A VOLUME ESTIMATE FOR THE SET OF STABLE LATTICES

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Abstract. We show that in high dimensions the set of stable lattices is almost of full measure in the space of unimodular lattices.

Let $G \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{R})$, $\Gamma \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{Z})$, and let $A \subset G$ denote the subgroup of diagonal matrices with positive entries. The quotient space $\mathcal{L}_n \stackrel{\text{def}}{=} G/\Gamma$ is naturally identified with the space of unimodular lattices in $\mathbb{R}^n$, and the group G (and any of it subgroups) acts via left translations, or equivalently, by acting on lattices via its linear action on $\mathbb{R}^n$. A lattice Λ is called *stable* if for any subgroup $\Delta \subset \Lambda$, one has $\mathrm{vol}\,(\Delta \otimes \mathbb{R}/\Delta) \geq 1$ (in the literature the term *semi-stable* is also used), and we denote the set of stable lattices by $\mathcal{S}^{(n)}$.

A central problem is to understand the orbits of the A-action on $\mathcal{L}_n$. In [SW] we proved that for any lattice $\Lambda \in \mathcal{L}_n$, the orbit-closure $\overline{A\Lambda}$ contains a stable lattice. This result reduces the proof of Minkowski's conjecture on the product of inhomogeneous linear forms to that of estimating the Euclidean covering radius of stable lattices (see [SW] for details). Understanding stable lattices is therefore a natural problem due to its connection both with well-studied problems in the geometry of numbers, and with dynamics of the A-action. Although $\mathcal{S}^{(n)}$ is compact (while $\mathcal{L}_n$ is not), in this note we show that $\mathcal{S}^{(n)}$ has almost full measure with respect to the natural probability measure on $\mathcal{L}_n$, for large n. Moreover the convergence to full measure is very fast. This answers a question we were asked by G. Harder, and can be viewed as a manifestation of the concentration of mass along the equator in high dimensional Euclidean balls.

We will prove the following.

Theorem 1. *Let m denote the G-invariant probability measure on $\mathcal{L}_n$ derived from Haar measure on G, and let $\mathcal{S}^{(n)}$ denote the subset of stable lattices in $\mathcal{L}_n$. Then there is a constant $C > 0$ such that for all sufficiently large n,*

$$m\left(\mathcal{L}_n \smallsetminus \mathcal{S}^{(n)}\right) \leq \left(\frac{C}{n}\right)^{\frac{n-1}{2}}.$$

In particular $m\left(\mathcal{S}^{(n)}\right) \longrightarrow 1$ as $n \to \infty$.

For $\Lambda \in \mathcal{L}_n$ and a subgroup $\Delta \subset \Lambda$, we denote by $r(\Delta)$ its *rank* and by $|\Delta|$ its *covolume* in the Euclidean subspace $\Delta \otimes \mathbb{R} \subset \mathbb{R}^n$. For $k = 1, \ldots, n-1$ let us denote $\mathcal{V}_k(\Lambda) \stackrel{\text{def}}{=} \left\{ |\Delta|^{1/k} : \Delta \subset \Lambda, r(\Delta) = k \right\}$ and $\alpha_k(\Lambda) = \min \mathcal{V}_k(\Lambda)$ so that Λ is stable if and only if $\alpha_k(\Lambda) \geq 1$ for $k = 1, \ldots, n-1$. Let

$$\mathcal{S}_k^{(n)}(t) \stackrel{\text{def}}{=} \{x \in \mathcal{L}_n : \alpha_k(x) \geq t\}, \quad \mathcal{S}_k^{(n)} \stackrel{\text{def}}{=} \mathcal{S}_k^{(n)}(1).$$

With this notation $\mathcal{S}^{(n)} = \bigcap_{k=1}^{n-1} \mathcal{S}_k^{(n)}$. We will show:

Proposition 2. *There is $C > 0$ such that for all sufficiently large n, and all $k \in \{1, \ldots, n-1\}$,*

$$m\left(\mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}\right) \leq \frac{1}{n} \left(\frac{C}{n}\right)^{\frac{k(n-k)}{2}}. \tag{1}$$

