetilde{q},\beta}}(\gamma) = \mathrm{hol}^{(y)}_{\widetilde{q}}(\gamma),$$

and since this holds for every closed loop γ, we must have $c = 0$ and $d = 1$, i.e. $g \in U$. Then by (4.9), $x = s(dy)_q$ for some $s \in \mathbb{R}$. Since $w = \beta - x$ is now a tremor on a surface with an aperiodic horizontal foliation, which evaluates to zero when applied to any element of absolute homology, by Proposition 2.4 we have $w = 0$, and $\beta = s(dy)_q$.

For the second assertion, suppose by contradiction that $\mathrm{trem}_\beta(q) \in \mathcal{L}$ for some $q \in \mathcal{L}$ with $\mathcal{L} = \overline{Uq}$ and $\beta \in \mathcal{T}_q$ an essential tremor. Let K be a bounded open subset of $\mathcal{L}$ and let $\varepsilon > 0$ be as in the first assertion. The translated set $g_t Uq$ is also dense in $\mathcal{L}$, and $g_t u\,\mathrm{trem}_\beta(q) \in \mathcal{L}$ for any $u \in U$. By Proposition 6.2, $g_{-t}u\,\mathrm{trem}_\beta(q) = \mathrm{trem}_{e^{-t}\beta}(g_{-t}uq)$. Taking t large enough so that $|L|_q(e^{-t}\beta) < \varepsilon$, and choosing u so that $g_{-t}uq \in K$, we get a contradiction to the choice of ε. □

Corollary 8.2. *Suppose $\mathcal{L}$ is a rank-one locus, $q_1, q_2 \in \mathcal{L}$ are horizontally minimal and have dense U-orbits, and for $i = 1, 2$ there are $\beta_i \in \mathcal{T}_{q_i}$ such that $\mathrm{trem}_{\beta_1}(q_1) = \mathrm{trem}_{\beta_2}(q_2)$. Then there is $u \in U$ such that $uq_1 = q_2$. Furthermore, if β_1 and β_2 are balanced then $q_1 = q_2$ and β_1 is obtained from β_2 by applying a translation equivalence.*

Proof. Let $q_3 = \mathrm{trem}_{\beta_i}(q_i)$, let M_3 be the underlying surface, and let $\varphi : S \to M_3$ be a marking map representing $\widetilde{q}_3 \in \pi^{-1}(q_3)$. For $i = 1, 2$, let

$$\widetilde{\beta}_i = \varphi_i^*(\beta_i) \in H^1(S, \Sigma; \mathbb{R}_x)$$

be the cohomology classes for which

$$\mathrm{trem}_{\widetilde{\beta}_i}(\widetilde{q}_i) = \widetilde{q}_3 \ \text{ and } \ \widetilde{q}_i \in \pi^{-1}(q_i).$$

By Proposition 4.14 we have $\mathrm{trem}_{\widetilde{\beta}_1 - \widetilde{\beta}_2}(\widetilde{q}_1) = \widetilde{q}_2$. It follows from Proposition 8.1 that $\widetilde{\beta}_1 - \widetilde{\beta}_2 = s_0 (dy)_{q_1}$ for some $s_0 \in \mathbb{R}$, i.e. $\mathrm{trem}_{\widetilde{\beta}_1 - \widetilde{\beta}_2}(\widetilde{q}_1) = u_{s_0}\widetilde{q}_1$ and $u_{s_0} q_1 = q_2$. If β_1, β_2 are balanced then

$$s_0 = \int_{M_{q_1}} dx \wedge s_0 \, dy = \int_{M_{q_1}} dx \wedge (\beta_1 - \beta_2) = L_{q_1}(\beta_1) - L_{q_1}(\beta_2) = 0,$$

and this implies that $q_1 = q_2$. Now considering the expression (4.8) giving $\mathrm{dev}(\mathrm{trem}_\beta(\widetilde{q}))$, we see that the only possible ambiguity in the choice of $\widetilde{\beta}_i$ for which $\mathrm{trem}_{\widetilde{\beta}_1}(\widetilde{q}) = \mathrm{trem}_{\widetilde{\beta}_2}(\widetilde{q})$ is if $\widetilde{\beta}_1, \widetilde{\beta}_2 \in H^1(S, \Sigma; \mathbb{R}_x)$ are exchanged by the action of $\varphi^{-1} \circ h \circ \varphi$, where h is a translation equivalence of the underlying surface M_q. This gives the last assertion. □

We can use Proposition 8.1 to construct examples fulfilling property (III) in the discussion preceding the formulation of Theorem 1.6; namely we will use the rank-one locus $\mathcal{L} = \mathcal{E}$. We remark that in the introduction we explicitly required that q admit a tremor which is both essential and absolutely continuous. In fact this assumption is redundant, that is for surfaces in $\mathcal{E}$, foliation cocycles are absolutely continuous. More precisely we have:

Lemma 8.3. *For each aperiodic $q \in \mathcal{E}$, and any $\beta \in \mathcal{T}_q$,*

$$|L|_q(\beta) \leqslant 1 \implies \beta \text{ is 2-absolutely continuous.} \tag{8.1}$$

Recall that (4.5) gives that if $\beta \in \mathcal{T}_q^{RN}(2)$ then $|L|_q(\beta) \leqslant 2$.

Proof. First suppose $\beta = \beta_\nu \in C_q^+$ with $L_q(\beta) = 1$. By Proposition 3.4 there is c_1 such that $\nu + \iota_* \nu = c_1 (dy)_q$. Since

$$\int_{M_q} dx \wedge dy = 1 = L_q(\beta) = \int_{M_q} dx \wedge \nu = \int_{M_q} dx \wedge d\iota_* \nu,$$

we must have $c_1 = 2$, i.e.

$$(dy)_q = \frac{1}{2} d\nu + \frac{1}{2} d\iota_* \nu.$$

This implies that $\beta \in C_q^{+,RN}(c)$ for $c = 2$. For a general $\beta \in \mathcal{T}_q$, with $|L|_q(\beta) \leqslant 1$, write $\beta = \beta_{\nu^+} - \beta_{\nu^-}$, with $\beta_{\nu^\pm} \in C_q^+$ and repeat the argument. For any transverse positive arc γ we have $\int_\gamma d\nu^\pm \in \left[0, 2\int_\gamma dy\right]$, which implies (4.3) with $c = 2$.

$\square$

8.2. **Nested orbit closures.** Theorems 1.6 and 1.8 both exhibit one-parameter families of distinct orbit-closures for the U-action (see (1.7) and (1.9)). This property is proved using the following general statement.

Proposition 8.4. *Let $F = \mathcal{E}$, let $c > 0$, and let F'' be the set defined by (7.2). Let $\mathfrak{F}_1$ be a subset of F'' containing an essential tremor of a surface q_0 in $\mathcal{E}$ with a dense U-orbit. For each $\rho > 0$ define*

$$\mathfrak{F}_\rho \stackrel{\mathrm{def}}{=} \left\{\mathrm{trem}_{\rho\beta}(q) : q \in \mathcal{E},\ \beta \in \mathcal{T}_q^{(0)},\ \mathrm{trem}_\beta(q) \in \mathfrak{F}_1\right\}. \tag{8.2}$$

Then for $0 < \rho_1 < \rho_2$ we have $\mathfrak{F}_{\rho_1} \neq \mathfrak{F}_{\rho_2}$.

Proof. By Corollary 6.1, replacing q_0 with an element in its U-orbit, there is no loss of generality in assuming that $\mathfrak{F}_1$ contains an essential balanced tremor of q_0. Thus if we define

$$\mathcal{T}_{q_0}^{(0)}(\rho) \stackrel{\mathrm{def}}{=} \left\{\beta \in \mathcal{T}_{q_0}^{(0)} : \mathrm{trem}_\beta(q_0) \in \mathfrak{F}_\rho\right\},$$

then $\mathcal{T}_{q_0}^{(0)}(1)$ contains a nonzero vector. Clearly for all $\rho > 0$ we have $\mathcal{T}_{q_0}^{(0)}(\rho) = \rho\mathcal{T}_{q_0}^{(0)}(1)$, so each of the sets $\mathcal{T}_{q_0}^{(0)}(\rho)$ contain nonzero vectors as well. By (7.2) and Corollary 8.2, the sets $\mathcal{T}_{q_0}^{(0)}(\rho)$ are bounded for each ρ. Now suppose by contradiction that for $\rho_1 < \rho_2$ we have $\mathfrak{F}_{\rho_1} = \mathfrak{F}_{\rho_2}$. Then

$$\mathcal{T}_{q_0}^{(0)}(\rho_1) = \mathcal{T}_{q_0}^{(0)}(\rho_2) = \frac{\rho_2}{\rho_1}\mathcal{T}_{q_0}^{(0)}(\rho_1).$$

But $\frac{\rho_2}{\rho_1} > 1$ and a bounded subset of $\mathcal{T}_{q_0}^{(0)}$ cannot be invariant under a nontrivial dilation if it contains nonzero points. This is a contradiction.

$\square$

Proof of Theorem 1.6. We will find a surface satisfying conditions (I), (II) and (III) of the theorem. It was shown by Katok and Stepin [KS] that there is a surface $q \in \mathcal{E}$ with a horizontal foliation which is not uniquely ergodic and has no horizontal saddle connection (Veech [Ve1] proved an equivalent result on $\mathbb{Z}_2$-skew products of rotations, see [MaTa]). Thus the underlying surface M_q satisfies condition (II). To see that q satisfies condition (III) we apply Proposition 8.1 to the rank-one locus $\mathcal{E}$.

To see that q satisfies condition (I), we use [BSW, Thm. 10.1], which states that the U-orbit of every point in $\mathcal{E}$ is generic for some measure;

furthermore, the result identifies the measure. In the terminology of [BSW], the G-invariant 'flat' measure on $\mathcal{E}$ is the measure of type 7. The last bullet point of the theorem states that a surface is equidistributed with respect to flat measure if it has no horizontal saddle connection and is not the result of applying a real-Rel flow to a lattice surface. However lattice surfaces without horizontal saddle connections have a uniquely ergodic horizontal foliation ([Ve4]) and the horizontal foliation is preserved under real-Rel deformations. This implies that q cannot be a real-Rel deformation of a lattice surface.

For the proof of the second assertion, equation (1.7), we combine Propositions 6.4 and 8.4. Namely, we let $q_r = \mathrm{trem}_{r,\beta}(q)$ be as in the statement of the Theorem and define

$$\hat{\mathfrak{F}}_\rho \stackrel{\mathrm{def}}{=} \overline{Uq_\rho} \quad \text{and} \quad \mathfrak{F}_\rho \stackrel{\mathrm{def}}{=} \left\{\mathrm{trem}_{\rho\beta}(q) : q \in \mathcal{E},\ \beta \in \mathcal{T}_q^{(0)},\ \mathrm{trem}_\beta(q) \in \hat{\mathfrak{F}}_1\right\}.$$

Recall the $\mathbb{R}^*$-action multiplying elements of $\mathcal{T}_q$ by positive scalars (see §6.2). Since q_r is obtained from q_1 using the $\mathbb{R}^*$-action with parameter r, by naturality of the $\mathbb{R}^*$-action (see Proposition 6.4) we obtain that $\hat{\mathfrak{F}}_\rho = \mathfrak{F}_\rho$. So $\hat{\mathfrak{F}}_{r_1} \subsetneq \hat{\mathfrak{F}}_{r_2}$ for $r_1 < r_2$, and (1.7) follows by Proposition 6.4. ∎

Remark 8.5. *As we remarked in the introduction (see Remark 1.7), Theorem 1.6 remains valid for other eigenform loci $\mathcal{E}_D$ in place of $\mathcal{E} = \mathcal{E}_4$. Indeed, the results of [BSW] used above are valid for all eigenform loci, and to prove the existence of surfaces in $\mathcal{E}_D$ whose horizontal foliations are minimal but not ergodic, one can use [CM] in place of [KS]. Thus the proof given above goes through with obvious modifications. Finally we note that Lemma 8.3 is also true for other eigenform loci, provided the constant 2 on the right hand side of (8.1) is replaced with an appropriate constant depending on the discriminant D. We leave the details to the reader.*

8.3. **Erratic divergence of nearby horocycle orbits.** A crucial ingredient in Ratner's measure classification theorem is the polynomial divergence of nearby trajectories for unipotent flows. As we have seen in Corollary 2.7 there is a quadratic upper bound on the distance between two nearby horocycle trajectories in a stratum $\mathcal{H}$, with respect to the sup-norm distance. Such upper bounds can also be obtained in the homogeneous space setting, but in that setting they are accompanied by complementary lower bounds. Namely, Ratner used the fact that if $\{u_s\}$ is a unipotent flow on a homogeneous space X, for some metric d on X we have (see e.g. [M, Cor. 1.5.18]):

($*$) *for any* $\varepsilon > 0$ *and every* $K \subset X$ *compact, there is* $\delta > 0$ *such that if* $x_1, x_2 \in X$ *and for some* $T > 0$ *we have*

$$|\{s \in [0,T] : d(u_s x_1, u_s x_2) < \delta, u_s x_1 \in K\}| \geqslant \frac{T}{2},$$

then for all $s \in [0,T]$ *for which* $u_s x_1 \in K$ *we have* $d(u_s x_1, u_s x_2) < \varepsilon$.

Our proof of Theorem 1.6 shows that ($*$) fails for strata and in fact we have:

Theorem 8.6. *There is a stratum* $\mathcal{H}$*, a compact set* $K \subset \mathcal{H}$*,* $\varepsilon > 0$*, and* $q_1, q_2 \in \mathcal{H}$*, so that for any* $\delta > 0$*,*

$$\liminf_{T\to\infty} \frac{1}{T} |\{s \in [0,T] : \mathrm{dist}(u_s q_1, u_s q_2) < \delta,\ u_s q_1 \in K\}| > \frac{1}{2}, \tag{8.3}$$

but the set

$$\{s \geqslant 0 : u_s q_1 \in K \text{ and } \mathrm{dist}(u_s q_1, u_s q_2) \geqslant \varepsilon\} \tag{8.4}$$

is nonempty.

Proof. Take $q_1 \in \mathcal{L}$ for some $\mathcal{L}$ as in the proof of Theorem 1.6, where q_1 admits an essential tremor, and is generic for the G-invariant measure on $\mathcal{L}$, and let q_2 be a balanced essential tremor of q_1. Let $0 < \varepsilon < \mathrm{dist}(q_1, q_2)$, so that (8.4) holds. Claim 2 in the proof of Theorem 1.5(iii) implies (8.3). □

Remark 8.7. *The construction in §10 exhibits a stronger contrast to* ($*$)*: it gives examples in which* (8.3) *holds while* (8.4) *is unbounded.*

9. Existence of non-generic surfaces

In this section we will prove Theorem 1.4. Let B be the upper-triangular group. We will need the following useful consequence of the interaction of tremors with the B-action.

Theorem 9.1. *Let* $\mathcal{H}$ *be a stratum of translation surfaces and let* $\mathcal{L} \subsetneq \mathcal{H}$ *be a* G*-invariant locus such that there is* $q \in \mathcal{L}$ *with* $\overline{Gq} = \mathcal{L}$ *and such that* q *admits an essential absolutely continuous tremor which does not belong to* $\mathcal{L}$*. Then the closure of the set*

$$\bigcup_{q' \in Bq} \{\mathrm{trem}_\beta(q') : \beta \in C^+_{q'} \text{ is an essential absolutely continuous tremor}\} \tag{9.1}$$

is G*-invariant and contains a* G*-invariant locus* $\mathcal{L}'$ *with* $\dim \mathcal{L}' > \dim \mathcal{L}$*.*

In particular, if $\mathcal{L} = \mathcal{E} \subset \mathcal{H}(1,1)$*, then the set in* (9.1) *is dense in* $\mathcal{H}(1,1)$*.*

Proof. Let Ω be the set in equation (9.1), and let F be the closure of Ω. By assumption there is $q \in \mathcal{L}$ and an absolutely continuous $\beta \in C_q^+ \smallsetminus T_q(\mathcal{L})$, and hence for $\varepsilon > 0$ sufficiently small, the curve

$$t \mapsto q(t) \stackrel{\text{def}}{=} \text{trem}_{t\beta}(q), \quad t \in (-\varepsilon, \varepsilon)$$

satisfies $q(t) \in \Omega \smallsetminus \mathcal{L}$ for $t \neq 0$ and $q = \lim_{t \to 0} q(t)$; i.e., $q \in \overline{\Omega \smallsetminus \mathcal{L}}$. By Proposition 6.2, Ω is B-invariant, and hence so is F. According to [EMM, Thm. 2.1], any B-invariant closed set is G-invariant, and is a finite disjoint union of G-invariant loci. This implies that $\mathcal{L} = \overline{Bq} \subset F$, and also that we can write $F = F_1 \sqcup \cdots \sqcup F_k$ where each F_i is a closed G-invariant locus supporting an ergodic G-invariant measure, and for $i \neq j$ we have $F_i \not\subset F_j$. There is an i so that $\mathcal{L} \subset F_i$, and we claim $\mathcal{L} \subsetneq F_i$. Suppose $\mathcal{L} = F_i$ and let $q(t)$ as above. Then for sufficiently small $t > 0$ we have $q(t) \notin F_i$. So there is some j such that F_j contains a sequence $q(t_n)$ with $t_n > 0$ and $t_n \to 0$. Since F_j is closed we find that $q \in F_j$. But since $F_i = \overline{Gq}$ and F_j is G-invariant and closed, we obtain that $F_i \subset F_j$, a contradiction proving the claim.

Thus if we set $\mathcal{L}' \stackrel{\text{def}}{=} F_i$ we have $\mathcal{L} \subsetneq \mathcal{L}'$, and since both $\mathcal{L}$ and $\mathcal{L}'$ are manifolds and each is the support of a smooth ergodic measure, we must have $\dim \mathcal{L} < \dim \mathcal{L}'$, as claimed. To prove the second assertion, that $\mathcal{L}' = \mathcal{H}(1,1)$ we note that by McMullen's classification [McM1], there are no G-invariant loci $\mathcal{L}'$ satisfying $\mathcal{E} \subsetneq \mathcal{L}' \subsetneq \mathcal{H}(1,1)$. ☐

Proof of Theorem 1.4. First we claim that a dense set of surfaces in $\mathcal{H}(1,1)$ are generic for $\mu_1 = \mu_{\mathcal{E}}$, the natural measure on $\mathcal{E}$. By Theorem 1.5(iii) it suffices to show that tremors of surfaces in $\mathcal{E}$ with no horizontal saddle connections are dense in $\mathcal{H}(1,1)$. By Theorem 9.1 it suffices to show that there exists a surface in $\mathcal{E}$ with no horizontal saddle connections that admits an essential tremor. Theorem 1.6 establishes this, and the claim is proved.

We now use a Baire category argument. Let μ_2 be the natural flat measure on the entire stratum $\mathcal{H}(1,1)$. Let f be a compactly supported continuous function with $\int f d\mu_1 \neq \int f d\mu_2$, and let $\varepsilon > 0$ be small enough so that

$$2\varepsilon < \left| \int f d\mu_1 - \int f d\mu_2 \right|.$$

For $j = 1, 2$ and $T > 0$ let

$$\mathcal{C}_{j,T} \stackrel{\text{def}}{=} \left\{ q \in \mathcal{H}(1,1) : \left| \frac{1}{T} \int_0^T f(u_s q) ds - \int f d\mu_j \right| < \varepsilon \right\}$$

(which is an open subset of $\mathcal{H}(1,1)$), and let

$$\mathcal{C}_j \stackrel{\mathrm{def}}{=} \bigcap_{n\in\mathbb{N}} \bigcup_{T\geqslant n} \mathcal{C}_{j,T}.$$

If q is generic for μ_j then $q \in \mathcal{C}_{j,T}$ for all T sufficiently large. Since generic surfaces for μ_j are dense in $\mathcal{H}(1,1)$, each $\mathcal{C}_j$ is a dense G_δ-subset of $\mathcal{H}(1,1)$. By definition, for $q \in \mathcal{C}_j$ we have a subsequence $T_n \to \infty$ such that $\frac{1}{T_n}\int_0^{T_n} f(u_s q)ds$ converges to a number L with $|L - \int f d\mu_j| \leqslant \varepsilon$. In particular, any $q \in \mathcal{C}_1 \cap \mathcal{C}_2$ satisfies (1.2). For the last assertion, note that the set of surfaces with a dense orbit under the diagonal group $\{g_t\}$, in either forward or backward time, is also a dense G_δ subset, and so intersects $\mathcal{C}_1 \cap \mathcal{C}_2$ nontrivially. □

10. A new horocycle orbit closure

In this section we will prove Theorem 1.8. We first show the inclusion between the two subsets of $\mathcal{H}(1,1)$ described in equation (1.8), namely we show that

$$\begin{aligned} &\overline{\{\mathrm{trem}_\beta(q) : q \in \mathcal{E} \text{ is aperiodic},\ \beta \in \mathcal{T}_q,\ |L|_q(\beta) \leqslant a\}} \\ \subset &\{\mathrm{trem}_\beta(q) : q \in \mathcal{E},\ \beta \in \mathcal{T}_q,\ |L|_q(\beta) \leqslant a\}. \end{aligned} \tag{10.1}$$

To see this note that Proposition 7.1 and Lemma 8.3 imply that the first set is contained in the closed set

$$\left\{\mathrm{trem}_\beta(q) : q \in \mathcal{E},\ \beta \in \mathcal{T}^{RN}(2a)\right\}.$$

Corollary 4.4 implies that any limit point must satisfy $|L|_q(\beta) \leqslant a$.

