A VOLUME ESTIMATE FOR THE SET OF STABLE LATTICES

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Abstract. We show that in high dimensions the set of stable lattices is almost of full measure in the space of unimodular lattices.

Let $G \overset{\text{def}}{=} \text{SL}_n(\mathbb{R})$, $\Gamma \overset{\text{def}}{=} \text{SL}_n(\mathbb{Z})$, and let $A \subset G$ denote the subgroup of diagonal matrices with positive entries. The quotient space $\mathcal{L}_n \overset{\text{def}}{=} G/\Gamma$ is naturally identified with the space of unimodular lattices in $\mathbb{R}^n$, and the group $G$ (and any of it subgroups) acts via left translations, or equivalently, by acting on lattices via its linear action on $\mathbb{R}^n$. A lattice $\Lambda$ is called stable if for any subgroup $\Delta \subset \Lambda$, one has $\text{vol}(\Delta \otimes \mathbb{R}/\Delta) \geq 1$ (in the literature the term semi-stable is also used), and we denote the set of stable lattices by $\mathcal{S}^{(n)}$.

A central problem is to understand the orbits of the $A$-action on $\mathcal{L}_n$. In [SW] we proved that for any lattice $\Lambda \in \mathcal{L}_n$, the orbit-closure $A\Lambda$ contains a stable lattice. This result reduces the proof of Minkowski’s conjecture on the product of inhomogeneous linear forms to that of estimating the Euclidean covering radius of stable lattices (see [SW] for details). Understanding stable lattices is therefore a natural problem due to its connection both with well-studied problems in the geometry of numbers, and with dynamics of the $A$-action. Although $\mathcal{S}^{(n)}$ is compact (while $\mathcal{L}_n$ is not), in this note we show that $\mathcal{S}^{(n)}$ has almost full measure with respect to the natural probability measure on $\mathcal{L}_n$, for large $n$. Moreover the convergence to full measure is very fast. This answers a question we were asked by G. Harder, and can be viewed as a manifestation of the concentration of mass along the equator in high dimensional Euclidean balls.

We will prove the following.

Theorem 1. Let $m$ denote the $G$-invariant probability measure on $\mathcal{L}_n$ derived from Haar measure on $G$, and let $\mathcal{S}^{(n)}$ denote the subset of stable lattices in $\mathcal{L}_n$. Then there is a constant $C > 0$ such that for all sufficiently large $n$,

$$m \left( \mathcal{L}_n \setminus \mathcal{S}^{(n)} \right) \leq \left( \frac{C}{n} \right)^{\frac{n-1}{n}}.$$

In particular $m \left( \mathcal{S}^{(n)} \right) \to 1$ as $n \to \infty$. 

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For $\Lambda \in \mathcal{L}_n$ and a subgroup $\Delta \subset \Lambda$, we denote by $r(\Delta)$ its rank and by $|\Delta|$ its covolume in the Euclidean subspace $\Delta \otimes \mathbb{R} \subset \mathbb{R}^n$. For $k = 1, \ldots, n-1$ let us denote $\mathcal{V}_k(\Lambda) \overset{\text{def}}{=} \{ |\Delta|^{1/k} : \Delta \subset \Lambda, r(\Delta) = k \}$ and $\alpha_k(\Lambda) = \min \mathcal{V}_k(\Lambda)$ so that $\Lambda$ is stable if and only if $\alpha_k(\Lambda) \geq 1$ for $k = 1, \ldots, n-1$. Let

$$V_k(\Lambda) \overset{\text{def}}{=} \{ \frac{|\Delta|^{1/k}}{k} : \Delta \subset \Lambda, r(\Delta) = k \}$$

and $\alpha_k(\Lambda) = \min V_k(\Lambda)$ so that $\Lambda$ is stable if and only if $\alpha_k(\Lambda) \geq 1$ for $k = 1, \ldots, n-1$. Let

$$S_k(t) \overset{\text{def}}{=} \{ x \in \mathcal{L}_n : \alpha_k(x) \geq t \}, \quad S_k^{(n)} \overset{\text{def}}{=} S_k(S_k(1)).$$

With this notation $S^{(n)} = \bigcap_{k=1}^{n-1} S_k^{(n)}$. We will show:

**Proposition 2.** There is $C > 0$ such that for all sufficiently large $n$, and all $k \in \{1, \ldots, n-1\}$,

$$m(\mathcal{L}_n \setminus S_k^{(n)}) \leq \frac{1}{n} \left( \frac{C}{n} \right)^{\frac{k(n-k)}{n}}.$$  \hspace{1cm} (1)

