VEECH'S DICHOTOMY AND THE LATTICE PROPERTY

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Abstract. Veech showed that if a translation surface has a stabilizer which is a lattice in $\mathrm{SL}(2,\mathbb{R})$ then any direction for the corresponding constant slope flow is either completely periodic or uniquely ergodic. We show that the converse does not hold: there are translation surfaces which satisfy Veech's dichotomy but for which the corresponding stabilizer subgroup is not a lattice. The construction relies on work of Hubert and Schmidt.

1. Introduction

In a celebrated paper [Ve1] Veech gave precise information about the billiard flow in regular polygons by establishing a link with the properties of the affine automorphism group of the related translation surfaces. Fix a regular polygon P in the plane. A starting point in P and an angle θ determine a billiard trajectory. Veech showed that for any angle θ one of two possibilities occurs:

(1) all trajectories in direction θ are either periodic or, if they hit a vertex in forward time, they also hit a vertex in backward time;
(2) no trajectory in direction θ hits a vertex in forward and backward time and all infinite trajectories are uniformly distributed.

A regular polygon has the property that the angles are rational multiples of π. We call a polygon with this property a rational polygon. Associated to each rational polygon there is a translation surface and the billiard trajectories correspond to geodesics on this translation surface [FoKi]. General references for translation surfaces are [Vo, MaTa]. A direction in which all infinite trajectories are dense is said to be *minimal*. A direction in which all infinite trajectories are uniformly distributed is said to be *uniquely ergodic*. A direction in which all orbits are periodic or saddle connections is said to be *completely periodic*. A translation surface satisfies the *Veech dichotomy* if each direction is either completely periodic or uniquely ergodic.

Veech found a sufficient condition for the Veech dichotomy, in terms of the affine automorphism group of the translation surface. An affine automorphism of a translation surface is a selfmap with constant derivative. Mapping an automorphism to its derivative gives a homomorphism from the group of affine automorphisms to $G = \mathrm{SL}(2, \mathbb{R})$. The image of the affine automorphism group under this map is called the Veech group. Veech showed that the Veech dichotomy holds when the Veech group is a lattice in G. We call such translation surfaces *lattice surfaces* though they are also called *Veech surfaces* in the literature. Examples of lattice polygons known prior to Veech include the square (see [HW]) and arithmetic examples of [Gu]. At the current time there are many known examples of translation surfaces which have the lattice property and many known examples of translation surfaces which do not have the lattice property. This raises the question of whether there might be other ways to establish the Veech dichotomy, or if the lattice property is in fact equivalent to the Veech dichotomy. The purpose of this note is to show that:

Theorem 1. [thm: Veech does not imply Veech] *There is a translation surface which satisfies the Veech dichotomy but is not a lattice surface.*

The proof of Theorem 1 relies on previous work of several authors. In the remainder of this introduction we will review these ideas and reduce the proof of the theorem to some observations about loci of branched covers.

A useful technique for constructing new translation surfaces from old is the branched cover construction. If $\pi : \widetilde{S} \to S$ is a branched cover and S has a translation structure then $\widetilde{S}$ has a natural translation structure where the charts are obtained by pulling back the charts for S. The inverse images of cone points of S are cone points of $\widetilde{S}$ but in addition the ramification points of the map π become cone points for $\widetilde{S}$. The relation between the Veech group of S and that of $\widetilde{S}$ was studied by Hubert and Schmidt [HuSc], building on previous work of several authors (see [Ve2, GuJu, Vo, GuHuSc]). Hubert and Schmidt found examples where S is a lattice surface but the Veech group of $\widetilde{S}$ is not finitely generated. For their construction they define the notion of a *non-periodic connection point* on a translation surface. This is a point p which has an infinite orbit under the Veech group, and which has the property that if one continues any segment from a cone point to p one must reach a singularity. Hubert and Schmidt show that there are surfaces in genus 2 and 3 containing non-periodic connection points, and that if $\widetilde{S} \to S$ is branched over non-periodic connection points in S then the Veech group of $\widetilde{S}$ is infinitely generated. They also observe that these examples satisfy the following weak form of the Veech dichotomy: every saddle connection direction is completely periodic. According to [FoKi] in the remaining directions the flow is minimal. These examples are the starting point for our analysis. To summarize we have the following:

Proposition 2 (Hubert-Schmidt)**.** *[prop: HS] There are lattice surfaces of genus 2 and 3 containing infinitely many non-periodic connection points. If X is such a translation surface, and $\widetilde{X}$ is a translation surface obtained by forming a cover of X branched only at non-periodic connection points, then $\widetilde{X}$ is not a lattice surface, yet has the property that any direction is either completely periodic or minimal.*

We will show that, provided the branching takes place *over a single point*, the flow in the minimal directions is in fact uniquely ergodic and thus these surfaces satisfy the full Veech dichotomy though their Veech groups are not lattices. The next proposition shows that such a surface exists.

Proposition 3. *[prop: construction] For a surface S of genus $g \geq 1$ and $d \geq 3$ there is a surface $\widetilde{S}$ and a branched covering map $\pi : \widetilde{S} \to S$ of degree d where the branch locus is a single point.*

This Proposition is not new (see e.g. [FaKr, §IV.9]) but we have included a direct proof for the reader's convenience. Taking $g = 2$ and $d = 3$ the construction gives a surface of genus 5. Fixing the branch point at an aperiodic connection point as in Proposition 2 we obtain the surface of smallest genus for which our arguments show that it satisfies the Veech dichotomy without the lattice property. It would be interesting to know the minimal genus in which this phenomenon can occur.

In order to prove that minimal directions on such a branched cover are uniquely ergodic, we will use a fundamental result of Masur [Ma]. Masur's theorem involves a moduli space for translation structures which is most easily described in the language of quadratic differentials. A translation structure on a surface S gives rise to a conformal structure on S. If we are given a Riemann surface then the information needed to reconstruct the translation structure is contained in a holomorphic quadratic differential with finite area. (The particular quadratic differentials that come from translation structures correspond to squares of Abelian differentials.) A useful way to construct a moduli space for quadratic differentials on a surface of a fixed genus g is to start with the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. For each surface S in $\mathcal{M}_g$ the collection of holomorphic quadratic differentials on S with finite area is a vector space. The unit area quadratic differentials correspond to a sphere in this vector space. We define $\mathcal{Q}_g$ to be the total space of this sphere bundle over $\mathcal{M}_g$. Also of importance for us will be $\mathcal{M}_{g,b}$ (resp. $\mathcal{Q}_{g,b}$), the moduli space of (quadratic differentials over) Riemann surfaces of genus g with b marked points.