For the last assertion of the Theorem, note that the inclusion in equation (1.9) is obvious from the first line of equation (1.8), and the naturality of the $\mathbb{R}^*$-action (Proposition 6.4). The inclusion is proper by Theorem 1.6.

It remains to show the existence of a surface q_1 for which we have equality in equation (1.8), namely for which

$$\overline{Uq_1} = \overline{\{\mathrm{trem}_\beta(q) : q \in \mathcal{E} \text{ is aperiodic},\ \beta \in \mathcal{T}_q,\ |L|_q(\beta) \leqslant a\}}. \tag{10.2}$$

Before doing this, we set up some notation to be used throughout this section and describe our strategy. We partition $\mathcal{E}$ into the following subsets:

$$\begin{aligned} \mathcal{E}^{(\mathrm{per})} &= \{q \in \mathcal{E} : M_q \text{ is horizontally periodic}\}, \\ \mathcal{E}^{(\mathrm{tor})} &= \{q \in \mathcal{E} : M_q \text{ is two tori glued along a horizontal slit}\} \smallsetminus \mathcal{E}^{(\mathrm{per})}, \\ \mathcal{E}^{(\mathrm{min})} &= \mathcal{E} \smallsetminus \left(\mathcal{E}^{(\mathrm{per})} \cup \mathcal{E}^{(\mathrm{tor})}\right) \\ &= \{q \in \mathcal{E} : \text{all infinite horizontal trajectories are dense}\}. \end{aligned}$$

Note that the set of aperiodic surfaces in $\mathcal{E}$ is precisely $\mathcal{E}^{(\mathrm{tor})} \cup \mathcal{E}^{(\mathrm{min})}$. It is easy to check that the sets $\mathcal{E}^{(\mathrm{per})}$ and $\mathcal{E}^{(\mathrm{min})}$ are both dense in $\mathcal{E}$; this follows easily from [MaTa, Thms 4.1 & 1.8]. The set $\mathcal{E}^{(\mathrm{tor})}$ is also dense — this can be derived from [EMM], or in a more elementary fashion from Proposition 3.5(2), see the proof of Proposition 10.2. We further partition $\mathcal{E}^{(\mathrm{tor})}$ according to the length of the slit:

$$\mathcal{E}^{(\mathrm{tor},H)} = \left\{q \in \mathcal{E}^{(\mathrm{tor})} : M_q \text{ is two tori glued along a horizontal slit of length } H\right\}.$$

Although the individual sets $\mathcal{E}^{(\mathrm{tor},H)}$ are not dense in $\mathcal{E}$, for each $H_0 > 0$ the union $\bigcup_{H>H_0} \mathcal{E}^{(\mathrm{tor},H)}$ is dense in $\mathcal{E}$.

Now for positive parameters a and H we define subsets of $\mathcal{H}(1,1)$:

$$\begin{aligned}
\mathcal{SF}^{(\mathrm{min})}_{(\leqslant a)} &= \left\{\mathrm{trem}_\beta(q) : q \in \mathcal{E}^{(\mathrm{min})},\ \beta \in \mathcal{T}_q,\ |L|_q(\beta) \leqslant a\right\}\\
\mathcal{SF}^{(\mathrm{tor})}_{(\leqslant a)} &= \left\{\mathrm{trem}_\beta(q) : q \in \mathcal{E}^{(\mathrm{tor})},\ \beta \in \mathcal{T}_q,\ |L|_q(\beta) \leqslant a\right\}\\
\mathcal{SF}_{(\leqslant a)} &= \mathcal{SF}^{(\mathrm{min})}_{(\leqslant a)} \cup \mathcal{SF}^{(\mathrm{tor})}_{(\leqslant a)}\\
\mathcal{SF}^{(\mathrm{tor},H)}_{(\leqslant a)} &= \left\{\mathrm{trem}_\beta(q) \in \mathcal{SF}^{(\mathrm{tor})}_{(\leqslant a)} : q \in \mathcal{E}^{(\mathrm{tor},H)}\right\}.
\end{aligned}$$

To lighten the notation, in the remainder of this section we will denote the closure $\overline{\mathcal{SF}_{(\leqslant a)}}$ by $\overline{\mathcal{SF}}$. The letters $\mathcal{SF}$ stand for 'spiky fish', and one can think of $\overline{\mathcal{SF}} \smallsetminus \mathcal{E}$ as the spikes of the spiky fish. For $q \in \mathcal{E}^{(\mathrm{tor})} \cup \mathcal{E}^{(\mathrm{min})}$, denote by $C_q^{+,\mathrm{erg}}$ the extreme rays in the cone of foliation cocycles. If the horizontal direction is not uniquely ergodic on M_q then Proposition 3.4 shows that $C_q^{+,\mathrm{erg}}$ consists of two rays interchanged by the involution ι. Further denote

$$\begin{aligned}
\mathcal{SF}^{(\mathrm{min})}_{(=a)} &= \left\{\mathrm{trem}_\beta(q) : q \in \mathcal{E}^{(\mathrm{min})}, \beta \in C_q^{+,\mathrm{erg}}, L_q(\beta) = a\right\}\\
\mathcal{SF}^{(\mathrm{tor})}_{(=a)} &= \left\{\mathrm{trem}_\beta(q) : q \in \mathcal{E}^{(\mathrm{tor})}, \beta \in C_q^{+,\mathrm{erg}}, L_q(\beta) = a\right\}\\
\mathcal{SF}_{(=a)} &= \mathcal{SF}^{(\mathrm{min})}_{(=a)} \cup \mathcal{SF}^{(\mathrm{tor})}_{(=a)}\\
\mathcal{SF}^{(\mathrm{tor},H)}_{(=a)} &= \left\{\mathrm{trem}_\beta(q) : q \in \mathcal{E}^{(\mathrm{tor},H)}, \beta \in C_q^{+,\mathrm{erg}}, L_q(\beta) = a\right\}.
\end{aligned}$$

Note that for $\beta \in C_q^{+,\mathrm{erg}}$, $L_q(\beta) = |L|_q(\beta)$.

With this terminology it is clear that equation (10.2) (and hence Theorem 1.8) follows from:

Theorem 10.1. *For any $a > 0$ there is $q_1 \in \mathcal{SF}^{(\mathrm{min})}_{(=a)}$, such that $\overline{Uq_1} = \overline{\mathcal{SF}} = \overline{\mathcal{SF}^{(\mathrm{tor})}_{(\leqslant a)}}$.*

The proof of Theorem 10.1 will make use of the following intermediate statements. Throughout this section, dist refers to the sup-norm distance discussed in §2.6. We will restrict dist to $\overline{\mathcal{SF}}$, in particular the balls which will appear in the proof are subsets of $\overline{\mathcal{SF}}$.

Proposition 10.2. *For any $q \in \mathcal{SF}^{(\min)}_{(\leqslant a)}$ and any $\varepsilon > 0$ there is $q' \in \mathcal{SF}^{(\mathrm{tor})}_{(\leqslant a)}$ such that* $\mathrm{dist}(q, q') < \varepsilon$.

Proposition 10.3. *For any $a > 0$, any $q \in \mathcal{SF}^{(\mathrm{tor})}_{(\leqslant a)}$ and any $\varepsilon > 0$ there is an H_0 such that for each $H > H_0$ there is a $q' \in \mathcal{SF}^{(\mathrm{tor},H)}_{(=a)}$ such that* $\mathrm{dist}(q, q') < \varepsilon$.

Note that the approximation described in Proposition 10.3 needs to accomplish two goals: approximate a tremor with total mass at most a by tremors of total mass exactly a; and do so with a prescribed slit length H.

Proposition 10.4. *For positive constants a and H and any $q \in \mathcal{SF}^{(\mathrm{tor},H)}_{(=a)}$ the set $\overline{Uq}$ contains all of $\mathcal{SF}^{(\mathrm{tor},H)}_{(=a)}$.*

Proof of Theorem 10.1 assuming Propositions 10.2, 10.3 and 10.4. The equality $\overline{\mathcal{SF}} = \overline{\mathcal{SF}^{(\mathrm{tor})}_{(\leqslant a)}}$ is clear from Proposition 10.2. We will prove:

(i) There is $q_1 \in \overline{\mathcal{SF}}$ for which the orbit Uq_1 is dense in $\overline{\mathcal{SF}}$.
(ii) Any q_1 as in (i) satisfies $q_1 = \mathrm{trem}_\beta(q)$ for some $q \in \mathcal{E}^{(\min)}$ and $\beta \in C_q^{+,\mathrm{erg}}$ with $L_q(\beta) = a$.

To prove (i), we will use the strategy of proof of the Baire category theorem. Given $\varepsilon > 0$ and a compact set $K \subset \overline{\mathcal{SF}}$, let $\mathcal{V}_{K,\varepsilon}$ denote the set of points in $\overline{\mathcal{SF}}$ whose U-orbit is ε-dense in K. By continuity of the horocycle flow and compactness of K, one sees that $\mathcal{V}_{K,\varepsilon}$ is open. We will show that $\mathcal{V}_{K,\varepsilon}$ is not empty. To see this, note that by Proposition 10.2, given a compact $K \subset \overline{\mathcal{SF}}$ and $\varepsilon > 0$ there is a finite set $F \subset \mathcal{SF}^{(\mathrm{tor})}_{(\leqslant a)}$ which is $\varepsilon/2$-dense in K. For $p \in F$, let $H_0 = H_0(p)$ be the constant given in Proposition 10.3, where we substitute p for q and replace ε with $\varepsilon/2$. Let $H > \max_{p \in F} H_0(p)$. Then for each p there is $q'_p \in \mathcal{SF}^{(\mathrm{tor},H)}_{(=a)}$ such that $\mathrm{dist}(p, q'_p) < \varepsilon/2$. Finally by Proposition 10.4, for any $q \in \mathcal{SF}^{(\mathrm{tor},H)}_{(=a)}$, the closure of Uq contains all of the q'_p. Thus the orbit Uq comes within distance $\varepsilon/2$ of each $p \in F$ and in particular is ε-dense in K.

Now let $K_1 \subset K_2 \subset \cdots$ be an exhaustion of $\overline{\mathcal{SF}}$ by compact sets with nonempty interiors. Let ε_1 be larger than the diameter of K_1, and let B_0 be a closed ball in the interior of $\mathcal{V}_{K_1,\varepsilon_1}$. We will iteratively choose $\varepsilon_1, \varepsilon_2, \ldots$ so that $\varepsilon_n \searrow 0$ and for all n, $B_n = B_0 \cap \bigcap_{j=1}^n \overline{\mathcal{V}}_{K_j,\varepsilon_j}$ has nonempty interior. Assuming we have chosen such $\varepsilon_1, \ldots, \varepsilon_n$, let

$\varepsilon_{n+1} < \varepsilon_n/2$ be small enough so that B_n contains a closed ball of radius ε_{n+1}. We have shown that $\mathcal{V}_{K_{n+1},\varepsilon_{n+1}} \neq \varnothing$, so by definition it intersects the interior of B_n. Since $\mathcal{V}_{K_{n+1},\varepsilon_{n+1}}$ is open, $B_{n+1} = \overline{\mathcal{V}}_{K_{n+1},\varepsilon_{n+1}} \cap B_n$ has nonempty interior. In particular the sets B_n form a nested sequence of compact sets satisfying the finite intersection property, so have a nontrivial intersection. Since $\varepsilon_n \searrow 0$ and $\bigcup K_n = \overline{\mathcal{SF}}$, for any point $q_1 \in \bigcap B_n$, we will have $\overline{Uq_1} = \overline{\mathcal{SF}}$ and assertion (i) is proved.

To prove assertion (ii), recall from (10.1) that $\overline{\mathcal{SF}}_{(\leqslant a)}$ is contained in the set

$$\mathcal{S}_{(\leqslant a)} \stackrel{\mathrm{def}}{=} \{\mathrm{trem}_\beta(q) : q \in \mathcal{E},\ \beta \in \mathcal{T}_q,\ |L|_q(\beta) \leqslant a\}.$$

Thus q_1 is of the form $\mathrm{trem}_\beta(q)$ for some $q \in \mathcal{E}$ and $\beta \in \mathcal{T}_q$ with $|L|_q(\beta) \leqslant a$. We cannot have $q \in \mathcal{E}^{(\mathrm{tor})} \cup \mathcal{E}^{(\mathrm{per})}$ since in both of these cases M_q would have a horizontal saddle connection of some length H, hence so would q_1, and hence any surface in $\overline{Uq_1}$ would have a horizontal saddle connection of length at most H. This would contradict the fact that Uq_1 is dense in $\mathcal{SF}_{(\leqslant a)}$. So we must have $q \in \mathcal{E}^{(\min)}$, and moreover q has no horizontal saddle connection. Similarly, β is not a multiple of the canonical foliation cocycle $(dy)_q$, because this would imply via (4.9) that $Uq_1 \subset \mathcal{E}$. In particular M_q is not horizontally uniquely ergodic.

Let ν_1 and $\nu_2 = \iota_*\nu_1$ be the ergodic transverse measures for the horizontal straightline flow on M_q, normalized so that $L_q(\beta_i) = 1$, where $\beta_i \stackrel{\mathrm{def}}{=} \beta_{\nu_i}$ for $i = 1, 2$ and write $\beta = a_1\beta_1 + a_2\beta_2$ where $|a_1| + |a_2| \leqslant a$. We can assume with no loss of generality that $a_2 \geqslant a_1$. Since β is not a multiple of $(dy)_q = \frac{1}{2}\nu_1 + \frac{1}{2}\nu_2$, we have $a_2 > a_1$. Defining $s = 2a_1$ and using (4.13) we get

$$\begin{aligned} \mathrm{trem}_\beta(q) &= \mathrm{trem}_{a_1\beta_1 + a_2\beta_2}(q) \\ &= \mathrm{trem}_{a_1\left(2\mathrm{hol}_q^{(y)} - \beta_2\right) + a_2\beta_2}(q) \\ &= \mathrm{trem}_{(a_2 - a_1)\beta_2}(u_s q) \end{aligned} \tag{10.3}$$

and this shows that we may replace q with $u_s q$ and β with $(a_2 - a_1)\beta_2$, which is an element of $C^{+,\mathrm{erg}}_{u_s q}$. So we assume that $\beta \in C^{+,\mathrm{erg}}_q$ and $L_q(\beta) \leqslant a$. Suppose $L_q(\beta) = a' < a$, then writing $\rho = \frac{a}{a'} > 1$ and letting $q_1 = \mathrm{trem}_\beta(q)$ and $q_2 = \mathrm{trem}_{\rho\beta}(q) \in \mathcal{SF}_{(\leqslant a)} = \overline{Uq_1}$, Proposition 6.4 implies that

$$\mathcal{SF}_{(\leqslant \rho a)} = \overline{Uq_2} \subset \overline{Uq_1} = \mathcal{SF}_{(\leqslant a)} \subset \mathcal{SF}_{(\leqslant \rho a)},$$

and thus $\mathcal{SF}_{(\leqslant \rho a)} = \mathcal{SF}_{(\leqslant a)}$. This contradicts Proposition 8.4, and hence $L_q(\beta) = a$. We have shown that there is $q_1 \in \overline{\mathcal{SF}}$ with $\overline{Uq_1} = \overline{\mathcal{SF}}$, and moreover q_1 must be in $\mathcal{SF}^{(\min)}_{(=a)}$, proving the theorem. □

We proceed with the proofs of Propositions 10.2, 10.4 and 10.3. As we will see now, the main ingredient for proving Proposition 10.2 is Proposition 3.5.

Proof of Proposition 10.2. By Proposition 4.8, it is enough to show that for any q in $\mathcal{E}^{(\mathrm{min})}$, any $\beta \in \mathcal{T}_q$, and any $\varepsilon' > 0$, there is $q_1 \in \mathcal{E}^{(\mathrm{tor})}$ and $\beta_1 \in C^+_{q_1}$, such that $\mathrm{dist}(q, q_1) < \varepsilon'$ and $\|\beta - \beta_1\| < \varepsilon'$. Here $\|\cdot\|$ is some norm on $H^1(S, \Sigma; \mathbb{R}_x)$, and we identify the cones C^+_q and $C^+_{q_1}$ with subsets of this vector space by choosing a marking and using period coordinates. We would like to use Proposition 3.5 (iii) and take $q_1 = r_{-\theta_j} q$, where $r_{-\theta_j}$ is the rotation of M_q which makes the slit σ_j horizontal, and for β_1 take the cohomology class corresponding to restriction of Lebesgue measure to a torus on M_{q_1} which is a connected component of the complement of the horizontal slit; i.e. the rotation of A_j. It is clear that for large j this choice would fulfill all our requirements, except perhaps the requirement that $q_1 \in \mathcal{E}^{(\mathrm{tor})}$. Namely it could be the case that the two translation equivalent slit tori which appear in Proposition 3.5 are periodic in direction θ_j. If this were to happen, we recall that M_{q_1} is presented as two tori glued along a horizontal slit, but the tori are horizontally periodic, so a small perturbation of these tori (in the space of tori $\mathcal{H}(0)$) will make them horizontally aperiodic. Pulling back to $\mathcal{E}$, i.e. regluing the aperiodic tori along the same slit, we get a new surface q_1' which is not horizontally periodic and can be made arbitrarily close to q_1. The cohomology class β_1' corresponding to the restriction of Lebesgue measure to one of the two aperiodic tori can be made arbitrarily close to β_1, completing the proof. □

Proposition 10.4 follows from a classical result of Hedlund [H] asserting that any horizontally aperiodic surface has a dense U-orbit in the space of tori $\mathcal{H}(0) \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$.

Proof of Proposition 10.4. Note that each surface q in $\mathcal{E}^{(\mathrm{tor},H)}$ has a splitting into two translation equivalent tori A_1 and A_2 glued along a horizontal slit of length H, and interchanged by the map ι of Proposition 3.1. The two rays in $C^{+,\mathrm{erg}}_q$ correspond, up to multiplication by scalars, to the restriction of the transverse measure $(dy)_q$ to each of the two tori. Thus if we set $s = 2a$, then each $q' \in \mathcal{SF}^{(\mathrm{tor},H)}_{(=a)}$ is obtained by a 'subsurface shear' of a surface in $\mathcal{E}^{(\mathrm{tor},H)}$, namely by applying u_s to one of the tori A_i and not changing the other torus — see Figures 8 and 9. The reason for taking $s = 2a$ is that the area of each of the A_i is exactly $1/2$. This description implies in particular that $\mathcal{SF}^{(\mathrm{tor},H)}_{(=a)}$ is the image of $\mathcal{E}^{(\mathrm{tor},H)}$ under a continuous map commuting with the

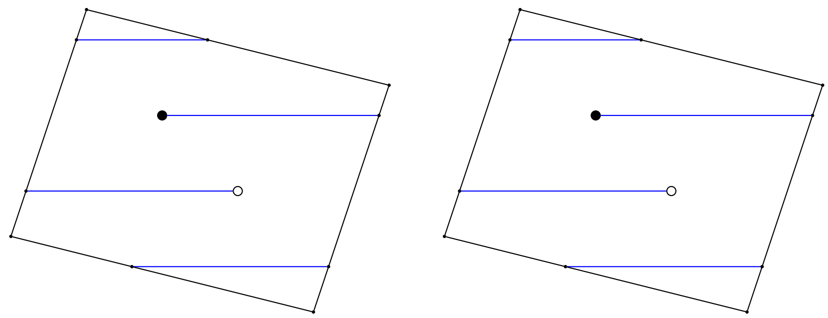

FIGURE 8. A surface $M_q \in \mathcal{E}$ obtained by gluing two identical horizontally aperiodic tori along a horizontal slit (in blue).