**Proof of Theorem 1.** For $n > C$, the largest value of $\left( \frac{C}{n} \right)^{\frac{k(n-k)}{n}}$ is attained when $k = 1$ and $k = n-1$. Therefore (1) implies

$$m(\mathcal{L}_n \setminus S^{(n)}) = m(\mathcal{L}_n \setminus \bigcap_{k=1}^{n-1} S_k^{(n)}) = m\left( \bigcup_{k=1}^{n-1} \mathcal{L}_n \setminus S_k^{(n)} \right) \leq \frac{n-2}{n} \left( \frac{C}{n} \right)^{\frac{n-1}{2}} \leq \left( \frac{C}{n} \right)^{\frac{n-1}{2}}.$$  \hspace{1cm} \hfill \Box

We will also show:

**Proposition 3.** There is $C_1 > 0$ such that if we set

$$t_k = t(n, k) \overset{\text{def}}{=} \left( \frac{n}{C_1} \right)^{\frac{n-k}{2n}},$$  \hspace{1cm} (2)

then

$$\max_{k=1, \ldots, n-1} m(\mathcal{L}_n \setminus S_k^{(n)}(t_k)) = o\left( \frac{1}{n} \right).$$

In particular, $m\left( \bigcap_{k=1}^{n-1} S_k^{(n)}(t_k) \right) \to_{n \to \infty} 1$.

**Remarks.** 1. Let us define $\bar{\alpha}_{n,k} \overset{\text{def}}{=} \sup \{ \alpha_k(\Lambda) : \Lambda \in \mathcal{L}_n \}$. These quantities are powers of the so-called Rankin constants or generalized Hermite constants usually denoted by $\gamma_{n,k}$ (see [Thu98]), namely they are related by

$$\bar{\alpha}_{n,k}^{2k} = \gamma_{n,k}. \hspace{1cm} (3)$$

The origin of this exponent $2k$ is the $1/k$ in the definition of $\mathcal{V}_k$, which we have imposed so that the functions $\alpha_k$ behave nicely with respect to homothety. This normalization has the additional advantage that the growth rate of the different $\bar{\alpha}_{n,k}$ (as a function of $n$) becomes the same for all $k$. Namely [Thu98, Cor. 2] and (3) show that $\log \bar{\alpha}_{n,k} = \frac{1}{2} \log n + O(1)$ (where the implicit constant depends on $k$).
2. It seems plausible that most lattices come close to realizing the Rankin constants, that is, for any $\varepsilon > 0$,

$$\lim_{n \to \infty} m (\{ \Lambda \in \mathcal{L}_n : \forall k, \alpha_k(\Lambda) > \tau_{n,k} - \varepsilon \}) = 1.$$  

Combined with the result of Thunder mentioned above, Proposition 3 may be viewed as supporting evidence for such a conjecture.

3. We take this opportunity to formulate an analogous question regarding the covering radius; that is, is it true that for any $\varepsilon > 0$,

$$\lim_{n \to \infty} m \left( \Lambda \in \mathcal{L}_n : \text{covrad}(\Lambda) < \inf_{\Lambda' \in \mathcal{L}_n} \text{covrad}(\Lambda') + \varepsilon \right) = 1,$$

where

$$\text{covrad}(\Lambda) = \inf \{ r > 0 : \mathbb{R}^n = \Lambda + B(0,r) \}$$

and $B(0,r)$ is the Euclidean ball of radius $r$ around the origin.

The proof of Propositions 2 and 3 relies on Thunder’s work and on a variant of Siegel’s formula [Sie45] which relates the Lebesgue measure on $\mathbb{R}^n$ and the measure $m$ on $\mathcal{L}_n$. We now review Siegel’s method and Thunder’s results.

In the sequel we consider $n \geq 2$ and $k \in \{1, \ldots, n-1\}$ as fixed and omit, unless there is risk of confusion, the symbols $n$ and $k$ from the notation.

Consider the (set valued) map $\Phi = \Phi_k^{(n)}$ that assigns to each lattice $\Lambda \in \mathcal{L}_n$ the following subset of the exterior power of $\bigwedge^k \mathbb{R}^n$:

$$\Phi(\Lambda) \overset{\text{def}}{=} \{ \pm w_\Delta : \Delta \subset \Lambda \text{ a primitive subgroup with } r(\Delta) = k \},$$

where $w_\Delta = v_1 \wedge \cdots \wedge v_k$ and $\{v_i\}_{i=1}^k$ is a basis for $\Delta$ (note that $w_\Delta$ is well-defined up to sign, and $\Phi(\Lambda)$ contains both possible choices). Let

$$\mathcal{V} = \mathcal{V}_k^{(n)} \overset{\text{def}}{=} \{ v_1 \wedge \cdots \wedge v_k : v_i \in \mathbb{R}^n \} \setminus \{0\}$$

be the variety of pure tensors in $\bigwedge^k \mathbb{R}^n$. For any compactly supported bounded Riemann integrable function $f$ on $\mathcal{V}$ set

$$\hat{f} : \mathcal{L}_n \to \mathbb{R}, \quad \hat{f}(\Lambda) \overset{\text{def}}{=} \sum_{w \in \Phi(\Lambda)} f(w). \quad (4)$$