There is a natural action of G on $\mathcal{Q}_g$. We set

$$a_t = \left(\begin{array}{cc} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{array} \right), \quad r_\theta = \left(\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right).$$

A trajectory $\{a_t x : t \geq 0\}$ is called *divergent* if the map $t \mapsto a_t x$ is proper; in other words, the trajectory eventually leaves any compact subset of X. Masur established a relation between the unique ergodicity of the translation flow in the direction θ and the dynamics of the geodesic trajectory $\{a_t r_\theta(q) : t \geq 0\}$, namely:

Proposition 4 (Masur)**.** *[prop: Masur condition] If $q \in \mathcal{Q}$ and the translation flow in the direction θ is not uniquely ergodic then $\{a_t r_\theta(q) : t \geq 0\}$ is divergent.*

Veech used Masur's theorem in his proof that the lattice property implies the Veech dichotomy. When q is a lattice surface with Veech group Γ, we can identify the G-orbit of q with the coset space G/Γ and we observe that the inclusion from the coset space to $\mathcal{Q}_g$ is a proper map. We can view this coset space as the unit tangent bundle of the hyperbolic orbifold $M = SO(2,\mathbb{R})\backslash G/\Gamma$. We can identify the flow a_t with the geodesic flow on this tangent bundle. We can write a hyperbolic orbifold of finite volume as a union of a compact set and a finite number of cusps. Veech showed that the saddle connection directions correspond to geodesics which diverge to cusps and that all other geodesics return infinitely often to the compact piece. In particular for lattice surfaces Masur's condition is necessary and sufficient for unique ergodicity. We summarize this discussion in the following proposition.

Proposition 5 (Veech)**.** *[prop: lattice] For a lattice surface q there is a compact subset $C \subset \Gamma\backslash G$ so that for every minimal direction θ, $a_t r_\theta(q)$ returns to C infinitely often.*

Let $\pi : \widetilde{S} \to S$ be a branched cover and let $\Sigma \subset S$ be the branch points. Below we will introduce the 'locus of branched covers' corresponding to π. This is a moduli space parametrizing the Riemann surfaces $\widetilde{X}$ with quadratic differentials, for which there is a Riemann surface X and a holomorphic map $p : \widetilde{X} \to X$ pulling back the quadratic differential on X to that on $\widetilde{X}$. Furthermore the cover p is *topologically equivalent* to the cover π in that there are diffeomorphisms

$h : S \to X$ and $\widetilde{h} : \widetilde{S} \to \widetilde{X}$ so that $h\pi = p\widetilde{h}$. We will need the following three Propositions:

Proposition 6. [prop: covers] *For a branched cover $\pi : \widetilde{S} \to S$ with b branch points the collection of holomorphic branched covers topologically equivalent to π can be parametrized by a topological space $\mathcal{Q}(\pi)$ with a G-action. We have continuous maps $P : \mathcal{Q}(\pi) \to \mathcal{Q}_{g,b}$ and $R : \mathcal{Q}(\pi) \to \mathcal{Q}_{\widetilde{g}}$ which are G-equivariant.*

Proposition 7. [prop: P proper] *The map P is proper.*

There is a surjective continuous 'forgetful' map $\bar{\Phi}_{g,n} : \mathcal{Q}_{g,n} \to \mathcal{Q}_g$ which maps a surface together with a collection of marked points to the surface alone.

Proposition 8. [prop: proper] *The forgetful map $\bar{\Phi}_{g,1} : \mathcal{Q}_{g,1} \to \mathcal{Q}_g$ is proper for $g \geq 2$.*

Proof of Theorem 1 assuming Propositions 3, 6, 7, and 8. Using Propositions 2 and 3, let $\pi : \widetilde{X} \to X$ be a cover branched at a single non-periodic connection point x. Let θ be a direction for which the flow on $\widetilde{X}$ is minimal. We need to show that the flow in direction θ is in fact uniquely ergodic. By Masur's theorem we need to show that $\{a_t r_\theta(\widetilde{X}) : t \geq 0\}$ is not divergent when viewed as a path in $\mathcal{Q}_{\widetilde{g}}$. The pair $(\widetilde{X}, X)$ corresponds to a point in $\mathcal{Q}(\pi)$ which maps by P to the surface X in $\mathcal{Q}_{g,1}$ with the marked point x.

$$
\begin{tikzcd}
\mathcal{Q}(\pi) \arrow[d, "P"'] \arrow[dr, "R"] & \\
\mathcal{Q}_{g,1} \arrow[d, "\bar{\Phi}_{g,1}"'] & \mathcal{Q}_{\widetilde{g}} \\
\mathcal{Q}_g &
\end{tikzcd}
$$

This pair maps via $\bar{\Phi}_{g,1}$ to the surface X in $\mathcal{Q}_g$. The path $\{a_t r_\theta(X) : t \geq 0\}$ returns infinitely often to a compact set $C \subset \mathcal{Q}_g$ by Proposition 5. The map P is proper by Proposition 7, and $\bar{\Phi}_{g,1}$ is proper by Proposition 8. Therefore $C' = \left(\bar{\Phi}_{g,1} \circ P\right)^{-1}(C)$ is compact and the path $a_t r_\theta((\widetilde{X}, X))$ returns infinitely often to C'. Applying R we see that $a_t r_\theta(\widetilde{X}) \subset \mathcal{Q}_{\widetilde{g}}$ returns infinitely often to the compact set $R(C')$ so by Masur's theorem the flow in direction θ on $\widetilde{X}$ is uniquely ergodic. $\square$

To explain intuitively the maps in Proposition 6 and why they should be proper, note that we might naively expect to be able to define a map $\mathcal{Q}_{g,b} \to \mathcal{Q}_{\widetilde{g}}$ by 'placing the branch points of π at the marked points and pulling back q via π'. This definition might not make sense however due to the different equivalence relations imposed in the definitions of $\mathcal{Q}_{g,b}$ and $\mathcal{Q}_{\widetilde{g}}$. To make this idea meaningful we resolve the ambiguity by passing to finite covers. An intuitive explanation of why the map in Proposition 8 is proper, is that the forgetful map 'forgets the location of the marked point', and thus the fiber is the compact underlying surface; when $b \geq 2$ the fiber of $\bar{\Phi}_{g,b}$ is the b-fold power of the surface with the diagonals removed, and is not compact. This explains why it is important to branch over a single point. By recent work of Cheung, Hubert and Masur [CHM], surfaces obtained by degree 2 covers of genus 2 surfaces, which necessarily have more than one branch point, admit minimal non-uniquely-ergodic directions.