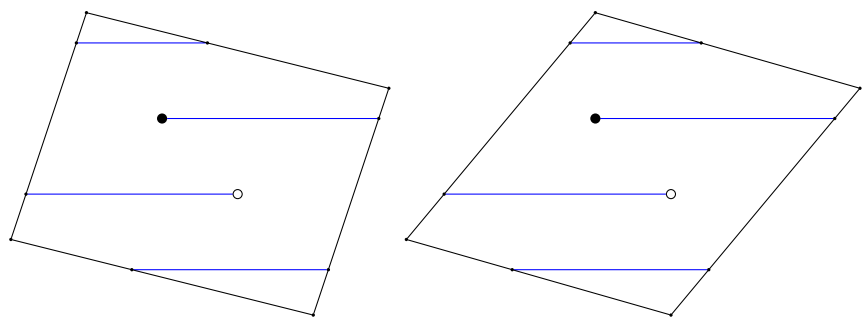

FIGURE 9. Applying a tremor in $C_q^{+,\mathrm{erg}}$ to M_q amounts to applying a horocycle shear to one of the two tori. The resulting surface is not in $\mathcal{E}$. Note that the length of the slit is unchanged.

U-action. So it suffices to show that the U-orbit of any $q \in \mathcal{E}^{(\mathrm{tor},H)}$ is dense in $\mathcal{E}^{(\mathrm{tor},H)}$.

We do this by defining a U-equivariant inclusion of $\mathcal{H}(0)^{(\mathrm{tor})}$, the set of tori that are horizontally aperiodic, into $\mathcal{E}^{(\mathrm{tor},H)}$, and using the previously mentioned theorem of Hedlund. Note that any surface in $\mathcal{E}^{(\mathrm{tor},H)}$ is obtained from a surface $M_{q'}$ for $q' \in \mathcal{H}(0)^{(\mathrm{tor})}$ by forming two copies of $M_{q'}$ and gluing them along a slit of length H starting at the marked point (the fact that the surface is aperiodic ensures that the slit exists). This defines a U-equivariant map $\mathcal{H}(0)^{(\mathrm{tor})} \to \mathcal{E}^{(\mathrm{tor},H)}$, which is continuous when $\mathcal{H}(0)^{(\mathrm{tor})}$ is equipped with its topology as a subset of $\mathcal{H}(0)$. Thus to complete the proof it suffices to show that any surface in $\mathcal{H}(0)^{(\mathrm{tor})}$ has a U-orbit which is dense in $\mathcal{H}(0)$ — which is Hedlund's theorem. $\square$

10.1. **Controlling tremors using checkerboards.** In order to prove Proposition 10.3 we will (among other things) have to deal with the following situation. Given $q \in \mathcal{E}$ and $\beta \in C_q^{+,\mathrm{erg}}$, with $L_q(\beta) < a$, we would like to find a surface $M_{q'}$ and $\beta' \in C_{q'}^{+,\mathrm{erg}}$, such that $L_{q'}(\beta') = a$ and $\mathrm{trem}_\beta(q)$ is close to $\mathrm{trem}_{\beta'}(q')$. We find q' close to the horocycle orbit of q. More specifically, we will choose s so that $q_0 = u_{-s}q$ and $\beta_0 = \beta + s\ \mathrm{hol}_q^{(y)}$ satisfy $\mathrm{trem}_\beta(q) = \mathrm{trem}_{\beta_0}(q_0)$ and $L_{q_0}(\beta_0) = a$, and take q' close to q_0. This transforms our problem into finding $\beta' \in C_{q'}^{+,\mathrm{erg}}$ which closely approximates $\beta_0 \in C_{q_0}^+$, where β_0 is not ergodic but rather is a nontrivial convex combination of $\mathrm{hol}_{q_0}^{(y)}$ and an ergodic foliation cocycle.

Controlling such convex combinations is achieved using what we will refer to informally as a ‘checkerboard pattern’. A checkerboard on a torus T is a pair of non-parallel line segments σ_1 and σ_2 on T which form the boundary of a finite collection of polygons, which can be colored in two colors so that no two adjacent polygons have the same color (see Figures 10 and 11). If we equip two identical tori T_1, T_2 with checkerboard patterns defined by the same lines σ_1, σ_2, and in which the colors in the coloring are swapped, we can form a surface M in $\mathcal{E}$ by gluing T_1 to T_2 in two different ways, namely along each of the σ_i. Both of these gluings give the same surface M, but it is decomposed as a union of two tori glued along a slit in two different ways (see Proposition 3.2). One decomposition is into the original tori T_1 and T_2, and the other is into the unions T_1', T_2' of parallelograms of a fixed color. Our interest will be in the ‘area imbalance’ of the checkerboard, which is the difference between the areas of $T_1 \cap T_1'$ and $T_2 \cap T_1'$. Informally, the area imbalance tells us how close these two decompositions are to each other.

In our application the lines σ_1 and σ_2 will both be nearly horizontal. Taking the normalized restriction $\mathrm{Leb}|_{T_1'}$ to one of the tori in the decomposition $M = T_1' \cup T_2'$ gives an ergodic foliation cocycle for the flow in the direction of σ_2, and the checkerboard picture shows that it closely approximates a nontrivial convex combination of the two ergodic components of the other foliation cocycle, in the direction of σ_1, namely the one coming from the normalized restrictions $\mathrm{Leb}|_{T_1}, \mathrm{Leb}|_{T_2}$. Controlling the coefficients in this convex combination amounts to controlling the area imbalance parameter, and this will be achieved below in Lemma 10.6, item (IV).

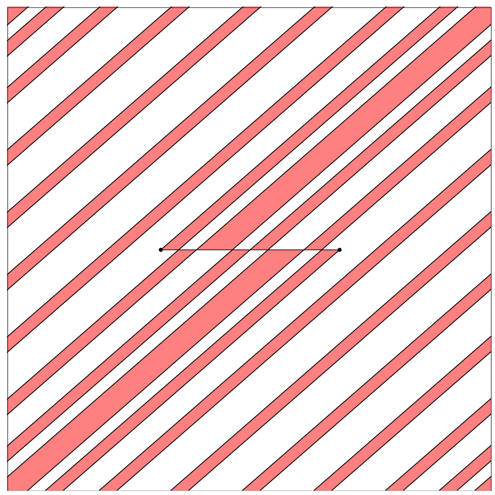

FIGURE 10. A checkerboard: when the σ_i (drawn in black) are long and orthogonal, the torus will be partitioned into small rectangles of alternating colors. The difference between the areas occupied by the colors is the *area imbalance*.

Checkerboards were originally introduced by Masur and Smillie in order to provide a geometric way to understand Veech's examples of surfaces with a minimal and non-ergodic horizontal foliation, see [MaTa, p. 1039 & Fig. 7]. We now proceed to a more precise discussion.

Let $p \in \mathcal{H}(0,0)$ be a torus with two marked points ξ_1 and ξ_2. Let $T = T_p$ be the underlying surface. Let σ_1, σ_2 be two non-parallel saddle connections on p from ξ_1 to ξ_2. Let $\bar{\sigma}_2$ be the segment obtained by reversing the orientation on σ_2, and let σ be the concatenation of σ_1 and $\bar{\sigma}_2$ so that σ is a closed loop on T. We have:

Lemma 10.5. *The following are equivalent:*

(i) *The loop σ is homologous to zero in $H_1(T; \mathbb{Z}/2\mathbb{Z})$.*

(ii) *It is possible to color the connected components of $T \smallsetminus \sigma$ with two colors so that components which are adjacent along a segment forming part of σ have different colors.*

(iii) *For $i = 1, 2$ let M_i be the surface obtained from the slit construction applied to σ_i (as in §3.1). Then M_1 and M_2 are translation equivalent.*

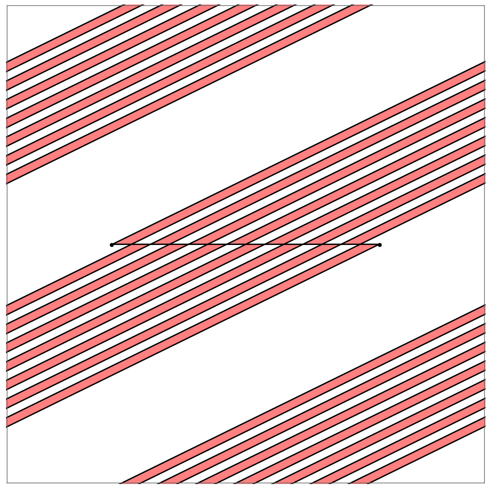

FIGURE 11. A key feature of this checkerboard is that the non-horizontal black segment crosses the horizontal segment immediately adjacent to its previous crossing, leading to strips of equal width and length.

Proof. The equivalence of (i) and (iii) follows from Proposition 3.2. We now show that (ii) is equivalent to the triviality of the class represented by σ. Consider the $\mathbb{Z}/2\mathbb{Z}$ valued 1-cochain Poincaré dual to σ. This cochain represents a trivial cocycle if and only if it is the coboundary of a $\mathbb{Z}/2\mathbb{Z}$-valued function. Associating colors to the values of such a function as in Figure 10 we have the checkerboard picture. Specifically being a coboundary with $\mathbb{Z}/2\mathbb{Z}$ coefficients means that two regions have the same color iff a generic path crosses σ an even number of times to get from one to the other. $\square$

Assume that σ_1 and σ_2 cross each other an odd number of times and satisfy the conditions of Lemma 10.5, let A be the area of T and let A_1, A_2 be the areas of the regions colored by the two colors in the coloring in (ii) above, so that $A_1 + A_2 = A$. We will refer to the quantity $\frac{|A_1 - A_2|}{A}$ as the *area imbalance* of the subdivision given by σ_1, σ_2 (note that when T_p has area one this is the same as $|A_1 - A_2|$).

We will need the following two lemmas on tori.

Lemma 10.6. *Suppose T is a torus for which the horizontal direction is aperiodic. Given $c \in [0, 1)$, a horizontal segment σ_1 on T, and $\eta > 0$, there is H_0 such that for any $H > H_0$, there is a second segment σ_2 on T joining the two endpoints of σ_1 for which the following hold:*

(I) *The segments σ_1, σ_2 on T intersect an odd number of times and satisfy the conditions of Lemma 10.5;*

(II) *Let $\theta \in (-\pi, \pi)$ be the direction of σ_2. Then $|\theta| < \eta$ and the flow in direction θ is aperiodic on T;*

(III) *the length of σ_2 is in the interval $(H, (1+\eta)H)$;*

(IV) *the area imbalance of σ_1, σ_2 is in the interval $(c-\eta, c+\eta)$.*

Lemma 10.7. *Let T be a horizontally minimal torus, and let σ_1 be a horizontal segment on T. Let $\sigma_2^{(k)}$ be a sequence of straight segments in T in direction $\theta_k \neq 0$, connecting the endpoints of σ_1, so that the loop σ above satisfies the conditions in Lemma 10.5, and satisfying $\lim_{k\to\infty} \theta_k = 0$. Let $T^{(k)}$ be any one of the two monochromatic regions, in the checkerboard coloring 10.5(ii). Then for any piecewise smooth bounded curve $\gamma \subset T$, which is transverse to the horizontal foliation, we have*

$$\lim_{k\to\infty} \frac{1}{\mathrm{Leb}(T^{(k)})} \int_\gamma dy|_{T^{(k)}} = \frac{1}{\mathrm{Leb}(T)} \int_\gamma dy. \tag{10.4}$$

We will give the proof of Lemmas 10.6 and 10.7 at the end of this section. First we conclude the proof of Proposition 10.3 assuming their validity.

Proof of Proposition 10.3. Let q be as in the statement of Proposition 10.3, that is q is obtained from $p \in \mathcal{H}(0)$ with minimal horizontal foliation, and from parameters $H_1 > 0$ and $s_1, s_2 \in \mathbb{R}$ satisfying

$$|s_1| + |s_2| \leqslant 2a, \tag{10.5}$$

as follows. First put a horizontal segment σ_1 of length H_1 on the underlying torus $T = T_p$ giving rise to a surface in $\mathcal{H}(0,0)$. Then apply the slit construction described in §3.1 to obtain a surface M_{q_0} for $q_0 \in \mathcal{E}^{(\mathrm{tor})}$ which is a union of two tori T_1 and T_2 with minimal horizontal foliations, glued along a horizontal slit of length H_1. Rescale so that this surface has area one, i.e. each T_i has area $1/2$. Then for $i = 1, 2$, apply the horocycle shear map u_{s_i} to T_i, and glue the resulting aperiodic tori to each other to obtain M_q. In light of the factor 2 appearing in (10.5), $q = \mathrm{trem}_\beta(q_0)$ for $\beta \in \mathcal{T}_{q_0}$ satisfying $|L|_{q_0}(\beta) \leqslant a$, so $q \in \mathcal{SF}^{(\mathrm{tor})}_{(\leqslant a)}$, and all surfaces in $\mathcal{SF}^{(\mathrm{tor})}_{(\leqslant a)}$ can be described in this way.

By swapping the roles of T_1 and T_2, replacing p with $u_{-s}p$, where $s = 2a - (s_1 + s_2)$, and replacing s_i with $s_i + s$ for some $s \in \mathbb{R}$, we can assume that

$$0 \leqslant s_1 \leqslant s_2 \text{ and } s_1 + s_2 = 2a. \tag{10.6}$$

Let

$$c \stackrel{\mathrm{def}}{=} \frac{s_2 - s_1}{2a}. \tag{10.7}$$

Let $M_{q_0} \in \mathcal{E}^{(\mathrm{tor})}$ be the surface constructed as in the above discussion (starting with T and σ_1 as in the paragraph above (10.6)) so that $q = \mathrm{trem}_{s_1\beta_1+s_2\beta_2}(q_0)$, where $\beta_i = \beta_{\nu_i}$ is the cohomology class corresponding to the transverse measure ν_i obtained by restricting the canonical transverse measure $(dy)_{q_0}$ to each of the tori T_i, and the tori are glued along a slit of length H_1. Let $\mathcal{U}$ denote the ε-ball around q. Our goal is to show that $\mathcal{U}$ contains some q' which is also a tremor of a surface $q'_0 \in \mathcal{E}^{(\mathrm{tor})}$, but for which the parameters s_1 and s_2 and the the slit length H are prescribed. More precisely $M_{q'_0}$ is built from two minimal tori T' and T'' glued along a horizontal slit of length H, $M_{q'}$ is obtained by applying the horocycle flow u_{2a} to T' and leaving T'' fixed (since T' has area $\frac{1}{2}$ this will give a tremor of total variation exactly a), and we need to carry the construction out for all $H > H_0$ where H_0 is allowed to depend on $\mathcal{U}$. Here T' is the component satisfying

$$A_2 = \mathrm{Leb}(T' \cap T_2) \geqslant A_1 = \mathrm{Leb}(T' \cap T_1). \tag{10.8}$$

We obtain q'_0 as follows. Using Lemma 10.6, we find σ_2 satisfying conditions (I–IV), for η sufficiently small (to be determined below). Define $q'_0 \stackrel{\mathrm{def}}{=} gq_0$ where $g \in \mathrm{SL}_2(\mathbb{R})$ is the composition of a small rotation and small diagonal matrix, moving the slit $\bar{\sigma}_2$ projecting to σ_2 to a horizontal slit of the required length H. Note that in light of (II) and (III), g is close to the identity in the sense that we can bound the norm $\|g - \mathrm{Id}\|$ with a bound which goes to zero as $\eta \to 0$, so that by choosing η small we can make $\mathrm{dist}(q_0, q'_0)$ as small as we wish.

Recall q is obtained from q_0 by shearing the two tori T_i (for $i = 1, 2$) by u_{s_i}. Define q' to be the surface obtained from q'_0 by shearing the torus T' by u_{2a}. We now show using (II) and (IV) that by making η small and H large we can ensure that $q' \in \mathcal{U}$. To see this, we will work in period coordinates, which by Proposition 2.5 gives the same topology as dist. We will choose a marking map $\varphi : S \to M_{q_0}$ and use it to define an explicit basis for $H_1(S, \Sigma)$, by pulling back a basis of $H_1(M_{q_0}, \Sigma_{q_0})$. Then we will show that for all η small enough and H large enough, when evaluating hol_q and $\mathrm{hol}_{q'}$ on the elements α of this basis, the differences $\|\mathrm{hol}_q(\alpha) - \mathrm{hol}_{q'}(\alpha)\|$ can be made as small as we wish. The basis is described as follows. For $i = 1, 2$, let $\alpha_1^{(i)}, \alpha_2^{(i)}$ be straight segments in T_i generating the homology, so that $\left\{\alpha_j^{(i)} : i, j = 1, 2\right\} \cup \{\bar{\sigma}_1\}$ form a basis for $H_1(M_{q_0}, \Sigma_{q_0}; \mathbb{Z})$. We now compute the holonomy vectors of these elements, corresponding to q and q'.

By the description of q from the preceding paragraph, and since $\alpha_j^{(i)} \subset T_i$, we have

$$\mathrm{hol}_q\left(\alpha_j^{(i)}\right) = u_{s_i}\mathrm{hol}_{q_0}\left(\alpha_j^{(i)}\right) = \mathrm{hol}_{q_0}\left(\alpha_j^{(i)}\right) + s_i\begin{pmatrix}\mathrm{hol}_{q_0}^{(y)}\left(\alpha_j^{(i)}\right)\\ 0\end{pmatrix} \tag{10.9}$$

and

$$\mathrm{hol}_q\left(\bar{\sigma}_1\right) = \mathrm{hol}_{q_0}\left(\bar{\sigma}_1\right). \tag{10.10}$$

Now let ν' be the transverse measure given by restricting the canonical transverse measure $(dy)_{q_0'}$ to T'. Then by the description of q' from the preceding paragraph we also have that

$$\mathrm{hol}_{q'}\left(\alpha_j^{(i)}\right) = \mathrm{hol}_{q_0'}\left(\alpha_j^{(i)}\right) + 2a\begin{pmatrix}\nu'\left(\alpha_j^{(i)}\right)\\ 0\end{pmatrix} \tag{10.11}$$

and

$$\mathrm{hol}_{q'}\left(\bar{\sigma}_1\right) = \mathrm{hol}_{q_0'}\left(\bar{\sigma}_1\right) + 2a\begin{pmatrix}\nu'\left(\bar{\sigma}_1\right)\\ 0\end{pmatrix}. \tag{10.12}$$

We want to show that by making η small, we can make the difference between (10.9) and (10.11), as well as the difference between (10.10) and (10.12), as small as we like.

Let μ' be the restriction of Lebesgue measure to T' so that, in the notation of Proposition 2.3, we have $\mu' = \mu_{\nu'}$, a (positive) measure with total variation $\frac{1}{2}$. Using the definition of the area imbalance and (10.8), we see that the area imbalance is $4A_2 - 1 = 1 - 4A_1$. This implies

$$\mu'(T_i) = A_i = \frac{1}{4}\left(1 + (-1)^i \cdot \text{area imbalance}\right) \quad (i = 1, 2).$$

Therefore, using equation (10.6), the choice of c in (10.7), along with (IV), we have

$$4a\mu'(T_i) \asymp a\left(1 + (-1)^i c\right) = a\left(\frac{s_1 + s_2}{2a} + (-1)^i\frac{s_2 - s_1}{2a}\right) = s_i,$$

where by $A \asymp B$ we mean that A can be made arbitrarily close to B by choosing η small enough. By (II), choosing η small forces θ to be close to 0, which is a uniquely ergodic direction on T_i. We apply Lemma 10.7, with $T = T_i, \gamma = \alpha_j^{(i)}$, and with $T^{(k)}$ any sequence of T' as above corresponding to $\eta \to 0$. We obtain that the second summands on the right hand side of (10.9) and (10.11) can be made close to each other by taking η sufficiently small.

