HOROCYCLE DYNAMICS IN RANK ONE INVARIANT SUBVARIETIES I: WEAK MEASURE CLASSIFICATION AND EQUIDISTRIBUTION

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Abstract. Let $\mathcal{M}$ be an invariant subvariety in the moduli space of translation surfaces. We contribute to the study of the dynamical properties of the horocycle flow on $\mathcal{M}$. In the context of dynamics on the moduli space of translation surfaces, we introduce the notion of a 'weak classification of horocycle invariant measures' and we study its consequences. Among them, we prove genericity of orbits and related uniform equidistribution results, asymptotic equidistribution of sequences of pushed measures, and counting of saddle connection holonomies. As an example, we show that invariant varieties of rank one, Rel-dimension one and related spaces obtained by adding marked points satisfy the 'weak classification of horocycle invariant measures'. Our results extend prior results obtained by Eskin-Masur-Schmoll, Eskin-Marklof-Morris, and Bainbridge-Smillie-Weiss.

Contents

1. Introduction

Any stratum of translation surfaces $\mathcal{H}$ is endowed with an action of the group $G \stackrel{\mathrm{def}}{=} \mathrm{SL}_2(\mathbb{R})$, and the restriction of this action to the subgroup

$$U = \{u_s : s \in \mathbb{R}\}, \quad \text{where } u_s \stackrel{\mathrm{def}}{=} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \tag{1}$$

is called the *horocycle flow*. It is a longstanding open problem to understand the invariant measures for the U-action on $\mathcal{H}$. It has been known for a while, that a 'sufficiently nice' classification of such measures would have many interesting dynamical and geometrical consequences. See [Zor06, HMSZ06], and see [CSW20] for a survey of recent results. This paper contributes to the investigation of this problem in two ways. Firstly, we obtain a partial classification of horocycle invariant ergodic measures, which we call *weak classification*, in the setting of *rank-one invariant subvarieties of Rel-dimension one*, or spaces obtained from them by adding marked points. All of these notions will be defined in §2. Secondly, we explore the implications of a weak classification result in rank-one invariant subvarieties (of arbitrary Rel-dimension) and show that it implies genericity of orbits, equidistribution of circle averages, and many additional equidistribution results.

We do not know whether the weak classification result holds in general rank-one invariant subvarieties; our approach crucially uses the Rel-dimension one assumption — see Remark 3.5. Nevertheless, in a follow-up paper [CWY], we extend the classification of U-orbit closures to general rank-one invariant subvarietes (omitting the restriction of one-dimensional Rel). We do so without obtaining a weak measure classification, but rather by an induction on the dimension of Rel. Although the current paper uses completely different techniques, it provides several inputs that are used in [CWY]. In [CWY], it will be useful to know which of our results hold for products of strata, and related spaces obtained by marking points. We will comment on this throughout the text.

In the remainder of this introduction we give an initial description of the main results and discuss the proofs. Precise statements of results require some preparations and will be given in the body of the text. Let $\mathcal{M}$ be an orbit-closure for the action of G. As was shown in a celebrated paper of Eskin, Mirzakhani and Mohammadi [EMM15], such orbit-closures are (unit-area subloci of) affine orbifolds in period coordinates, and carry a wealth of additional structure. Following [AW], we refer to these orbit closures as *invariant subvarieties*. A fundamental invariant of an invariant subvariety is its *rank*, introduced in [Wri15a]. We will be interested in the rank-one case, which is characterized by the fact that the foliation into G-orbits and the Rel foliation (see §2.3) are complementary foliations, in the sense that a neighborhood of a point can be described in terms of the G-action and the Rel-foliation. A further invariant is the *Rel-dimension* (the complex dimension of leaves of the Rel foliation). Now suppose $\mathcal{N} \subset \mathcal{M}$ is an invariant subvariety. It was shown in [EM18] that $\mathcal{N}$ is the support of a unique G-invariant Radon probability measure, which has a geometric description in terms of the affine structure of $\mathcal{N}$. We denote this measure by $m_{\mathcal{N}}$. It is well-known that $m_{\mathcal{N}}$ is ergodic for the action of U and, as first observed by Calta [Cal04] and explained in §2.5, such a measure can be pushed by real Rel to create more U-invariant ergodic measures. We say that $\mathcal{M}$ satisfies the *weak classification of U-invariant measures* if any ergodic U-invariant probability Radon measure measure on $\mathcal{M}$, that assigns measure zero to the set of surfaces with horizontal saddle connections, is obtained by this construction. In Theorem 3.4 we prove that invariant subvarieties of rank one and with Rel-dimension one satisfy weak classification. We remark that all invariant subvarieties modeled on a complex vector space of dimension 3 are of rank one with Rel-dimension one and thus satisfy the hypotheses of Theorem 3.4; explicit examples are given in Corollaries 3.6, 3.9 and 3.10.

Theorem 6.1 gives severe restrictions on the collection of weak-* limits of measures obtained by averaging along a U-orbit in a rank-one invariant subvariety. This theorem uses the rank-one structure and is false in general, as shown in [CSW20]. We then explore the implications of weak classification in the setting of rank-one invariant subvarieties (omitting the assumption that the Rel-dimension is one). Under these conditions, we can derive the following results. In Theorem 6.2 we prove that the U-orbit of any surface without horizontal saddle connections is generic for some U-ergodic invariant measure. Moreover, for a given test function f and a given error ε, genericity holds uniformly outside certain compact 'singular sets', see Theorem 6.6. This result is an analogue of [DM93, Thm. 3]. We show in Theorem 7.3 that for any U-invariant ergodic measure μ, the pushforwards $g_{t*}\mu$ converge to $m_{\mathcal{N}}$ for some invariant subvariety $\mathcal{N}$, as $t \to \infty$ and to $m_{\mathcal{N}'}$ for some invariant subvariety $\mathcal{N}' \subset \mathcal{N}$, as $t \to -\infty$ (unless μ assigns positive measure to the set of surfaces with horizontal saddle connections, in which case they diverge in the space of probability measures). Here

$$g_t \stackrel{\text{def}}{=} \operatorname{diag}(e^t, e^{-t}) \tag{2}$$

denotes the geodesic flow. In Theorem 7.1 we obtain a similar result for pushing a G-invariant measure by a real Rel flow. Finally we show in Theorem 7.4 that in a rank one locus satisfying the weak classification of U-invariant measures, pushed circle averages $\frac{1}{2\pi}\int_0^{2\pi} g_{t*}r_\theta\delta_q$ converge as $t \to \infty$, to the G-invariant measure on $\mathcal{N} = \overline{Gq}$. Here r_θ is the rotation matrix by angle θ. As a consequence, every surface satisfies quadratic growth of saddle connection holonomies. We give another application to a counting problem arising from billiards, in §8.

Our technique revisits and adapts arguments of Ratner [Rat91] for proving the measure classification of unipotent flows on homogeneous spaces, and subsequent work of Dani and Margulis [DM93] for harnessing this in order to obtain equidistribution results. The first papers in which these ideas were adapted to translation surfaces are [EMS03, EMWM06], in which a weak classification of U-invariant measures was proved, and equidistribution for circle averages was deduced, in the context of branched covers of Veech surfaces. In [CW10] and [BSW16], measure classification results were obtained in eigenform loci in genus two, which are particular cases of a rank-one invariant subvariety with Rel-dimension one. Specifically, the paper [BSW16] proves a complete measure classification result, which is stronger than our 'weak classification', as it also classifies measures for which typical surfaces do have horizontal saddle connections. From this strong measure classification, equidistribution results are deduced, namely a strong version of Theorem 6.2, and analogues of Theorems 7.3, 7.1 and 7.4. Our work adapts the techniques in [BSW16] to our more general setting and establishes that most of these equidistribution results, and additional ones, can be proved assuming a weaker measure classification result. A key technical input is Proposition 5.6, which improves [BSW16, Proof of Thm.

11.1, Step 3] and uniformly bounds the time horocycle trajectories starting in a given compact set can spend close to certain singular sets.

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2. Strata of translation surfaces

In this section we collect some standard definitions and set our notations. See [MT02, Zor06, FM14, Wri15b, BSW16] for helpful surveys and additional introductory material.

2.1. **Translation surfaces and developing map.** Let S be a compact and oriented topological surface of genus g, let Σ be a finite subset of S and let $\kappa = (k_\xi)_{\xi\in\Sigma}$ be an integer partition of $2g-2$. That is, $\sum_{\xi\in\Sigma} k_\xi = 2g-2$ and k_ξ are non-negative integers. We emphasize that $\kappa_\xi = 0$ is allowed, in which case ξ is called a *marked point.* A *singularity-labeled translation surface of type* κ is a triple $q = (X_q, \omega_q, \mathfrak{z}_q)$ where X_q is a compact and connected Riemann surface of genus g, ω_q is a non-zero holomorphic 1-form on X_q whose set of zeroes is denoted by Σ_q and $\mathfrak{z}_q : \Sigma \to X_q$ is an injective map whose image contains Σ_q and such that $\mathfrak{z}_q(\xi)$ has order k_ξ. Where it will not cause any confusion we will omit the words 'singularity-labeled' and refer to q simply as a *translation surface of type* κ. A *marking* on a translation surface q is a homeomorphism $\varphi : S \to X_q$ such that $\varphi(\xi) = \mathfrak{z}_q(\xi)$ for $\xi \in \Sigma$, and a pair (q, φ) is called a *marked translation surface with labeled singularities.* Two marked translation surfaces (q_1, φ_1) and (q_2, φ_2) are isomorphic if there is a biholomorphism $\phi : X_{q_1} \to X_{q_2}$ such that $\phi^*\omega_{q_2} = \omega_{q_1}$, $\phi \circ \mathfrak{z}_{q_1} = \mathfrak{z}_{q_2}$ and $\varphi_2^{-1} \circ \phi \circ \varphi_1$ is isotopic to the identity rel Σ. Let $\mathcal{H}_{\mathrm{m}}$ be the space of isomorphism classes of marked translation surfaces of type κ with labeled singularities and let $\mathrm{Mod}(S, \Sigma)$ be the mapping class group of S that preserves Σ point-wise. It acts by precomposition on the space $\mathcal{H}_{\mathrm{m}}$ and we denote by $\mathcal{H}$ the quotient and by $\pi : \mathcal{H}_{\mathrm{m}} \to \mathcal{H}$ the quotient map. The space $\mathcal{H} = \mathcal{H}(\kappa)$ is the *moduli space of translation surfaces of type* κ *with labeled singularities.* There is a structure of linear manifold on $\mathcal{H}_{\mathrm{m}}$ that turns the following map into a local linear isomorphism.

$$\mathrm{dev} : \mathcal{H}_{\mathrm{m}} \to H^1(S, \Sigma; \mathbb{R}^2), \quad \mathrm{dev}\left((q, \varphi)\right) \stackrel{\mathrm{def}}{=} \left(\gamma \mapsto \int_{\varphi(\gamma)} \omega_q\right).$$

Each homeomorphism of S preserving Σ acts on homology and this induces an action of the group $\mathrm{Mod}(S, \Sigma)$ on $H^1(S, \Sigma; \mathbb{R}^2)$. For this action, the map dev is $\mathrm{Mod}(S, \Sigma)$-equivariant. The space $\mathcal{H}$ can be endowed with the quotient topology and inherits an orbifold structure from the one on $\mathcal{H}_{\mathrm{m}}$.

The *area* of a translation surface is the integral of the volume form induced by the 1-form, i.e.

$$\mathrm{Area}(q) = \frac{1}{2\mathbf{i}} \int_S \omega_q \wedge \overline{\omega}_q.$$

We denote the subset of area-one surfaces in $\mathcal{H}_{\mathrm{m}}$ and $\mathcal{H}$ by $\mathcal{H}_{\mathrm{m}}^{(1)}$ and $\mathcal{H}^{(1)}$ respectively.

Associated with a translation surface is a translation atlas on X_q, obtained by locally integrating ω_q. A *saddle connection* on a translation surface q is a path $\gamma : [0,1] \to X_q$ such that $\gamma(0), \gamma(1) \in \Sigma_q, \gamma(t) \notin \Sigma_q$ for $t \in (0,1)$, and $t \mapsto \gamma(t)$ is linear in planar charts. We refer to the vector $\int_\gamma \omega_q \in \mathbb{C} \cong \mathbb{R}^2$ as the *holonomy* of γ with respect to q, and denote it by $\mathrm{hol}_q(\gamma)$.

2.2. **G-action, invariant subvarieties.** As in the introduction, throughout this paper we set $G \stackrel{\mathrm{def}}{=} \mathrm{SL}_2(\mathbb{R})$. The space $\mathcal{H}_{\mathrm{m}}$ is endowed with a G-action, and the developing map is equivariant with respect to the natural action of G by post-composition on $H^1(S,\Sigma;\mathbb{R}^2)$. More precisely, for any (q,φ) and any $g \in G$, we have

$$\mathrm{dev}(g(q,\varphi)) = g\,\mathrm{dev}(q,\varphi),$$

and thus, for any saddle connection δ on q and any $g \in G$,

$$g\,\mathrm{hol}_q(\delta) = \mathrm{hol}_{gq}(\delta). \tag{3}$$

The G-action preserves the area of a translation surface. Besides the group U defined in (1), we will consider

$$A \stackrel{\mathrm{def}}{=} \{g_t : t \in \mathbb{R}\} \quad \text{and} \quad B \stackrel{\mathrm{def}}{=} AU, \tag{4}$$

where g_t is as in (2). The action of A will be referred to as the *geodesic flow.*

Definition 2.1 (Invariant subvariety)**.**

- *An* equilinear manifold *is a connected component of some* $\mathrm{dev}^{-1}(V)$ *where* $V \subset H^1(S,\Sigma;\mathbb{C}) \cong H^1(S,\Sigma;\mathbb{R}^2)$ *is a* $\mathbb{C}$*-linear subspace defined over* $\mathbb{R}$ *with* $\dim_{\mathbb{C}}(V) = d$*, and such that if* $\gamma \in \mathrm{Mod}(S,\Sigma)$ *preserves* V*, then* $\gamma|_V$ *has determinant* ± 1*. See* [SSWY].
- *A* d-dimensional equilinear subspace *is a subset* $\mathcal{M} \subset \mathcal{H}$ *such that* $\pi^{-1}(\mathcal{M})$ *is a locally finite union of d-dimensional equilinear manifolds.*
- *A d-dimensional* invariant subvariety *is a d-dimensional equilinear subspace which cannot be presented as a finite union of proper d-dimensional equilinear subspaces.*

By the work of Eskin, Mirzakhani and Mohammadi ([EM18] and [EMM15]), the orbit-closures for the $\mathrm{GL}_2^+(\mathbb{R})$-action are exactly the invariant subvarieties, closed invariant sets are always finite unions of invariant subvarieties, and the orbit-closures and invariant measures for G coincide with the orbit-closures and invariant measures for its subgroup P. What we call 'invariant subvariety' was called 'affine invariant manifold' in [EMM15]; our choice of terminology follows [AW]. Note also that Definition 2.1 differs slightly from the definitions given in [EMM15] but will be more convenient for us, see [SSWY] for a more detailed discussion of this definition.