In the remainder of the paper we make this intuitive picture rigorous by constructing moduli spaces of branched coverings of quadratic differentials by quadratic differentials and establishing the properties we need. The paradigm that we follow for the construction of moduli spaces is quite standard. Moduli spaces of branched covers of Riemann surfaces were constructed by [Nat] using a similar paradigm. A novel feature of our construction is that we use the language of orbifold fibrations which is very well suited to our purposes. Along the way we will prove Propositions 3, 6, 7, and 8.

2. Orbifolds and Moduli Spaces

An issue that arises in the construction of moduli spaces is the fact that some of spaces that arise are quotients of non-free group actions. This is often resolved by considering Teichmüller spaces which play the role of universal covering spaces of moduli spaces. For more information on the general construction of Teichmüller spaces see [E]. This solution will not work for us because we are interested in showing that certain maps are proper and this property is not preserved after passing to the universal cover. Instead we will make use of the notion of *orbifold* and *orbifold fibration* as introduced in [Th, §13.2].

We recall these notions. An orbifold is a topological space X with a system of 'orbifold charts' $\phi_j : U_j \to X$ where for each index j there is an open set U_j in Euclidean space and a finite group Γ_j acting smoothly on U_j so that ϕ_j induces an injective map $U_j/\Gamma_j \to X$ (the complete definition prescribes how these charts fit together, see [Th] or the definition of orbifold in Wikipedia.) Orbifolds are produced by group actions on manifolds.

Proposition 9. [prop: quotient] *If Γ acts properly discontinuously on a manifold M then the quotient has an orbifold structure.*

See [Th, Prop. 13.2.1]. A key point in the proof is that for any point $x \in M$ we can find a neighborhood U_x so that if $\gamma \in \Gamma$ satisfies $\gamma(U_x) \cap U_x \neq \emptyset$ then $\gamma(x) = x$ and $\gamma(U_x) = U_x$. The orbifold charts arise by composing the inclusion $\iota : U_x \to M$ with the projection $M \to M/\Gamma$. If I_x denotes the isotropy group of x then I_x is finite and the chart induces a homeomorphism from U_x/I_x to its image in M/Γ.

A map $p : X \to Y$ between orbifolds is an *orbifold fibration with generic fiber F* if each point $y \in Y$ has a neighborhood $U = \widetilde{U}/\Lambda$ so that $p^{-1}(U) = (\widetilde{U} \times F)/\Lambda$ where the action of Λ on $\widetilde{U} \times F$ preserves the factors and the action of Λ on the first factor agrees with the previous action of Λ on $\widetilde{U}$. See [Th, Def. 13.4.1]. An orbifold covering space is an orbifold fibration for which the fiber is discrete.

Proposition 10. [prop: fibration quotient] *Suppose $p : E \to B$ is an orbifold fibration with generic fiber F, and that Γ acts on E and B so that p is equivariant. Assume that Γ acts properly discontinuously on E. Let Γ_0 be the subgroup of Γ acting trivially on B and assume that Γ/Γ_0 acts properly discontinuously on B and that the induced map $q : E/\Gamma_0 \to B$ is an orbifold fibration. Then the induced map $\bar{p} : E/\Gamma \to B/\Gamma$ is an orbifold fibration with generic fiber F/Γ_0.*

Proof. For each $x \in B$ we have a neighborhood U_x as discussed above. By assumption the projection $q : E/\Gamma_0 \to B$ is a fibration. Let $V_x \subset E/\Gamma_0$ denote the set $q^{-1}(U_x)$. By replacing U_x with a smaller set if necessary we can assume that V_x is diffeomorphic to $U_x \times (F/\Gamma_0)$. Let I_x denote the isotropy group of x in Γ/Γ_0. The group I_x is finite and acts on $U_x \times (F/\Gamma_0)$ preserving the projection onto the first factor. Since the action of Γ/Γ_0 on B is faithful a generic point has a trivial stabilizer so the generic fiber is F/Γ_0. $\square$

We will be dealing with closed surfaces with marked points. In order to apply a result of [EE] in Proposition 11 we assume that when we are dealing with the torus at least one point is marked, and when we are dealing with the sphere at least 3 points are marked.

We begin by sketching the construction of the standard moduli space following [E]. Let S be a fixed smooth surface of genus g. Consider the collection of pairs (X, h) where X is a Riemann surface of genus g and h is a diffeomorphism from S to X. Two pairs (X_1, h_1) and (X_2, h_2)

are equivalent if there is a holomorphic map $f : X_1 \to X_2$ such that $fh_1 = h_2$.

$$\begin{tikzcd} S \arrow[r, "h_1"] \arrow[dr, "h_2"'] & X_1 \arrow[d, "f"] \\ & X_2 \end{tikzcd}$$

Let $\mathrm{Conf}(S)$ denote the set of equivalence classes of pairs. We can identify this set with the set of conformal structures on S. Let $\mathrm{Diffeo}(S)$ denote the group of orientation preserving diffeomorphisms of S. There is a natural action of $\mathrm{Diffeo}(S)$ on the set $\mathrm{Conf}(S)$ where the diffeomorphism $\gamma \in \mathrm{Diffeo}(S)$ sends (X, h) to $(X, h\gamma)$. Set-theoretically, the moduli space $\mathcal{M}_g = \mathcal{M}(S)$ is the quotient $\mathrm{Conf}(S)/\mathrm{Diffeo}(S)$.