Furthermore, since $\mathrm{dist}(q_0, q_0') \asymp 0$, we have

$$\left\|\mathrm{hol}_{q_0'}\left(\alpha_j^{(i)}\right) - \mathrm{hol}_{q_0}\left(\alpha_j^{(i)}\right)\right\| \asymp \left\|\mathrm{hol}_{q_0'}\left(\bar{\sigma}_1\right) - \mathrm{hol}_{q_0}\left(\bar{\sigma}_1\right)\right\|.$$

Thus for η small enough we can make the difference between (10.9) and (10.11) as small as we like. We also have

$$\nu'(\bar{\sigma}_1) \leqslant \int_{\bar{\sigma}_1} (dy)_{q_0'} = |\sin(\theta)|\ell(\bar{\sigma}_1),$$

where $\ell(\bar{\sigma}_1)$ denotes the length of $\bar{\sigma}_1$. Thus by (II) and (10.12),

$$\|\mathrm{hol}_{q_0'}(\bar{\sigma}_1) - \mathrm{hol}_{q'}(\bar{\sigma}_1)\| \asymp 0.$$

Putting these estimates together we see that the difference between (10.10) and (10.12) can also be made as small as we like. $\square$

Proof of Lemma 10.6. Let T_0 be the standard torus $\mathbb{R}^2/\mathbb{Z}^2$, and let

$$\psi : T \to T_0$$

be an affine homeomorphism. Since the horizontal direction is aperiodic on T, ψ maps σ_1 to a segment on $\hat{\sigma}_1 \stackrel{\mathrm{def}}{=} \psi(\sigma)$ on T_0 with holonomy $(x, \alpha x)$ for some $\alpha \notin \mathbb{Q}$ and $x > 0$. Let ξ_1, ξ_2 be the endpoints of $\hat{\sigma}_1$ in T_0. We will choose k an even positive integer, and a simple closed curve ℓ from ξ_1 to ξ_1, and let $\hat{\sigma}_2$ be the shortest curve homotopic to the concatenation of k copies of ℓ, followed by one copy of $\hat{\sigma}_1$. Also we will denote $\sigma_2 = \psi^{-1}(\hat{\sigma}_2)$. Since k is even, the curve σ of Lemma 10.5 is homologous to an even multiple of $\psi^{-1}(\ell)$ and thus (I) holds. The choice of the curve ℓ corresponds to the choice of $(m, n) \in \mathbb{Z}^2$ with $\gcd(m, n) = 1$. Since α is irrational, the linear form $(m, n) \mapsto m\alpha - n$ assumes a dense set of values on pairs $(m, n) \in \mathbb{Z}^2$ with $\gcd(m, n) = 1$ (see [CE] for a stronger statement). We choose m, n so that

$$|x(m\alpha - n) - (1 - c)| < \eta. \tag{10.13}$$

We can make this choice with m, n large enough, so that the direction of ℓ approaches the direction of slope α. Note that for all k, the direction of $\hat{\sigma}_2$ is closer to the direction of $\hat{\sigma}_1$ than the direction of ℓ, and this means that the direction θ of σ_2 is nearly horizontal. Hence for such (m, n) and all large k, $|\theta|$ is small. Because $\alpha \notin \mathbb{Q}$ the slope of σ_2 is irrational and so we have (II). As we incrementally increase $k \in 2\mathbb{N}$, the length of $\hat{\sigma}_2$ increases by approximately twice the length of ℓ. So for all large enough H, we can find k so that (III) holds.

We now verify (IV), which requires describing the region and coloring given by σ_1 and σ_2 as in Lemma 10.5. (It may be helpful to consult Figure 11, which has 15 intersections between the curves, counting the intial and terminal points, 7 red strips and 6 white strips.) We will work in T_0 instead of T. The holonomy of $\hat{\sigma}_2$ is $k(m, n) + (x, x\alpha)$. The curves $\hat{\sigma}_1$ and $\hat{\sigma}_2$ intersect in $k + 1$ points (including ξ_1, ξ_2) and these intersection points divide each $\hat{\sigma}_i$ into k equal length pieces. Consecutive pieces of the division of $\hat{\sigma}_2$ bound strips of the coloring given by Lemma 10.5. So we obtain a region R composed of $k-1$ strips of alternating color where each strip is a flat parallelogram with sides $\frac{1}{k}(k(m,n)+(x,x\alpha))$ and $\frac{1}{k}(x,x\alpha)$. As $k-1$ is odd, the areas of all but one of these strips cancel out. This gives that the contribution of R to the area imbalance of R is equal to the area A of one strip. We have

$$A = \left|\det\begin{pmatrix} m+\frac{x}{k} & \frac{x}{k} \\ n+\frac{x\alpha}{k} & \frac{x\alpha}{k}\end{pmatrix}\right| = \frac{|m\alpha - n|}{k}.$$

The complement of R has one color and area $1-(k-1)A$. This implies that the total area imbalance is

$$1-(k-1)A - A = 1-kA = 1-x|m\alpha-n|.$$

So (IV) follows from (10.13), and the proof is complete. □

Proof of Lemma 10.7. Let $\mu_0 = \frac{1}{\mathrm{Leb}(T)}\mathrm{Leb}$ be normalized Lebesgue measure on T. Since we have assumed that T is horizontally minimal, and minimal straightline flows on tori are uniquely ergodic, μ_0 is the unique Borel probability measure on T invariant under horizontal straightline flow.

For each k define a measure $\mu_k = \frac{1}{\mathrm{Leb}(T^{(k)})}$ (the normalized restriction of Lebesgue measure to $T^{(k)}$). We claim that μ_k converges weak-* to μ_0, as $k \to \infty$. Indeed, let $\Upsilon_k^{(x)}(t)$ denote the image of $x \in T$ under straightline flow in direction θ_k to time t. We can write μ_k as a convex combination of normalized length measures along segments

$$\left\{\Upsilon_k^{(x)}(t) : t \in [0,S]\right\},$$

for $x \in \sigma_1$ and with S the first return time of x to σ_1 along its orbit in direction θ_k (that is, segments passing parallel to the long sides in the parallelograms of the checkerboard pattern). The length of these segments goes to infinity and their direction becomes more and more horizontal as $k \to \infty$. By unique ergodicity, for any continuous test function f on T, any $\varepsilon > 0$, and any sufficiently large S (independent of x),

$$\left|\frac{1}{S}\int_0^S f\left(\Upsilon_0^{(x)}(t)\right)\,dt - \int f\,d\mu_0\right| < \frac{\varepsilon}{2},$$

and by uniform continuity of f, for any fixed S and all large enough k,

$$\left|\frac{1}{S}\int_0^S f\left(\Upsilon_0^{(x)}(t)\right)\,dt - \frac{1}{S}\int_0^S f\left(\Upsilon_k^{(x)}(t)\right)\,dt\right| < \frac{\varepsilon}{2}.$$

Putting these together we get $\mu_k \to \mu_0$.

We can now recover the integrals appearing in (10.4) from μ_0 and μ_k, as follows. Let $\bar{\gamma} \subset T$ denote the image of γ, and let $r > 0$ be small enough so that for all large enough k, the maps

$$\bar{\gamma} \times [0, r] \to T, \quad (x, t) \mapsto \Upsilon_k^{(x)}(t)$$

are injective, and their image does not intersect σ_1. For $k \geqslant 0$, let

$$A_k \stackrel{\text{def}}{=} \bigcup_{x \in \bar{\gamma}} \left\{ \Upsilon_k^{(x)}(t) : t \in [0, r] \right\}.$$

Then by Fubini's formula for Lebesgue measure, we have

$$\frac{1}{\mathrm{Leb}(T)} \int_\gamma dy = \frac{1}{r}\, \mu_0(A_0) \quad \text{and} \quad \frac{1}{\mathrm{Leb}(T^{(k)})} \int_\gamma dy|_{T^{(k)}} = \frac{1}{r}\, \mu_k(A_k). \tag{10.14}$$

The Lebesgue measure of ∂A_0 is zero, and hence by weak-$*$ convergence,

$$\lim_{k \to \infty} \mu_k(A_0) = \mu_0(A_0).$$

Also, the symmetric difference $A_0 \triangle A_k$ satisfies $\mu_k(A_0 \triangle A_k) \to_{k \to \infty} 0$, as can be shown by an elementary argument which we leave to the reader. This shows that

$$\lim_{k \to \infty} \mu_k(A_k) = \mu_0(A_0).$$

Together with (10.14), this implies (10.4). □

11. Non-integer Hausdorff dimension

The purpose of this section and the following one is to prove Theorem 1.9. Throughout this section we use the notation of §10. We briefly explain the basic idea of the proof. We can think of a neighborhood of $\mathcal{E}$ as being modelled on a neighborhood of the zero section in the total space of the normal bundle $\mathscr{N}(\mathcal{E})$ (see Corollary 3.3). Thus we can think of $\mathcal{SF}_{(\leqslant a)}$ as a subset of the total space of $\mathscr{N}(\mathcal{E})$. For all $q \in \mathcal{E}$, the intersection of $(\mathscr{N}(\mathcal{E}))_q$ with $\mathcal{SF}_{(\leqslant a)}$ is either a point or a line segment, contained in the two-dimensional space $(\mathscr{N}_x(\mathcal{E}))_q$. By [CHM] the set of $q \in \mathcal{E}$, for which this set is not a point has Hausdorff dimension 4.5.

Obtaining the lower bound is easier, and we use Proposition 11.1 to say that the Hausdorff dimension is at least 4.5 +1. Obtaining the upper bound is more involved, occupying §11.2 and §12. We denote the Hausdorff dimension of a subset A of a metric space X by $\dim A$. We will use the following well-known facts about Hausdorff dimension (see e.g. [Fa, Mat]):

Proposition 11.1. *Let X and X' be metric spaces.*

(1) *If* $f : X \to X'$ *is a Lipschitz map then* $\dim X \geqslant \dim f(X)$. *In particular, Hausdorff dimension is invariant under bi-Lipschitz homeomorphisms.*
(2) *For a countable collection* $X_1, X_2, \dots$ *of subsets of* X *we have* $\dim \bigcup X_i = \sup_i \dim X_i$.
(3) *Let* A *and* B *be subsets of Euclidean space and let* $X \subset A \times B$ *be such that for all* $a \in A$, $\dim\{b \in B : (a, b) \in X\} \geqslant d$. *Then*

$$\dim X \geqslant \dim A + d. \tag{11.1}$$

In particular $\dim(A \times B) \geqslant \dim A + \dim B$.

Note that when stating Theorem 1.9 we did not specify a metric on $\mathcal{H}(1,1)$. For concreteness one can take the metric to be the metric dist defined in §2.6, but note that in view of items (1) and (2) of Proposition 11.1, the Hausdorff dimension of a set with respect to two different metrics on $\mathcal{H}(1,1)$ is equal, as long as they are mutually bi-Lipschitz on compact sets. We will use this fact repeatedly.

The next definition fixes an identification of an open set in the stratum with cohomology via periodic coordinates. This is helpful for working with the metric dist. Formally, let $\mathcal{U} \subset \mathcal{H}$ be an open set and $\pi \colon \mathcal{H}_{\mathrm{m}} \to \mathcal{H}$ be the forgetful map of §2.1. In this section, we say that $\mathcal{U}$ is an *adapted neighborhood* if it is precompact, and there is a triangulation of S such that a connected component of $\pi^{-1}(\mathcal{U})$ is contained in V_τ, where V_τ is described in §2.2. Additionally we will say that a relatively open $\mathcal{U} \subset \mathcal{E}$ is an *adapted neighborhood (in* $\mathcal{E}$*)* if it is the intersection of an adapted neighborhood in $\mathcal{H}(1,1)$, with the locus $\mathcal{E}$.

11.1. **Proof of lower bound.** We use the notation introduced in §10, and begin with the proof of the easier half of the theorem.

Proof of lower bound in Theorem 1.9. For each $\delta > 0$, we will define a subset $X_0 \subset \mathcal{SF}_{(\leqslant a)}$, subsets $X_1 \subset \mathcal{E}$, $X_2 \subset \mathbb{R}$, and a surjective Lipschitz map $f : X_0 \to X_1 \times X_2$, where $\dim X_1 \geqslant 4.5 - \delta$ and $\dim X_2 = 1$. The statement will then follow via Proposition 11.1.

Let $\mathcal{U} \subset \mathcal{H}(1,1)$ be an adapted neighborhood, so that we can identify $\mathcal{U}$ with an open subset of $H^1(S, \Sigma; \mathbb{R}^2)$. Fix a norm $\|\cdot\|$ on $H^1(S, \Sigma; \mathbb{R}_x)$ which is invariant under translation equivalence arising from the orbifold group of $\mathcal{E}$, as in Proposition 2.1. According to Corollaries 6.1 and 8.2, for any $q' \in \mathcal{SF}^{(\min)}_{(\leqslant a)}$ there is a unique $q = q(q') \in \mathcal{E}^{(\min)}$ and a unique $\beta = \beta(q') \in \mathcal{T}^{(0)}_q$ (up to translation equivalence) such that $q' = \mathrm{trem}_{q,\beta}$. Define

$$\bar{f} : \mathcal{SF}^{(\min)}_{(\leqslant a)} \to \mathcal{E}^{(\min)} \times \mathbb{R}_{\geqslant 0} \text{ by } \bar{f}(q') \overset{\mathrm{def}}{=} \big(q(q'), \|\beta(q')\|\big). \tag{11.2}$$

Note that because translation equivalences preserve $\|\cdot\|$ this is well-defined. By Corollaries 4.5 and 3.3 we have that $\beta(q') \in \mathscr{N}_x(\mathcal{E})$ for all q', where $\mathscr{N}_x(\mathcal{E})$ is a flat subbundle.

We claim that by making $\mathcal{U}$ small enough, $\bar{f}$ restricted to $\mathcal{U}$ is a Lipschitz map (where we use the metric arising from dist on the domain and first summand of the range of $\bar{f}$). Indeed, by the continuity of the map in (2.9), and the fact that $\mathcal{U}$ is precompact, the metric dist is bi-Lipschitz to the metric $\mathrm{dist}'(q_1, q_2) \stackrel{\mathrm{def}}{=} \|\mathrm{hol}(\widetilde{q}_1) - \mathrm{hol}(\widetilde{q}_2)\|$ arising from period coordinates and the chosen norm $\|\cdot\|$ (when $\widetilde{q}_i \in \pi^{-1}(q_i)$ belong to some fixed connected lift of $\mathcal{U}$). Furthermore, if $\mathcal{U}$ is small enough, then for the projections introduced in §2.3, we have from Corollary 3.3 that

$$\mathrm{hol}(q(q')) = P^+(\mathrm{hol}(q')) \quad \text{and} \quad \mathrm{hol}(\beta(q')) = P^-(\mathrm{hol}(q'));$$

that is, in period coordinates on $\mathcal{U}$, $\bar{f}$ is obtained by writing a vector in $H^1(S, \Sigma; \mathbb{R}^2)$ in terms of its coordinates with respect to the two factors in a direct sum decomposition, composed with taking the norm on the second coordinate. This is clearly a Lipschitz map with respect to dist'.

Fix $\eta > 0$ and set

$$\begin{aligned} X_1^{(\eta)} &\stackrel{\mathrm{def}}{=} \left\{ q \in \mathcal{E}^{(\min)} : \text{ there is } \beta \in \mathcal{T}_q^{(0)} \text{ with } |L|_q(\beta) \leqslant a \text{ and } \|\beta\| = \eta \right\}, \\ X_0 &\stackrel{\mathrm{def}}{=} \left\{ q' \in \mathcal{SF}^{(\min)}_{(\leqslant a)} : q(q') \in X_1^{(\eta)},\ \|\beta(q')\| \leqslant \eta \right\}, \\ X_2 &\stackrel{\mathrm{def}}{=} [0, \eta], \end{aligned}$$

and define

$$f : X_0 \to X_1^{(\eta)} \times X_2, \quad f \stackrel{\mathrm{def}}{=} \bar{f}|_{X_0}.$$

Then f is Lipschitz on the intersection of X_0 with any compact set, and the definitions ensure that f is surjective. So it remains to show that for $\eta > 0$ small enough we have

$$\dim X_1^{(\eta)} \geqslant 4.5 - \delta. \tag{11.3}$$

Let

$$X_1 = \left\{ q \in \mathcal{E}^{(\min)} : \text{horizontal flow on } M_q \text{ is not uniquely ergodic} \right\}.$$

Since $X_1 = \bigcup_{\eta > 0} X_1^{(\eta)}$, by Proposition 11.1 (2) it suffices to show that $\dim X_1 \geqslant 4.5$. This is deduced from work of Cheung, Hubert and Masur as follows. By the general theory of local cross-sections (see e.g. [MSY]), the action of the group $\{r_\theta : \theta \in \mathbb{S}^1\}$ on $\mathcal{E}$ admits a cross-section, that is, we can parameterize a small neighborhood in $\mathcal{E}$ by $(q, \theta) \mapsto r_\theta q$, where q ranges over a 4-dimensional smooth manifold $\mathcal{V}$,

θ ranges over an open set in $\mathbb{S}^1$, and the parameterizing map is Bi-Lipschitz. Thus these coordinates identify a neighborhood in $\mathcal{E}$ with a Cartesian product $\mathcal{V} \times I$ where I is an interval in $\mathbb{S}^1$. It is shown in [CHM] that $\mathcal{V}$ contains a Borel subset A of full measure, such that for each $q \in A$ there is a subset $\Theta_q \subset \mathbb{S}^1$ so that for $q \in A$, $\theta \in \Theta_q$ we have $r_\theta q \in X_1$, and $\dim \Theta_q = 0.5$. Proposition 11.1, item (1) and formula (11.1) now imply (11.3). □

Remark 11.2. *We remark without proof that the map $\bar{f}$ introduced in (11.2) would not be Lipschitz if we defined the second coordinate to be $|L|_q(\beta)$. Indeed, if we were to define $\bar{f}$ in this way and extend it to tremors of surfaces in $\mathcal{SF}^{(\mathrm{tor})}_{(\leqslant a)}$, then Proposition 10.3 would show that $\bar{f}$ is not even continuous. Also, it is likely that $\bar{f}$ is not bi-Lipschitz, and this is part of the challenge in proving the upper bound.*

11.2. **Proof of upper bound.** We now begin the proof of the upper bound, starting with a brief guide to its proof. In order to cover $\mathcal{SF}_{(\leqslant a)}$ efficiently, we will view a subset of this set as lying in a product space, namely a local trivialization of the bundle $\mathscr{N}(\mathcal{E})$ as in the proof of the lower bound. To efficiently cover $\mathcal{SF}_{(\leqslant a)}$ in this product space we find convex sets $J_i \subset \mathcal{E}$ so that the fixed-size tremors of points in J_i vary in a controlled way.

Proposition 11.3 gives an upper bound for the Hausdorff dimension that fits this strategy. The remainder of this section is devoted to proving the upper bound assuming this proposition, that is showing that $\mathcal{SF}_{(\leqslant a)}$ satisfies the assumptions of Proposition 11.3, and in §12 we prove the proposition.

11.2.1. *Preparations for the upper bound: general result for efficient covers.* We begin with our general result for exploiting efficient covers of convex sets. Let $Y \subset \mathbb{R}^d$ and let $|Y|$ denote the Lebesgue measure of Y. Let $\mathcal{N}^{(\varepsilon)}(Y)$ denote the ε-neighborhood of Y, that is $\mathcal{N}^{(\varepsilon)}(Y) = \bigcup_{y \in Y} B(y, \varepsilon)$. The *inradius* of $Y \subset \mathbb{R}^d$ is defined to be the supremum of $r \geqslant 0$ such that Y contains a ball of radius r.

Proposition 11.3. *Let $P_1 \subset \mathbb{R}^d$, $P_2 \subset \mathbb{R}^2$ be balls. Let $Z \subset P_1 \times P_2$, and $\{Z(t) : t \in \mathbb{N}\}$ be a collection of subsets of $P_1 \times P_2$, such that for any $T > 0$, $Z \subset \bigcup_{t=T}^{\infty} Z(t)$. Assume furthermore that there are positive constants c_1, c_2, and $\delta < 1$ and that for each $t \in \mathbb{N}$, $Z(t)$ is a finite disjoint union of sets $X_i(t) \times Y_i(t)$, with $X_i(t) \subset P_1$, $Y_i(t) \subset P_2$, for which the following hold:*

(i) *Each $X_i(t)$ is contained in a convex set $J_i(t)$ such that the $J_i(t)$ are pairwise disjoint, and each has inradius at least $c_1 e^{-2t}$.*

(ii) *Each* $Y_i(t)$ *is a rectangle whose shorter side has length at most* $c_2 e^{-2t}$.

(iii) $\left|\bigcup_i \mathcal{N}^{(e^{-2t})}(X_i(t))\right| \leqslant c_2 e^{-\delta t}$.

Then

$$\dim Z \leqslant d + 1 - \frac{\delta}{5}. \tag{11.4}$$

To obtain an upper bound on the Hausdorff dimension of $\mathcal{SF}_{(\leqslant a)}$, we will verify the assumptions of Proposition 11.3, with $d = 5$. In our setup, a small adapted neighborhood $\mathcal{U} \subset \mathcal{E}$ will play the role of a neighborhood in $\mathbb{R}^5$, and the 2-dimensional subspace $\mathscr{N}_x(\mathcal{E})$ will play the role of $\mathbb{R}^2$.

11.2.2. *Preparations for upper bound: transverse systems.* In order to verify hypotheses (i) and (ii) of Proposition 11.3 we need to choose convex sets in $\mathcal{E}$ so that the $\mathscr{N}_x(\mathcal{E})$ fibers intersected with $\mathcal{SF}_{(\leqslant a)}$ vary in a controlled way. To do this, we now get good approximations for the cone of foliation cocycles which will be constant on our convex subsets of $\mathcal{E}$. Our strategy will be to define convex regions, on which the horizontal flow is combinatorially similar up to some fixed time. Arguments like this are standard when using Rauzy-Veech induction. In our setup it will be more convenient to use transverse systems, which we now introduce. The advantage of transverse systems is that they have a more transparent interaction with the geodesic flow $\{g_t\}$. See [MW2, §2] for a related construction.