Proof of Theorem 1. For $n > C$, the largest value of $\left(\frac{C}{n}\right)^{\frac{k(n-k)}{2}}$ is attained when $k = 1$ and $k = n-1$. Therefore (1) implies

$$\begin{aligned} m\left(\mathcal{L}_n \smallsetminus \mathcal{S}^{(n)}\right) &= m\left(\mathcal{L}_n \smallsetminus \bigcap_{k=1}^{n-1} \mathcal{S}_k^{(n)}\right) = m\left(\bigcup_{k=1}^{n-1} \mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}\right) \\ &\leq \frac{n-2}{n}\left(\frac{C}{n}\right)^{\frac{n-1}{2}} \leq \left(\frac{C}{n}\right)^{\frac{n-1}{2}}. \end{aligned}$$

□

We will also show:

Proposition 3. *There is $C_1 > 0$ such that if we set*

$$t_k = t(n,k) \stackrel{\text{def}}{=} \left(\frac{n}{C_1}\right)^{\frac{n-k}{2n}}, \tag{2}$$

then

$$\max_{k=1,\ldots,n-1} m\left(\mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}(t_k)\right) = o\left(\frac{1}{n}\right).$$

In particular, $m\left(\bigcap_{k=1}^{n-1} \mathcal{S}_k^{(n)}(t_k)\right) \to_{n\to\infty} 1$.

Remarks. 1. Let us define $\bar{\alpha}_{n,k} \stackrel{\text{def}}{=} \sup\{\alpha_k(\Lambda) : \Lambda \in \mathcal{L}_n\}$. These quantities are powers of the so-called *Rankin constants* or *generalized Hermite constants* usually denoted by $\gamma_{n,k}$ (see [Thu98]), namely they are related by

$$\bar{\alpha}_{n,k}^{2k} = \gamma_{n,k}. \tag{3}$$

The origin of this exponent $2k$ is the $1/k$ in the definition of $\mathcal{V}_k$, which we have imposed so that the functions α_k behave nicely with respect to homothety. This normalization has the additional advantage that the growth rate of the different $\bar{\alpha}_{n,k}$ (as a function of n) becomes the same for all k. Namely [Thu98, Cor. 2] and (3) show that $\log \bar{\alpha}_{n,k} = \frac{1}{2}\log n + O(1)$ (where the implicit constant depends on k).

2. It seems plausible that most lattices come close to realizing the Rankin constants, that is, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} m\left(\{\Lambda \in \mathcal{L}_n : \forall k,\, \alpha_k(\Lambda) > \overline{\alpha}_{n,k} - \varepsilon\}\right) = 1.$$

Combined with the result of Thunder mentioned above, Proposition 3 may be viewed as supporting evidence for such a conjecture.

3. We take this opportunity to formulate an analogous question regarding the *covering radius*; that is, is it true that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} m\left\{\Lambda \in \mathcal{L}_n : \operatorname{covrad}(\Lambda) < \inf_{\Lambda' \in \mathcal{L}_n} \operatorname{covrad}(\Lambda') + \varepsilon\right\} = 1,$$

where

$$\operatorname{covrad}(\Lambda) = \inf\{r > 0 : \mathbb{R}^n = \Lambda + B(0,r)\}$$

and $B(0,r)$ is the Euclidean ball of radius r around the origin.

The proof of Propositions 2 and 3 relies on Thunder's work and on a variant of Siegel's formula [Sie45] which relates the Lebesgue measure on $\mathbb{R}^n$ and the measure m on $\mathcal{L}_n$. We now review Siegel's method and Thunder's results.

In the sequel we consider $n \geq 2$ and $k \in \{1, \dots, n-1\}$ as fixed and omit, unless there is risk of confusion, the symbols n and k from the notation. Consider the (set valued) map $\Phi = \Phi_k^{(n)}$ that assigns to each lattice $\Lambda \in \mathcal{L}_n$ the following subset of the exterior power of $\bigwedge^k \mathbb{R}^n$:

$$\Phi(\Lambda) \stackrel{\text{def}}{=} \{\pm w_\Delta : \Delta \subset \Lambda \text{ a primitive subgroup with } r(\Delta) = k\},$$

where $w_\Delta \stackrel{\text{def}}{=} v_1 \wedge \cdots \wedge v_k$ and $\{v_i\}_{i=1}^k$ is a basis for Δ (note that w_Δ is well-defined up to sign, and $\Phi(\Lambda)$ contains both possible choices). Let

$$\mathscr{V} = \mathscr{V}_k^{(n)} \stackrel{\text{def}}{=} \{v_1 \wedge \cdots \wedge v_k : v_i \in \mathbb{R}^n\} \smallsetminus \{0\}$$

be the variety of pure tensors in $\bigwedge^k \mathbb{R}^n$. For any compactly supported bounded Riemann integrable[1] function f on $\mathscr{V}$ set

$$\hat{f} : \mathcal{L}_n \to \mathbb{R}, \qquad \hat{f}(\Lambda) \stackrel{\text{def}}{=} \sum_{w \in \Phi(\Lambda)} f(w). \tag{4}$$