In fact the space $\mathcal{M}(S)$ has an orbifold structure but this structure is not apparent from the quotient construction above. In order to establish the existence of the orbifold structure it is useful to form the quotient in two stages. Let $\mathrm{Diffeo}_0(S) \subset \mathrm{Diffeo}(S)$ denote the normal subgroup of diffeomorphisms isotopic to the identity. The quotient $\mathrm{Conf}(S)/\mathrm{Diffeo}_0(S)$ is Teichmüller space, and will be denoted by $\mathcal{T}_g$ or $\mathcal{T}(S)$. Teichmüller space has a natural smooth manifold structure and in fact it is diffeomorphic to $\mathbb{R}^n$ (see [E, §11]). We define the mapping class group to be $\mathrm{Mod}(S) = \mathrm{Diffeo}(S)/\mathrm{Diffeo}_0(S)$. The group $\mathrm{Mod}(S)$ acts properly discontinuously on $\mathcal{T}(S)$. We can recover the moduli space as $\mathcal{M}(S) = \mathcal{T}(S)/\mathrm{Mod}(S)$. Since it is the quotient of a smooth manifold by a properly discontinuous smooth action, $\mathcal{M}(S)$ has the structure of an orbifold (see Proposition 9).

A modification of this construction produces moduli spaces of Riemann surfaces with finite sets of marked points [E, §5]. In this case we start with a surface S and a finite set $\Sigma \subset S$ with cardinality b. Let $\mathrm{Diffeo}(S, \Sigma) \subset \mathrm{Diffeo}(S)$ be the subgroup of diffeomorphisms that fix Σ. Let $\mathrm{Diffeo}_0(S, \Sigma)$ be the normal subgroup consisting of diffeomorphisms which are isotopic to the identity relative to Σ. We define $\mathcal{T}(S, \Sigma)$ to be $\mathrm{Conf}(S)/\mathrm{Diffeo}_0(S, \Sigma)$. Once again $\mathcal{T}(S, \Sigma)$ has a natural smooth manifold structure and is diffeomorphic to $\mathbb{R}^m$. Define $\mathrm{Mod}(S, \Sigma) = \mathrm{Diffeo}(S, \Sigma)/\mathrm{Diffeo}_0(S, \Sigma)$. $\mathrm{Mod}(S, \Sigma)$ acts properly discontinuously on $\mathcal{T}(S, \Sigma)$ and we let

$$\mathcal{M}(S, \Sigma) = \mathcal{T}(S, \Sigma)/\mathrm{Mod}(S, \Sigma).$$

This space has a natural orbifold structure. The topology of the spaces $\mathcal{T}(S, \Sigma)$ and $\mathcal{M}(S, \Sigma)$ as well as the isomorphism type of the groups $\mathrm{Mod}(S, \Sigma)$ depend only on g and b and are usually written as $\mathcal{T}_{g,b}$, $\mathcal{M}_{g,b}$ and $\mathrm{Mod}_{g,b}$.

The inclusion $\mathrm{Diffeo}(S,\Sigma) \subset \mathrm{Diffeo}(S)$ defines a surjective map

$$\Phi_{g,b} : \mathcal{M}_{g,b} = \mathrm{Conf}(S)/\mathrm{Diffeo}(S,\Sigma) \to \mathcal{M}_g = \mathrm{Conf}(S)/\mathrm{Diffeo}(S).$$

We call $\Phi_{g,b}$ the *forgetful map.* This map can be shown to be an orbifold fibration in general but we only need the special case described in the following result.

Proposition 11. [prop: fibration1] *For $g \geq 2$, the map $\Phi_{g,1} : \mathcal{M}_{g,1} \to \mathcal{M}_g$ is an orbifold fibration and the generic fiber is the surface with genus g.*

Proof. Let S be a surface of genus g and let Σ be a subset with a single element z. Let $\Phi = \Phi_{g,1}$ and let $\widetilde{\Phi}$ be the natural map from $\mathcal{T}_{g,1}$ to $\mathcal{T}_g$. The map $\widetilde{\Phi}$ was shown to be a fibration with fiber $\widetilde{S}$ in [B].

Since the inclusion of $\mathrm{Diffeo}(S,\Sigma)$ into $\mathrm{Diffeo}(S)$ maps $\mathrm{Diffeo}_0(S,\Sigma)$ into $\mathrm{Diffeo}_0(S)$, it defines a homomorphism from $\mathrm{Mod}_{g,b}$ to Mod_g. This homomorphism is surjective since any diffeomorphism is isotopic to one which fixes Σ. Write $\Gamma = \mathrm{Mod}_{g,1}$ and let Γ_0 denote the kernel of the homomorphism from Γ to Mod_g. An element $\gamma_0 \in \Gamma_0$ is represented by a diffeomorphism which fixes z and is isotopic to the identity. The track of the point z under such an isotopy is a loop based at z, so each isotopy from γ_0 to the identity gives rise to an element of $\pi_1(S)$. The group $\mathrm{Diffeo}_0(S)$ is contractible by [EE] so any two isotopies from γ_0 to the identity are connected by a one parameter family of isotopies from γ_0 to the identity. It follows that the corresponding element of $\pi_1(S)$ is well defined and we obtain a homomorphism from Γ_0 to $\pi_1(S)$. Using the isotopy extension theorem we can show that this map is an isomorphism. If we act by Γ_0 then the induced projection on the quotient space $\mathcal{T}_{g,1}/\Gamma_0 \to \mathcal{T}_g$ is a fibration with fiber S (see [EF]), known as the universal curve. According to Proposition 10 the fibration $\Phi : \mathcal{T}_{g,1}/\Gamma \to \mathcal{T}_g/\Gamma$ is an orbifold fibration with generic fiber S. □

We can modify the previous construction to produce the moduli space of unit area quadratic differentials, $\mathcal{Q}(S) = \mathcal{Q}_g$. Consider pairs (X,h) where X is a Riemann surface with a holomorphic quadratic differential with area 1 and $h : S \to X$ is a diffeomorphism. Two pairs (X_1,h_1) and (X_2,h_2) are equivalent if there is a holomorphic map $f : X_1 \to X_2$ such that $fh_1 = h_2$ and f *preserves the quadratic differential.* We denote by $\mathrm{Quad}(S)$ the set of equivalence classes. There is a natural action of $\mathrm{Diffeo}(S)$ on $\mathrm{Quad}(S)$ and the quotient (as a set) is $\mathcal{Q}_g$. As before it is useful to deal with an intermediate quotient. We define $\widetilde{\mathcal{Q}}_g = \mathrm{Quad}(S)/\mathrm{Diffeo}_0(S)$. We call it the Teichmüller space of quadratic differentials. This space can be identified with the cotangent bundle of Teichmüller space and the projection map of the cotangent bundle can be identified with the map which maps a marked quadratic differential to its underlying marked Riemann surface. The collection of unit area quadratic differentials can be identified with the unit sphere bundle in this cotangent bundle with respect to an appropriate metric [E]. Since the Teichmüller space of quadratic differentials is the total space of a fiber bundle with smooth fiber and smooth base it is a smooth manifold.