Let $\widetilde{q} \in \mathcal{H}_{\mathrm{m}}$ and let M_q be the underlying translation surface. A *transverse system* on M_q is a finite collection of disjoint arcs of finite length which are transverse to the horizontal foliation on M_q, do not contain points of Σ, and intersect every horizontal leaf (see [MW2, Fig. 2.1]). The arcs may contain points of Σ in their closure. For example, if the horizontal foliation on M_q is minimal then σ could be any short vertical arc not passing through singularities, and if M_q is aperiodic and ε is an arbitrary positive number, σ could be the union of downward pointing vertical prongs of length ε starting at all singular points (and where the singular points at their extremities are not considered a part of the prong).

We now define some structures associated with a transverse system. We mark one point on each connected component of σ. A σ*-almost horizontal segment* is a continuous oriented path ℓ from σ to σ, which starts and ends at marked points, is a concatenation of an edge along σ, a piece of a horizontal leaf in $M_q \smallsetminus \Sigma_q$ which does not intersect σ in its interior, and another edge along σ. The orientation of a σ-almost horizontal segment is the one given by rightward motion along horizontal leaves. Two σ-almost horizontal segments are said to be *isotopy equivalent* if they are homotopic with fixed endpoints, and where the homotopy is through σ-almost horizontal segments. Up to isotopy equivalence there are only finitely many σ-almost horizontal segments. A *σ-almost horizontal loop* is a continuous oriented loop which is a concatenation of σ-almost horizontal segments, where the orientation of the loop is consistent with the orientation of each of the segments. We say that a σ-almost horizontal loop is *reduced* if it intersects each connected component of σ at most once. With each σ-almost horizontal loop γ we associate a cohomology class $\beta_\gamma \in H^1(M_q, \Sigma_q; \mathbb{R})$ via Poincaré duality.

We will need the following:

Lemma 11.4. *Let M_q be a horizontally minimal surface. Then for any transverse system σ, the cohomology classes corresponding to all σ-almost horizontal loops generate $H^1(M_q, \Sigma; \mathbb{Z})$.*

Proof. The union of σ-almost horizontal segments in one isotopy equivalence class is the union of sub-arcs of σ and a topological disc foliated by parallel horizontal segments. The union of these topological discs gives a presentation of $M_q \smallsetminus \Sigma$ as a cell complex. We call it the *cell complex associated with* σ (see [MW2, §2.4]). This generalizes the well-known Veech zippered rectangles construction [Ve3]; namely the zippered rectangle construction arises when σ has one connected component, and the two endpoints of σ are mapped by the horizontal straightline flow to singular points in forward time. In the zippered rectangle case, a proof of the Lemma is given in [Y, §4.5].

Since we have assumed that M_q is horizontally minimal, any open subinterval $\sigma' \subset \sigma$ can serve as a transverse system. We choose $\sigma' \subset \sigma$ so that it satisfies the conditions mentioned above, namely, the cell complex associated with σ' is a zippered rectangle construction. Since the σ'-almost horizontal loops are a subset of the σ-almost horizontal loops, the statement for σ follows from the statement for σ'. $\square$

Given a marking map $S \to M_q$ we can think of each β_γ as an element of $H^1(S, \Sigma; \mathbb{R})$. We denote by $C_q^+(\sigma)$ the convex cone over all of the β_γ, that is

$$C_q^+(\sigma) = \mathrm{conv}\left(\{t\beta_\gamma : \gamma \text{ is a } \sigma\text{-almost horizontal loop on } M_q \text{ and } t > 0\}\right).$$

Note that $C_q^+(\sigma)$ is a finitely generated cone. Indeed, if we let $\mathscr{L} = \mathscr{L}_{q,\sigma}$ denote the reduced σ-almost horizontal loops, then $C_q^+(\sigma)$ is the convex cone generated by $\beta_\gamma, \ \gamma \in \mathscr{L}$. Since β_γ only depends on the homotopy class of γ, and there are only finitely many isotopy classes of σ-almost horizontal segments, this shows the finite generation of $C_q^+(\sigma)$.

Let C_q^+ be the cone of foliation cocycles as in §2.5. Clearly, if $\sigma \subset \sigma'$ are transverse systems then $C_q^+(\sigma) \subset C_q^+(\sigma')$. We have the following standard fact.

Proposition 11.5. *Suppose M_q has no horizontal saddle connections and let $\sigma_1 \supset \sigma_2 \supset \cdots$ be a nested sequence of transverse systems for the horizontal foliation on M_q, with total length going to zero. Then*

$$C_q^+ \subset \bigcap_{n=1}^{\infty} C_q^+(\sigma_n). \tag{11.5}$$

Remark 11.6. *In fact we have equality in* (11.5)*. In this paper we only need the inclusion stated above. The reverse inclusion can be proved along the lines of* [MW2, Proof of Thm. 1.1]*; for a similar result in the context of interval exchange transformations, see* [Ve2, Lemma 1.5].

Proof of Proposition 11.5. We need to show that $C_q^+ \subset C_q^+(\sigma_n)$ for every n. We use the Birkhoff ergodic theorem. Take an ergodic invariant probability measure μ for the straightline flow on M_q, let ν be a transverse measure corresponding to μ as in Proposition 2.3, and let β_ν be the corresponding foliation cocycle. Since M_q has no horizontal saddle connections, ν is non-atomic, the horizontal straighline flow on M_q is minimal, and C_q^+ is the convex cone generated by the foliation cocycles β_ν arising in this way. Take a horizontal leaf ℓ which lies on a generic horizontal straightline trajectory for μ. This implies that ℓ intersects any transverse system infinitely many times. Genericity means that for a transverse arc γ, $\nu(\gamma) = \lim_{S\to\infty} \frac{1}{S}\#(\gamma \cap \ell_S)$, where ℓ_S is a piece of the leaf starting at some fixed point on ℓ and of length S (and the limit exists). Let σ_n' be a connected component of σ_n which intersects ℓ infinitely many times. Then we can find a sequence of intersections of ℓ and σ_n' such that the horizontal lengths of subsegments of ℓ between consecutive intersections grow longer and longer. Closing up these segments along σ_n' gives longer and longer σ_n-almost horizontal loops, and taking the Poincaré dual of a renormalized sum of a large number of them gives a sequence approaching ν (as can be seen by evaluating these sums on closed loops γ). This implies $\beta_\nu \in C_q^+(\sigma_n)$. □

11.2.3. *Transverse systems in $\mathcal{E}$ and $\mathcal{H}(1,1)$.* We now specialize to $\mathcal{H}(1,1)$ and specify the collection of transverse systems $\{\sigma_n\}$ explicitly. Recall our convention that singularities for a surface in $\mathcal{H}(1,1)$ are labeled. Each $q \in \mathcal{H}(1,1)$ has two vertical prongs issuing from the first singular point in a downward direction, and we denote by $\bar{\sigma}_t$

the union of the corresponding vertical segments of length e^{-t}. On any compact subset of $\mathcal{H}(1,1)$ there is a lower bound on the length of a shortest saddle connection, and so for t large enough the vertical prongs do not hit singular points and so $\bar{\sigma}_t$ is well-defined. If M_q is horizontally minimal then each horizontal leaf intersects $\bar{\sigma}_t$ and in particular each horizontal separatrix starting at a singularity has a first intersection with $\bar{\sigma}_t$. Denote by $\varepsilon = \varepsilon(q,t)$ the maximal length, along $\bar{\sigma}_t$, of a segment starting at a singularity and ending at the first intersection of a horizontal separatrix ξ with $\bar{\sigma}_t$. Let $\hat{\sigma}_t \subset \bar{\sigma}_t$ be the union of the two vertical prongs taken of length ε. Note that $\hat{\sigma}_t$ is a transverse system on M_q if M_q is horizontally minimal, but some non-minimal surfaces have horizontal leaves that miss $\hat{\sigma}_t$.

Fix an adapted neighborhood $\mathcal{U}$, and recall that by choosing a connected component of $\pi^{-1}(\mathcal{U})$, we can equip all $q \in \mathcal{U}$ with a marking map (up to equivalence), and this identifies each C_q^+ with a cone in $H^1(S,\Sigma;\mathbb{R}_x)$. For those $q \in \mathcal{U}$ for which M_q has no horizontal saddle connections, the marking map also determines the cone $C_q^+(\hat{\sigma}_t)$ as a cone in $H^1(S,\Sigma;\mathbb{R}_x)$. We denote it by $\widetilde{C}_q^+(t)$ in order to lighten the notation. Since $\hat{\sigma}_t$ is invariant under the map ι, this identification does not depend on the choice of the marking map (within its equivalence class). As in Corollary 3.3 let $H^1(S,\Sigma;\mathbb{R}^2) = T(\mathcal{E}) \oplus \mathscr{N}(\mathcal{E})$ be the decomposition into ι invariant and anti-invariant classes. By Corollary 4.5, a balanced signed foliation cocycle belongs to $\mathscr{N}_x(\mathcal{E})$. As in the proof of Proposition 3.5, let $\bar{\pi} : \mathcal{E} \to \mathcal{H}(0)$ be the projection which maps a surface $q \in \mathcal{E}$ to the torus $M_q/\langle\iota\rangle$, and forgets the marked point (one of the two endpoints of the slit) corresponding to the second singular point of M_q.

The area-one condition in the definition of $\mathcal{E}$ means that $\mathcal{E}$ is not a linear space. For our proof we will need to cover $\mathcal{E}$ by convex subsets, and in order to make the notion of convexity meaningful we work locally, as follows. Recall that $\mathcal{U} \subset \mathcal{E}$ is an adapted neighborhood (in $\mathcal{E}$) if it is the intersection of $\mathcal{E}$ with an adapted neighborhood in the stratum. In this case there is a triangulation τ of S such that any connected component of $\pi^{-1}(\mathcal{U})$ is contained in the intersection of the set V_τ (as in §2.2) with the fixed point set of the involution described in Proposition 3.1, and with the locus of area-one surfaces. Let $q \in \mathcal{U}$ and fix a marking map of $\varphi : S \to q$ representing a surface $\widetilde{q} \in V_\tau$. Let $\Phi = \Phi_q$ be the map which sends $x \in T_q(\mathcal{E})$ to the surface q' satisfying $\mathrm{hol}(\widetilde{q}') = c(\mathrm{hol}(\widetilde{q}) + x)$, where $\widetilde{q}'$ is given by the marking map determined by φ and τ (see §2.2) and the rescaling factor c is chosen so that the surface q' has area one. A *convex adapted neighborhood* of

q is $\Phi(\mathcal{W})$ where $\mathcal{W}$ is an open convex subset of $T_q(\mathcal{E})$ so that $\Phi|_{\mathcal{W}}$ is a homeomorphism onto its image, which is contained in $\mathcal{U}$. When discussing diameters, convex sets, etc., we will do this with respect to the linear structure on $\mathcal{W}$. We say that a collection $\mathcal{J}$ of convex subsets of a convex adapted neighborhood is a *weak convex partition* if the interiors $\{J^\circ : J \in \mathcal{J}\}$ are disjoint, and the union of closures $\bigcup_{J\in\mathcal{J}} \bar{J}$ covers all horizontally minimal surfaces in $\mathcal{U}$.

It is clear from definitions that for $t \in \mathbb{R}$,

$$\widetilde{C}^+_{g_{-t}q}(0) = g_{-t}\left(\widetilde{C}^+_q(t)\right). \tag{11.6}$$

Let $\widetilde{\mathcal{E}}_{\mathrm{m}} = \pi^{-1}(\mathcal{E})$, and let $\widetilde{\mathcal{E}}_{\mathrm{m},t}$ denote the surfaces in $\widetilde{\mathcal{E}}_{\mathrm{m}}$ which have no vertical saddle connections of length at most e^{-t}, and for which every horizontal straightline leaf intersects $\hat{\sigma}_t$. Note that for these surfaces, the cone $\widetilde{C}^+_q(t)$ is well-defined, that the set of horizontally minimal marked surfaces in $\widetilde{\mathcal{E}}$ is contained in $\bigcup_{t>0}\widetilde{\mathcal{E}}_{\mathrm{m},t}$, and that a collection of horizontally minimal marked surfaces belonging to a compact subset of $\widetilde{\mathcal{E}}$ is contained in $\widetilde{\mathcal{E}}_{\mathrm{m},t}$ for all t small enough. For each t we define a partition $\mathcal{J}_t$ of $\widetilde{\mathcal{E}}_{\mathrm{m},t}$ into t-equivalence classes, with the property that t-equivalent surfaces $\widetilde{q}_1, \widetilde{q}_2$ have $\hat{\sigma}_t$-almost horizontal segments which are homotopic and have the same intersection pattern with $\hat{\sigma}_t$.

More precisely, we let $\xi_1(q), \ldots, \xi_k(q)$ be the horizontal segments from Σ to $\hat{\sigma}_t$ on M_q, not intersecting Σ in their interiors, and continued along $\hat{\sigma}_t$ so that they end at points of Σ. In the situation at hand, of surfaces in $\mathcal{H}(1,1)$, we have $k = 8$ since there are four horizontal prongs issuing from each of the two singularities. By choice of the orientations, we have

$$\mathrm{hol}^{(y)}_q(\xi_j(q)) > 0 \quad \text{for each } j. \tag{11.7}$$

Since $\hat{\sigma}_t$ and the collection of $\xi_j(\widetilde{q})$ is invariant under the involution ι, there are two indices j realizing the maximum in the definition of $\varepsilon(q,t)$, and we permute indices so that $\xi_2 = \iota(\xi_1)$ and

$$\mathrm{hol}^{(y)}_q(\xi_1(q)) = \mathrm{hol}^{(y)}_q(\xi_2(q)) = \max_j \mathrm{hol}^{(y)}_q(\xi_j(q)) \leqslant e^{-t}. \tag{11.8}$$

We add two more segments ξ_9, ξ_{10} which are continuations of ξ_1, ξ_2 from their intersection with $\hat{\sigma}_t$, to the interior of $\hat{\sigma}_t$, and we switch the orientation of ξ_9, ξ_{10} so that (11.7) continues to hold.

We choose an equivalence class of marking maps $\widetilde{q} \in \pi^{-1}(q)$. By ι-invariance we can think of the $\xi_j(\widetilde{q})$ as representing paths on the topological marking surface (S, Σ). We say that $\widetilde{q}_1$ and $\widetilde{q}_2$ are *t-equivalent* if, possibly after permuting the indices j, for $i = 1, 2$ the paths $\xi_j(\widetilde{q}_i)$ represent the same homotopy classes when pulled back to S, (11.7) and

(11.8) continue to hold, and the order of intersections of the ξ_j with each connected component of $\hat{\sigma}_t$ is the same.

Recall the cell complex associated with $\hat{\sigma}_t$, discussed in the proof of Lemma 11.4. This complex gives a polygon decomposition of M_q into rectangles, with vertical and horizontal sides being subsegments of $\hat{\sigma}_t$ and concatenations of some of the ξ_j. From this it is easy to see that any $\hat{\sigma}_t$-almost horizontal segment on M_q is homotopic to a concatenation of some of the ξ_j. This implies that if $\widetilde{q}_1, \widetilde{q}_2$ are t-equivalent then there is a bijection between the homotopy classes represented by their $\hat{\sigma}_t$-almost horizontal segments, which preserves the order in which they intersect the transverse system.

Note that the definition of t-equivalence only involved the intersection pattern of certain horizontal and vertical lines on the surface. From this, and the rescaling properties of the geodesic flow, we obtain the equivariance property

$$\widetilde{q} \in J \in \mathcal{J}_t \iff g_{-t}\widetilde{q} \in g_{-t}(J) \in \mathcal{J}_0. \tag{11.9}$$

Lemma 11.7. *Let $\mathcal{U} \subset \mathcal{E}$ be a convex adapted neighborhood, and let $\mathcal{V} \subset \mathcal{E}_{\mathrm{m}}$ be a connected component of $\pi^{-1}(\mathcal{U})$. Then for all large enough t, the partition*

$$\left\{\pi\left(\mathcal{V} \cap \bar{J}\right) : J \in \mathcal{J}_t\right\} \tag{11.10}$$

is a weak convex partition of $\mathcal{U}$. For $J \in \mathcal{J}_t$, surfaces in the boundary $\bar{J} \smallsetminus (\bar{J})^\circ$ have horizontal saddle connections, and are either horizontally non-minimal, or horizontally uniquely ergodic.

Lemma 11.7 gives some geometrical control over the elements of the partition $\mathcal{J}_t$; and in light of Proposition 11.5, the same partition can also be used in order to control the direction of foliation cocycles.

Proof. Since $\mathcal{U}$ is precompact, there is a lower bound on the length of a vertical saddle connection of surfaces in $\mathcal{U}$, so for all large enough t, $\mathcal{U} \cap \pi(\widetilde{\mathcal{E}}_{\mathrm{m},t})$ contains the set of horizontally minimal surfaces in $\mathcal{U}$. Since the sets $J \in \mathcal{J}_t$ give a partition of $\mathcal{E}_{\mathrm{m},t}$, in order to show that the sets in (11.10) form a weak convex partition of $\mathcal{U}$, we only need to show that each of the sets in (11.10) is convex, and that the interiors of these sets are disjoint.

By construction of $\mathcal{V}$ and $\mathcal{U}$, the map $\pi|_{\mathcal{V}} : \mathcal{V} \to \mathcal{U}$ is injective, modulo the local group, and for each $q \in \mathcal{U}$ we denote by $\widetilde{q}$ its preimage in $\mathcal{V}$. Then $\widetilde{q}'$ belongs to the t-equivalence class J of $\widetilde{q}$ if the following hold:

- all horizontal leaves on the underlying surface $M_{q'}$ intersect the transverse system $\hat{\sigma}_t$;

- formulas (11.7), (11.8) hold for q' (possibly up to permutation);
- for all i, j,

$$\mathrm{hol}^{(y)}_{\widetilde{q}}(\xi_i) > \mathrm{hol}^{(y)}_{\widetilde{q}}(\xi_j) \iff \mathrm{hol}^{(y)}_{\widetilde{q}'}(\xi_i) > \mathrm{hol}^{(y)}_{\widetilde{q}'}(\xi_j). \tag{11.11}$$

The first of these conditions holds if the horizontal foliation on $M_{q'}$ is minimal, which holds for a dense set of surfaces. Conditions (11.7) and (11.11) involve inequalities between holonomies and thus give convex conditions in period coordinates. Therefore the set $(\bar{J})^\circ$ is precisely the set of surfaces satisfying the inequalities in (11.7) and (11.11). This implies that the sets $\{\bar{J} : J \in \mathcal{J}_t\}$ are convex, and their interiors $\{(\bar{J})^\circ : J \in \mathcal{J}_t\}$ are disjoint.

For the last assertion, let $q \in \bar{J} \smallsetminus (\bar{J})^\circ$. Then on M_q there are two ξ_i, ξ_j with the same vertical holonomy; their concatenation gives a horizontal saddle connection. Applying the translation automorphism ι we get at least two horizontal saddle connections on M_q, and now results about surfaces in eigenform loci, summarized in [BSW, Thm. 7.13], show that there are three possibilities for the horizontal foliation: M_q could have a horizontal cylinder decomposition, could be made of two horizontally minimal tori glued along a slit, or could be horizontally uniquely ergodic. ☐

We note that Lemma 11.7 remains true, with a very similar proof, if $\mathcal{E}$ is replaced by any G-invariant locus, and $\hat{\sigma}_t$ is replaced with any transverse system satisfying the equivariance property (11.9). We now use the additional structure of $\mathcal{E}$ in order to state and prove bounds on the objects associated with a transverse system.

Lemma 11.8. *Let $\mathcal{U} \subset \mathcal{E}$ be a convex adapted neighborhood, let $\mathcal{J}_t$ be the partitions as in Lemma 11.7, let $K_1 \subset \mathcal{H}(0)$ be compact, and let $a > 0$. If $q \in \mathcal{U} \cap \mathcal{E}^{(\min)}$ is horizontally minimal then there are positive constants c_1 and c_2 (depending on q) such that if $t > 0$ satisfies $g_{-t}\bar{\pi}(q) \in K_1$ (where $\bar{\pi} : \mathcal{E} \to \mathcal{H}(0)$ is the projection defined at the beginning of §11.2.3), then the following hold:*

(a) *The length of each $\hat{\sigma}_t$-almost horizontal loop is at least $c_1 e^t$, and the inradius of J is at least $c_1 e^{-2t}$, where $J \in \mathcal{J}_t$ is the partition element containing q.*

Suppose furthermore that q is not horizontally uniquely ergodic, let P^- be the projection onto the orthocomplement of involution invariant classes as in §2.3, and let $\widetilde{C}^+_q(t) = C^+_q(\hat{\sigma}_t)$ as above. Then

(b)
$$P^-\left(\left\{\beta \in \widetilde{C}^+_q(t) : L_q(\beta) \leqslant a\right\}\right) \tag{11.12}$$
is contained in a rectangle with diameter in the interval $[c_1, c_2]$.