Let $\mathcal{M}$ be a d-dimensional invariant subvariety. An d-dimensional equilinear manifold L contained in $\pi^{-1}(\mathcal{M})$ is called a *lift of* $\mathcal{M}$. By definition, $\pi^{-1}(\mathcal{M})$ is the union of the lifts of $\mathcal{M}$ and $\mathrm{Mod}(S,\Sigma)$ permutes these lifts. The last requirement in Definition 2.1 is referred to as *irreducibility.* It can be checked (see [SY]) that irreducibility is equivalent to the requirement that $\mathrm{Mod}(S,\Sigma)$ acts transitively on the set of lifts of $\mathcal{M}$.

Let $\mathcal{M}^{(1)}$ be the area-one locus in $\mathcal{M}$. Using Lebesgue measures on the lifts of $\mathcal{M}$ and a 'cone construction', one can construct a natural smooth G-invariant Radon probability measure of full support on $\mathcal{M}^{(1)}$. For more details on this construction, using the language above, see [SSWY]. We will denote this measure by $m_{\mathcal{M}}$. It was shown in [EM18] that all G-invariant ergodic measures on strata arise in this way, and that $m_{\mathcal{M}}$ is the unique G-invariant ergodic measure with $\operatorname{supp} m_{\mathcal{M}} = \mathcal{M}^{(1)}$.

2.3. **Rel foliation, Rel vector fields, rank.** We recall the following construction (see [Zor06, MW14, McM14, BSW16]). Let $\mathrm{Res} : H^1(S, \Sigma; \mathbb{R}^2) \to H^1(S; \mathbb{R}^2)$ be the canonical cocycle restriction map and denote its kernel by $\mathfrak{R}$. Clearly $\mathfrak{R}$ is a $\mathbb{C}$-linear subspace defined over $\mathbb{R}$. It will be useful to have an explicit set of generators for $\mathfrak{R}$. For any $\xi \in \Sigma$, denote by ξ^* the element of $H^1(S, \Sigma; \mathbb{R}^2)$ that is dual (with respect to the intersection form) to the cycle of $H_1(S \smallsetminus \Sigma; \mathbb{Z})$ represented by a contractible loop on S that circles counterclockwise the point ξ. The elements $\{\xi^* : \xi \in \Sigma\}$ are a set of generators for $\mathfrak{R}$, and they satisfy one relation $\sum_{\xi \in \Sigma} \xi^* = 0$. In particular $\dim_{\mathbb{C}}(\mathfrak{R}) = n - 1$, where n is the cardinality of Σ.

There is a foliation of $H^1(S, \Sigma; \mathbb{R}^2)$ by translates of $\mathfrak{R}$ which can be pulled back by the developing map to give a foliation of $\mathcal{H}_{\mathrm{m}}$. Since the map $\mathrm{dev} \circ \mathrm{Res}$ is $\mathrm{Mod}(S, \Sigma)$-equivariant, there is an induced foliation of $\mathcal{H}$. The area of two translation surfaces which are in the same leaf for this foliation are the same and thus the foliation restricts to give a foliation of $\mathcal{H}^{(1)}$. We call this foliation (on any one of the spaces $\mathcal{H}_{\mathrm{m}}, \mathcal{H}$, or $\mathcal{H}^{(1)}$) the *Rel foliation*; it has also been called the kernel foliation, the isoperiodic foliation, and the absolute period foliation.

Recall our convention that the singularities of surfaces in $\mathcal{H}$ are labeled. This implies that elements of $\mathrm{Mod}(S, \Sigma)$ do not permute the singularities and act trivially on $\mathfrak{R}$. For any $v \in \mathfrak{R}$, the constant vector field v can be lifted to a vector field on $\mathcal{H}_{\mathrm{m}}$ by dev and since singularities are not permuted by $\mathrm{Mod}(S, \Sigma)$, this vector field descends to a well-defined vector field on $\mathcal{H}$, which is called the Rel vector field associated to v and denoted $\vec{v}$.

Definition 2.2. *Let $q \in \mathcal{H}$ and let $v \in \mathfrak{R}$. We say that $\mathrm{Rel}_v(q)$ is well-defined and equal to q' if there is a smooth path $\gamma : I \to \mathcal{H}$, where I is an interval that contains 0 and 1, and such that $\gamma(0) = q$, $\gamma(1) = q'$ and $\forall t_0 \in I$, $\frac{d}{dt}|_{t=t_0} \gamma = \vec{v}$. We denote by Δ the subset of $\mathcal{H} \times \mathfrak{R}$ consisting of pairs (q, v) for which $\mathrm{Rel}_v(q)$ is well-defined.*

For fixed $v \in \mathfrak{R}$, we denote by $\mathcal{H}'_v$ the set of surfaces $q \in \mathcal{H}$ such that $(q, v) \in \Delta$; and for any $v \in \mathfrak{R}_{\mathcal{M}}$, we denote by $\mathcal{M}'_v$ the set of surfaces $q \in \mathcal{M}$ such that $(q, v) \in \Delta$.

We summarize some properties of this partially defined map (see [BSW16, §4] for proofs). The set Δ is open and the map

$$\Delta \to \mathcal{H}, \qquad (q, v) \mapsto \mathrm{Rel}_v(q)$$

is continuous. In particular (see [BSW16, Prop. 4.6]), for sequences $q_j \in \mathcal{H}, g_j \in G$, $v_j \in \mathfrak{R}$, for which the limits $q_\infty = \lim_{j \to \infty} q_j$, $g_\infty = \lim_{j \to \infty} g_j$, $v_\infty = \lim_{j \to \infty} v_j$ exist, we have

$$\text{if } \mathrm{Rel}_{v_\infty}(g_\infty q_\infty) \text{ is well-defined, then it is equal to } \lim_{j \to \infty} \mathrm{Rel}_{v_j}(g_j q_j). \tag{5}$$

The group G acts on $\mathfrak{R}$ by restriction of its action by post-composition on $H^1(S, \Sigma; \mathbb{C})$, and for any $g \in G$,

$$(q, v) \in \Delta \iff (gq, gv) \in \Delta, \quad \text{and} \quad \mathrm{Rel}_{gv}(gq) = g\mathrm{Rel}_v(q). \tag{6}$$

The set Δ is completely described in [MW14, BSW16] in terms of saddle connections. In particular, if $n = 2$ and q has no saddle connection whose holonomy is parallel to v, then $(q, v) \in \Delta$.

Let $\mathcal{M} \subset \mathcal{H}$ be an invariant subvariety, let L be one of its lifts and let V be the $\mathbb{C}$-linear subspace of $H^1(S, \Sigma; \mathbb{C})$ defined over $\mathbb{R}$ on which L is modeled, i.e., L coincides with a connected component of $\mathrm{dev}^{-1}(V)$. We define

$$\mathfrak{R}_L \stackrel{\mathrm{def}}{=} \mathfrak{R} \cap V.$$

Since by assumption, $\mathrm{Mod}(S, \Sigma)$ acts transitively on the lifts of $\mathcal{M}$ and $\mathrm{Mod}(S, \Sigma)$ acts trivially on $\mathfrak{R}$, it is easy to see that if L' is another lift then $\mathfrak{R}_L = \mathfrak{R}_{L'}$. We thus denote by $\mathfrak{R}_{\mathcal{M}}$ this common subspace. By construction, for any $q \in \mathcal{M}$ and $v \in \mathfrak{R}_{\mathcal{M}}$ we have $\mathrm{Rel}_v(q) \in \mathcal{M}$ whenever the latter is well-defined, and $\mathfrak{R}_{\mathcal{M}}$ is the biggest linear subspace of $\mathfrak{R}$ with this property. Whenever $\mathcal{N} \subset \mathcal{M}$ is an inclusion of invariant subvarieties, we have $\mathfrak{R}_{\mathcal{N}} \subset \mathfrak{R}_{\mathcal{M}}$. We refer to the number $\dim_{\mathbb{C}}(\mathfrak{R}_{\mathcal{M}})$ as the *Rel-dimension of* $\mathcal{M}$.

Let $V \subset H^1(S, \Sigma; \mathbb{C})$ be the local model of a lift L of $\mathcal{M}$. It follows from [AEM17, Theorem 1.4] that $\mathrm{Res}(V)$ is a symplectic subspace of $H^1(S; \mathbb{C})$ with respect to the symplectic structure induced by the cup product, and in particular, that $\dim_{\mathbb{C}}(\mathrm{Res}(V))$ is even. In particular this implies that for any affine subvariety $\mathcal{M}$, $\dim_{\mathbb{C}}(\mathcal{M}) - \dim_{\mathbb{C}}(\mathfrak{R}_{\mathcal{M}})$ is an even number.

Definition 2.3 (Rank)**.** *The* rank *of* $\mathcal{M}$ *is defined to be* $\frac{1}{2}\left(\dim_{\mathbb{C}}(\mathcal{M}) - \dim_{\mathbb{C}}(\mathfrak{R}_{\mathcal{M}})\right)$.

The rank is a fundamental invariant of an invariant subvariety. See [Wri15a] for additional characterizations. In this paper we will be particularly interested in the case $\mathrm{rank}(\mathcal{M}) = 1$. It follows from the discussion preceding Definition 2.3 that if $\dim_{\mathbb{C}}(\mathcal{M}) = 2$ then necessarily $\mathcal{M}^{(1)}$ is a closed G-orbit, and that being a rank-one invariant subvariety of Rel-dimension one is equivalent to the condition $\dim_{\mathbb{C}}(\mathcal{M}) = 3$.

Let L be a lift of $\mathcal{M}$ and let $\Gamma \subset \mathrm{Mod}(S, \Sigma)$ be the stabilizer of L. The group Γ acts properly discontinuously on L and thus the quotient space L/Γ inherits an orbifold structure. We define the *self-intersection locus of* $\mathcal{M}$ as the set $X_{\mathcal{M}}$ of points in $\mathcal{M}$ that are in the image by π of the intersection of at least two distinct lifts. We also define the *orbifold locus* $O_{\mathcal{M}}$ as the image by π of the set of points in L that have a non-trivial stabilizer in Γ. This does not depend on the choice of the lift L. If we denote by $F : L/\Gamma \to \mathcal{M}$ the quotient map induced π, then $F^{-1}(X_{\mathcal{M}})$ coincides with the locus where F is not injective and $F^{-1}(O_{\mathcal{M}})$ coincides with the orbifold locus of L/Γ in the usual sense. Self-intersections and orbifolds points in invariant subvarieties cannot be ruled out in general. See [EM18]. These complications do not arise for rank-one loci:

Proposition 2.4. *If* $\mathcal{M}$ *is a rank one invariant subvariety then its orbifold locus and its self-intersection locus are empty.*

Proof. Let L be a lift of $\mathcal{M}$, let $V \subset H^1(S, \Sigma; \mathbb{C})$ be the linear subspace on which it is modelled and let $(q, f) \in L$. Since V is defined over $\mathbb{R}$, the real and imaginary parts $\mathrm{Re}(\mathrm{dev}(q, f))$ and $\mathrm{Im}(\mathrm{dev}(q, f))$ both belong to V and a dimension count shows that

$$V = \mathrm{span}\left(\mathrm{Re}(\mathrm{dev}(q, f)), \mathrm{Im}(\mathrm{dev}(q, f))\right) \oplus \mathfrak{R}_{\mathcal{M}}. \tag{7}$$

Denote by $\Gamma \subset \mathrm{Mod}(S, \Sigma)$ the stabilizer of L and let $\gamma \in \Gamma$ be such that $(q, f) \cdot \gamma = (q, f)$. Since dev is Γ-equivariant and the action of Γ on V preserves the $\mathbb{R}$-structure, γ acts trivially on $\mathrm{span}\left(\mathrm{Re}(\mathrm{dev}(q, f)), \mathrm{Im}(\mathrm{dev}(q, f))\right)$. Recalling our convention that singular points are labeled, we thus find that γ acts trivially on V, using the formula (7). This implies that γ acts trivially on L and thus the orbifold locus of $\mathcal{M}$ is trivial.