The group $\mathrm{Mod}(S)$ acts smoothly and properly discontinuously on $\widetilde{\mathcal{Q}}_g$. We define the moduli space of quadratic differentials

$$\mathcal{Q}_g = \widetilde{\mathcal{Q}}_g / \mathrm{Mod}(S)$$

This space has an orbifold structure by Proposition 9.

We can define versions of these spaces for surfaces with finite sets of marked points following the same paradigm. Note that we are not assuming any relation between the marked points and the quadratic differential. In particular the collection of marked points need not contain the collection of zeros and poles of the quadratic differential. We define $\widetilde{\mathcal{Q}}_{g,b} = \widetilde{\mathcal{Q}}(S, \Sigma)$ to be the quotient $\mathrm{Quad}(S, \Sigma)/\mathrm{Diffeo}_0(S, \Sigma)$. This space has a manifold structure. We can see this by looking at the projection p_b from $\widetilde{\mathcal{Q}}(S, \Sigma)$ to $\widetilde{\mathcal{T}}(S, \Sigma)$ which takes a quadratic differential on a marked surface to the underlying marked surface. This map is the projection map of a fiber bundle where the fiber is the (topological) sphere of unit area quadratic differentials on the surface.

The space

$$\mathcal{Q}_{g,b} = \widetilde{\mathcal{Q}}(S, \Sigma)/ \mathrm{Mod}(S, \Sigma)$$

has an orbifold structure. We define the forgetful map $\bar{\Phi}_{g,b} : \mathcal{Q}_{g,b} \to \mathcal{Q}_g$ using the inclusion $\mathrm{Mod}(S, \Sigma) \subset \mathrm{Mod}(S)$.

Proposition 12. [prop: fibration2] *The natural maps $p_0 : \mathcal{Q}_g \to \mathcal{M}_g$ and $p_b : \mathcal{Q}_{g,b} \to \mathcal{M}_{g,b}$ are orbifold fibrations with the sphere as generic fiber. They satisfy $p_0 \bar{\Phi}_{g,b} = \Phi_{g,b} p_b$.*

Proof. The projection p_0 is mapping class group equivariant. The quotient spaces are the moduli space of unit area quadratic differentials and the moduli space of Riemann surfaces. There is an induced map between the quotient spaces which is just the map which projects a quadratic differential to its underlying Riemann surface. According to Proposition 10 the induced projection map on the quotient spaces is an orbifold fibration with sphere as generic fiber (in the terminology of Proposition 10 the group Γ_0 is trivial). The argument for p_b is the same. The second assertion is clear from definitions. $\square$

Orbifold fibrations are useful for us because of the following result, whose proof is immediate from the definitions and is omitted.

Proposition 13. *[prop: fibration3] An orbifold fibration with compact generic fiber is a proper map.*

We now have the tools we need for the following.

Proof of Proposition 8.

$$\begin{tikzcd} \mathcal{Q}_{g,1} \arrow[r, "\bar{\Phi}_{g,1}"] \arrow[d, "p_1"'] & \mathcal{Q}_g \arrow[d, "p_0"] \\ \mathcal{M}_{g,1} \arrow[r, "\Phi_{g,1}"] & \mathcal{M}_g \end{tikzcd}$$

The map p_1 mapping quadratic differentials to their underlying Riemann surfaces is proper because it is an orbifold fibration with sphere as generic fiber. By Proposition 11, the fiber of the forgetful map $\Phi_{g,1}$ is the surface of genus g. Since the fiber is compact the map $\Phi_{g,1}$ is proper. Thus $\Phi_{g,1}p_1$ is proper and since the diagram commutes, so is $p_0\bar{\Phi}_{g,1}$. In particular $\bar{\Phi}_{g,1}$ is proper. □

3. The moduli space of a branched cover

Let $\pi : \widetilde{S} \to S$ be a branched covering map, where $\widetilde{S}$ is a surface of genus $\widetilde{g}$, and let $\Sigma \subset S$ be the set of branch points, i.e. the points over which π is not a covering map. We want to follow the same paradigm to construct a moduli space of branched covering maps of Riemann surfaces or holomorphic quadratic differentials topologically equivalent to π. We will consider the case of quadratic differentials because that is what we will need but the same construction gives a moduli space of branched covers of Riemann surfaces as in [Nat].

Consider quintuples $(p, \widetilde{X}, X, \widetilde{h}, h)$ where $\widetilde{X}$ and X are Riemann surfaces with holomorphic quadratic differentials, $p : \widetilde{X} \to X$ is a holomorphic branched covering map which pulls back the quadratic differential on X to that on $\widetilde{X}$, $\widetilde{h} : \widetilde{S} \to \widetilde{X}$ and $h : S \to X$ are diffeomorphisms such that $h\pi = p\widetilde{h}$.

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\widetilde{h}"] \arrow[d, "\pi"'] & \widetilde{X} \arrow[d, "p"] \\ S \arrow[r, "h"] & X \end{tikzcd}$$

We say $(p_1, \widetilde{X}_1, X_1, \widetilde{h}_1, h_1)$ and $(p_2, \widetilde{X}_2, X_2, \widetilde{h}_2, h_2)$ are equivalent if there are holomorphic equivalences $\widetilde{w} : \widetilde{X}_1 \to \widetilde{X}_2$ and $w : X_1 \to X_2$

which preserve quadratic differentials so that $p_2\widetilde{w} = wp_1$, $\widetilde{w}\widetilde{h}_1 = \widetilde{h}_2$ and $wh_1 = h_2$. Let $\mathrm{Quad}(\pi)$ denote the set of equivalence classes.

$$\begin{tikzcd}
\widetilde{S} \arrow[r, "\widetilde{h}_1"] \arrow[d, "\pi"'] & \widetilde{X}_1 \arrow[r, "\widetilde{w}"] \arrow[d, "p_1"'] & \widetilde{X}_2 \arrow[d, "p_2"] & \widetilde{S} \arrow[l, "\widetilde{h}_2"'] \arrow[d, "\pi"] \\
S \arrow[r, "h_1"] & X_1 \arrow[r, "w"] & X_2 & S \arrow[l, "h_2"']
\end{tikzcd}$$