(c) *The rectangle in (b) can be chosen so that one of its sides has length bounded above by $c_2 e^{-2t}$.*

Proof. In order to obtain the bounds in (a), note that the existence of a short σ-almost horizontal segment implies the existence of a short saddle connection. Note also that the transverse system $\hat{\sigma}_t$ is the preimage under $\bar{\pi}$ of a transverse system σ_0 on the torus $\bar{\pi}(M_q)$. Using the affine comparison map $\psi_{g_{-t}}$ corresponding to g_{-t} as in §2.4, we can consider the image of this transverse system on $g_{-t}\bar{\pi}(q)$. If $g_{-t}\bar{\pi}(q) \in K_1$ there exists c_1' depending only on K_1 so that any almost-horizontal loop, with respect to a transverse system of bounded length, has length at least c_1'. Considering the effect of the map $\psi_{g_{-t}}^{-1}$, we obtain the required lower bound on the length of a $\hat{\sigma}_t$-almost horizontal segment on M_q. Now take some lower bound c_1'' for the inradius of an element J in the partition $\mathcal{J}_0$, satisfying $\bar{\pi} \circ \pi(J) \cap K_1 \neq \varnothing$. Such a lower bound exists because K_1 is compact and the collection $\mathcal{J}_0$ is locally finite. By (11.9), we can pull back to $\mathcal{J}_t$ using g_t and use (2.12) to obtain the lower bound of $c_1'' e^{-2t}$ on the inradius of elements of $\mathcal{J}_t$. Taking $c_1 = \min(c_1', c_1'')$ we obtain (a).

We now prove assertion (b). Note that here c_1, c_2 are allowed to depend on q. The continuity of L_q, the fact that $\widetilde{C}_q^+(t)$ is a finitely generated convex cone, and the fact that $L_q(\beta) > 0$ when β is a σ-almost horizontal loop, imply that the set $\left\{\beta \in \widetilde{C}_q^+(t) : L_q(\beta) \leqslant a\right\}$ is compact. Hence so is the set appearing in (11.12). Boundedness follows from the properness of the metric dist. Since q admits an essential tremor, there is $\beta_0 \in \widetilde{C}_q^+$ for which $P^-(\beta_0) \neq 0$ and this implies the lower bound in (b).

The proof of (c) combines the upper bound in (b), the effect of renormalization by the flow g_t, and the fact that the action of g_t preserves the Lebesgue measure on $\mathscr{N}_x(\mathcal{E})$, the real part of the normal bundle. In the proof of (c) we will write $A \ll B$ if A and B are two quantities depending on several parameters and $A \leqslant CB$ for some constant C (the implicit constant) independent of these parameters. In this proof the implicit constant is allowed to depend on q but not on t.

It follows from Proposition 2.2 and Corollaries 3.3 and 4.5, that the projection P^- is defined over $\mathbb{Q}$. This implies that P^1 maps the lattice of $\mathbb{Z}$-points $H^1(S;\mathbb{Z}_x)$ to a sublattice Λ in $\mathscr{N}_x(\mathcal{E},\mathbb{Z}) \stackrel{\text{def}}{=} \mathscr{N}_x(\mathcal{E}) \cap H^1(S,\Sigma;\mathbb{Z})$.

Let M_t be the underlying surface of $g_{-t}q$ and denote by $\psi_t : M_q \to M_t$ the affine comparison map defined in §2.4. Let $\mathscr{L}(q)$ and $\mathscr{L}(g_{-t}q)$

denote respectively the set of reduced $\hat{\sigma}_t$- (resp., $\psi_t(\hat{\sigma}_t)$-) almost horizontal loops on q (resp., on $g_{-t}q$). By Lemma 11.4, for $\mathscr{L}$ equal to either of $\mathscr{L}(q)$ and $\mathscr{L}(g_{-t}q)$, we have that $\{\beta_\gamma : \gamma \in \mathscr{L}\}$ contains a basis of $H^1(S;\mathbb{Z})$, and hence the projection $P^-(\{\beta_\gamma : \gamma \in \mathscr{L}\})$ generates Λ. Let Ψ_t be the map $q \mapsto g_{-t}q$. By choosing a marking map $\varphi : S \to M_q$ and using $\psi_t \circ \varphi$ as a marking map for M_t, this induces a map $\bar{\Psi}_t : H^1(S,\Sigma;\mathbb{R}^2) \to H^1(S,\Sigma;\mathbb{R}^2)$. Since the map ι of Proposition 3.1 commutes with the map ψ_t, the map P^- commutes with Ψ_t, and hence we have the following diagram:

$$\begin{array}{ccc} H^1(S,\Sigma;\mathbb{R}_x) \cong T_q\mathcal{U} & \xrightarrow{\bar{\Psi}_t} & H^1(S,\Sigma;\mathbb{R}_x) \cong T_{g_{-t}q}\mathcal{H} \\ \Big\downarrow P^- & & \Big\downarrow P^- \\ \mathscr{N}_x(\mathcal{E}) & \xrightarrow{\bar{\Psi}_t|_{\mathscr{N}_x(\mathcal{E})}} & \mathscr{N}_x(\mathcal{E}) \end{array}$$

The preceding discussion shows that $\bar{\Psi}_t(\Lambda) = \Lambda$, and therefore

$$\left|\det\left(\bar{\Psi}_t|_{\mathscr{N}_x(\mathcal{E})}\right)\right| = 1. \tag{11.13}$$

Similarly to (11.6), we have an equivariance relation

$$\mathscr{L}(q) = \bar{\Psi}_t^{-1}(\mathscr{L}(g_{-t}q)).$$

Also, as in Proposition 6.2, we have that for $\beta \in \mathcal{T}_q$, if we set $\beta' \stackrel{\text{def}}{=} \bar{\Psi}_t(\beta)$ then $L_{g_{-t}q}(\beta') = e^{-t}L_q(\beta)$. This gives

$$\begin{aligned} & P^-\left(\left\{\beta \in \widetilde{C}_q^+(t) : L_q(\beta) \leqslant a\right\}\right) \\ =& \bar{\Psi}_t^{-1} \circ P^- \circ \bar{\Psi}_t\left(\left\{\beta \in \widetilde{C}_q^+(t) : L_q(\beta) \leqslant a\right\}\right) \\ =& \bar{\Psi}_t^{-1} \circ P^-\left(\left\{\beta' \in \widetilde{C}_{g_{-t}q}^+(0) : L_{g_{-t}q}(\beta') \leqslant e^{-t}a\right\}\right) \\ \subset& \bar{\Psi}_t^{-1}\left(\left\{\beta'' \in \mathscr{N}_x(\mathcal{E}) : \|\beta''\| \ll e^{-t}a\right\}\right), \end{aligned}$$

where the bound in the last inclusion follows from Proposition 6.7 and Lemma 8.3, and the fact that $\bar{\pi}(g_{-t}q) \in K_1$ and on compact sets, the metric dist is bi-Lipschitz to any norm in period coordinates. Thus, using (11.13), the set in the left hand side of (11.12) is a convex subset of $\mathscr{N}_x(\mathcal{E})$ of area $\ll e^{-2t}$. On the other hand, by (b), it contains a vector of length $\gg 1$. This means that it is contained in a rectangle whose small sidelength is $\ll e^{-2t}$, as claimed. □

11.2.4. *Preparations for proving the upper bound: Nondivergence estimates.* Masur's criterion states that if the vertical foliation on a surface M_q is not uniquely ergodic then $g_tq \to \infty$ as $t \to \infty$. In this paper we are dealing with horizontal foliations so we have that if the horizontal foliation on M_q is not uniquely ergodic then $g_{-t}q \to \infty$ as $t \to \infty$; i.e., the backward trajectory eventually leaves every compact set. The following result gives (for a fixed surface) an upper bound for the measure of directions in which the orbit has escaped a large compact set by a fixed time.

Proposition 11.9 (Athreya)**.** *For any stratum $\mathcal{H}$ there is $\delta > 0$, and a compact subset $K \subset \mathcal{H}$ such that for any compact set $Q \subset \mathcal{H}$ and any $T_0 > 0$ there is $C > 0$ so that for all $q \in Q$ and all $T > 0$, we have*

$$\left|\left\{\theta \in \mathbb{S}^1 : \forall t \in [T_0, T_0 + T],\ g_{-t} r_\theta q \notin K\right\}\right| \leqslant C e^{-\delta T}.$$

The formulation given above is stronger than the statement of [At, Thm. 2.2]. Namely, in [At], the constant C is allowed to depend on q, while we claim that C can be chosen uniformly over the compact set Q. One can check that the stronger Proposition 11.9 follows from the proof given in [At]. Alternatively, one can derive it from [AAEKMU, Prop. 3.7]. Indeed, in the notation of [AAEKMU], set $\delta = \frac{2}{3}$, $a < 2^{-\frac{5}{2}} C_1^{-\frac{3}{2}}$ and $C = a^{-2T_0} C(x)$, and note that $C(x)$ is uniform when x ranges over a compact set, and for $N > \frac{2T_0}{t}$ we have

$$Z\left(X_{\leqslant M}, N, 1, \frac{2}{3}\right) \supset \{q : \alpha(g_t q) \leqslant M \text{ for all } T_0 \leqslant t \leqslant N\}.$$

Remark 11.10. *Proposition 11.9 is convenient for our covering arguments because if we take a compact set K' whose interior contains K, and slightly larger, and if $g_{-t} q \notin K'$ for all $t \in [T_0, T + T_0]$,then for q' in a small neighborhood of q we have $g_{-t} q' \notin K$ for all $t \in [T_0, T + T_0]$. Applying Proposition 11.9 to K' we have exponential decay (in T) of the measure of a neighborhood of the set we are covering.*

11.2.5. *Proof of upper bound.* We now prove the upper bound, assuming Proposition 11.3, which will be proved in the next section.

Proof of the upper bound in Theorem 1.9. We divide the argument into steps.

Step 1: Reduction to $\mathcal{SF}^{(\min)}_{(\leqslant a)} \cap \mathrm{NUE}$.

For each $H_0 > 0$, the set $\overline{\bigcup_{H \leqslant H_0} \mathcal{E}^{(\mathrm{tor}, H)}}$ is a proper submanifold of $\mathcal{E}$ with boundary (in the closure we pick up surfaces made of identical periodic tori glued along an embedded slit). On $\bigcup_{H \leqslant H_0} \mathcal{E}^{(\mathrm{tor}, H)}$, if $\beta \in \mathcal{T}_q^{(0)}$ satisfies $|L|_q(\beta) = s$, the map $(q, s) \mapsto \mathrm{trem}_\beta(q)$ performs symmetric horocycle shears in opposite directions corresponding to u_s and u_{-s} on the two tori which are connected components of the complement of the slit. Therefore this map is locally Lipschitz for the metric coming from any norm in period coordinates, and as in the proof of the lower bound (see the discussion of the map $\bar{f}$), this means it is locally Lipschitz for dist. Thus by Proposition 11.1, taking the union over all $H_0 \in \mathbb{N}$, the subset of $\mathcal{SF}_{(\leqslant a)}$ consisting of tremors of surfaces in $\mathcal{E}^{(\mathrm{tor})} \cup \mathcal{E}^{(\mathrm{per})}$ has Hausdorff dimension at most 5. So we need only bound the Hausdorff dimension of the set of surfaces $\mathrm{trem}_\beta(q)$ where q is horizontally minimal and non-uniquely ergodic, i.e., bound the dimension of the essential tremors in $\mathcal{SF}_{(\leqslant a)}$. Note that by Lemma 11.7, the collections of such surfaces is covered by the sets $\{(\bar{J})^\circ : J \in \mathcal{J}_t\}$ for all sufficiently large t.

Step 2: A countable cover. In light of Proposition 11.1(2), it is enough to cover $\mathcal{SF}_{(\leqslant a)}$ by countably many subsets, and give a uniform upper bound on the Hausdorff dimension of each. The countable collection we will use, which is denoted below by Z, exhausts the set of essential tremors $\mathcal{SF}^{(\min)}_{(\leqslant a)} \cap \mathrm{NUE}$, and depends on several parameters: the adapted subset in $\mathcal{E}$ containing the surface q for which $\mathrm{trem}_\beta(q) \in SF^{(\min)}_{(\leqslant a)}$, the return time under g_{-t} to a certain compact set K', and constants coming from Lemma 11.8.

To make this precise, define

$$\mathcal{E}' \stackrel{\mathrm{def}}{=} \{q \in \mathcal{E} : M_q \text{ admits an essential tremor}\},$$

and write $\mathcal{H}$ for $\mathcal{H}(1,1)$. Let $\delta > 0$ and $K \subset \mathcal{H}$ be a compact set as in Proposition 11.9. We assume with no loss of generality that $\delta < 1$. Let dist be the metric of §2.6 and let

$$K' \stackrel{\mathrm{def}}{=} \{q \in \mathcal{H}(1,1) : \mathrm{dist}(q, K) \leqslant 1\}.$$

By Proposition 2.5, K' is compact.

We can cover $\mathcal{E}'$ with countably many convex adapted neighborhoods with compact closures. Given such a convex adapted neighborhood $\mathcal{U} \subset \mathcal{E}$, and given a parameter $T_0 > 0$, let $C = C(\mathcal{U}, T_0)$ be as in Proposition 11.9 with $Q \stackrel{\mathrm{def}}{=} \overline{\mathcal{U}}$. If $q \in \mathcal{U} \cap \mathcal{E}'$ and $\beta \in \mathcal{T}_q^{(0)}$, there are $c_1 = c_1(q)$, $c_2 = c_2(q)$ so the conclusions of Lemma 11.8 are satisfied. Masur's criterion [MaTa] applied to the horizontal foliation of M_q implies that the trajectory $\{g_{-t}q : t > 0\}$ is divergent, and in particular, there is $T_1(q)$ such that for all $t \geqslant T_1(q)$, $g_{-t}q \notin K'$. For each $\mathcal{U}$ in the above countable collection, each $T_0 \in \mathbb{N}$, and each $c \in \mathbb{N}$ with $c \geqslant C(\mathcal{U}, T_0)\, e^{\delta T_0}$, let $Z = Z(\mathcal{U}, T_0, c)$ denote the set of tremors $\mathrm{trem}_\beta(q)$ where $q \in \mathcal{U} \cap \mathcal{E}'$ and $\beta \in \mathcal{T}_q^{(0)}$ satisfy the bounds

$$|L|_q(\beta) \leqslant a, \quad T_1(q) \leqslant T_0, \quad c_2(q) \leqslant c, \quad c_1(q) \geqslant \frac{1}{c}.$$

Then in light of Proposition 11.1(2) it suffices to show that

$$\dim Z \leqslant 6 - \frac{\delta}{5}. \tag{11.14}$$

Step 3: Applying Proposition 11.3.

Let $K_1 \subset \mathcal{H}(0)$ be a compact set so that for each $q \in \mathcal{H}(0)$ for which the horizontal foliation is aperiodic, the set of return times $\{t \in \mathbb{N} : g_{-t}q \in K_1\}$ is unbounded. The choice of K_1 ensures that for any $T > 0$,

$$Z \subset \bigcup_{t \in \mathbb{N},\, t \geqslant T_0} Z(t),$$

where

$$Z(t) \stackrel{\mathrm{def}}{=} \left\{\mathrm{trem}_{q,\beta} \in Z : q \in \mathcal{U} \cap \mathcal{E}',\, \beta \in \mathcal{T}_q^{(0)},\, g_{-t}\bar{\pi}(q) \in K_1\right\}.$$

Let

$$X(t) \stackrel{\mathrm{def}}{=} \{q \in Z \cap \mathcal{E}' : g_{-t}\bar{\pi}(q) \in K_1\}.$$

We now check that all the conditions of Proposition 11.3 are satisfied. We first check (iii). By (2.12) and the definition of K' we see that for any $q_0 \in \mathcal{N}^{(e^{-2t})}(q)$ and $t \geqslant T_1(q_0)$ we must have $g_{-t}q_0 \notin K$. Thus if $\mu_{\mathcal{E}}$ denotes the flat measure on $\mathcal{E}$, Proposition 11.9 and a Fubini argument show that for each $t \in \mathbb{N}$,

$$\mu_{\mathcal{E}}\left(\mathcal{N}^{(e^{-2t})}\left(\{q \in \mathcal{U} \cap \mathcal{E}' : T_1(q) \leqslant T_0\}\right)\right) \leqslant Ce^{\delta T_0}\, e^{-\delta t}, \tag{11.15}$$

where $C = C(\mathcal{U}, T_0)$.

We now check conditions (i) and (ii). Using Lemmas 11.7 and 11.8, for each t define finitely many convex sets $J_i(t)$ of inradius at least $c_1 e^{-2t}$ which cover $X(t)$ and for which the map $q \mapsto \widetilde{C}_q^+(t)$ is constant on $J_i(t)$, and set

$$X_i(t) \stackrel{\mathrm{def}}{=} X(t) \cap J_i(t)$$

and

$$Y_i(t) \stackrel{\mathrm{def}}{=} \bigcup_{q \in X_i(t)} P^-\left(\left\{\beta \in \widetilde{C}_q^+(t) : L_q(\beta) \leqslant a\right\}\right).$$

With these definitions, it follows from Lemma 11.8 (with $c_2 = c = 1/c_1$) that all conditions of Proposition 11.3 are satisfied and the result follows. $\square$

12. Effective covers of convex sets

In this section we prove Proposition 11.3. First, we briefly outline the idea of the proof. The main difficulty is to find efficient covers of $\bigcup_i X_i(t)$ by small balls of a fixed radius. If the intersection of a ball with one of the sets $J_i(t)$ appearing in (i) has significant measure, it will contribute significantly to our cover, and it follows from (iii) that the number of such balls is not too large (see (12.7)). The subset of $\bigcup_i X_i(t)$ not covered by such balls requires more work, and in particular, the key technical result Corollary 12.3.

In this section the notation $|A|$ may mean one of several different things: if $A \subset \mathbb{R}^d$ then $|A|$ denotes the Lebesgue measure of A. Let $\mathbb{S}^{d-1}$ denote the $d-1$ dimensional unit sphere in $\mathbb{R}^d$, then for $A \subset \mathbb{S}^{d-1}$, $|A|$ denotes the measure of A with respect to the unique rotation invariant probability measure on $\mathbb{S}^{d-1}$. If $A \subset \mathbb{R}^d \times \mathbb{S}^{d-1}$, then $|A|$ denotes the measure of A with respect to the product of these measures.

The next Proposition contains the main geometric idea, and implies Corollary 12.3 via standard covering arguments for Euclidean spaces. The Proposition provides power law savings for the measure of the subset of a convex set K for which the ball centered at such a point intersects K in small measure.

Proposition 12.1. *For any $d \geqslant 2$ there are positive constants c, C, depending only on d, such that for any compact convex set $K \subset \mathbb{R}^d$ with inradius $R > 0$, and any $\varepsilon \in (0,1)$, the set*

$$K^{(\varepsilon)} \stackrel{\text{def}}{=} \left\{x \in K : |B(x, \varepsilon R) \cap K| \leqslant c(\varepsilon R)^d\right\}$$

satisfies

$$\left|K^{(\varepsilon)}\right| \leqslant C\varepsilon^2 |K|.$$

We briefly discuss the proposition and its proof. Observe that the condition of being in $K^{(\varepsilon)}$ is more restrictive than being near the boundary of K. For example, if K is a line segment then $K^{(\varepsilon)}$ is empty for small enough ε. It turns out to be useful to think of convex sets in two dimensions, and the main idea of the proof is to reduce the problem to a two-dimensional statement via polar coordinates. The two-dimensional case is proved by comparing the measure of a 'bulk' (which is denoted by K' in the proof) to a quantity that bounds $K^{(\varepsilon)}$.