Let now L' be another lift of $\mathcal{M}$ that contains (q, f). By (7), the local model of L' coincides with V and thus $L = L'$. This proves that the self-intersection locus of $\mathcal{M}$ is also trivial. $\square$

2.4. **Real Rel vector fields and real Rel flow.** Following [BSW16], we write $\mathbb{R}^2 = \mathbb{R}_x \oplus \mathbb{R}_y$ for the canonical splitting into x and y coordinates. This induces the splitting

$$H^1\left(S, \Sigma; \mathbb{R}^2\right) = H^1(S, \Sigma; \mathbb{R}_x) \oplus H^1(S, \Sigma; \mathbb{R}_y), \tag{8}$$

where $H^1(S, \Sigma; \mathbb{R}_x) \cong H^1(S, \Sigma; \mathbb{R}) \cong H^1(S, \Sigma; \mathbb{R}_y)$. The subspace $H^1(S, \Sigma; \mathbb{R}_x)$ shall be referred to as the *horizontal space.* If $v \in H^1(S, \Sigma; \mathbb{R}_x)$, it will be convenient to denote by $\mathbf{i} \cdot v$ the corresponding vector in $H^1(S, \Sigma; \mathbb{R}_y)$; that is, with respect to the splitting (8), $(x, 0)$ projects to $x \in H^1(S, \Sigma; \mathbb{R}_x)$ and $(0, x)$ projects to $\mathbf{i} \cdot x \in H^1(S, \Sigma; \mathbb{R}_y)$. We denote

$$Z \stackrel{\text{def}}{=} \mathfrak{R} \cap H^1(S, \Sigma; \mathbb{R}_x) \quad \text{ and } Z_{\mathcal{M}} \stackrel{\text{def}}{=} \mathfrak{R}_{\mathcal{M}} \cap H^1(S, \Sigma; \mathbb{R}_x)$$

where $\mathcal{M} \subset \mathcal{H}$ is an invariant subvariety. If $v \in Z$, the vector field $\vec{v}$ is called a *real Rel vector field.* For any $r > 0$, we denote by $\mathcal{M}_r$ the set of surfaces in $\mathcal{M}$ which do not have a horizontal saddle connection strictly shorter than r, and by $\mathcal{M}_\infty$ the set of surfaces in $\mathcal{M}$ that do not have horizontal saddle connections. With respect to the action of G by post-composition on $\mathfrak{R}$, the space $\mathfrak{R}_{\mathcal{M}}$ is invariant, and the subgroup $B \subset G$ defined in (4) leaves the subspace $Z_{\mathcal{M}}$ invariant. We can use this action to define semi-direct products, as follows

Definition 2.5. *Let $\mathcal{M} \subset \mathcal{H}$ be an invariant subvariety. We define*

$$N_{\mathcal{M}} \stackrel{\text{def}}{=} B \ltimes Z_{\mathcal{M}} \quad \text{and} \quad L_{\mathcal{M}} \stackrel{\text{def}}{=} G \ltimes \mathfrak{R}_{\mathcal{M}}.$$

The set $\mathcal{M}_\infty$ is a dense G_δ subset of $\mathcal{M}$. Since B preserves horizontal saddle connections and real Rel is defined on surfaces without horizontal saddle connections, the map

$$N_{\mathcal{M}} \times \mathcal{M}_\infty \ni (b, z, q) \mapsto \mathrm{Rel}_z(bq)$$

is well-defined. Moreover, it satisfies the axioms of a group action (see [MW14]). In particular, for $z \in Z_{\mathcal{M}}$, the map $(t, q) \mapsto \mathrm{Rel}_{tz}(q)$ defines a *real Rel flow* on $\mathcal{M}_\infty$, and this flow commutes with the horocycle flow. On the other hand, the same map defined on the larger set $L_{\mathcal{M}} \times \mathcal{M}$ need not satisfy the axioms of a group action, even if well-defined. In order to stress this point, for general $q \in \mathcal{M}$ and $(g, v) \in L_{\mathcal{M}}$ for which $\mathrm{Rel}_v(gq)$ is well-defined, we will write $gq \mathbin{\oplus} v \stackrel{\text{def}}{=} \mathrm{Rel}_v(gq)$.

2.5. **Real Rel pushes of ergodic U-invariant measures.** For fixed $q \in \mathcal{H}$, let $Z^{(q)}$ denote the set of $z \in Z$ for which $\mathrm{Rel}_z(q)$ is well-defined. By (6), $Z^{(q)} = Z^{(uq)}$ for any $u \in U$ and $q \in \mathcal{H}$, and hence, for any ergodic U-invariant probability measure μ on $\mathcal{H}$, there is a set $Z^{(\mu)} \subset Z$ such that $Z^{(q)} = Z^{(\mu)}$ for μ-a.e. $q \in \mathcal{H}$. Let $z \in Z^{(\mu)}$. Denoting by $\mathcal{B}(\mathcal{M})$ the Borel σ-algebra on $\mathcal{M}$, we define a measure by

$$\forall X \in \mathcal{B}(\mathcal{M}), \quad \mathrm{Rel}_{z*}\mu(X) = \mu(\mathrm{Rel}_{-z}(\mathcal{H}'_{-z} \cap X)). \tag{9}$$

The measure $\mathrm{Rel}_{z*}\mu$ is also U-invariant and ergodic. In particular, let $\mathcal{M} \subset \mathcal{H}$ be an invariant subvariety. If μ is an ergodic U-invariant measure supported on $\mathcal{M}$

and $z \in Z^{(\mu)} \cap Z_{\mathcal{M}}$, then $\mathrm{Rel}_{z*}\mu$ is also an ergodic U-invariant measure supported on $\mathcal{M}$. We finally denote

$$N_\mu \stackrel{\text{def}}{=} \left\{(b,z) \in N_{\mathcal{M}} : z \in Z^{(b_*\mu)} \text{ and } \mathrm{Rel}_{z*}(b_*\mu) = \mu\right\}. \tag{10}$$

It follows from [BSW16, Cor. 4.20] that N_μ is a closed subgroup of $N_{\mathcal{M}}$, and thus N_μ°, the connected component of the identity in N_μ, is a connected Lie group. We recall that a point $q \in \mathcal{H}$ is generic for an ergodic U-invariant measure μ if for any compactly supported continuous function $f : \mathcal{M} \to \mathbb{R}$,

$$\frac{1}{T}\int_0^T f(u_s q)\, ds \underset{T\to\infty}{\longrightarrow} \int_{\mathcal{H}} f \, d\mu.$$

The set of generic points of μ, which we denote by Ω_μ, has full measure for μ. We will need the following:

Proposition 2.6 ([BSW16], Cor. 4.24, Prop. 4.27 & Cor. 4.28)**.** *Let μ be an ergodic U-invariant measure. Then:*

- *$Z^{(\mu)} \subset Z^{(q)}$ for any $q \in \Omega_\mu$.*
- *If $q, q' \in \Omega_\mu$ and $n = (b,z) \in N_{\mathcal{M}}$ satisfy that $z \in Z^{(b_*\mu)}$ and $q' = \mathrm{Rel}_z(bq)$, then $n \in N_\mu$.*
- *If $n = (b,z) \in N_{\mathcal{M}}$ with $z \in Z^{(\mu)}$, and $q \in \Omega_\mu$, then $\mathrm{Rel}_z(bq) \in \Omega_{\mu'}$ for $\mu' = \mathrm{Rel}_{z*}(b_*\mu)$.*

We will also need the following result, which is proved in [CW10, §6] for eigenform loci in genus 2, and is a special case of a general result proved in [SSWY]:

Proposition 2.7. *Let $\mathcal{M}$ be a rank-one invariant subvariety and let μ be a U-invariant ergodic probability measure on $\mathcal{M}$. Suppose that μ assigns zero measure to surfaces with horizontal saddle connections, and that $\mathrm{Rel}_{x*}\mu = \mu$ for every $x \in Z_{\mathcal{M}}$. Then $\mu = m_{\mathcal{M}}$.*

Remark 2.8. *The proof of Proposition 2.7 given in* [SSWY] *also be works for rank-one invariant subvarieties in products of strata, provided one has that the measure $m_{\mathcal{M}}$ is mixing for the diagonal g_t-action.*

2.6. **Products of strata.** For $i = 1, \ldots, k$, let $\mathcal{H}_i$ be a stratum and let $\mathcal{H}_{i,\mathrm{m}}$ be the corresponding stratum of marked surfaces. The group G acts diagonally on the products $H_1 \times \cdots \times \mathcal{H}_k$ and $\mathcal{H}_{1,\mathrm{m}} \times \cdots \times \mathcal{H}_{k,\mathrm{m}}$. It is interesting the study of the dynamics of the G-action and U-action to this more general setup. Moreover, as discussed in [MW17] and as we will also see in [CWY], passing to a suitable boundary of an invariant subvariety naturally lead to invariant subvarieties in products of strata, as well as their finite covers or quotients (obtained e.g. by introducing nodes on different components). Therefore this extension is valuable even if one is only interested in understanding the dynamics on a single $\mathcal{H}$. In this section, and throughout the paper, we record some notation and some simple observations that are likely to be useful in future work where this more general setup will be considered.

It is likely that G-orbit closures on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_k$ can also be classified. Partial results in this direction have been announced by Brown, Eskin, Filip and Rodriguez-Hertz, and see [MW17, Conj. 2.10] and [CW, §5] for formulations of a conjectural classification. If we let

$$\mathrm{dev} = \mathrm{dev}_1 \times \cdots \times \mathrm{dev}_k,$$

where $\mathrm{dev}_i : \mathcal{H}_{i,\mathrm{m}} \to H^1(S_i, \Sigma_i; \mathbb{R}^2)$ is the developing map associated with the i-the stratum $\mathcal{H}_i$, then one can define an invariant subvariety as in Defionition 2.1, and it is likely that any orbit-closure for the G-action on a product of strata is obtained by restricting an invariant subvariety to a subset where the area of the component surfaces in each $\mathcal{H}_i$ are fixed constants. With this extended definition of an invariant subvariety, one can define the rank, Rel space, and Rel-dimension of an invariant subvariety in a product of strata exactly as in §2.3.

Example 2.9. *Note that there is no reason for an invariant subvariety $\mathcal{M} \subset \mathcal{H}_1 \times \mathcal{H}_2$ to be a product of invariant subvarieties $\mathcal{M}_1, \mathcal{M}_2$. Indeed, there are non-obvious joinings between invariant subvarieties, even in rank one. As an example, consider the Bouw-Möller surfaces $q_{m,n}$ whose Veech group is the (m, n, ∞) Hecke triangle group (see* [BM10, Hoo13]*). For all but finitely many k, the surfaces $q_{2,k}$ and $q_{k,k}$ are primitive, belong to different strata, and their Veech groups are commensurable. This means that if we denote by $\mathcal{H}_1(k), \mathcal{H}_2(k)$ the stratum containing $q_{2,k}$ and $q_{k,k}$ respectively, then $\mathcal{H}_1(k) \times \mathcal{H}_2(k)$ contains a closed G-orbit which projects onto the closed orbits of $q_{2,k}$ and $q_{k,k}$ respectively, but $q_{2,k}$ and $q_{k,k}$ do not share a common quotient.*

Notational conventions. We will be interested in the horocycle dynamics on invariant subvarieties of rank one. Since the horocycle flow preserves the area of a surface, it is sufficient to consider the area one locus in an invariant subvariety. *From now on, to simplify notation, we will restrict attention to area-one surfaces and we will write $\mathcal{H}$ for $\mathcal{H}^{(1)}$, $\mathcal{M}$ for $\mathcal{M}^{(1)}$, etc.* Also, although some statements are valid in greater generality, *from now until the end of the paper, $\mathcal{M}$ will denote a rank one invariant subvariety in $\mathcal{H}$.*

3. Horocycle dynamics in rank-one, Rel-one invariant subvarieties and compact extension

This section is concerned with the classification of ergodic U-invariant measures supported on invariant subvarieties of rank one and Rel-dimension one and related compact extensions where marked points are added.

3.1. **The drift argument.**

The following key argument is modeled on Ratner's work and was adapted to the present context in [EMM06, BSW16].

Proposition 3.1. *Let μ be an ergodic U-invariant probability measure on $\mathcal{M}$. If μ is not supported on a closed U-orbit then* $\dim(N_\mu^\circ) \geqslant 2$.

Remark 3.2. *Contrary to* [BSW16, Prop 9.4]*, we do not need to assume that μ does not assign positive measure to the set of surfaces with horizontal saddle connection. We also note that the proof given below works for the U-action on a product of strata.*

Proof. Let $\varepsilon > 0$ and let $K \subset \Omega_\mu$ be a compact subset such that $\mu(K) > 1 - \varepsilon$. Such a compact set exists by inner regularity of the measure μ. By combining Birkhoff's ergodic theorem and Egorov's theorem, one obtains that there is another compact set $K_0 \subset \mathcal{M}$ and T_0 large enough so that for any $T > T_0$ and any $q \in K_0$,

$$\frac{1}{T} |\{t \in [0, T] : u_t q \in K\}| > 1 - 2\varepsilon. \tag{11}$$

Let r denote the Rel-dimension of $\mathcal{M}$, and choose a G-equivariant linear isomorphism $\mathfrak{R}_{\mathcal{M}} \simeq \mathbb{R}^{2r}$ as follows. We write $\mathbf{z} \in \mathfrak{R}_{\mathcal{M}}$ as $\mathbf{z} = (\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{i} \cdot \mathbf{y}$ where $\mathbf{x} = (x_i)_{i=1}^r$ and $\mathbf{y} = (y_i)_{i=1}^r$ belong to $\mathbb{R}^r$, and we set

$$\mathbf{z} = \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_r \\ y_r \end{pmatrix} \quad \text{and} \quad g\mathbf{z} = \begin{pmatrix} g\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\ \vdots \\ g\begin{pmatrix} x_r \\ y_r \end{pmatrix} \end{pmatrix},$$

where on the right-hand side, g acts linearly on each of the r copies of $\mathbb{C} \cong \mathbb{R}^2$. In terms of the discussion in §2.3, each $\mathbf{z}_j = x_j + \mathbf{i} \cdot y_j$ is the integral of the holomorphic 1-form on a linear combination of the cycles ξ^*. Under this isomorphism, we have $Z_{\mathcal{M}} = \{(\mathbf{x}, \mathbf{y}) : \mathbf{y} = 0\} \simeq \mathbb{R}^r$.