We define $\mathrm{Diffeo}(\pi)$ to be the set of pairs of diffeomorphisms $(\widetilde{\gamma}, \gamma)$ where $\gamma : S \to S$ and $\widetilde{\gamma} : \widetilde{S} \to \widetilde{S}$ such that $\gamma\pi = \pi\widetilde{\gamma}$ and γ fixes Σ. We obtain a group structure by defining multiplication in $\mathrm{Diffeo}(\pi)$ to be coordinatewise composition. We define $\mathrm{Diffeo}_0(\pi)$ to be the set of pairs $(\widetilde{\gamma}, \gamma)$ isotopic to $(\mathrm{id}, \mathrm{id})$ via an *isotopy respecting* π, i.e. parametrized families $H_t, \widetilde{H}_t$ of diffeomorphisms of S and $\widetilde{S}$ respectively, such that H_t fixes Σ and $H_0 = \mathrm{id}$, $H_1 = \gamma$, $\widetilde{H}_0 = \mathrm{id}$, $\widetilde{H}_1 = \widetilde{\gamma}$ and $\pi\widetilde{H}_t = H_t\pi$ for each t. We define $\mathrm{Mod}(\pi) = \mathrm{Diffeo}(\pi)/\mathrm{Diffeo}_0(\pi)$ and $\widetilde{\mathcal{Q}}(\pi) = \mathrm{Quad}(\pi)/\mathrm{Diffeo}_0(\pi)$. There is a natural action of $\mathrm{Mod}(\pi)$ on $\widetilde{\mathcal{Q}}(\pi)$, and we denote the quotient by $\mathcal{Q}(\pi)$. Below we will equip $\mathcal{Q}(\pi)$ with a natural topology.

Let b be the cardinality of Σ and let d the degree of the cover π. Observe that the definition of $\mathrm{Diffeo}_0(\pi)$ ensures that if $(\widetilde{\gamma}, \gamma) \in \mathrm{Diffeo}(\pi)$ (resp. $\mathrm{Diffeo}_0(\pi)$) then $\gamma \in \mathrm{Diffeo}(S, \Sigma)$ (resp. $\mathrm{Diffeo}_0(S, \Sigma)$). Let $\widetilde{P} : \mathrm{Quad}(\pi) \to \mathrm{Quad}(S, \Sigma)$ be the map which takes the quintuple $(p, \widetilde{X}, X, \widetilde{h}, h)$ to the pair (X, h). It follows that $\widetilde{P}$ induces a map on equivalence classes $\widetilde{\mathcal{Q}}(\pi) \to \widetilde{\mathcal{Q}}_{g,b}$ which we also denote by $\widetilde{P}$. It also follows that the homomorphism

$$H : \mathrm{Mod}(\pi) \to \mathrm{Mod}(S, \Sigma), \quad H(\widetilde{\gamma}, \gamma) = \gamma$$

is well-defined. H gives us a $\mathrm{Mod}(\pi)$-action on $\widetilde{\mathcal{Q}}_{g,b}$ and $\widetilde{P}$ is $\mathrm{Mod}(\pi)$-equivariant.

Let $z \in S \smallsetminus \Sigma$, $\widetilde{z} \in \pi^{-1}(z)$, $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$, $\Gamma \subset \pi_1(S \smallsetminus \Sigma, z)$ be the image of $\pi_1(\widetilde{S} \smallsetminus \widetilde{\Sigma}, \widetilde{z})$ under the projection π, and let $N(\Gamma)$ be the normalizer of Γ in $\pi_1(S \smallsetminus \Sigma, z)$.

Lemma 14. [lem: finite ker] $\ker H$ *is finite, and can be identified with* $N(\Gamma)/\Gamma$.

Proof. Suppose $(\widetilde{\gamma}, \gamma)$ represents an element of $\mathrm{Mod}(\pi)$ which is in $\ker H$ so that γ is isotopic to the identity via an isotopy fixing Σ. The map $\pi|_{\widetilde{S} \smallsetminus \widetilde{\Sigma}}$ is a covering map, so lifting the isotopy to $\widetilde{S} \smallsetminus \widetilde{\Sigma}$ we obtain that $\widetilde{\gamma}$ is isotopic to a diffeomorphism $\widetilde{\gamma}_1 : \widetilde{S} \smallsetminus \widetilde{\Sigma} \to \widetilde{S} \smallsetminus \widetilde{\Sigma}$ covering the identity. The collection of such diffeomorphisms forms a group

F which we can identify with the deck group $N(\Gamma)/\Gamma$ (see [Ha, Prop. 1.39]). This is a finite group since Γ is of finite index (its index is d, the degree of the cover). Moreover all elements of the deck group can arise. By the unique lifting property [Ha, Prop. 1.34], $F \cap \mathrm{Diffeo}_0(\widetilde{S}) = \{\mathrm{id}\}$, i.e., the inclusion of F in $\mathrm{Mod}(\widetilde{S})$ induces an inclusion of groups. □

Let $\widetilde{\mathcal{Q}}'(\pi)$ denote the quotient of $\widetilde{\mathcal{Q}}(\pi)$ by $\ker H$ (considered at this point only as a set with no additional structure). By equivariance, two quintuples which differ by an element of $\ker H$ map to the same point under $\widetilde{P}$, so that $\widetilde{P}$ induces a map $\widetilde{P}' : \widetilde{\mathcal{Q}}'(\pi) \to \widetilde{\mathcal{Q}}(S, \Sigma)$.

Lemma 15. *The map $\widetilde{P}'$ is bijective.* [lem: bijective]

Proof. We begin by proving surjectivity. Given (X, h) we need to produce $\widetilde{X}$, p and $\widetilde{h}$ so that the quintuple $(p, \widetilde{X}, X, \widetilde{h}, h)$ represents an element of $\widetilde{\mathcal{Q}}(\pi)$. We define $\widetilde{X}$ to be the surface $\widetilde{S}$ with the Riemann surface structure and quadratic differential structure pulled back via the branched cover $h\pi$. We take p to be the map $h\pi$ and we take $\widetilde{h}$ to be the identity.