Then it is known (see [Wei82, Lemma 2.4.2]) that the (finite) sum (4) defines a function in $L^1(\mathcal{L}_n, m)$. This allows us to define a Radon measure $\theta = \theta_k^{(n)}$ on $\mathscr{V}$ by the formula

$$\int_{\mathscr{V}} f d\theta \stackrel{\text{def}}{=} \int_{\mathcal{L}_n} \hat{f}\, dm, \text{ for } \ f \in C_c(\mathscr{V}). \tag{5}$$

Write $G = G_n \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{R})$. There is a natural transitive action of G_n on $\mathscr{V}$ and the stabilizer of $e_1 \wedge \cdots \wedge e_k$ is the subgroup

$$H = H_k^{(n)} \stackrel{\text{def}}{=} \left\{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in G : A \in G_k, D \in G_{n-k}\right\}.$$

[1] i.e. the measure of points at which f is not continuous is zero.

We therefore obtain an identification $\mathscr{V} \simeq G/H$ and view θ as a measure on G/H. It is well-known (see e.g. [Wei82]) that up to a proportionality constant there exists a unique G-invariant measure $m_{G/H}$ on G/H; moreover, given Haar measures m_G, m_H on G and H respectively, there is a unique normalization of $m_{G/H}$ such that for any $f \in L^1(G, m_G)$

$$\int_G f\, dm_G = \int_{G/H} \int_H f(gh) dm_H(h) dm_{G/H}(gH). \tag{6}$$

We choose the Haar measure m_G so that it descends to our probability measure m on $\mathcal{L}_n$; similarly, we choose the Haar measure m_H so that the periodic orbit $H\mathbb{Z}^n \subset \mathcal{L}_n$ has volume 1. These choices of Haar measures determine our measure $m_{G/H}$ unequivocally. It is clear from the defining formula (5) that θ is G-invariant and therefore the two measures $m_{G/H}, \theta$ are proportional. In fact (see [Sie45] for the case $k = 1$ and [Wei82, Lemma 2.4.2] for the general case),

$$m_{G/H} = \theta. \tag{7}$$

For $t > 0$, let $\chi = \chi_t : \mathscr{V} \to \mathbb{R}$ be the restriction to $\mathscr{V}$ of the characteristic function of the ball of radius t around the origin, in $\bigwedge^k \mathbb{R}^n$, with respect to the natural inner product obtained from the Euclidean inner product on $\mathbb{R}^n$. Note that $\hat{\chi}(x) = 0$ if and only if $x \in \mathcal{S}_k^{(n)}\left(t^{1/k}\right)$ and furthermore, $\hat{\chi}(x) \geq 1$ if $x \in \mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}\left(t^{1/k}\right)$. It follows that

$$m\left(\mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}(t)\right) \leq \int_{\mathcal{L}_n} \widehat{(\chi_{t^k})} dm = \int_{\mathscr{V}} \chi_{t^k} d\theta. \tag{8}$$

Let V_j denote the volume of the Euclidean unit ball in $\mathbb{R}^j$ and let ζ denote the Riemann zeta function. We will use an unconventional convention $\zeta(1) = 1$, which will make our formulae simpler. For $j \geq 1$, define

$$R(j) \stackrel{\text{def}}{=} \frac{j^2 V_j}{\zeta(j)} \quad \text{and} \quad B(n,k) \stackrel{\text{def}}{=} \frac{\prod_{j=1}^n R(j)}{\prod_{j=1}^k R(j) \prod_{j=1}^{n-k} R(j)}.$$

The following is [Thu98, Lemma 5]:

Theorem 4 (Thunder). *For $t > 0$, we have $\int_{\mathscr{V}} \chi_t\, dm_{G/H} = B(n,k)\frac{t^n}{n}$.*

(Note that in Thunder's notation, by [Thu98, §4], $c(n,k) = B(n,k)/n$.)

We will need to bound $B(n,k)$.