If μ is not supported on a closed U-orbit, there is a point $q \in K_0$ and a sequence of points $q_n \in K_0 \smallsetminus Uq$ that converges to q. For n large enough, we can find $g_n \in G$ and $W_n \in \mathfrak{R}_{\mathcal{M}}$ such that

$$(g_n, W_n) \in L_{\mathcal{M}} \smallsetminus U, \quad (g_n, W_n) \to (id, 0) \quad \text{and} \quad q_n = g_n q \mathbin{+\!\!\!\!\bigcirc} W_n.$$

Write

$$g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \quad \text{and} \quad W_n = (\mathbf{x}_n, \mathbf{y}_n). \tag{12}$$

Let $s_n(t) = \frac{a_n t - b_n}{d_n - c_n t}$. A computation using (6) shows that for any $t > 0$,

$$u_{s_n(t)} q_n = \begin{pmatrix} \frac{1}{d_n - c_n t} & 0 \\ c_n & d_n - c_n t \end{pmatrix} u_t q \mathbin{+\!\!\!\!\bigcirc} (\mathbf{x}_n + s_n(t)\mathbf{y}_n, \mathbf{y}_n).$$

Let $\varepsilon' > 0$, and define

$$I_n \stackrel{\text{def}}{=} \{t \in \mathbb{R} : \max(|\ln(d_n - c_n t)|, \|\mathbf{x}_n + s_n(t)\mathbf{y}_n\|) < \varepsilon'\},$$

where $\|\cdot\|$ denotes the sup-norm on $\mathfrak{R}_{\mathcal{M}}$ with respect to our chosen coordinates.

If n is large enough, the set I_n is an interval that contains 0 and, up to choosing a smaller ε' if needed, we can assume that for any $t \in I_n$, we have:

$$\frac{1}{2}|d_n - c_n t - 1| < |\ln(d_n - c_n t)| < 2|d_n - c_n t - 1| \tag{13}$$

and

$$\frac{1}{2}|t| < |s_n(t)| < 2|t|. \tag{14}$$

We want to find $t \in I_n$ such that

$$u_{s_n(t)} q_n \text{ and } u_t q \text{ are both in } K \tag{15}$$

and

$$\frac{\varepsilon'}{16} \leqslant \max(|\ln(d_n - c_n t)|, \|\mathbf{x}_n + s_n(t)\mathbf{y}_n\|) \leqslant \varepsilon'. \tag{16}$$

Notice that if n is large enough then $T_n \stackrel{\text{def}}{=} \sup(I_n)$ is larger than T_0 and using (13), we have :

$$\frac{1}{T_n}\left|\left\{t \in [0, T_n] : \max(|\ln(d_n - c_n t)|, \|\mathbf{x}_n + s_n(t)\mathbf{y}_n\|) < \frac{\varepsilon'}{16}\right\}\right| < \frac{1}{2}.$$

Finally, the set of times $t \in [0, T_n]$ for which $u_t q$ is in K has measure at least $(1-2\varepsilon)T_n$ and likewise for $u_t q_n$. Using (13) again, we deduce that the set of times $t \in [0, T_n]$ for which $u_{s_n(t)} q_n$ belongs to K has measure at least $(1-4\varepsilon)T_n$. Consequently, whenever ε is small enough, for all n large enough (depending on T_0 and hence on ε), there exists $t = t_n$ satisfying the desired properties (15) and (16). We choose ε' small enough to that $gp \mathbin{\text{⧺}} W$ is well defined for any g, W such that $(g - id, W)$ has coefficients smaller that ε' and any $p \in K$. Passing to a converging subsequence in $u_{s_n(t_n)} q_n$ and $u_{t_n} q$ and invoking (5), we obtain a pair of generic points p and p' in K such that $p' = gp \mathbin{\text{⧺}} W$ with

$$g = \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix} \text{ and } W = (\mathbf{x}, 0), \tag{17}$$

and

$$\frac{\varepsilon'}{16} \leqslant \max(|a|, \|\mathbf{x}\|) \leqslant \varepsilon'.$$

From Proposition 2.6 we have that $(g, W) \in N_\mu$. Since ε' was arbitrary, this shows that $N_\mu \smallsetminus U$ contains elements arbitrarily close to the identity. Since N_μ is a Lie subgroup of $N_{\mathcal{N}}$, this implies $\dim N_\mu^\circ > 1$. □

For future reference, we record some additional information which the proof of Proposition 3.1 gives.

Corollary 3.3. *Let μ be a horocycle invariant measure on $\mathcal{M}$ which is not supported on a closed U-orbit, and let $\varepsilon > 0$. Let $\mathfrak{R}_{\mathcal{M}} = \mathfrak{R}_0 \oplus \mathfrak{R}_1$ be a direct sum decomposition into G-invariant subspaces, and let $\mathbf{y} = \mathbf{y}^{(0)} + \mathbf{y}^{(1)}$ be the corresponding decomposition of $\mathbf{y}$ for $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathfrak{R}_{\mathcal{M}}$.*

(i) *Suppose that there does not exist a pair of elements g, W as in* (17)*, such that*
$$(g, W) \in N_\mu, \ a \neq 0 \ \ \textit{and} \ \ \max(|a|, \|\mathbf{x}\|) < \varepsilon.$$
Then for any sequence (g_n, W_n) as in (12)*, satisfying the conditions appearing in the proof, we have $c_n = o(\|\mathbf{y}_n\|)$.*

(ii) *Suppose that there does not exist a pair of elements g, W as in* (17)*, such that*
$$(g, W) \in N_\mu, \ a = 0 \ \ \textit{and} \ \ 0 < \|\mathbf{x}\| < \varepsilon.$$
Then there is $C > 0$ so that for any sequence (g_n, W_n) as in (12)*, satisfying the conditions appearing in the proof, for all large enough n we have*
$$\frac{1}{C} < \frac{|c_n|}{\|\mathbf{y}_n\|} < C.$$

(iii) *Suppose that there does not exist an element $W \in \mathfrak{R}_0 \cap N_\mu$ with $0 < \|W\| < \varepsilon$. Then for any sequence (g_n, W_n) as in* (12)*, satisfying the conditions appearing in the proof, we have*
$$\left\|\mathbf{y}_n^{(0)}\right\| = O\left(\max\left(|c_n|, \left\|\mathbf{y}_n^{(1)}\right\|\right)\right). \tag{18}$$

Proof. The proof given above starts with $\varepsilon > 0$ and sequences $q_n = g_n q \mathbin{\text{⧺}} W_n$, and shows the existence of (g, W) as in (17) such that $(g, W) \in N_\mu$ and $\max(|a|, \|\mathbf{x}\|) < \varepsilon$. It follows from the assumption in (i) that for any such sequence (g_n, W_n) we must have $a = 0$. Hence, along any subsequence taken in the proof, we have

$$\ln(d_n - c_n t_n) \to -a = 0, \ \ \mathbf{x}_n + s_n(t_n)\mathbf{y}_n \to \mathbf{x} \ \text{ and } \ \|\mathbf{x}\| \geqslant \frac{\varepsilon'}{8}.$$

Since $\mathbf{x}_n \to 0$, $d_n \to 1$ and t_n is comparable to $s_n(t_n)$, this can only happen if $c_n = o(\|\mathbf{y}_n\|)$.

Under the assumption in (ii), for any subsequence taken in the proof we can take a further subsequence along which

$$-a = \lim_{n\to\infty} \ln(d_n - c_n t_n) \quad \textit{and} \quad \mathbf{x} = \lim_{n\to\infty}(\mathbf{x}_n + s_n(t_n)\mathbf{y}_n) = \lim_{n\to\infty} s_n(t_n)\mathbf{y}_n, \tag{19}$$

with $a, \mathbf{x}$ both nonzero. This implies that

$$\lim_{n\to\infty} \frac{t_n c_n}{\|s_n(t_n)\mathbf{y}_n\|} = \frac{1-e^{-a}}{\|\mathbf{x}\|} \neq 0 \tag{20}$$

and (ii) follows from (14), i.e. from the fact that $s_n(t_n)$ and t_n are comparable. Under the assumption in (iii), if (18) fails for some sequence (g_n, W_n) as in the proof, then $\max\left(|c_n|, \|\mathbf{y}_n^{(1)}\|\right) = o(\|\mathbf{y}_n^{(0)}\|)$ along a subsequence. In particular $\mathbf{y}_n^{(0)}$ converges to $\mathbf{x}$ and thus $\mathbf{x}^{(1)} = 0$. Using (20) one more time we also get $a = 0$. It follows that $(g, W) \in \mathfrak{R}_0$, a contradiction. □

3.2. **Saddle connection free U-invariant ergodic measures.** We will now deduce from Proposition 3.1 that invariant subvarieties of rank one and Rel-dimension one satisfy the weak classification of U-invariant measures.

The set of surfaces with horizontal saddle connections is invariant under U. In particular, if μ is an ergodic U-invariant measure, the measure of this set is either 1 or 0. In the second case, we say μ is a *saddle connection free measure.*

Theorem 3.4. *Suppose $\mathcal{M}$ is of rank one and of Rel-dimension one. Let μ be a saddle connection free ergodic U-invariant probability measure on $\mathcal{M}$. Then there is some invariant subvariety $\mathcal{N} \subset \mathcal{M}$ and some $x \in Z_{\mathcal{M}}$ such that $\mu = \mathrm{Rel}_{x*} m_{\mathcal{N}}$.*

Proof. It is well-known (see e.g. [Vee89]) that the normalized length measure on a periodic U-orbit is not saddle connection free. Thus, by Proposition 3.1 there are $a \geqslant 0$, $w \in Z_M$ and a homomorphism $\rho : \mathbb{R} \to N_\mu^\circ \smallsetminus U$ such that

$$\left.\frac{d}{dt}\right|_{t=0} \rho(t) = \left(\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right), w\right), \quad \text{where } (a, w) \neq (0,0). \tag{21}$$

If $a = 0$, we have that $\rho(t) = (id, tw)$ and thus $\rho(t)_*\mu = \mathrm{Rel}_{tw*}\mu = \mu$. Therefore μ is invariant under both U and Rel_{tw} for all t. Since $\mathcal{M}$ is of Rel-dimension 1, we deduce via Proposition 2.7 that $\mu = m_{\mathcal{M}}$. If $a \neq 0$ we have

$$\rho(t) = \left(\begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix}, e^{at}w - w\right)$$

and it follows via (6) that the measure $\mu_0 \stackrel{\text{def}}{=} \mathrm{Rel}_{-w*}\mu$ is invariant under the upper triangular subgroup B. Here we have used the assumption that μ assigns zero measure to surfaces without horizontal saddle connections, since it implies $Z_{\mathcal{M}} \subset Z^{(\mu)}$, that is $\mathrm{Rel}_{-w}(q)$ is well-defined for every $w \in Z_{\mathcal{M}}$, for μ-a.e. q. It follows from [EM18] that there is an invariant subvariety $\mathcal{N} \subset \mathcal{M}$ such that $\mu_0 = m_{\mathcal{N}}$ and thus $\mu = \mathrm{Rel}_{w*} m_{\mathcal{N}}$. □

Remark 3.5. *The drift argument relies on a fine analysis of the displacement between two horocycle trajectories of nearby points. In order to remove the assumption that $\mathcal{M}$ is of Rel-dimension one, it is tempting to argue inductively, replacing the horocycle flow by a well-chosen flow in the group N_μ in order to pick up additional* *invariance of the measure. However, to compute the displacement between two nearby trajectories under this new flow, one would need the 'group-action property' (6), with G replaced by larger groups $G \ltimes \mathfrak{R}_0$, for $\mathfrak{R}_0 \subset \mathfrak{R}$ (even the weaker property $\mathrm{Rel}_{x_1+x_2}(q) = \mathrm{Rel}_{x_1}(\mathrm{Rel}_{x_2}(q))$ for suitable $x_1, x_2 \in \mathfrak{R}$ might suffice). These relations are not true in general.*

We now list some examples to which our results apply. For relevant definitions, see [McM06]. Let $D \equiv 0 \mod 4$ or $D \equiv 1 \mod 4$, let $\mathcal{O}_D$ be the quadratic order of discriminant D and let ΩE_D be the space of Prym eigenforms for real multiplication by $\mathcal{O}_D$. Let $\Omega E_D^{odd}(2,2)$ and $\Omega E_D(2,1,1)$ be the intersection of ΩE_D with the strata $\mathcal{H}^{odd}(2,2)$ and $\mathcal{H}(2,1,1)$. It follows from [LN18] that these spaces are not empty whenever $D \not\equiv 5 \bmod 8$, in which case there are at most two connected components. Any such connected component is a rank 1 invariant subvariety of Rel-dimension 1, see [LN16, Lemma 3.1]. Thus, Theorem 3.4 has the following corollary:

Corollary 3.6. *Let $D \equiv 0, 1 \mod 4$, $D \not\equiv 5 \mod 8$ be a positive integer that is not a square and let $\mathcal{M}$ be a connected component of $\Omega E_D^{odd}(2,2)$ or $\Omega E_D(2,1,1)$. Then $\mathcal{M}$ satisfies the weak classification of U-invariant measures.*

Let $\mathcal{M}$ be a connected component of ΩE_D. In the case where D is a square, $\mathcal{M}$ consists of branched covers of flat tori and the fact that $\mathcal{M}$ satisfies the weak classification of U-invariant measures follows from the work of Eskin, Markloff and Morris [EMWM06, Theorem 2.1]. When D is not a square, $\mathcal{M}$ is a nonarithmetic invariant subvariety, which means that its field of definition is a non trivial algebraic extension of $\mathbb{Q}$. The examples mentioned in Corollary 3.6 do not arise via a covering construction and thus provide the first non-trivial examples in genus 3 of invariant subvarieties in which a classification of U-invariant measures is known. It is worth mentioning that any nonarithmetic rank 1 invariant subvariety of Rel-dimension 1 in $\mathcal{H}^{odd}(2,2)$ arises in this way, see [Ygo20]. The other Prym eigenform loci in genus 3 are $\Omega(\kappa) \stackrel{\text{def}}{=} \Omega E_D \cap \mathcal{H}(\kappa)$ for $\kappa \in \{4^{odd}, 4^{hyp}, (2,2)^{hyp}, (1,1,1,1)\}$. The first three are a finite union of Teichmüller curves (*i.e.,* satisfy $dim(\mathcal{M}) = 2$), see [LN18, Prop 2.3] for a proof in the case $(2,2)^{hyp}$, and the latter has Rel-dimension 2. In genus 4 and 5, no explicit description of the connected components of the Prym eigenform loci is known besides the case $\mathcal{H}(6)$ that is addressed in [LN20]. Nevertheless, we can still use Prym eigenforms to produce more examples of rank one invariant subvarieties of Rel-dimension 1. We recall the following that lists all the potential rank one invariant subvarieties of Rel-dimension 1 arising from Prym eigenforms:

Lemma 3.7 ([LN16, Lemma 3.1])**.** *Let $D \equiv 0, 1 \mod 4$, let $g = 4, 5$, let κ be an integer partition of $2g - 2$ such that $\Omega E_D(\kappa) \neq \varnothing$ and let $\mathcal{M}$ be a connected component of $\Omega E_D(\kappa)$. Then $\mathcal{M}$ is a rank one invariant subvariety and its Rel-dimension is 1 if, and only if, κ belongs to the following list:*

$$(3,3), (2,2,2)^{even}, (1,1,4), (1,1,2,2), (4,4)^{even}.$$

Proposition 3.8. *For any κ as in Lemma 3.7, there is $D \equiv 0, 1 \mod 4$ such that D is not a square and $\Omega E_D(\kappa) \neq \varnothing$.*

Proof. It is explained in [LN16] that each κ as in Lemma 3.7 is associated with a rank 2 invariant subvariety denoted by $Prym(\kappa)$, and the latter can easily be verified to be arithmetic. It thus follows from [EFW18, Theorem 1.7] that this affine variety itself contains a nonarithmetic rank one invariant subvariety $\mathcal{M}$. It is not hard to see that $\mathcal{M}$ must contain a surface q that contains a hyperbolic matrix A in its Veech group. See for instance [Wri14, Remark 4.2]. It follows from [McM06, Theorem 3.5] that $q \in \Omega E_D(\kappa)$ with $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(tr(A))$. If $tr(A)$ is rational then it follows from [McM03, Theorem 9.8] that q covers a torus, in contradiction with the fact that $\mathcal{M}$ is nonarithmetic. □

Using Theorem 3.4, we get:

Corollary 3.9. *Let $\mathcal{M}$ be a connected component of $\Omega E_D(\kappa)$ where κ and D are as in Proposition 3.8. Then $\mathcal{M}$ satisfies the weak classification of U-invariant measures.*

The Pentagon variety described in [EMMW20] yields another infinite family of quadratic rank one invariant subvarieties of Rel-dimension 1 contained in $\mathcal{H}(2,2,1,1)$. It is explained in [EMMW20, Appendix B] that the intersection of the Pentagon variety with $\mathcal{H}(2,2,1,1)$ is a non-empty arithmetic rank two invariant subvariety of Rel-dimension 1. Using once more [EFW18, Theorem 1.7], we deduce the existence of infinitely many quadratic rank one invariant subvarieties of Rel-dimension 1.

Corollary 3.10. *Let $\mathcal{M}$ be one of the infinitely many quadratic rank one invariant subvarieties contained in the intersection of the Pentagon variety with $\mathcal{H}(2,2,1,1)$. Then $\mathcal{M}$ satisfies the weak classification of U-invariant measures.*

4. Adding marked points

Let Σ be a collection of singularities and let $\kappa = (k_\xi)_{\xi\in\Sigma}$ be a partition of $2g-2$. We will be interested in the case that some of the k_ξ are zero (and thus correspond to marked points). Let

$$\Sigma_0 \stackrel{\text{def}}{=} \{\xi \in \Sigma : k_\sigma \neq 0\} \tag{22}$$

be the subset of 'non-removable' singularities, let $\mathfrak{S} \subset \Sigma$ be a subset containing Σ_0, and $\kappa_{\mathfrak{S}} \stackrel{\text{def}}{=} (k_\xi)_{\xi\in\mathfrak{S}}$. By definition of Σ_0, $\kappa_{\mathfrak{S}}$ is also an integer partition of $2g-2$. Let $\mathcal{H}_{\mathfrak{S}} \stackrel{\text{def}}{=} \mathcal{H}(\kappa_{\mathfrak{S}})$ be the moduli space of singularity-labeled translation surfaces of type $\kappa_{\mathfrak{S}}$. We also consider the space $\mathcal{H}_{\mathrm{m}}(\kappa_{\mathfrak{S}})$ of marked surfaces of type $\kappa_{\mathfrak{S}}$ and we denote by $\pi_{\mathfrak{S}} : \mathcal{H}_{\mathrm{m}}(\kappa_{\mathfrak{S}}) \to \mathcal{H}_{\mathfrak{S}}$ the canonical projection that forgets the marking. We define the forgetful map

$$p_{\mathfrak{S}} : \mathcal{H} \to \mathcal{H}_{\mathfrak{S}}, \quad q \mapsto (X_q, \omega_q, \mathfrak{z}_q|_{\mathfrak{S}}), \tag{23}$$

and the corresponding map at the level of marked surfaces

$$\tilde{p}_{\mathfrak{S}} : \mathcal{H}_{\mathrm{m}} \to \mathcal{H}_{\mathrm{m}}(\kappa_{\mathfrak{S}}), \quad (q, f) \mapsto ((X_q, \omega_q, \mathfrak{z}_q|_{\mathfrak{S}}), f). \tag{24}$$

These two maps satisfy

$$\pi_{\mathfrak{S}} \circ \tilde{p}_{\mathfrak{S}} = p_{\mathfrak{S}} \circ \pi, \tag{25}$$

and we have:

Proposition 4.1. *The fibers of $\tilde{p}_{\mathfrak{S}}$ are connected.*

Proof. Let (q_1, f_1) and (q_2, f_2) be two marked surfaces in $\mathcal{H}_{\mathrm{m}}$ that have the same image under $\tilde{p}_{\mathfrak{S}}$. By definition, this means that there is biholomorphism $\varphi : X_{q_1} \to X_{q_2}$ such that $\varphi^*\omega_{q_2} = \omega_{q_1}$ and $\varphi \circ \mathfrak{z}_{q_1}|_{\mathfrak{S}} = \mathfrak{z}_{q_2}|_{\mathfrak{S}}$, as well as an isotopy $(\psi_t : S \to S)_{t\in[0,1]}$ that connects the identify map of S to $f_2^{-1} \circ \varphi \circ f_1$ without moving the points in $\mathfrak{S}$. Let $\mathfrak{z}_t : \Sigma \to X_{q_2}$ be the map $(f_2 \circ \psi_t)|_\Sigma$. Notice in particular that this map coincides with $\mathfrak{z}_2$ on $\mathfrak{S}$. Define

$$\gamma : [0,1] \to \mathcal{H}_{\mathrm{m}}, \quad t \mapsto (X_{q_2}, \omega_{q_2}, \mathfrak{z}_t, f_2 \circ \psi_t).$$

The path γ connects (q_2, f_2) to (q_1, f_1) while staying in a single fiber of $\tilde{p}_{\mathfrak{S}}$. $\square$

If we let $\mathrm{Res}_{\mathfrak{S}} : H^1(S,\Sigma;\mathbb{C}) \to H^1(S,\mathfrak{S};\mathbb{C})$ be the canonical restriction map induced by the inclusion $\mathfrak{S} \subset \Sigma$, we have

$$\mathrm{dev} \circ \tilde{p}_{\mathfrak{S}} = \mathrm{Res}_{\mathfrak{S}} \circ \mathrm{dev}, \tag{26}$$

where we denote the developing maps of $\mathcal{H}_{\mathrm{m}}$ and $\mathcal{H}_{\mathrm{m}}(\kappa_{\mathfrak{S}})$ by the same symbol dev. Denote

$$\mathfrak{R}_{\mathfrak{S}} \stackrel{\mathrm{def}}{=} \ker \mathrm{Res}_{\mathfrak{S}} \quad \text{and} \quad Z_{\mathfrak{S}} \stackrel{\mathrm{def}}{=} \mathfrak{R}_{\mathfrak{S}} \cap Z.$$

If $v \in H^1(S,\Sigma;\mathbb{C})$, we will denote by $v|_{\mathfrak{S}}$ the image of v under $\mathrm{Res}_{\mathfrak{S}}$. Let $\{\xi^* : \xi \in \Sigma\} \subset \mathfrak{R}$ be as in §2.3. Any element of $\mathfrak{R}_{\mathfrak{S}}$ (resp. $Z_{\mathfrak{S}}$) is a linear combination with complex (resp. real) coefficients of the ξ^* where ξ ranges over $\Sigma \smallsetminus \mathfrak{S}$.

If $\xi \in \mathfrak{S}$, we can think of ξ^* simultaneously as an element of $H^1(S,\Sigma;\mathbb{C})$ and $H^1(S,\mathfrak{S};\mathbb{C})$, and this identification defines a natural embedding

$$\iota : \ker\left(H^1(S,\mathfrak{S};\mathbb{R}^2) \to H^1(S;\mathbb{R}^2)\right) \to \mathfrak{R}; \tag{27}$$

that is, between the Rel subspaces corresponding to $\mathcal{H}_{\mathfrak{S}}$ and $\mathcal{H}$.

Proposition 4.2. *Let $\mathcal{N} \subset \mathcal{H}_{\mathfrak{S}}$ be an invariant subvariety. Then $\mathcal{M} \stackrel{\mathrm{def}}{=} p_{\mathfrak{S}}^{-1}(\mathcal{N})$ is an invariant subvariety. Furthermore, if μ is a U-invariant ergodic Radon probability measure on $\mathcal{M}$ that assigns zero mass to the set of surfaces with horizontal saddle connections, is invariant under Rel_v for any $v \in Z_{\mathfrak{S}}$, and such that ${p_{\mathfrak{S}}}_*\mu = m_{\mathcal{N}}$, then $\mu = m_{\mathcal{M}}$.*

Proof. We start by proving that $\mathcal{M}$ is an invariant variety. Let $Y \subset \mathcal{H}_{\mathrm{m}}(\kappa_{\mathfrak{S}})$ be a lift of $\mathcal{N}$ and let W be the linear subspace of $H^1(S,\mathfrak{S};\mathbb{C})$ on which it is modeled. Let $V \subset H^1(S,\Sigma;\mathbb{C})$ be the pre-image of W under the map $\mathrm{Res}_{\mathfrak{S}}$ and let X be the pre-image of Y under $\tilde{p}_{\mathfrak{S}}$. We claim that X is an equilinear manifold. Indeed, it follows from Proposition 4.1 that X is connected and it follows from (26) that X is open in $\mathrm{dev}^{-1}(V)$. Since X is closed, we conclude that X coincides with a connected component of $\mathrm{dev}^{-1}(V)$. Since $\mathrm{Mod}(S,\Sigma)$ acts trivially on $\mathfrak{R}$, the induced action on V of the stabilizer in $\mathrm{Mod}(S,\Sigma)$ of X acts by determinant 1 endomorphisms. This proves that X is an equilinear manifold. We deduce from (25) that

$$\pi^{-1}(\mathcal{M}) = \bigcup \tilde{p}_{\mathfrak{S}}^{-1}(Y),$$

where Y ranges over the set of lifts of $\mathcal{N}$. Furthermore, this union is locally finite as the collection of lifts $\mathcal{N}$ is itself locally finite. This proves that $\mathcal{M}$ is an invariant variety, and it is easily verified that any lift of $\mathcal{M}$ is of the form $\tilde{p}_{\mathfrak{S}}^{-1}(Y)$, where Y is a lift of $\mathcal{N}$.

The second assertion follows from Proposition 2.7. To see this, note that the embedding ι defined in (27) induces an embedding of the spaces $\mathfrak{R}_{\mathcal{N}}$ and $Z_{\mathcal{N}}$ in the spaces $\mathfrak{R}_{\mathcal{M}}$ and $Z_{\mathcal{M}}$ respectively, satisfying

$$p_{\mathfrak{S}} \circ \mathrm{Rel}_{\iota(x)} = \mathrm{Rel}_x \circ p_{\mathfrak{S}} \text{ for all } x \in Z_{\mathcal{N}}, \tag{28}$$

and

$$Z_{\mathcal{M}} = \iota(Z_{\mathcal{N}}) \oplus Z_{\mathfrak{S}}. \tag{29}$$

We have assumed that μ is invariant under $Z_{\mathfrak{S}}$, and that $p_{\mathfrak{S}*}\mu = m_{\mathcal{N}}$, which is invariant under $Z_{\mathcal{N}}$. Considering the disintegration of μ into its conditional measures with respect to the map $p_{\mathfrak{S}}$, which is unique up to a null set, and using (28) and (29), we see that μ is invariant under Rel_x for any $x \in Z_{\mathcal{M}}$. □

Theorem 4.3. *Let $\mathcal{N} \subset \mathcal{H}_{\mathfrak{S}}$ be an invariant subvariety and let μ be an ergodic U-invariant measure on $\mathcal{H}$ such that $p_{\mathfrak{S}*}\mu = m_{\mathcal{N}}$. Then there is an invariant subvariety $\mathcal{M} \subset \mathcal{H}$ and $x \in Z$ such that $\mu = \mathrm{Rel}_{x*} m_{\mathcal{M}}$.*

The proof will require the drift argument presented in §3.1.

Proof. Let μ be an ergodic U-invariant measure, let N_μ be the group as in (10) and let N_μ° be its connected component of identity. We will first prove Theorem 4.3 under the additional assumption that

$$N_\mu^\circ \cap Z_{\mathfrak{S}} = \{0\}. \tag{30}$$

We claim that for any $\varepsilon > 0$, we can find a compact set $L \subset \mathcal{H}$ with $\mu(L) > 1-\varepsilon$ that intersects every fiber of $p_{\mathfrak{S}}$ in at most a finite set. Indeed, by inner regularity of μ, let K be a compact set in Ω_μ of measure bigger than $1-\varepsilon$. Combining Egorov's theorem and Birkhoff's ergodic theorem, we can find a compact set $L \subset \Omega_\mu$ and $T_0 > 0$ so that for any $T > T_0$ and any $q \in L$,

$$\frac{1}{T}|\{s \in [0,T] : u_s q \in K\}| > 1 - 2\varepsilon. \tag{31}$$

Suppose there is $q \in \mathcal{N}$ such that $L_q \stackrel{\mathrm{def}}{=} L \cap p_{\mathfrak{S}}^{-1}(\{q\})$ is infinite. By compactness, there is sequence of points $q_n \in L_q$ converging to a point $q_\infty \in L_q$ with $q_n \neq q_\infty$ for all n. It follows from (26) that for all n large enough, there is $w_n \in \mathfrak{R}_{\mathfrak{S}} \smallsetminus \{0\}$ such that $q_n = q_\infty \mathbin{+\!\!\!\!\!\circ}\, w_n$ and $w_n \to 0$. Applying the drift argument, in the form given in Corollary 3.3(iii), and with $\mathfrak{R}_0 = \mathfrak{R}_{\mathfrak{S}}$, $\mathfrak{R}_1 = \iota(\mathfrak{R}_{\mathcal{N}})$, we see that N_μ contains nontrivial elements arbitrarily close to the identity and belonging to $Z_{\mathfrak{S}}$. This contradiction to (30) proves the claim.