We now prove injectivity. Say we are given $(p_1, \widetilde{X}_1, X_2, \widetilde{h}_1, h_1)$ and $(p_2, \widetilde{X}_2, X_2, \widetilde{h}_2, h_2)$ for which (X_1, h_1) and (X_2, h_2) are equivalent in $\widetilde{Q}(S, \Sigma)$, so there is a holomorphic $f : X_1 \to X_2$ preserving the quadratic differentials with $fh_1 = h_2$. Pick a basepoint z in $S \smallsetminus \Sigma$, and let $\widetilde{z} \in \pi^{-1}(z)$. Let $\widetilde{x}_1 = \widetilde{h}_1(\widetilde{z})$ and $\widetilde{x}_2 = \widetilde{h}_2(\widetilde{z})$. Let $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$, $\widetilde{\Sigma}_i = p_i^{-1}(h(\Sigma))$ for $i = 1, 2$. By chasing the diagram below we see that

$$(fp_1)_* \pi_1 \left(\widetilde{X}_1 \smallsetminus \widetilde{\Sigma}_1, \widetilde{x}_1\right) = (h_2\pi)_* \pi_1 \left(\widetilde{S} \smallsetminus \widetilde{\Sigma}, \widetilde{z}\right) = p_{2*} \pi_1 \left(\widetilde{X}_2 \smallsetminus \widetilde{\Sigma}_2, \widetilde{x}_2\right).$$

The Galois correspondence for covering spaces shows that these two covering spaces are equivalent so there is $\widetilde{f} : \widetilde{X}_1 \smallsetminus \widetilde{\Sigma}_1 \to \widetilde{X}_2 \smallsetminus \widetilde{\Sigma}_2$ taking $\widetilde{x}_1$ to $\widetilde{x}_2$ such that $p_1 = p_2 \widetilde{f}$. We can extend $\widetilde{f}$ to a map $\widetilde{X}_1 \to \widetilde{X}_2$, which we continue to denote by $\widetilde{f}$. Since both $\widetilde{X}_1$ and $\widetilde{X}_2$ have the quadratic differential structure obtained by pulling back the quadratic differential from X the map $\widetilde{f}$ preserves the quadratic differential structure so $(p_1, \widetilde{X}_1, X_1, \widetilde{h}_1, h_1), (p_2, \widetilde{X}_2, X_2, \widetilde{h}_2, h_2)$ are equivalent.

$$\begin{tikzcd}
\widetilde{S} \arrow[r, "\widetilde{h}_1"] \arrow[d, "\pi"'] & \widetilde{X}_1 \arrow[r, dashed, "\widetilde{f}"] \arrow[d, "p_1"'] & \widetilde{X}_2 \arrow[d, "p_2"] & \widetilde{S} \arrow[l, "\widetilde{h}_2"'] \arrow[d, "\pi"] \\
S \arrow[r, "h_1"] & X_1 \arrow[r, "f"] & X_2 & S \arrow[l, "h_2"']
\end{tikzcd}$$

□

Since $\widetilde{\mathcal{Q}}_{g,b}$ is a manifold, we can use $\widetilde{P}$ to induce a manifold topology on $\widetilde{\mathcal{Q}}'(\pi)$ as well. Taking quotients by the action, the map $\widetilde{P}'$ induces a map $P : \mathcal{Q}(\pi) \to \mathcal{Q}(S, \Sigma)$. Our next objective is to analyze this map.

We begin with some general observations on diffeomorphisms and fundamental groups. Let Y be a surface, and let $z \in Y$. Let $\mathrm{Aut}(\pi_1(Y, z))$ (resp. $\mathrm{Inn}(\pi_1(Y, z))$) be the group of (inner) automorphisms of $\pi_1(Y, z)$. Define $\mathrm{Out}(Y) = \mathrm{Out}(\pi_1(Y, z))$ to be

$$\mathrm{Aut}(\pi_1(Y, z))/\mathrm{Inn}(\pi_1(Y, z)).$$

There is a well-defined homomorphism from $\mathrm{Mod}(Y)$ to $\mathrm{Out}(Y)$ which we now describe. Given $f : Y \to Y$, set $w = f(z)$, and consider the isomorphism $f_* : \pi_1(Y, z) \to \pi_1(Y, w)$. Given a path ρ from w to z there is a canonical map $\rho_* : \pi_1(Y, w) \to \pi_1(Y, z)$. If we change the path ρ to a path σ then the two identifications differ by the inner automorphism corresponding to conjugation by $\rho \cdot \sigma^{-1}$. In particular the composition $\rho_* f_*$ determines an element of Out, which is independent of the path ρ and depends only on the isotopy class of f.

Lemma 16. *[lem: cofinite im] The image of H is of finite index in* $\mathrm{Mod}(S, \Sigma)$.

Proof. By definition, the image of H consists of those elements $\gamma \in \mathrm{Diffeo}(S, \Sigma)$ for which there is a lift $\widetilde{\gamma} \in \mathrm{Diffeo}(\widetilde{S})$ with $\gamma\pi = \pi\widetilde{\gamma}$. We claim that it suffices to consider this lifting problem for $Y = S \smallsetminus \Sigma$ and $\widetilde{Y} = \widetilde{S} \smallsetminus \widetilde{\Sigma}$ in place of S and $\widetilde{S}$. Indeed, if $\gamma \in \mathrm{Diffeo}(S, \Sigma)$ and $f = \gamma|_Y$ has a lift $\widetilde{f} : \widetilde{Y} \to \widetilde{Y}$, since $\widetilde{S}$ is the completion of $\widetilde{Y}$ with respect to a complex structure for which $\widetilde{f}$ is holomorphic, we can continuously extend $\widetilde{f}$ to obtain a lift of $\widetilde{\gamma}$ defined on $\widetilde{S}$.

Let $\widetilde{z}$ be a point in $\widetilde{Y}$ that maps to z and let $\widetilde{w}$ be a point that maps to w. Let $\Gamma_{\widetilde{z}}$ be the group $p_*(\pi_1(\widetilde{Y}, \widetilde{z})) \subset \pi_1(Y, z)$ and let $\Gamma_{\widetilde{w}}$ be the group $p_*(\pi_1(\widetilde{Y}, \widetilde{w})) \subset \pi_1(Y, w)$. The Galois theory of covering spaces tells us that the diffeomorphism $f : Y \to Y$ lifts to a diffeomorphism $\widetilde{f} : \widetilde{Y} \to \widetilde{Y}$ taking $\widetilde{z}$ to $\widetilde{w}$ if and only if $f_*(\Gamma_{\widetilde{z}}) = \Gamma_{\widetilde{w}}$. If we want to know whether f has any lift to a map $\widetilde{f}$ then we need to consider all possible values for $\widetilde{w}' = f(\widetilde{z})$. As $\widetilde{w}'$ ranges over points in $p^{-1}(w)$ the groups $\Gamma_{\widetilde{w}'}$ range over all conjugates of $\Gamma_{\widetilde{w}}$. Thus f lifts if and only if f_* maps $\Gamma_{\widetilde{z}}$ to a conjugate of $\Gamma_{\widetilde{w}}$.