Lemma 5. *There is $C > 0$ so that for all large enough n and all $k = 1, \ldots, n-1$,*

$$B(n,k) \leq \left(\frac{C}{n}\right)^{\frac{k(n-k)}{2}}. \tag{9}$$

Proof. In this proof $c_0, c_1, \ldots$ are constants independent of n, k, j. Because of the symmetry $B(n,k) = B(n, n-k)$ it is enough to prove (9) with $k \le \frac{n}{2}$. Using the formula $V_j = \frac{\pi^{j/2}}{\Gamma(\frac{j}{2}+1)}$ we obtain

$$B(n,k) = \prod_{j=1}^{k} \frac{R(n-k+j)}{R(j)} = \prod_{j=1}^{k} \frac{\zeta(j)(n-k+j)^2 \frac{\pi^{(n-k+j)/2}}{\Gamma(\frac{n-k+j}{2}+1)}}{\zeta(n-k+j) j^2 \frac{\pi^{j/2}}{\Gamma(\frac{j}{2}+1)}}$$

$$= \prod_{j=1}^{k} \frac{\zeta(j)}{\zeta(n-k+j)} \cdot \left(\frac{n-k+j}{j}\right)^2 \cdot \pi^{\frac{n-k}{2}} \cdot \frac{\Gamma(\frac{j}{2}+1)}{\Gamma(\frac{n-k+j}{2}+1)}.$$

Note that $\zeta(s) \ge 1$ is a decreasing function of $s > 1$, so (recalling our convention $\zeta(1) = 1$) $\frac{\zeta(j)}{\zeta(n-k+j)} \le c_0 \stackrel{\text{def}}{=} \zeta(2)$. It follows that for all large enough n and for any $1 \le j \le k$,

$$\frac{\zeta(j)}{\zeta(n-k+j)} \cdot \left(\frac{n-k+j}{j}\right)^2 \cdot \pi^{\frac{n-k}{2}} \le c_0 n^2 \pi^{\frac{n-k}{2}} \le 4^{\frac{n-k}{2}}. \tag{10}$$

According to Stirling's formula, there are positive constants c_1, c_2 such that for all $x \ge 2$,

$$c_1 \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x \le \Gamma(x) \le c_2 \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x.$$

We set $u \stackrel{\text{def}}{=} \frac{j}{2} + 1$ and $v \stackrel{\text{def}}{=} \frac{n-k}{2}$, so that $u + v \ge \frac{n-1}{4}$, and obtain

$$\frac{\Gamma(\frac{j}{2}+1)}{\Gamma(\frac{n-k+j}{2}+1)} = \frac{\Gamma(u)}{\Gamma(u+v)} \le \frac{c_2}{c_1} \sqrt{\frac{u+v}{u}} \frac{u^u}{(u+v)^{u+v}} \frac{e^{u+v}}{e^u}$$

$$\le c_3 e^v \frac{u^{u-1/2}}{(u+v)^{u+v-1/2}} = c_3 \left(\frac{e}{u+v}\right)^v \frac{1}{(1+\frac{v}{u})^{u-1/2}}, \tag{11}$$

$$\le c_3 \left(\frac{4e}{n-1}\right)^{\frac{n-k}{2}}.$$

Using (10) and (11) we obtain

$$B(n,k) \le \left[c_3 4^{\frac{n-k}{2}} \left(\frac{4e}{n-1}\right)^{\frac{n-k}{2}}\right]^k = \left[c_3 \left(\frac{16e}{n-1}\right)^{\frac{n-k}{2}}\right]^k.$$

So taking $C > 16c_3 e$ we obtain (9) for all large enough n. $\square$

Proof of Propositions 2 and 3. Let C be as in Lemma 5 and let $C_1 > C$. For Proposition 3, using (8), (7) and Theorem 4, for all sufficiently large n we have

$$m\left(\mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}(t_k)\right) \le B(n,k) \frac{t_k^{kn}}{n}$$

$$\le \frac{1}{n} \left(\frac{C}{n}\right)^{\frac{k(n-k)}{2}} \left(\frac{n}{C_1}\right)^{\frac{k(n-k)}{2}} = \frac{1}{n} \left(\frac{C}{C_1}\right)^{\frac{k(n-k)}{2}}.$$

Multiplying by n and taking the maximum over k we obtain

$$n \max_{k=1,\dots,n} m\left(\mathcal{L}_n \smallsetminus \mathcal{S}_k^{(n)}(t_k)\right) \leq \left(\frac{C}{C_1}\right)^{\frac{n-1}{2}} \to_{n\to\infty} 0.$$

The proof of Proposition 2 is identical using $t = 1$ instead of t_k. □

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