Then it is known (see [Wei82, Lemma 2.4.2]) that the (finite) sum (4) defines a function in $L^1(\mathcal{L}_n, m)$. This allows us to define a Radon measure $\theta = \theta_k^{(n)}$ on $\mathcal{V}$ by the formula

$$\int_{\mathcal{V}} f d\theta \overset{\text{def}}{=} \int_{\mathcal{L}_n} \hat{f} dm, \quad \text{for } f \in C_c(\mathcal{V}). \quad (5)$$

Write $G = G_n^{(\text{def})} = \text{SL}_n(\mathbb{R})$. There is a natural transitive action of $G_n$ on $\mathcal{V}$ and the stabilizer of $e_1 \wedge \cdots \wedge e_k$ is the subgroup

$$H = H_k^{(n)} \overset{\text{def}}{=} \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in G : A \in G_k, D \in G_{n-k} \right\}.$$
We therefore obtain an identification $\mathcal{V} \simeq G/H$ and view $\theta$ as a measure on $G/H$. It is well-known (see e.g. [Wei82]) that up to a proportionality constant there exists a unique $G$-invariant measure $m_{G/H}$ on $G/H$; moreover, given Haar measures $m_G, m_H$ on $G$ and $H$ respectively, there is a unique normalization of $m_{G/H}$ such that for any $f \in L^1(G, m_G)$

$$
\int_G f \, dm_G = \int_{G/H} \int_H f(gh) \, dm_H(h) \, dm_{G/H}(gH).
$$

(6)

We choose the Haar measure $m_G$ so that it descends to our probability measure $m$ on $\mathcal{L}_n$; similarly, we choose the Haar measure $m_H$ so that the periodic orbit $H \mathbb{Z}^n \subset \mathcal{L}_n$ has volume 1. These choices of Haar measures determine our measure $m_{G/H}$ unequivocally. It is clear from the defining formula (5) that $\theta$ is $G$-invariant and therefore the two measures $m_{G/H}, \theta$ are proportional. In fact (see [Sie45] for the case $k = 1$ and [Wei82, Lemma 2.4.2] for the general case),

$$
m_{G/H} = \theta.
$$

(7)

For $t > 0$, let $\chi = \chi_t : \mathcal{V} \to \mathbb{R}$ be the restriction to $\mathcal{V}$ of the characteristic function of the ball of radius $t$ around the origin, in $\bigwedge^k \mathbb{R}^n$, with respect to the natural inner product obtained from the Euclidean inner product on $\mathbb{R}^n$. Note that $\hat{\chi}(x) = 0$ if and only if $x \in S^{(n)}_k (t^{1/k})$ and furthermore, $\hat{\chi}(x) \geq 1$ if $x \in \mathcal{L}_n \setminus S^{(n)}_k (t^{1/k})$. It follows that

$$
m \left( \mathcal{L}_n \setminus S^{(n)}_k(t) \right) \leq \int_{\mathcal{L}_n} (\hat{\chi}_t)^k \, dm = \int_{\mathcal{V}} \chi_t \, d\theta.
$$

(8)

Let $V_j$ denote the volume of the Euclidean unit ball in $\mathbb{R}^j$ and let $\zeta$ denote the Riemann zeta function. We will use an unconventional convention $\zeta(1) = 1$, which will make our formulæ simpler. For $j \geq 1$, define

$$
R(j) \overset{\text{def}}{=} \frac{j^2 V_j}{\zeta(j)} \quad \text{and} \quad B(n, k) \overset{\text{def}}{=} \frac{\prod_{j=1}^n R(j)}{\prod_{j=1}^k R(j) \prod_{j=1}^{n-k} R(j)}.
$$

The following is [Thu98, Lemma 5]:

**Theorem 4** (Thunder). For $t > 0$, we have $\int_{\mathcal{V}} \chi_t \, dm_{G/H} = B(n, k) \frac{\zeta}{n}$.

(Note that in Thunder’s notation, by [Thu98, §4], $c(n, k) = B(n, k) / n$.)