As before let d denote the degree of the covering which is also the index of $\Gamma_{\widetilde{z}}$. Let $\mathcal{F}_d$ be the collection of subgroups of $\pi_1(Y, z)$ of index d, and let $\mathcal{F}'_d$ be the set of conjugacy classes in $\mathcal{F}_d$. The group Aut acts on $\mathcal{F}_d$ and on $\mathcal{F}'_d$ and its subgroup Inn acts trivially on $\mathcal{F}'_d$ so Out acts on $\mathcal{F}'_d$.

If we choose a path ρ from w to z then $\rho_*(\Gamma_{\widetilde{w}})$ is conjugate to $\Gamma_{\widetilde{z}}$ so f lifts to an $\widetilde{f}$ if an only if the element of Aut corresponding to f fixes the conjugacy class of $\Gamma_{\widetilde{z}}$ in $\mathcal{F}'_d$. This shows that the image of H corresponds to the stabilizer of the conjugacy class of $\Gamma_{\widetilde{z}}$ in $\mathcal{F}'_d$. Since the set $\mathcal{F}'_d$ is finite the image of this homomorphism has finite index. □

Corollary 17. *The moduli space $\mathcal{Q}(\pi)$ has a natural orbifold structure and $P : \mathcal{Q}(\pi) \to \mathcal{Q}(S, \Sigma)$ is a finite covering of orbifolds.*

Proof. Since $\mathrm{Mod}(S, \Sigma)$ acts properly discontinuously on $\widetilde{\mathcal{Q}}(S, \Sigma)$, by Lemma 15 and equivariance $\mathrm{Mod}(\pi)$ acts properly discontinuously on the manifold $\widetilde{\mathcal{Q}}'(\pi)$. Proposition 9 implies that $\mathcal{Q}(\pi)$ is an orbifold. The degree of P is the index of the image of H in $\mathrm{Mod}(S, \Sigma)$ which by Lemma 16 is finite. □

Proof of Proposition 7. The map P is an orbifold fibration with finite generic fiber. In particular the generic fiber is compact. By Proposition 13 the map P is proper. □

Proof of Proposition 6. Let $\widetilde{R} : \mathrm{Quad}(\pi) \to \mathcal{Q}(\widetilde{S})$ be the map which sends the class of the quintuple $(p, \widetilde{X}, X, \widetilde{h}, h)$ to the class of the pair $(\widetilde{X}, \widetilde{h})$. As explained the group $\mathrm{Diffeo}(\pi)$ acts on $\mathrm{Quad}(\pi)$. The group $\mathrm{Diffeo}(\pi)$ also acts on $\mathrm{Quad}(\widetilde{S})$ via the natural map from $\mathrm{Diffeo}(\pi)$ to $\mathrm{Diffeo}(\widetilde{S})$. It follows that $\widetilde{R}$ descends to a well-defined map $R : \mathcal{Q}(\pi) \to \mathcal{Q}(\widetilde{S})$, which is continuous since $\widetilde{R}$ factors through $\widetilde{\mathcal{Q}}'(\pi)$, and the topology on $\mathcal{Q}(\pi)$ is defined via Lemma 15.

The required maps have been constructed. The only thing that remains is to discuss the G-action. In the case of the space of quadratic differentials there is a natural action of G. We begin by describing this action on the set of pairs (X, h). We think of a quadratic differential as being given by a set of charts mapping open subsets of X to open subsets of $\mathbb{C}$. Then the action of $g \in G$ is obtained by identifying $\mathbb{C}$ with $\mathbb{R}^2$ and post-composing the charts with the map $g : \mathbb{R}^2 \to \mathbb{R}^2$. The action of $\mathrm{Diffeo}(S)$ is by pre-composition on the charts, hence these actions commute, and therefore we have a well-defined action of G on the quotient space. The same reasoning implies that there is a well-defined G-action on $\mathcal{Q}(\pi)$, and since P and R both descend from maps defined on $\mathrm{Quad}(\pi)$, they are G-equivariant. □

Proof of Proposition 3. Let $x \in S$ and let U be a small disk around x and let y be a point of S outside U. If we are given a finite-index subgroup Γ of $\pi_1 (S \smallsetminus \{x\}, y)$ we can construct a branched cover of S

by constructing the corresponding cover of $S \smallsetminus \{x\}$ and then taking the metric completion. This branched cover will have non-trivial branching if a loop obtained by travelling from y to the boundary of U, traversing the boundary of U once and then returning to y along the same path is not in Γ. The fundamental group of $S \smallsetminus \{x\}$ is a free group on $2g$ generators. Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be loops on S so that a loop γ corresponding to the boundary of U is given by the product of commutators $\alpha_1\beta_1{\alpha_1}^{-1}{\beta_1}^{-1} \cdots \alpha_g\beta_g{\alpha_g}^{-1}{\beta_g}^{-1}$. Let D_d be the dihedral group of order $2d$ generated by a and b where $a^2 = b^2 = (ab)^d = 1$. Define a homomorphism $h : \pi_1(S \smallsetminus \{x\}, y) \to D_d$ so that $h(\alpha_1) = a$, $h(\beta_1) = b$ and $h(\alpha_j) = h(\beta_j) = 1$ for $j > 1$. This homomorphism has the property that $h(\gamma) = aba^{-1}b^{-1} = (ab)^2$. This element is not the identity since $d > 2$. Let $F \subset D_d$ be the subgroup containing a and 1. Let $\Gamma \subset \pi_1(S \smallsetminus \{x\}, y)$ be $h^{-1}(F)$. The group Γ has index d, hence the covering space of $S \smallsetminus \{x\}$ corresponding to Γ has degree d. Let $\widetilde{S}$ be the branched cover obtained by completing this covering space and let π be the projection. The branching at x is nontrivial because the loop γ corresponds to an element of $\pi_1(S \smallsetminus \{x\}, y)$ which is not in Γ. $\square$

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