To simplify notation, we write $\nu \stackrel{\mathrm{def}}{=} m_{\mathcal{N}} = p_{\mathfrak{S}*}\mu$. Let $q \mapsto \nu_q$ be the fiber-wise measures obtained by disintegration of μ along the fibers of $p_{\mathfrak{S}}$. The fact that μ is U-invariant implies that there is a ν-conull set $E \subset \mathcal{N}$ such that for any $s \in \mathbb{R}$ and any $q \in E$ we have

$$u_{s*}\nu_q = \nu_{u_s q}. \tag{32}$$

Let $\nu_q = \nu_q^{(at)} + \nu_q^{(na)}$ be the decomposition of ν_q into its atomic and non-atomic parts. That is, $\nu_q^{(at)}$ is the restriction of ν_q to its collection of atoms. By (32) and the ergodicity of ν we find that for ν-a.e. q, either $\nu_q = \nu_q^{(at)}$ or $\nu_q = \nu_q^{(na)}$. Suppose by contradiction that the latter holds, and consider once again the compact set L. We have shown that $\nu(L) > 0$ and L_q is finite for any $q \in \mathcal{N}$. Then we have $\nu_q(L) = \nu_q^{(na)}(L) = 0$ for ν-a.e. q, and thus $\mu(L) = \int_{\mathcal{N}} \nu_q(L)\, d\nu(q) = 0$, a contradiction.

Thus there is a conull set $E \subset \mathcal{N}$ such that ν_q is an atomic measure whenever $q \in E$. By ergodicity and (32) it also follows that the number of atoms, which we denote by N, and their mass can be assumed to be constant on E. Thus each atom has mass $1/N$, and we denote by $\{q^{(1)}, \ldots, q^{(N)}\}$ the atoms of ν_q.

We now claim that for any compact L of $\mathcal{M}$ with $\mu(L) > 1 - \frac{1}{N}$, there are $q \in L$, a sequence of times $s_n > 0$ converging to 0 and a sequence of $w_n \in \mathfrak{R}_{\mathfrak{S}}$ converging to 0 such that

$$q_n = v_{s_n} q \mathbin{+\!\!\!\!\!\bigcirc} w_n \in L, \tag{33}$$

where

$$v_s = \left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right).$$

Indeed, let L be such a compact set. For any $q \in E$ we either have $\nu_q(L) = 1$ or $\nu_q(L) \leqslant 1 - \frac{1}{N}$. Since $\mu(L) > 1 - \frac{1}{N}$, the set $\{q \in E : \nu_q(L) = 1\}$ must have positive measure and by Luzin's theorem, it contains a set E_0 of positive measure so that the restriction of the measurable function $q \mapsto \left\{q^{(1)}, \dots, q^{(N)}\right\}$ to E_0 is continuous. As the measure ν is G-invariant, the set

$$E_1 \stackrel{\mathrm{def}}{=} \left\{q \in E_0 : v_{-\frac{1}{n}} q \in E_0 \text{ for infinitely many } n\right\}$$

has positive measure. Let $q \in E_1$, so that $q \in E_0$ and there is a decreasing sequence s_n converging to 0 such that $q_n \stackrel{\mathrm{def}}{=} v_{s_n} q \in E_0$. By definition of E_1, for any $j \in \{1, \dots, N\}$ the points $q_n^{(j)}$ are in L and so is $q^{(j)}$. Since $p_{\mathfrak{S}}$ is G-equivariant, we know that $v_{s_n} q^{(1)}$ is in the same fiber of $p_{\mathfrak{S}}$ as $\left\{q_n^{(1)}, \dots, q_n^{(N)}\right\}$. By continuity and up to extracting a further subsequence, there is a $j_0 \in \{1, \dots, N\}$ and $w_n \in Z_{\mathfrak{S}}$ such that $q_n^{(j_0)} = v_{s_n} q^{(1)} \mathbin{+\!\!\!\!\!\bigcirc} w_n$ and $w_n \to 0$, which proves the claim.

Now the drift argument shows that the measure μ is invariant under a one-parameter group ρ satisfying (21), and with $a \neq 0$. Indeed, in the notation of Corollary 3.3, and with $\mathfrak{R}_0 = \mathfrak{R}_{\mathfrak{S}}$, we have that $\mathbf{y}_n^{(1)} = 0$ for all n, and thus the claim follows from (30) and items (i) and (iii) of Corollary 3.3. Using B-invariance and [EM18] as in the proof of Theorem 3.4, we see that there is $x \in Z$ such that $\mu = \mathrm{Rel}_{x*} m_{\mathcal{M}}$ for some invariant subvariety $\mathcal{M} \subset \mathcal{H}$. This concludes the proof under the assumption (30).

When assumption (30) does not hold, we proceed as follows. Let $\mathfrak{S}' \subset \Sigma$ be a subset of minimal cardinality containing $\mathfrak{S}$, and for which

$$N_\mu^\circ \cap Z_{\mathfrak{S}'} = \{0\}. \tag{34}$$

Condition (34) is clearly satisfied for $\mathfrak{S}' = \Sigma$, so the collection of subsets for which (34) holds is not empty. We define the map

$$p_{\mathfrak{S}', \mathfrak{S}} : \mathcal{H}_{\mathfrak{S}'} \to \mathcal{H}_{\mathfrak{S}}, \ q \mapsto (X_q, \omega_q, \mathfrak{z}|_{\mathfrak{S}}).$$

Notice that this map is exactly the map defined in (23) in the case where the source is the stratum $\mathcal{H}(\kappa_{\mathfrak{S}'})$. By construction, we have $p_{\mathfrak{S}} = p_{\mathfrak{S}', \mathfrak{S}} \circ p_{\mathfrak{S}'}$. Let $\mathcal{N}' \stackrel{\mathrm{def}}{=} p_{\mathfrak{S}', \mathfrak{S}}^{-1}(\mathcal{N})$. It follows from Proposition 4.2 applied to $\mathcal{H}_{\mathfrak{S}'}$ and $p_{\mathfrak{S}', \mathfrak{S}}$ (in place of $\mathcal{H}$ and $p_{\mathfrak{S}}$) that $\mathcal{N}'$ is an invariant subvariety. To conclude the proof of Theorem 4.3, it is enough to prove that $p_{\mathfrak{S}'*}\mu = m_{\mathcal{N}'}$. Indeed, in that case we can repeat the argument presented in the first part of the proof, with $\mathfrak{S}'$ in place of $\mathfrak{S}$.

We now set $\nu \stackrel{\mathrm{def}}{=} p_{\mathfrak{S}'*}\mu$ and prove $\nu = m_{\mathcal{N}'}$. We shall use Proposition 4.2 applied once again to $p_{\mathfrak{S}', \mathfrak{S}}$ in place of $p_{\mathfrak{S}}$. Since $p_{\mathfrak{S}', \mathfrak{S}} \circ p_{\mathfrak{S}'} = p_{\mathfrak{S}}$ and by our assumption that $p_{\mathfrak{S}*}\mu = m_{\mathcal{N}}$, we have $p_{\mathfrak{S}', \mathfrak{S}*}\nu = m_{\mathcal{N}}$. By the second assertion of the Proposition, it is enough to show that the measure ν is invariant under the flow Rel_v for any $v \in Z_{\mathfrak{S}',\mathfrak{S}}$, where

$$Z_{\mathfrak{S}',\mathfrak{S}} \stackrel{\mathrm{def}}{=} \ker\Big(H^1(S,\mathfrak{S}';\mathbb{C}) \to H^1(S,\mathfrak{S};\mathbb{C})\Big).$$

Let $\{\xi^* : \xi \in \Sigma\}$ be as in §2.3. Then the space $Z_{\mathfrak{S}',\mathfrak{S}}$ is spanned by the restriction to $H^1(S,\mathfrak{S}';\mathbb{C})$ of the ξ^* (which formally is the vector $\mathrm{Res}_{\mathfrak{S}'}(\xi^*)$) where ξ ranges over the points in $\mathfrak{S}' \smallsetminus \mathfrak{S}$. Let $\xi_0 \in \mathfrak{S}' \smallsetminus \mathfrak{S}$. By the minimality of $\mathfrak{S}'$, we have that

$$\dim\left(N_\mu^\circ \cap Z_{\mathfrak{S}' \smallsetminus \{\xi_0\}}\right) > 0.$$

Let $x \in N_\mu^\circ \cap Z_{\mathfrak{S}' \smallsetminus \{\xi_0\}}$ be a non-zero element. We can write

$$x = \lambda_{\xi_0}\xi_0^* + \sum_{\xi \in \Sigma \smallsetminus \mathfrak{S}'} \lambda_\xi \xi^*$$

and we claim that $\lambda_{\xi_0} \neq 0$. Indeed, otherwise we would have that $x \in Z_{\mathfrak{S}'}$ which, in combination with (34), would imply $x = 0$. By rescaling we can assume $\lambda_{\xi_0} = 1$. Now, notice that the restriction to $H^1(S,\mathfrak{S}';\mathbb{C})$ of $\sum_{\xi \in \Sigma \smallsetminus \mathfrak{S}'} \lambda_\xi \xi^*$ is equal to 0 and thus we have $\mathrm{Res}_{\mathfrak{S}'}(x) = \mathrm{Res}_{\mathfrak{S}'}(\xi_0^*)$. It follows from (26) and the fact that $x \in N_\mu$ that

$$\mathrm{Rel}_{t\cdot \mathrm{Res}_{\mathfrak{S}}(\xi_0^*)}{}_*\nu = p_{\mathfrak{S}'*}(\mathrm{Rel}_{t\cdot x*}\mu) = p_{\mathfrak{S}'*}\mu = \nu,$$

hence ν is indeed invariant by Rel_v for any $v \in Z_{\mathfrak{S}',\mathfrak{S}}$. □

Remark 4.4. *The special case of Theorem 4.3 in which $\mathcal{N}$ is a closed $\mathrm{GL}_2^+(\mathbb{R})$-orbit is the main result of* [EMM06]*. In the same paper, the authors deduce that the G-orbit closure of a translation surface that is a translation branched cover of a Veech surface satisfies the weak classification of U-invariant measures. We recall that a translation cover of q_0 is a pair (q,ρ) where $\rho : X_q \to X_{q_0}$ is a holomorphic map such that $\rho^*\omega_{q_0} = \omega_q$ and that q_0 is a Veech surface if its G-orbit is closed. Analogously, it should be possible to deduce from Theorem 4.3 that the G-orbit-closure of a translation branched cover of a surface in a rank one invariant subvariety of Rel-dimension one, satisfies weak classification of U-invariant measures. This would require pinning down the relationship, as dynamical systems for the U-action, between the spaces $p_{\mathfrak{S}}^{-1}(\mathcal{N})$ of Proposition 4.2 (adding marked points to surfaces in an invariant subvariety $\mathcal{N}$), and the space of branched translation covers of a fixed topological type of surfaces in $\mathcal{N}$. See* [SW08, Section 3] *for a related discussion of the* moduli space of a branched cover.

For $\mathcal{M} = p_{\mathfrak{S}}^{-1}(\mathcal{N})$ as in Proposition 4.2, the invariant subvarieties $\mathcal{M}' \subset \mathcal{M}$ such that $p_{\mathfrak{S}}(\mathcal{M}') = \mathcal{N}$ correspond to finite $\mathrm{GL}_2^+(\mathbb{R})$-equivariant configurations of marked points over $\mathcal{N}$. Such configurations have been classified, see [EFW18, AW21].

As a corollary we obtain that adding marked points to invariant subvarieties that satisfy themselves the weak classification of U-invariant measures, results in more such invariant subvarieties. In particular, we have:

Corollary 4.5. *Let $\mathcal{M} \subset \mathcal{H}$ be an invariant subvariety such $p_{\Sigma_0}(\mathcal{M})$ has rank one and Rel-dimension one, where Σ_0 is as in* (22)*. Then $\mathcal{M}$ satisfies the weak classification of U-invariant measures.*

Remark 4.6. *The proofs of Theorems 3.4 and 4.3 given above would also work for products of strata, provided one had a suitable analogue of* [EMM15] *for the diagonal B-action, and an analogue of Proposition 2.7. See Remark 2.8.*

5. Nondivergence of the horocycle flow and frequency of passage near singular sets

The main results of this section are Proposition 5.6 and Proposition 5.8, which uniformly bound the amount of time a horocycle trajectory can spend either near the support of one of the measures described in §2.5, or near the set of surfaces with a horizontal saddle connection of bounded length. A special case of Proposition 5.6 was obtained in [BSW16, §10]. We also establish a similar result for large circles in Proposition 5.12 that is used for the proof of Theorem 7.4.

5.1. Singular sets and long horocycle trajectories.

By a well-known compactness criterion, the sets

$$\{q \in \mathcal{H} : \text{any saddle connection on } q \text{ has length } \geqslant \varepsilon\}$$

are an exhaustion of $\mathcal{H}$ by compact sets. We will be interested in giving upper bounds on the amount of time a horocycle trajectory can spend outside large compact sets. For measurable $A \subset \mathbb{R}$, we denote the Lebesgue measure of A by $|A|$. Let $\mathcal{M}$ be an invariant subvariety and let $\mathcal{M}_r$ denote the surfaces in $\mathcal{M}$ with no horizontal saddle connections of length less than r.