We will need to bound $B(n, k)$.

**Lemma 5.** There is $C > 0$ so that for all large enough $n$ and all $k = 1, \ldots, n-1$,

$$
B(n, k) \leq \left( \frac{C}{n} \right)^{\frac{k(n-k)}{2}}.
$$

(9)
Proof. In this proof $c_0, c_1, \ldots$ are constants independent of $n, k, j$. Because of the symmetry $B(n, k) = B(n, n - k)$ it is enough to prove (9) with $k \leq \frac{n}{2}$.

Using the formula $V_j = \frac{\pi^{j/2}}{\Gamma(\frac{j}{2} + 1)}$ we obtain

$$B(n, k) = \prod_{j=1}^{k} \frac{R(n - k + j)}{R(j)} = \prod_{j=1}^{k} \frac{\zeta(j)(n - k + j)^2 \frac{\pi^{(n-k+j)/2}}{\Gamma(\frac{(n-k+j)}{2} + 1)}}{\zeta(n - k + j)^2 \frac{\pi^{j/2}}{\Gamma(\frac{j}{2} + 1)}}$$

$$= \prod_{j=1}^{k} \frac{\zeta(j)}{\zeta(n - k + j)} \cdot \left(\frac{n - k + j}{j}\right)^2 \cdot \frac{n-k}{n} \cdot \frac{\Gamma(\frac{j}{2} + 1)}{\Gamma(\frac{n-k+j}{2} + 1)}.$$ 

Note that $\zeta(s) \geq 1$ is a decreasing function of $s > 1$, so (recalling our convention $\zeta(1) = 1$) $\frac{\zeta(j)}{\zeta(n-k+j)} \leq c_0 \overset{\text{def}}{=} \zeta(2)$. It follows that for all large enough $n$ and for any $1 \leq j \leq k$,

$$\frac{\zeta(j)}{\zeta(n-k+j)} \leq c_0 \zeta(2)^{j/n-k} \leq c_0 n^{2 \pi n-k} \leq 4^n.$$ (10)

According to Stirling’s formula, there are positive constants $c_1, c_2$ such that for all $x \geq 2$,

$$c_1 \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x \leq \Gamma(x) \leq c_2 \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x.$$ 

We set $u \overset{\text{def}}{=} \frac{j}{2} + 1$ and $v \overset{\text{def}}{=} \frac{n-k}{2}$, so that $u + v \geq \frac{n-1}{4}$, and obtain

$$\frac{\Gamma(\frac{j}{2} + 1)}{\Gamma(\frac{n-k+j}{2} + 1)} = \frac{\Gamma(u)}{\Gamma(u + v)} \leq \frac{c_2}{c_1} \sqrt{\frac{u+v}{u}} \frac{u^u}{u^u} \frac{e^{u+v}}{e^{u}}$$

$$\leq c_3 e^{v} \frac{u^{u-1/2}}{(u+v)^{u+v-1/2}} = c_3 \left(\frac{e}{u+v}\right)^v \frac{1}{(1 + \frac{v}{u})^{u-1/2}}.$$ (11)

Using (10) and (11) we obtain

$$B(n, k) \leq \left[c_3 4\frac{n-k}{n-1}\left(\frac{4e}{n-1}\right)^{\frac{n-k}{2}}\right]^k = \left[c_3 \left(\frac{16e}{n-1}\right)^{\frac{n-k}{2}}\right]^k.$$ 

So taking $C > 16c_3e$ we obtain (9) for all large enough $n$. \qed

Proof of Propositions 2 and 3. Let $C$ be as in Lemma 5 and let $C_1 > C$. For Proposition 3, using (8), (7) and Theorem 4, for all sufficiently large $n$ we have

$$m\left(\mathcal{L}_n \times S_k^{(n)}(t_k)\right) \leq B(n, k)^{\frac{k n}{n}}$$

$$\leq \frac{1}{n} \left(\frac{C}{n}\right)^{\frac{k(n-k)}{2}} \left(\frac{n}{C_1}\right)^{\frac{k(n-k)}{2}} = \frac{1}{n} \left(\frac{C}{C_1}\right)^{\frac{k(n-k)}{2}}.$$
Multiplying by $n$ and taking the maximum over $k$ we obtain
\[ n \max_{k=1, \ldots, n} m \left( \mathcal{L}_n \setminus S_k^{(n)}(t_k) \right) \leq \left( \frac{C}{C_1} \right)^{n-1} \to_{n \to \infty} 0. \]

The proof of Proposition 2 is identical using $t = 1$ instead of $t_k$. \qed

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**References**


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