Since the statement of Proposition 12.1 is invariant under homotheties, we can and will assume that $R = 1$. For $\psi \in \mathbb{S}^{d-1}$, and $x \in \mathbb{R}^d$, let $\tau_\psi(x) \stackrel{\text{def}}{=} \{x + s\psi : s \in \mathbb{R}\}$ be the line through x in direction ψ, and let

$$K^{(\varepsilon)}(\psi) \stackrel{\text{def}}{=} \left\{x \in K^{(\varepsilon)} : |\tau_\psi(x) \cap K| < \varepsilon\right\}.$$

Lemma 12.2. *For any $d \geqslant 2$ there is a positive constant c so that for any $\varepsilon \in (0,1)$, there is $\psi \in \mathbb{S}^{d-1}$ such that*

$$\left|K^{(\varepsilon)}(\psi)\right| \geqslant \frac{\left|K^{(\varepsilon)}\right|}{2}. \tag{12.1}$$

Proof. Let $c = \frac{1}{2^{d+2}d}$, and suppose $x \in K^{(\varepsilon)}$, so that $|B(x,\varepsilon) \cap K| \leqslant c\varepsilon^d$. For each $\theta \in \mathbb{S}^{d-1}$, we write

$$T_\theta(x) = |\tau_\theta(x) \cap K| \quad \text{and} \quad \rho(\theta) = \sup\{s > 0 : x + s\theta \in K\}.$$

Then $\max(\rho(\theta), \rho(-\theta)) \geqslant \frac{T_\theta(x)}{2}$. Computing the volume of $B(x,\varepsilon) \cap K$ in polar coordinates, we have

$$\begin{aligned} c\varepsilon^d \geqslant & |B(x,\varepsilon) \cap K| = \int_{\mathbb{S}^{d-1}} \int_0^{\rho(\theta)} r^{d-1} dr\, d\theta \\ \geqslant & \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_0^{\frac{T_\theta(x)}{2}} r^{d-1} dr d\theta \geqslant \frac{1}{2^{d+1}d} \int_{\mathbb{S}^{d-1}} T_\theta(x)^d d\theta. \end{aligned}$$

So by Markov's inequality and the choice of c,

$$|\{\theta \in \mathbb{S}^{d-1} : T_\theta(x) < \varepsilon\}| \geqslant \frac{1}{2}. \tag{12.2}$$

Now consider the set

$$A \stackrel{\text{def}}{=} \left\{(x,\theta) \in K^{(\varepsilon)} \times \mathbb{S}^{d-1} : T_\theta(x) < \varepsilon\right\}.$$

From (12.2) and Fubini we have

$$\frac{\left|K^{(\varepsilon)}\right|}{2} \leqslant |A| = \int_{\mathbb{S}^{d-1}} \left|K^{(\varepsilon)}(\theta)\right| d\theta.$$

Thus for some $\psi \in \mathbb{S}^{d-1}$ we have (12.1). □

Proof of Proposition 12.1. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the standard basis of $\mathbb{R}^d$ and let p_0 be a point for which $B(p_0, 1) \subset K$. Applying a rotation and a translation, we may assume that $p_0 = 0$ and $\psi = \mathbf{e}_d$, where ψ is as in Lemma 12.2. We will make computations in cylindrical coordinates, i.e. we will consider the sphere $\mathbb{S}^{d-2}$ as embedded in span $(\mathbf{e}_1, \ldots, \mathbf{e}_{d-1})$ and write vectors in $\mathbb{R}^d$ as $r\theta + z\mathbf{e}_d$. In these coordinates, d-dimensional Lebesgue measure is given by $\alpha r^{d-2}\, dr\, d\theta\, dz$, where $d\theta$ is the rotation invariant probability measure on $\mathbb{S}^{d-2}$ and $\alpha = \alpha_{d-1}$ is a constant. For each $\theta \in \mathbb{S}^{d-2}$, define

$$\rho_\theta = \sup\{r \in \mathbb{R} : r\theta \in K\} \text{ and } f_\theta(r) = |\tau_{\mathbf{e}_d}(r\theta) \cap K|,$$

i.e., $f_\theta(r)$ is the length of the intersection with K of the vertical line through $r\theta$. Let

$$K' = K \cap \left\{ r\theta + z\mathbf{e}_d : r \in \left[\frac{\rho_\theta}{3}, \frac{2\rho_\theta}{3}\right]\right\}.$$

Since K is convex, the function f_θ is concave, and since $B(0,1) \subset K$, $f_\theta(0) \geqslant 1$. This implies that whenever $r\theta + z\mathbf{e}_d \in K^{(\varepsilon)}(\mathbf{e}_d)$, $r \geqslant (1-\varepsilon)\rho_\theta$. Furthermore, whenever $r\theta + z\mathbf{e}_d \in K'$ we have $f_\theta(r) \geqslant \frac{1}{3}$. Clearly $f_\theta(r) \leqslant \varepsilon$ whenever there is z for which $r\theta + z \in K^{(\varepsilon)}$, and hence

$$\begin{aligned}
\left|K^{(\varepsilon)}(\mathbf{e}_d)\right| &\leqslant \alpha \int_{\mathbb{S}^{d-2}} \int_{(1-\varepsilon)\rho_\theta}^{\rho_\theta} \varepsilon r^{d-2} dr\, d\theta \\
&\leqslant \alpha\varepsilon \int_{\mathbb{S}^{d-2}} \int_{(1-\varepsilon)\rho_\theta}^{\rho_\theta} \rho_\theta^{d-2} dr\, d\theta = C'\alpha\varepsilon^2 \int_{\mathbb{S}^{d-2}} \int_{\frac{\rho_\theta}{3}}^{\frac{2\rho_\theta}{3}} r^{d-2} dr\, d\theta \\
&\leqslant C'\alpha\varepsilon^2 3 \int_{\mathbb{S}^{d-2}} \int_{\frac{\rho_\theta}{3}}^{\frac{2\rho_\theta}{3}} f_\theta(r) r^{d-2} dr\, d\theta = 3C'\varepsilon^2|K'|,
\end{aligned}$$

where

$$C' = \frac{3^{d-1}(d-1)}{2^{d-1}-1}.$$

Since $K' \subset K$, we have shown

$$\left|K^{(\varepsilon)}(\mathbf{e}_d)\right| \leqslant 3C'\varepsilon^2|K|. \tag{12.3}$$

Now taking $C = 6C'$, recalling that $\psi = \mathbf{e}_d$, and combining Lemma 12.2 with (12.3) we obtain the desired result. $\square$

Let $N(A, R)$ denote the minimal number of balls of radius R needed to cover $A \subset \mathbb{R}^d$.

Corollary 12.3. *For any $d \geqslant 2$ there exist positive constants $\bar{c}, \bar{C}$ so that if $K \subset \mathbb{R}^d$ is a convex set with inradius R then the set*

$$K^{(\varepsilon,\bar{c})} \stackrel{\text{def}}{=} \{x \in K : |B(x, \varepsilon R) \cap K| < \bar{c}\, |B(x, \varepsilon R)|\} \tag{12.4}$$

satisfies

$$N\left(K^{(\varepsilon,\bar{c})}, \varepsilon R\right) \leqslant \bar{C}\, |K|\, \varepsilon^{2-d} R^{-d}.$$

Proof. Let $K^{(\varepsilon)}, c, C$ be as in Proposition 12.1, and let $\bar{c}$ be small enough so that

$$\bar{c}\ |B(x, \varepsilon R)| < c \left(\frac{\varepsilon}{2}\, R\right)^d.$$

This choice ensures that if $x \in K^{(\varepsilon,\bar{c})}$ and $y \in B\left(x, \frac{\varepsilon}{2}R\right)$ then $y \in K^{(\varepsilon/2)}$; i.e., $B\left(x, \frac{\varepsilon}{2}R\right) \subset K^{(\varepsilon/2)}$. Let $B_1, \ldots, B_N$ be a minimal collection of balls of radius εR which cover $K^{(\varepsilon,\bar{c})}$ and have centers $x_1, \ldots, x_N$ in $K^{(\varepsilon,\bar{c})}$.

Then for each i, $\left|B_i \cap K^{(\varepsilon/2)}\right| \geqslant \left|B\left(x_i, \frac{\varepsilon}{2}R\right)\right| = \kappa\varepsilon^d R^d$ for a constant κ depending on d. By the Besicovitch covering theorem (see e.g. [Mat, Chap. 2]), each point in $K^{(\varepsilon,\bar{c})}$ is covered at most N_d times, where N_d is a number depending only on d. Therefore,

$$\begin{aligned} N\kappa\varepsilon^d R^d = & \sum_{i=1}^{N}\left|B\left(x_i, \frac{\varepsilon}{2}R\right)\right| \leqslant \sum_{i=1}^{N}\left|B_i \cap K^{(\varepsilon/2)}\right| \\ \leqslant & N_d\left|K^{(\varepsilon/2)}\right| \leqslant N_d C\frac{\varepsilon^2}{4}|K|, \end{aligned}$$

where we used Proposition 12.1 for the last inequality. Setting $\bar{C} = \frac{N_d C}{4\kappa}$ we obtain the required estimate. □

We are now ready for the

Proof of Proposition 11.3. For each $t \in \mathbb{N}$ we will find an efficient cover of $Z(t)$ by balls of radius $e^{-\left(2+\frac{\delta}{2}\right)t}$, from which we will derive the Hausdorff dimension bound. We will lighten the notation by writing $\hat{N}(P,t)$ for $N\left(P, e^{-\left(2+\frac{\delta}{2}\right)t}\right)$. We will continue with the notation $A \ll B$ used in the proof of Lemma 11.8, and write $A \asymp B$ if $A \ll B$ and $B \ll A$. In this proof the implicit constant is allowed to depend on $d, c_1, c_2, \delta, P_1, P_2$.

We claim that

$$\hat{N}(Z(t), t) \ll e^{\left(\left(2+\frac{\delta}{2}\right)(d+1)-\frac{\delta}{2}\right)t}. \tag{12.5}$$

To prove (12.5), we will find an efficient cover for each set $X_i(t)$ and each $Y_i(t)$, and combine them. By assumption (ii), $\hat{N}(Y_i(t), t) \ll e^{\left(2+\frac{\delta}{2}\right)t}\, e^{\frac{\delta}{2}t} = e^{(2+\delta)t}$ for each i. Indeed, the first term in this product comes from covering the long side, of length $\ll 1$, and the second term is needed for covering the short side of length $\ll e^{-2t}$. So it suffices to show

$$\sum_i \hat{N}(X_i(t), t) \ll e^{\left(\left(2+\frac{\delta}{2}\right)d-\delta\right)t}. \tag{12.6}$$

With the notation of (12.4) define

$$J_i'(t) \stackrel{\text{def}}{=} J_i^{\left(e^{-\frac{\delta}{2}t}, \bar{c}\right)}.$$

We will consider the sets $\bar{X}_i(t) = X_i(t) \smallsetminus J_i'(t)$ and $X_i(t) \cap J_i'(t)$ separately, finding efficient covers for each. If $x \in \bar{X}_i(t)$ then

$$\begin{aligned} & \left|B\left(x, e^{-\left(2+\frac{\delta}{2}\right)t}\right) \cap J_i(t) \cap \mathcal{N}^{\left(e^{-2t}\right)}(X_i(t))\right| \\ \asymp & \left|B\left(x, e^{-\left(2+\frac{\delta}{2}\right)t}\right)\right| \asymp e^{-d\left(2+\frac{\delta}{2}\right)t}. \end{aligned} \tag{12.7}$$

Let $\left\{B_j^{(i)}\right\}_j$ be a minimal collection of balls of radius $e^{-\left(2+\frac{\delta}{2}\right)t}$ centered at points in $\bar{X}_i(t)$ needed to cover $\bar{X}_i(t)$. By the Besicovitch covering theorem, the collection $\left\{B_j^{(i)}\right\}$ has bounded multiplicity, i.e. for each x and i, $\#\left\{j : x \in B_j^{(i)}\right\} \ll 1$. Since the $J_i(t)$ are disjoint, the collection $\mathcal{B}_t = \left\{B_j^{(i)} \cap J_i(t)\right\}_{i,j}$ is also of bounded multiplicity. Taking into account (12.7), we have

$$\sum_i \hat{N}\left(\bar{X}_i(t), t\right) \ll \#\mathcal{B}_t \ll e^{d\left(2+\frac{\delta}{2}\right)t} \left|\bigcup_i \mathcal{N}^{(e^{-2t})}(X_i(t))\right| \overset{\text{(iii)}}{\ll} e^{\left(d\left(2+\frac{\delta}{2}\right)-\delta\right)t}. \tag{12.8}$$

We also have from Corollary 12.3 (with $R = e^{-2t}$ and $\varepsilon = e^{-\frac{\delta}{2}t}$) that

$$\begin{aligned} \sum_i \hat{N}\left(J_i'(t), t\right) &\ll \sum_i e^{\frac{\delta}{2}(d-2)t} e^{2dt} |J_i(t)| \\ &\ll e^{\left(\left(2+\frac{\delta}{2}\right)d-\delta\right)t} \left|\bigcup_i J_i(t)\right| \ll e^{\left(\left(2+\frac{\delta}{2}\right)d-\delta\right)t}. \end{aligned} \tag{12.9}$$

Combining the estimates (12.8) and (12.9), we obtain (12.6), and thus (12.5).

We now prove (11.4). Let

$$s > d + 1 - \frac{\delta}{5}$$

and set

$$s' \overset{\text{def}}{=} \frac{\delta}{2} - \left(2 + \frac{\delta}{2}\right) \cdot \frac{\delta}{5} > 0 \tag{12.10}$$

(where we have used $\delta < 1$). We need to show that for any $\eta > 0$, we can cover Z by a collection of balls $\mathcal{B}$ of radius at most η, so that $\sum_{B\in\mathcal{B}} \operatorname{diam}(B)^s \ll 1$. To this end, choose T so that $e^{-\left(2+\frac{\delta}{2}\right)T} < \eta$. For each $t \geqslant T$ let $\mathcal{B}_t$ be a collection of $\hat{N}(Z(t), t)$ balls of radius $e^{-\left(2+\frac{\delta}{2}\right)t}$ covering $Z(t)$ and let $\mathcal{B} = \bigcup_t \mathcal{B}_t$. Then by (12.5) we have

$$\begin{aligned} \sum_{B\in\mathcal{B}} \operatorname{diam}(B)^s &\ll \sum_{t \geqslant T} \hat{N}(Z(t), t) e^{-\left(2+\frac{\delta}{2}\right)st} \\ &\ll \sum_{t\geqslant T} e^{\left(\left(2+\frac{\delta}{2}\right)(d+1)-\frac{\delta}{2}-\left(2+\frac{\delta}{2}\right)\left(d+1-\frac{\delta}{5}\right)\right)t} = \sum_{t\geqslant T} e^{-s't} \to_{T\to\infty} 0. \end{aligned}$$

So for large enough T we have our required cover. $\square$

13. Atomic tremors

In this section we complete the proof of Proposition 4.1 (and thus of Corollary 4.4). We recall that in §4.2, these results were already proved in a special case (namely assuming (4.10)), and that this special case is sufficient for the proofs of Theorems 1.5, 1.8 and 1.9.

We note that in the literature there are several different conventions regarding atomic transverse measures. Recall from the second paragraph of §2.5 that in this paper, atomic transverse measures can only be supported on loops of a certain kind. As we will now see, these loops arise on boundaries of cylinders, but also arise as 'ghosts of departed cylinders', that is loops comprised of finitely many horizontal saddle connections, which are not boundaries of cylinders, but might represent core curves of cylinders on nearby surfaces. We first define these loops precisely and give our definition of atomic transverse measures. We then explain why this is a natural definition from our point of view.

We say that a finite, cyclically ordered collection of horizontal saddle connections $\delta_1, \ldots, \delta_t$ *forms a loop* if the right endpoint of δ_i is the left endpoint of δ_{i+1} (addition mod t). Any singular point $\xi \in \Sigma$ of degree a, is contained in a neighborhood $\mathcal{U}_\xi$ naturally parameterized by polar coordinates $(r\cos\theta, r\sin\theta)$, for $0 \leqslant r < r_0$ and $\theta \in \mathbb{R}/(2\pi(a+1)\mathbb{Z})$, where $r = 0$ corresponds to ξ (see [BSW, §2.5]). If $\xi \in \Sigma$ is a right endpoint of δ_i and a left endpoint of δ_{i+1}, we can parameterize the intersections of δ_i, δ_{i+1} with $\mathcal{U}_\xi$ using polar coordinates, and the *i-th turning angle* is the difference in angle between δ_i and δ_{i+1}. The turning angle is well-defined modulo $2\pi(a+1)\mathbb{Z}$ and is an odd multiple of π. We say that the loop is *continuously extendable* if for each i the i-th turning angle is $\pm\pi$, and we say it is *primitive* if whenever we have a repetition $\delta_i = \delta_j$, $i \neq j$, we must have that the turning angle at both of the endpoints of δ_i differs in sign from that of δ_j. Thus on each surface there are only finitely many primitive continuously extendable loops.

In this paper we will always assume that the atomic part of a transverse measure is a finite linear combination of Dirac masses on periodic trajectories or on primitive continuously extendable loops. Furthermore, the mass of a saddle connection δ (i.e., the measure assigned to it by the transverse measure) in such an atomic measure is obtained as follows. Writing $\ell_k = \left(\delta_1^{(k)}, \ldots, \delta_{s_k}^{(k)}\right)$ for the primitive extendable loops in the support of the measure, there are numbers a_k such that the mass of δ is $\sum_k a_k \,\#\left\{i : \delta_i^{(k)} = \delta\right\}$.

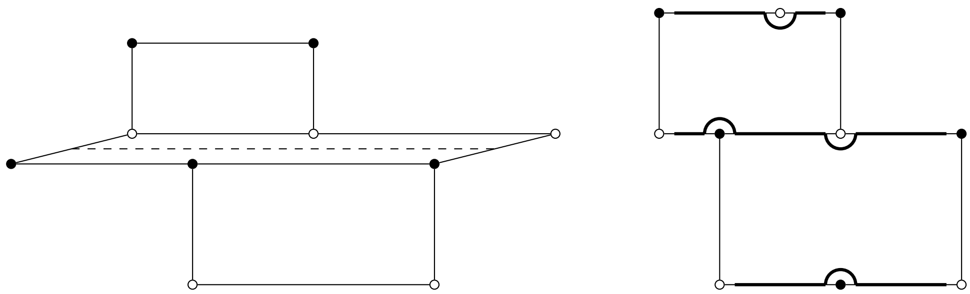

FIGURE 12. The union of all horizontal saddle connections on the surface shown on the right, is a continuously extendable loop. The half-circles extending this curve to the punctured surface are shown. This curve is the 'ghost' of a core curve of a cylinder on a nearby surface (shown on the left).

We now motivate this definition. The leaves of the horizontal foliation $\mathcal{F}$ on a surface M_q have a natural metric inherited from the 1-form dx on the plane, and we say that a leaf is *critical* if it is incomplete with respect to this metric (and then its completion contains a singularity). A bounded critical leaf is a horizontal saddle connection, a bounded non-critical leaf is periodic, and an unbounded critical leaf is isometric to a ray and is called a *separatrix*. Since transverse measures are assumed to be a system of finite measures, and infinite leaves (critical or noncritical leaves) have nontrivial accumulation points, the atoms of a transverse measure cannot be on infinite leaves. Thus the cases left to discuss are of transverse measures with atoms on periodic horizontal leaves and on horizontal saddle connections. We will be interested in transverse measures ν which are naturally associated with cohomology classes $\beta_\nu \in H^1(M_q, \Sigma_q; \mathbb{R})$ (recall from §1.2 that tremor mappings are determined by the cohomology class β_ν). As we explained in §2.5, if ν is non-atomic, it determines a 1-cochain on $H_1(M_q, \Sigma_q)$, and this gives the assignment $\nu \mapsto \beta_\nu$; the definition of tremor maps depends only on the cohomology class β_ν (see (1.5)). If ν is atomic we will define β_ν as a cohomology class rather than an explicit cochain. The *continuous extension* $\check{\delta}$ of δ is a continuous closed curve homotopic to δ with all its points in $S \smallsetminus \bigcup_\xi \mathcal{U}_\xi$, which is the same as δ outside the neighborhoods $\mathcal{U}_\xi$, and such that for each i, the intersection of δ_i, δ_{i+1} with $\mathcal{U}_\xi$ is replaced with a curve on $\partial\mathcal{U}_\xi$ corresponding to $r = r_0$ and θ in an interval of length π. See Figure 12.

The assigment $\nu \mapsto \beta_\nu$ is now defined as follows. By linearity, we only need to define this assignment in the case that ν assigns unit mass to one primitive continuously extendable loop or to one closed horizontal leaf. In the former case we let $\check{\delta}$ denote the continuous extension of the continuously extendable loop, and in the latter case we let $\check{\delta}$ denote the horizontal periodic loop supporting the measure. These are closed loops avoiding Σ_q so represent elements of $H_1(M_q \smallsetminus \Sigma_q)$, and hence, by Poincaré-Lefschetz duality, of $H^1(M_q, \Sigma_q; \mathbb{R})$.

The assignment $\nu \mapsto \beta_\nu$ is not injective, indeed atomic transverse measures supported on distinct homotopic closed horizontal curves yield the same cohomology class. It follows from Proposition 2.4 that for non-atomic tremors, horizontal cylinders are the only source of non-injectivity, that is, when q has no horizontal cylinders, the map $\nu \mapsto \beta_\nu$ is injective when restricted to non-atomic transverse measures. Furthermore, when q does have a horizontal cylinder C with area A, the Dirac atomic measure on any core curve of C defines the same cohomology class as $\frac{1}{A}dy|_C$. That is, atomic measures on periodic leaves represent cohomology classes that can also be represented by non-atomic transverse measures.