Theorem 5.1 (Quantitative nondivergence of the horocycle flow, [MW02])**.** *There are constants $C > 0$, ρ_0 and $\alpha > 0$ such that for every $q \in \mathcal{M}$, every $\rho \in (0, \rho_0]$ and every $T > 0$, if for every saddle connection δ on q we have $\max_{s\in[0,T]} \|\mathrm{hol}_{u_s q}(\delta)\| \geqslant \rho$, then for any $0 < \varepsilon < \rho$,*

$$|\{s \in [0, T] : u_s q \text{ has a saddle connection of length} < \varepsilon\}| < C \left(\frac{\varepsilon}{\rho}\right)^{\alpha} T. \tag{35}$$

In particular,

(I) *For any $\varepsilon > 0$ and any compact $L \subset \mathcal{M}$, there is a compact K such that for any $T > 0$ and any $q \in L$,*

$$\frac{1}{T}|\{s \in [0, T] : u_s q \notin K\}| < \varepsilon.$$

(II) *For any $\varepsilon > 0$ and $r > 0$, there is a compact $K \subset \mathcal{M}$ such that for any $q \in \mathcal{M}_r$ there is a $T_0 > 0$ such that for all $T > T_0$,*

$$\frac{1}{T}|\{s \in [0, T] : u_s q \notin K\}| < \varepsilon.$$

Statements (I) and (II) are also true if one replaces the interval $[0, T]$ with the interval $[-T, 0]$.

It will be useful to introduce a scalar product $\langle\cdot,\cdot\rangle$ on $Z_{\mathcal{M}}$ and we denote by $\|\cdot\|$ the induced norm on $Z_{\mathcal{M}}$. For $R > 0$ and an invariant subvariety $\mathcal{N} \subsetneq \mathcal{M}$, we denote

$$B^{\perp}_{\mathcal{N}}(R) \stackrel{\mathrm{def}}{=} \left\{x \in Z^{\perp}_{\mathcal{N}} : \|x\| \leqslant R\right\}. \tag{36}$$

From [MW14, Thm. 11.2] or [BSW16, Thm. 6.1] we have:

Lemma 5.2. *There is a constant $\sigma > 0$, depending on the scalar product on $Z_{\mathcal{M}}$, such that for any $R > 0$, if $x \in Z_{\mathcal{M}}$ with $\|x\| < R$ and $q \in \mathcal{M}_{\sigma R}$ then $\mathrm{Rel}_x(q)$ is well-defined.*

Proposition 5.3. *Let $\mathcal{N} \subset \mathcal{M}$ be an invariant subvariety and let Z' be any linear subspace of $Z_{\mathcal{M}}$ transverse to $Z_{\mathcal{N}}$. Then for any $R > 0$, there is $r > 0$ such that the map*

$$\mathcal{N}_r \times \{x \in Z' : \|x\| \leqslant R\} \to \mathcal{M}, \qquad (q, x) \mapsto \mathrm{Rel}_x(q) \tag{37}$$

is well-defined and injective.

Proof. Let K be a compact subset of $\mathcal{M}$ as in item (II) of Theorem 5.1 for $r = 1$ and $\varepsilon = \frac{1}{2}$. Since K is compact, there is $\ell \in (0, 1)$ such that any surface $q \in K$ does not have horizontal saddle connections short er than ℓ. For any $x \in Z'$, the vector field $\vec{x}$ is transverse to $\mathcal{N}$. From this, and from Lemma 5.2, we find that there is $\varepsilon' < \frac{\ell}{\sigma}$ such that the map

$$(K \cap \mathcal{N}) \times \{x \in Z' : \|x\| \leqslant \varepsilon'\} \to \mathcal{M}, \qquad (q, x) \mapsto \mathrm{Rel}_x(q)$$

is well-defined. Now it follows from Proposition 2.4, perhaps after making ε' smaller, that this map is injective.

Given $R > 0$, let $r \overset{\mathrm{def}}{=} \frac{R}{\varepsilon'}$ and $t \overset{\mathrm{def}}{=} \log r$, so that $e^t \varepsilon' = R$ and $r = e^t$. Suppose $q, q' \in \mathcal{N}_r$ and $x, x' \in \{z \in Z' : \|z\| \leqslant R\}$ are such that $\mathrm{Rel}_x(q) = \mathrm{Rel}_{x'}(q')$ (it follows from Lemma 5.2 and the choice of ε' that $\mathrm{Rel}_x(q)$ and $\mathrm{Rel}_{x'}(q')$ are well-defined). By (3) and (4), one has $g_{-t}q, g_{-t}q' \in \mathcal{N}_1$, and thus, by definition of K there is $s > 0$ such that $u_s g_{-t} q$ and $u_s g_{-t} q'$ both belong to K. By (6),

$$\mathrm{Rel}_{e^{-t}x}(u_s g_{-t} q) = u_s g_{-t} \mathrm{Rel}_x(q) = u_s g_{-t} \mathrm{Rel}_{x'}(q') = \mathrm{Rel}_{e^{-t}x'}(u_s g_{-t} q').$$

By definition $e^{-t}x$ and $e^{-t}x'$ belong to $\{z \in Z' : \|z\| \leqslant \varepsilon'\}$, and thus by the injectivity shown in the first part of the proof, $x = x'$ and $q = q'$. □

The following useful statement was derived from Theorem 5.1 in [BSW16, Prop. 10.5].

Proposition 5.4. *Let L be a compact subset of $\mathcal{M}_\infty$, $\varepsilon > 0$ and $r > 0$. There is an open neighborhood $\mathcal{W}$ of L and a compact $\widehat{\mathcal{W}} \subset \mathcal{M}_r$ containing $\mathcal{W}$ such that for any $q \in \mathcal{H}$ and any interval $I \subset \mathbb{R}$ for which $u_{s_0} q \in \mathcal{W}$ for some $s_0 \in I$, we have*

$$\left| \left\{ s \in I : u_s q \notin \widehat{\mathcal{W}} \right\} \right| < \varepsilon |I|.$$

By combining Proposition 5.3 and Proposition 5.4, we get the following:

Proposition 5.5. *Let $\mathcal{N} \subset \mathcal{M}$ be an invariant subvariety. For any $R > 0$, $\varepsilon > 0$ and any compact $K \subset \mathcal{N}_\infty$, there is $\delta > 0$, a neighborhood $\mathcal{W}$ of K and a compact set $\widehat{\mathcal{W}} \subset \mathcal{N}$ containing $\mathcal{W}$, such that:*

- *the map*

$$\widehat{\mathcal{W}} \times B^{\perp}_{\mathcal{N}}(R) \times B^{\perp}_{\mathcal{N}}(\delta) \to \mathcal{M}, \qquad (q, x, y) \mapsto \mathrm{Rel}_{x + \mathbf{i} \cdot y}(q)$$

  is well-defined, and a homeomorphism onto its image;
- *for any interval $I \subset \mathbb{R}$ and any $q \in \mathcal{M}$ the following is satisfied: if for some $s_0 \in I$ we have that $u_{s_0} q \in \mathcal{W}$, then*

$$\left| \left\{ s \in I : u_s q \notin \widehat{\mathcal{W}} \right\} \right| < \varepsilon |I|.$$

Proof. By Proposition 5.3, there exists $r > 0$ such that the map $\Phi : \mathcal{N}_r \times B^{\perp}_{\mathcal{N}}(R) \to \mathcal{M}$ defined by

$$\mathcal{N}_r \times B^{\perp}_{\mathcal{N}}(R) \to \mathcal{M}, \qquad (q, x) \mapsto \mathrm{Rel}_x(q) \tag{38}$$

is well-defined and injective. Let $\mathcal{W}$ and $\widehat{\mathcal{W}}$ be the sets provided by Proposition 5.4, corresponding to this r and to $L = K$, where K is as in the statement of the Proposition. Since $\widehat{\mathcal{W}} \subset \mathcal{N}_r$, it follows that the restriction of Φ to $\widehat{\mathcal{W}} \times B^{\perp}_{\mathcal{N}}(R)$ is still injective. Since $\widehat{\mathcal{W}}$ is compact, and since for any $y \in Z^{\perp}_{\mathcal{N}}$, $\mathbf{i} \cdot y$ is transverse to $\mathcal{N}$, there is $\delta_0 > 0$ such that for any $\delta \leqslant \delta_0$, the map

$$\widehat{\mathcal{W}} \times B^{\perp}_{\mathcal{N}}(R) \times B^{\perp}_{\mathcal{N}}(\delta) \to \mathcal{M}, \qquad (q, x, y) \mapsto \mathrm{Rel}_{\mathbf{i}\cdot y}(\mathrm{Rel}_x(q))$$

is well-defined, continuous and injective. It follows from [BSW16, Prop. 4.5] that for any $(q, x, y) \in \widehat{\mathcal{W}} \times B^{\perp}_{\mathcal{N}}(R) \times B^{\perp}_{\mathcal{N}}(\delta)$ we have $\mathrm{Rel}_{\mathbf{i}\cdot y}(\mathrm{Rel}_x(q)) = \mathrm{Rel}_{x+\mathbf{i}\cdot y}(q)$. Also, by Proposition 2.4, this is an open map. This concludes the proof. $\square$

For any $R > 0$ and any $K \subset \mathcal{N}$, denote

$$Z(K, R) \stackrel{\mathrm{def}}{=} \left\{ \mathrm{Rel}_x(q) : q \in K,\ x \in B^{\perp}_{\mathcal{N}}(R) \cap Z^{(q)} \right\}.$$

Proposition 5.6 (Frequency of passage near Rel pushes of invariant subvarieties)**.** *Let $\mathcal{N} \subsetneq \mathcal{M}$ be an invariant subvariety, let $r > 0$, let K be a compact subset of $\mathcal{N}_\infty$ and let $\varepsilon > 0$. There is a neighborhood $\mathcal{U}$ of $Z(K, r)$ and $R > r$ such that for any compact subset L of $\mathcal{M}_R \smallsetminus Z(\mathcal{N}, R)$, there is $T_0 > 0$ such that for any $T > T_0$ and any $q \in L$, we have*

$$\frac{1}{T} \left|\{s \in [0, T] : u_s q \in \mathcal{U}\}\right| < \varepsilon. \tag{39}$$

Formula (39) *remains true if we replace* $[0, T]$ *by* $[-T, 0]$.

Proof. Let R' be large enough so that

$$R' > \frac{(8 + \varepsilon)(r + 1)}{\varepsilon}. \tag{40}$$

Using Propositions 2.4 and 5.5, one obtains that there is an open neighborhood $\mathcal{W}$ of K in $\mathcal{N}$, a compact $\widehat{\mathcal{W}} \subset \mathcal{N}$ that contains $\mathcal{W}$, and $\delta > 0$, such that the map

$$\bar{\Phi} : \widehat{\mathcal{W}} \times B^{\perp}_{\mathcal{N}}(R') \times B^{\perp}_{\mathcal{N}}(\delta) \to \mathcal{M}, \qquad \bar{\Phi}(q, x, y) \stackrel{\mathrm{def}}{=} \mathrm{Rel}_{x+\mathbf{i}\cdot y}(q)$$

is well-defined and is a homeomorphism onto its image, and such that for any interval $I \subset \mathbb{R}$, any $s_0 \in I$ and any $q \in \mathcal{N}$ for which $u_{s_0} q \in \mathcal{W}$, we have

$$\left|\left\{ s \in I : u_s q \notin \widehat{\mathcal{W}} \right\}\right| < \frac{\varepsilon}{4} |I|. \tag{41}$$

Denote the interior of $B^{\perp}_{\mathcal{N}}(R')$ by $\mathrm{int}(B^{\perp}_{\mathcal{N}}(R'))$ (that is, use strict inequality in (36)), and let

$$\mathcal{V} \stackrel{\mathrm{def}}{=} \mathcal{W} \times \mathrm{int}(B^{\perp}_{\mathcal{N}}(r + 1)) \times \mathrm{int}(B^{\perp}_{\mathcal{N}}(\delta)) \quad \text{ and } \quad \mathcal{U} \stackrel{\mathrm{def}}{=} \bar{\Phi}(\mathcal{V}).$$

Since $\mathcal{M}$ has rank one, and since $\bar{\Phi}$ is a homeomorphism onto its image, $\mathcal{U}$ is open in $\mathcal{M}$. It contains $Z(K, r)$ by construction. Let $q \in \mathcal{M} \smallsetminus Z(\mathcal{N}, R')$ and let

$$I_{\mathcal{U},q} \stackrel{\mathrm{def}}{=} \{t > 0 : u_t q \in \mathcal{U}\}.$$

By injectivity of the map $\bar{\Phi}$, for any $t \in I_{\mathcal{U},q}$ there is a unique triple $(q_t, x_t, y_t) \in \mathcal{V}$ such that $u_t q = \bar{\Phi}(q_t, x_t, y_t)$. Since $q \notin Z(\mathcal{N}, R')$ we have $y_t \neq 0$. For any $t \in I_{\mathcal{U},q}$, let

$$\mathcal{J}_t = t + \{s \in \mathbb{R} : \|x_t + s y_t\| \leqslant R'\} \tag{42}$$

$$\mathcal{I}_t = t + \{s \in \mathbb{R} : \|x_t + s y_t\| < r + 1\}. \tag{43}$$

Since $\mathcal{I}_t$ and $\mathcal{J}_t$ are sub-level sets of real quadratic polynomials, they are intervals.