Let $\alpha \mapsto \beta_\nu(\cdot)$ be the evaluation map. It is clear from the definition in §2.5, that if ν is non-atomic and α is represented by a concatenation of horizontal saddle connections, then $\beta_\nu(\alpha) = 0$. With our definition of β_ν, we also have $\beta_\nu(\alpha) = 0$ if ν is atomic and α is a cycle represented by a horizontal closed path because the path may be homotoped away from horizontal saddle connections. However it is possible to have continuously extendable loops α and δ such that for the atomic tremor β_ν associated with δ we have $\beta_\nu(\alpha) \neq 0$.

If it happens that α is represented by a concatenation of positively oriented horizontal saddle connections and $s_0 \stackrel{\text{def}}{=} \beta_\nu(\alpha) \neq 0$, then the tremor to time $-s_0$ will not be defined; indeed, if the surface $q' = \mathrm{trem}_{-s_0\beta_\nu}(q)$ existed, then using (4.8), we would have $\mathrm{hol}_{q'}(\alpha) = (0, 0)$, which is impossible. For instance, in Figure 12, this situation will arise if α is the class represented by the horizontal saddle connections in the middle of the diagram. This shows why the requirement in Proposition 4.13 that the tremor is non-atomic, is essential. Finally we note that the positivity property $L_q(\beta_\nu) > 0$ (see the first paragraph of §4.1.2) extends to cocycles arising from atomic transverse measures.

13.1. **Refining an APD.** Our discussion of tremors for atomic transverse measures will rely on the construction in §4.2. Recall from §4.2 that an APD for q is a polygon decomposition of the underlying surface M_q, into triangles and quadrilaterals, such that the quadrilaterals contain a horizontal diagonal. We consider all edges of an APD as open, i.e. they do not contain their endpoints. In order to pay attention to atomic measures, we further subdivide each edge of an APD into finitely many subintervals by removing the points that lie on horizontal saddle connections. We will denote by J_i these open intervals lying on edges of an APD. We will refer to an APD whose edges have been additionally subdivided as above, as a *refined APD*. For each i, each polygon P with $J_i \subset \partial P$, and each $x \in J_i$, we define the opposite point $\mathrm{opp}_P(x)$ as in §4.2.

Let $J = J_{i_0}$ for some i_0, $J \subset \partial P$, and let $J' = \mathrm{opp}_P(J)$. Then J' is a union of either one or two of the intervals J_i, for $i \neq i_0$, depending on whether a point of J has an opposite point in Σ. In the former case we set $J_0 = J$ and in the latter case we set $J_0 = J \smallsetminus \mathrm{opp}_P^{-1}(\Sigma)$. With these definitions $\mathrm{opp}_P|_{J_0} : J_0 \to J'$ is a bijection. Note that each endpoint of J lies on a horizontal saddle connection or in Σ, and each endpoint of J_0 is either an endpoint of some J_i or lies on an infinite critical leaf.

Let ν be a transverse measure on M_q whose atoms, if any, are on non-critical periodic trajectories. It assigns a measure to each of the intervals J, J', J_0, and by our condition that any atoms lie on periodic trajectories, the restriction to J has the same mass as the restriction to J_0. The measures will be denoted by $\nu_J, \nu_{J'}, \nu_{J_0}$. They satisfy the invariance property (4.11).

Conversely, given a refined APD for a translation surface q, suppose we are given a collection of finite measures ν_J on the edges J as above, satisfying the invariance property. Since an infinite leaf has an accumulation point in one of the J, by the invariance property, any atoms of the measures ν_J lie on finite leaves. The points of M_q lying on horizontal saddle connections are not in any of the J's, and thus we can reconstruct from the ν_J a transverse measure all of whose atoms (if any) are on periodic trajectories. The cohomology class β corresponding to this transverse measure satisfies $\beta(E) = \sum_{J \in E} \nu_J(J)$ for any edge E of the APD.

13.2. **Proof of Proposition 4.1.** We will use the same proof strategy as in §4.2. Namely, we will use APD's to describe transverse measures as measures on the edges of the APD, and discuss what happens to measures when taking limits. In this section, we will have to be more careful in treating limits of atomic measures.

Proof of Proposition 4.1 under two simplifying assumptions. We first give the proof for atomic tremors which are limits of non-atomic ones, and with no escape of mass (as defined below). Let $\widetilde{q}_n \to \widetilde{q}$ and $\beta_n \to \beta$ be as in the statement of the Proposition, let $q_n = \pi(\widetilde{q}_n), q = \pi(q)$ be the projections to $\mathcal{H}$, and let M_{q_n}, M_q the underlying surfaces. As in §4.2, we can assume that $\widetilde{q}_n$ and $\widetilde{q}$ are represented by marking maps

$\varphi_n \to M_{q_n}$, $\varphi : S \to M_q$ such that $\varphi_n \circ \varphi^{-1}$ is piecewise affine with derivative tending to Id as $n \to \infty$. Let $K \subset M_q$ denote any one of the intervals J, J', J_0 in a refined APD for q. Our first simplifying assumption is that each $\beta_n = \beta_{\nu_n}$ for some non-atomic transverse measure ν_n.

Let $\nu_K^{(n)}$ denote the measure on K given by the pushforward of ν_n under $\varphi \circ \varphi_n^{-1}$, and denote the total variation of $\nu_K^{(n)}$ by $m_K^{(n)}$. This number can be expressed as the evaluation of β_n on a path $\sigma = \sigma_K$ from singular points to singular points that is a concatenation of K with parts of horizontal saddle connections. Since $\beta_n \to \beta$, we have $m_K^{(n)} \to_{n\to\infty} m_K = \beta(\sigma)$. Let $\widetilde{K} = \varphi^{-1}(K) \subset S$. Since K is open and not horizontal, $\widetilde{K}$ has a natural compactification $\bar{K}$ in which we add bottom and top endpoints $x_K^{\mathrm{b}}, x_K^{\mathrm{t}}$ to $\widetilde{K}$. Note that we consider $\bar{K}$ abstractly, and not as a subset of S. Because the $\nu_K^{(n)}$ are non-atomic, each measure $\nu_K^{(n)}$ can be viewed as a measure on the compact interval $\bar{K}$, assigning mass zero to endpoints. Passing to further subsequences, we can assume each sequence $\left(\nu_K^{(n)}\right)_n$ converges to a measure $\nu_{\bar{K}}$ on $\bar{K}$ such that $\nu_K = \nu_{\bar{K}}|_K$. We have

$$m_K = \nu_{\bar{K}}(\bar{K}) = \nu_K(\widetilde{K}) + \mathrm{e}_K^{\mathrm{b}} + \mathrm{e}_K^{\mathrm{t}}, \tag{13.1}$$

where the numbers $\mathrm{e}_K^{\mathrm{b}} = \nu_{\bar{K}}(x_K^{\mathrm{b}})$, $\mathrm{e}_K^{\mathrm{t}} = \nu_{\bar{K}}(x_K^{\mathrm{t}})$ record the escape of mass to endpoints. We can concretely express the $\mathrm{e}_K^{\mathrm{b,t}}$ by subdividing K into two half-intervals $K^{\mathrm{b}}, K^{\mathrm{t}}$ whose common endpoint is an interior point of K which has zero measure under ν_K. In these terms

$$\mathrm{e}_K^{\mathrm{b}} = \lim_{n\to\infty} \nu_K^{(n)}(K^{\mathrm{b}}) - \nu_K(K^{\mathrm{b}}) \tag{13.2}$$

(and this limit does not depend on the decomposition $K = K^{\mathrm{b}} \cup K^{\mathrm{t}}$).

Since the collection of measures $\{\nu_K\}$ satisfies the invariance property, it defines a transverse measure, and we let β' be the corresponding cohomology class.

Our second simplifying assumption is that there is no escape of mass, i.e. all the $\mathrm{e}_K^{\mathrm{b,t}}$ are 0. Using the fact that each β_n is non-atomic, for each edge E of the refined APD we have:

$$\beta(E) \leftarrow \beta_n(E) = \sum_K m_K^{(n)} \to \sum_K m_K \overset{(13.1)}{=} \sum_K \nu_K(\widetilde{K}) = \beta'(E),$$

where the sum ranges over open intervals $K \subset E$ covering all but finitely many points of E. In this case we have shown that $\beta = \beta'$ corresponds to a transverse measure, and we are done. This establishes the statement under our simplifying assumptions. ∎

In order to treat the case that some of the $e_K^{b,t}$ are positive, we will need to record additional information about the invariance property satisfied by the measures $\nu_{\bar{K}}$. Informally, if we have escape of mass to a singular point ξ, we will want to record the angular sector of length π at ξ, bounded by horizontal sides, to which the mass escaped. In order to formalize this, it will be useful to use boundary-marked surfaces (see [BSW, §2.5]). Let $\check{S} \to \check{q}$ be a blown up marked version of the marked surface $S \to q$. Let $\xi \in \Sigma$ and recall that $\check{q}$ replaces ξ with a circle parameterized by an angular variable θ taking values in $\mathbb{R}/(2(a+1)\pi\mathbb{Z})$, where a is the order of ξ. Each θ will be called a *prong at* ξ which can be thought of as the tangent direction of an infinitesimal line segment of angle $\theta \mod 2\pi\mathbb{Z}$ ending at ξ. The infinitesimal line is horizontal if and only if $\theta \in \pi\mathbb{Z}$. In a similar way we can blow up nonsingular points of S, replacing them with a circle parameterized by $\mathbb{R}/2\pi\mathbb{Z}$, and thus talk about the prongs at a regular point (this corresponds to a singularity of order $a = 0$). For each $k \in \mathbb{Z}/(2(a+1)\pi\mathbb{Z})$, and each ξ two prongs at ξ are called *bottom-adjacent* (resp. *top-adjacent*) if their angular parameter belongs to the same interval $[k\pi, (k+1)\pi]$ with k even (resp. odd), and adjacent if they are either bottom- or top-adjacent. By definition of an APD, at each ξ and each k, there is at least one edge E with an endpoint in $(k\pi, (k+1)\pi)$.

We have compactified the line segments K corresponding to J, J_0, J' as above by abstract points x_K^b, x_K^t, and these points map to points in $\check{S}$ by continuously continuing the embedding $\widetilde{K} \to \check{S}$. We will denote these points in $\check{S}$ by their angular parameters $\theta_K^{b,t}$ and call them *prongs of the APD*. Any point which is a regular point on the surface $\widetilde{q}$, and which is on the interior of an edge J (in the above notation, points of $J \smallsetminus J_0$), will only be the endpoint of one top prong and one bottom prong, and the adjacency classes of these prongs will be singletons. In order to keep the notation consistent we will still refer to these endpoints as prongs, although we do not need to mark these points or blow up $\widetilde{q}$ at these points.

Since the APD contains no horizontal segments, $\theta_K^{b,t} \notin \pi\mathbb{Z}$. Note that for k even (resp. odd), all prongs of the APD with angular parameter in $(k\pi, (k+1)\pi)$ are of form θ_K^b (resp. θ_K^t). In the preceding discussion (see formula (13.2)), we have associated to each of these prongs an 'escape of mass' quantity $e_K^{b,t}$.

Claim 1:

(1) The weights of prongs of the APD only depend on their adjacency class. More precisely, if K, K' are edges of the APD with

bottom- (resp. top-) adjacent prongs $\theta_K^{\mathrm{b}}, \theta_{K'}^{\mathrm{b}}$ (resp. $\theta_K^{\mathrm{t}}, \theta_{K'}^{\mathrm{t}}$) then $\mathrm{e}_K^{\mathrm{b}} = \mathrm{e}_{K'}^{\mathrm{b}}$ (resp. $\mathrm{e}_K^{\mathrm{t}} = \mathrm{e}_{K'}^{\mathrm{t}}$).

(2) For any horizontal saddle connection σ, let ξ_1, ξ_2 in S be consecutive points of σ lying on edges of the APD (the ξ_i could either be singular points or interior points of edges of the APD which are endpoints of subintervals K). For $i = 1, 2$, let $\theta_i^{(\sigma)}$ represent the two prongs of σ at ξ_i, and let K_i (resp. L_i) be intervals with prongs at ξ_i which are part of the APD, such that θ_{K_i} (resp. θ_{L_i}) is bottom- (resp. top-) adjacent to $\theta_i^{(\sigma)}$. Then

$$\mathrm{e}_{K_1}^{\mathrm{b}} + \mathrm{e}_{L_1}^{\mathrm{t}} = \mathrm{e}_{K_2}^{\mathrm{b}} + \mathrm{e}_{L_2}^{\mathrm{t}}. \tag{13.3}$$

(3) If a horizontal prong adjacent to $\theta_K^{\mathrm{b,t}}$ is on an infinite critical leaf then $\mathrm{e}_K^{\mathrm{b,t}} = 0$.

Proof of Claim 1: Because adjacent prongs are in the same $(k\pi, (k+1)\pi)$ interval of direction, they are exchanged by opp_P and so statement (1) follows from (13.2) and the invariance property of the measures ν_K. To see (2), note that the assumption that ξ_i are consecutive along σ means that K_1, L_1, K_2, L_2 are both subintervals of edges of one polygon P for the APD, with $\mathrm{opp}_P(K_1) = K_2$ and $\mathrm{opp}_P(L_1) = L_2$. By (13.2) we have

$$\mathrm{e}_{K_i}^{\mathrm{b}} + \mathrm{e}_{L_i}^{\mathrm{t}} = \lim_{n\to\infty} \left(\nu_{K_i}^{(n)}(K_i^{\mathrm{b}}) + \nu_{L_i}^{(n)}(L_i^{\mathrm{t}})\right) - \left(\nu_{K_i}(K_i^{\mathrm{b}}) + \nu_{L_i}(L_i^{\mathrm{t}})\right)$$

for each i, and (13.3) follows from the invariance property of each of the $\nu_K^{(n)}$ on $K_1^{\mathrm{b}}, L_1^{\mathrm{t}}, K_1^{\mathrm{b}} \cup L_1^{\mathrm{t}}$.

For (3), any critical leaf ℓ intersects some interval J of the APD in its interior infinitely many times. If $\mathrm{e}_K^{\mathrm{b,t}} \neq 0$ for a prong $\theta_K^{\mathrm{b,t}}$ adjacent to a prong defined by an endpoint of ℓ, we obtain infinitely many atoms in the interior of J, and by the invariance property, they all have the same ν_J-mass. This contradicts the finiteness of the measure ν_J. △

We can now interpret extendable loops for boundary marked surfaces using our notion of adjacency: an extendable loop is a loop formed as a concatenation of saddle connections which are bottom- or top-adjacent at each of their endpoints. Thus each meeting of consecutive saddle connections represents an adjacency class and we say that δ *represents* each of the classes defined by these meeting points. By (1), the loss of mass parameters $\mathrm{e}_K^{\mathrm{b,t}}$ assign numbers $\mathrm{e}_{\mathcal{A}}$ to each bottom/top adjacency class $\mathcal{A}$. The following claim shows that these numbers can be expressed in terms of extendable loops.

Claim 2: Suppose that some of the $\mathrm{e}_K^{\mathrm{b,t}}$ are positive. Then there is an integer $s \geqslant 0$, primitive extendable loops $\delta^{(j)}$, $j = 1, \dots, s$ and finitely many positive real numbers $c_1, \dots, c_s$ such that for each adjacency class $\mathcal{A}$,

$$\mathrm{e}_{\mathcal{A}} = \sum_{\delta^{(j)} \text{ represents } \mathcal{A}} c_j. \tag{13.4}$$

Proof of Claim 2. Choose the adjacency class $\mathcal{A}_1$ for which

$$\mathrm{e}_{\mathcal{A}_1} = \min\{\mathrm{e}_{\mathcal{A}} : \mathrm{e}_{\mathcal{A}} > 0\}.$$

By a straightforward induction, it suffices to show that M_q contains a primitive extendable loop, such that all the adjacency classes $\mathcal{A}$ represented by this loop satisfy $\mathrm{e}_{\mathcal{A}} \geqslant \mathrm{e}_{\mathcal{A}_1}$. To see this, let $\bar{\delta}_1$ be an outgoing (i.e., right-pointing) prong in $\mathcal{A}_1$. According to item (3) of Claim 1, $\bar{\delta}_1$ is the initial point of a horizontal saddle connection δ_1. Let $\mathcal{A}_2^{\mathrm{b,t}}$ be the two adjacency classes of the terminal point of δ_1. Then according to (13.3), at least one of $\mathrm{e}^{\mathrm{b,t}}_{\mathcal{A}_2^{\mathrm{b,t}}}$ is positive, and hence is bounded below by $\mathrm{e}_{\mathcal{A}_1}$. We choose δ_2 to lie on an outgoing prong of this adjacency class, and continue, finding $\delta_1, \delta_2, \dots$ which form an extendable loop, such that the adjacency class $\mathcal{A}_i$ represented by the meeting of δ_i, δ_{i+1} has $\mathrm{e}_{\mathcal{A}_i} \geqslant \mathrm{e}_{\mathcal{A}_1}$. We have completed the proof of the claim. △

Completion of the proof of Proposition 4.1. We continue with the notation used in the case of no escape of mass. Recall that β' denotes the cohomology class corresponding to the limiting transverse measure on the interior of edges of the APD. We still assume that $\beta = \lim_{n\to\infty} \beta_n$ is a limit of cohomology classes corresponding to non-atomic transverse measures. We now show

$$\beta - \beta' = \sum_{j=1}^{s} c_j \beta^{(j)}, \tag{13.5}$$

where for $j = 1, \dots, s$, $\beta^{(j)}$ is the class Poincaré dual to the primitive extendable loop $\delta^{(j)}$ provided by Claim 2. Indeed, it is enough to check this identity by evaluating on the paths $\alpha = \sigma_K$ introduced in the paragraph above (13.1), since such paths represent cycles which generate $H_1(M_q, \Sigma_q)$. For such paths, (13.5) is immediate from (13.1) and (13.4). Equation (13.5) completes the proof of Proposition 4.1, under the assumption that the β_n are non-atomic.

For the general case, for each n, write $\beta_n = \beta_n^{\mathrm{na}} + \beta_n^{\mathrm{at}}$ as a sum of cohomology classes represented by non-atomic and atomic tremors respectively. Since $L_{q_n}(\beta_n^{\mathrm{na}}) \leqslant L_{q_n}(\beta_n) \to L_q(\beta)$, the sequence $L_{q_n}(\beta_n^{\mathrm{na}})$ is bounded and hence the sequence $(\beta_n^{\mathrm{na}})_{n\in\mathbb{N}} \subset H^1(S, \Sigma; \mathbb{R})$ is also bounded. Therefore we can pass to subsequences to assume $\beta_n^{\mathrm{na}} \to_{n\to\infty} u_1$ and hence $\beta_n^{\mathrm{at}} \to_{n\to\infty} u_2 = \beta - u_1$. By what we have already shown, u_1 is a signed foliation cocycle. Now write β_n^{at} as a sum $\sum_{j=1}^{s_n} c_j^{(n)} I(\cdot, \check{\delta}_n^{(j)})$ where $\delta_n^{(j)}$ is an extendable loop. Since there is a bound on the number of horizontal saddle connections on a surface in a fixed stratum, we can pass to a subsequence to assume that $s = s_n$ is fixed independently of n and for each $j = 1, \ldots, s$, each $\delta_n^{(j)}$ passes through the same prongs in the same order. Passing to further subsequences we can assume that for each j, $c_j = \lim_{n\to\infty} c_j^{(n)}$ exists. Passing to further subsequences and re-indexing, there is $r \leqslant s$ such that for $j \leqslant r$, the total length of each $\delta_n^{(j)}$ is bounded independently of n, and for $j > r$, the total length of each $\delta_n^{(j)}$ tends to infinity.

Let $j \leqslant r$. Since the lengths of the $\delta_n^{(j)}$ are bounded we can pass to further subsequences to assume that the number of intersection points of each edge E of the APD with each curve $\delta_n^{(j)}$ is fixed. This implies that the sequence of cohomology classes $I(\cdot, \check{\delta}_n^{(j)})$ converges, and the limit is a signed foliation cocycle associated with an atomic transverse measure.

Now let $j > r$, and fix a refined APD for q. Then for all large enough n, the curve $\delta_n^{(j)}$ is longer than all horizontal cylinders or saddle connections on M_q. Thus for all large n, the $\delta_n^{(j)}$ give rise to measures on the sides E of the APD that assign zero measure to endpoints of segments. Thus we can repeat the analysis in the first part of the proof (assuming no escape of mass) to the atomic measures $\delta_n^{(j)}$. We find that their limit corresponds to a transverse measure, as required. □

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