We claim that if $t \in I_{\mathcal{U},q}$ and I is an interval that contains one of the endpoints of $\mathcal{J}_t$, we have

$$|\mathcal{I}_t \cap I| < \frac{\varepsilon}{4}|\mathcal{J}_t \cap I|. \tag{44}$$

Indeed, define $p(s) \stackrel{\text{def}}{=} \|x_t + sy_t\|^2 = \|x_t\|^2 + 2s\langle x_t, y_t\rangle + s^2\|y_t\|^2$ and for any $C \geqslant 0$, define

$$\Delta_C \stackrel{\text{def}}{=} 4\langle x_t, y_t\rangle^2 - 4\|y_t\|^2(\|x_t\|^2 - C^2)). \tag{45}$$

This is the discriminant of the real quadratic polynomial whose zeroes are points at which $\|x_t + sy_t\| = C$. Since $\|x_t + sy_t\| \geqslant 0$ for all s, we have $\Delta_0 \leqslant 0$ and since $R' > r + 1$, by the assumption that I contains one of the endpoints of $\mathcal{J}_t$ we have $\Delta_{r+1} > 0$. From the formula for the roots of a quadratic polynomial, for any $C \geqslant r + 1$ we have

$$p(s) \leqslant C^2 \iff s \in -\frac{\langle x_t, y_t\rangle}{\|y_t\|^2} + \left[\frac{-\sqrt{\Delta_C}}{2\|y_t\|^2}, \frac{\sqrt{\Delta_C}}{2\|y_t\|^2}\right]. \tag{46}$$

By construction, we have that $s + t \in \mathcal{J}_t$ if and only if $p(s) \leqslant R'^2$ and $s + t \in \mathcal{I}_t$ if and only if $p(s) < (r + 1)^2$. Remembering that $\Delta_{r+1} = \Delta_0 + 4\|y_t\|^2(r + 1)^2$ and that $\Delta_0 \leqslant 0$, we have

$$\frac{|\mathcal{I}_t \cap I|}{|\mathcal{J}_t \cap I|} \leqslant \frac{\sqrt{\Delta_{r+1}}}{\|y_t\|^2} \cdot \frac{2\|y_t\|^2}{\sqrt{\Delta_{R'}} - \sqrt{\Delta_{r+1}}}. \tag{47}$$

Indeed, (47) is obvious in case $\mathcal{I}_t \cap I = \varnothing$; in case this set is not empty, I contains a subinterval bounded by an endpoint of $\mathcal{I}_t$ and an endpoint of $\mathcal{J}_t$, so an interval in which $P(s) \in \big((r + 1)^2, R'^2\big]$. This gives a lower bound for the denominator $|\mathcal{J}_t \cap I|$ in (47), and as an upper bound for the numerator we use $|\mathcal{I}_t|$. We obtain

$$\begin{aligned}
\frac{|\mathcal{I}_t \cap I|}{|\mathcal{J}_t \cap I|} &\stackrel{(47)}{\leqslant} 2\frac{\sqrt{\Delta_{r+1}}}{\sqrt{\Delta_{R'}} - \sqrt{\Delta_{r+1}}} \\
&= 2\frac{\sqrt{\Delta_{r+1}}(\sqrt{\Delta_{R'}} + \sqrt{\Delta_{r+1}})}{4\|y_t\|^2(R'^2 - (r + 1)^2)} = \frac{\Delta_{r+1} + \sqrt{\Delta_{r+1}\Delta_{R'}}}{2\|y_t\|^2(R'^2 - (r + 1)^2)} \\
&\leqslant \frac{\Delta_{r+1} + \sqrt{\Delta_0^2 + 16\|y_t\|^4 R'^2(r + 1)^2}}{2\|y_t\|^2(R'^2 - (r + 1)^2)} \\
&\leqslant \frac{\Delta_{r+1} - \Delta_0 + 4\|y_t\|^2 R'(r + 1)}{2\|y_t\|^2(R'^2 - (r + 1)^2)} \\
&\leqslant 2\frac{(r + 1)^2 + R'(r + 1)}{R'^2 - r^2} = \frac{2(r + 1)}{R' - r} \stackrel{(40)}{<} \frac{\varepsilon}{4}.
\end{aligned}$$

This proves (44).

Now, we claim that if $s \in \mathcal{J}_t \smallsetminus \mathcal{I}_t$ is such that $u_s q \in \mathcal{U}$ then $u_{s-t}q_t \notin \widehat{\mathcal{W}}$. Indeed, we have $u_s q = \mathrm{Rel}_{x_t+(s-t)y_t+\mathbf{i}\cdot y_t}(u_{s-t}q)$ by (6) and thus, if $u_{s-t}q \in \widehat{\mathcal{W}}$, then $u_s q = \bar{\Phi}(u_{s-t}q, x_t + (s - t)y_t, y_t)$. But the fact that $s \notin \mathcal{I}_t$ yields a contradiction to the injectivity of $\bar{\Phi}$.

For I containing an endpoint of $\mathcal{J}_t$, we deduce

$$\begin{aligned}|\mathcal{J}_t \cap I_{\mathcal{U},q} \cap I| = & \ |\mathcal{I}_t \cap I_{\mathcal{U},q} \cap I| + |(\mathcal{J}_t \smallsetminus \mathcal{I}_t) \cap I_{\mathcal{U},q} \cap I| \\ \leqslant & \ |\mathcal{I}_t \cap I| + \left|\left\{ s \in \mathcal{J}_t \cap I : u_{s-t} q_t \notin \widehat{\mathcal{W}} \right\}\right| \\ \overset{(44),(41)}{<} & \ \frac{\varepsilon}{4}|\mathcal{J}_t \cap I| + \frac{\varepsilon}{4}|\mathcal{J}_t \cap I| = \frac{\varepsilon}{2}|\mathcal{J}_t \cap I|.\end{aligned}$$

Now, let σ be as in Lemma 5.2, let $c \overset{\mathrm{def}}{=} \max(1, \sigma)$ and let $R \overset{\mathrm{def}}{=} cR'$. We will show that $\mathcal{U}$ and R satisfy the required conclusions. Indeed, let L be a compact subset of $\mathcal{M}_R \smallsetminus Z(\mathcal{N}, R)$. For any $q \in L$, we will denote by $\mathcal{J}_t = \mathcal{J}_{q,t}$ the interval as in (42) to stress the dependence on q. We claim that there is $\delta_0 > 0$ such that for any $q \in L$ and for any $t \in I_{\mathcal{U},q}$, if $0 \in \mathcal{J}_{q,t}$ then $\|y_t\| \geqslant \delta_0$. Indeed, suppose by contradiction that there is a sequence $q_n \in L$ together with $t_n \in I_{\mathcal{U},q_n}$ such that $0 \in \mathcal{J}_{q,t_n}$. Let $(q'_n, x_n, y_n) \in \mathcal{V}$ such that $u_{t_n} q_n = \bar{\Phi}(q'_n, x_n, y_n)$ and assume that $\|y_n\| \to 0$. Since $0 \in \mathcal{J}_{q,t_n}$, we have

$$q_n = u_{-t_n} \bar{\Phi}(q'_n, x_n, y_n) = u_{-t_n} q'_n \mathbin{+\!\!\!\!\bigcirc} (x_n - t_n y_n + \mathbf{i} y_n)$$

with $\|x_n - t_n y_n\| \leqslant R'$. Passing to a subsequence, we can assume that q_n converges to some $q_\infty \in L$ and that $x_n - t_n y_n$ converges to some x_∞ satisfying $\|x_\infty\| \leqslant R'$. In particular, $q_\infty \mathbin{+\!\!\!\!\bigcirc} (-x_\infty, 0)$ is well-defined by Lemma 5.2 and the choices of R and R'. It follows from [BSW16, Prop 4.6] that $q_n \mathbin{+\!\!\!\!\bigcirc} -(x_n - t_n y_n, y_n)$ is a sequence in $\mathcal{N}$ that converges to $q_\infty \mathbin{+\!\!\!\!\bigcirc} (-x_\infty, 0)$. Since $\mathcal{N}$ is closed, this shows that q_∞ belongs to $Z(\mathcal{N}, R)$, which is a contradiction and our claim is proved.

Let $T > 0$, let $q \in L$ and let $t \in I_{\mathcal{U},q}$ satisfy $[0,T] \cap \mathcal{J}_{q,t} \neq \varnothing$. If $[0,T]$ does not contain one of the endpoints of $\mathcal{J}_{q,t}$, then it is contained in $\mathcal{J}_{q,t}$. In particular, $0 \in \mathcal{J}_{q,t}$ and thus $\|y_t\| \geqslant \delta_0$. Using the Cauchy-Schwarz inequality, it follows that

$$T \leqslant |\mathcal{J}_{q,t}| \overset{(46)}{\leqslant} \frac{\sqrt{\Delta_{R'}}}{\|y_t\|^2} \overset{(45)}{\leqslant} \frac{2R'}{\|y_t\|} \leqslant \frac{2R}{\delta_0}.$$

Let $T_0 \overset{\mathrm{def}}{=} \frac{2R}{\delta_0}$. We proved that for any $q \in L$, any $T > T_0$ and any $t \in I_{\mathcal{U},q}$ such that $[0,T] \cap \mathcal{J}_{q,t} \neq \varnothing$, we have that $[0,T]$ contains one of the endpoints of $\mathcal{J}_{q,t}$. The set $[0,T] \cap I_{\mathcal{U},q}$ is covered by the sets $(\mathcal{J}_{q,t})_{t \in I_{\mathcal{U},q}}$. From this cover, we can extract a subcover $(\mathcal{J}_j)$ such that any $q \in [0,T]$ is covered at most twice, and hence

$$\begin{aligned}|[0,T] \cap I_{\mathcal{U},q}| &\leqslant \sum_j |[0,T] \cap I_{\mathcal{U},q} \cap \mathcal{J}_j| \\ &< \frac{\varepsilon}{2} \sum_j |[0,T] \cap \mathcal{J}_j| \leqslant \varepsilon T.\end{aligned}$$

□

Remark 5.7. *As the above proof shows, the statement is true for the following values of R and T_0. For R we can take any number satisfying*

$$R > \frac{(8+\varepsilon)(r+1)\sigma}{\varepsilon},$$

for any $\sigma \geqslant 1$ which satisfies the conclusion of Lemma 5.2. For T_0 we can take any number satisfying $T_0 \geqslant \frac{2R}{\delta_0}$, where $\delta_0 > 0$ has the following property: suppose $\widehat{\mathcal{W}}$ satisfies the properties in the first paragraph of the proof, and $q_0 \in L$ can be written as $\mathrm{Rel}_{x+\mathbf{i}y}(q_1)$ for some $q_1 \in \widehat{\mathcal{W}}$ and some $x, y \in Z_{\mathcal{N}}^{\perp}$ with $\|x\| \leqslant R$. Then $\|y\| \geqslant \delta_0$.

Proposition 5.8 (Frequency of passage near the horizontal saddle connection locus)**.** *Let $\varepsilon > 0$ and let $r > 0$. There is an open set $\mathcal{U}$ that contains $\mathcal{M} \smallsetminus \mathcal{M}_r$ and $R > r$ such that for any compact subset L of $\mathcal{M}_R$, there is $T_0 > 0$ such for any $T > T_0$ and any $q \in L$, we have*

$$\frac{1}{T} \left|\{s \in [0,T] : u_s q \in \mathcal{U}\}\right| < \varepsilon. \tag{48}$$

The same conclusion holds if we replace $[0,T]$ *in* (48) *with* $[-T,0]$.

Proof. Let C, ρ_0 and α as in Theorem 5.1, let $\varepsilon' < \rho_0$ be small enough so that $C(\frac{\varepsilon'}{\rho_0})^\alpha < \varepsilon$, let $K \subset \mathcal{M}$ be the compact set of surfaces whose shortest saddle connection has length $\geqslant \varepsilon'$, let $t > \log(\frac{r}{\varepsilon'})$, let $R = \rho_0 e^t$ and finally let $\mathcal{U} = g_t(K^c)$. Notice that $\mathcal{M} \smallsetminus \mathcal{M}_r \subset \mathcal{U}$.

Now let L be any compact subset of $\mathcal{M}_R$. By compactness, there is $\eta > 0$ such that for any $q \in L$ and any saddle connection δ on q with $\mathrm{hol}_q(\delta) = (x,y)$, we either have $|x| > R$ or $|y| > \eta$. Let T_0' be large enough so that $T_0' e^t \eta \geqslant 2\rho_0$. This condition ensures for any $q \in L$, the length of any saddle connection on $g_{-t}q$ will eventually become larger than ρ_0 after applying the horocycle flow up to time T_0. It follows from Theorem 5.1 that for any $T > T_0'$ and any $q \in L$

$$\frac{1}{T} \left|\{s \in [0,T] : u_s g_{-t} q \in K^c\}\right| < \varepsilon.$$

Finally, let $T_0 = e^{2t} T_0'$, let $q \in L$ and let $T > T_0$. Then we have

$$\begin{aligned} \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{U}}(u_s q)\, ds &= \frac{1}{T}\int_0^T \mathbf{1}_{K^c}(g_{-t} u_s q)\, ds \\ =&\frac{1}{Te^{-2t}} \left|\left\{s \in \left[0, Te^{-2t}\right] : u_s g_{-t} q \notin K\right\}\right| < \varepsilon. \end{aligned}$$

□

5.2. **Singular sets and large circles.** We extend the Euclidean norm on Z_M to a Euclidean norm on $\mathfrak{R}_{\mathcal{M}}$, *i.e.,* for any $v = x + \mathbf{i}\cdot y \in \mathfrak{R}_{\mathcal{M}}$, we let $\|v\| = \sqrt{\|x\|^2 + \|y\|^2}$. Notice that r_θ acts on $\mathfrak{R}_{\mathcal{M}}$ by isometry.

Proposition 5.9. *Let $\mathcal{N} \subset \mathcal{M}$ be an invariant subvariety and let $K_0 \subset \mathcal{M} \smallsetminus \mathcal{N}$ be compact. There exists $\tilde{\delta} > 0$ such that for any $q \in K_0$ and for all $t > 0$, if there exists $\theta \in [0, 2\pi]$ such that $g_t r_\theta q = q' \mathbin{+\!\!\!\!\!\bigcirc} v$ with $q' \in \mathcal{N}$ and $v \in \mathfrak{R}_{\mathcal{N}}^{\perp}$, then $\|g_{-t} v\| \geqslant \tilde{\delta}$.*

Proof. Since $K_0 \cap \mathcal{N} = \varnothing$, K_0 is compact and $\mathcal{N}$ is closed, it follows by continuity of Rel flows that there is $\tilde{\delta} > 0$ such that for any $w \in \mathfrak{R}_{\mathcal{N}}^{\perp}$ with $\|w\| < \tilde{\delta}$ and any $q \in K_0$ we have that $\mathrm{Rel}_w(q)$ is well-defined and does not belong to $\mathcal{N}$. Let $t > 0$ and assume that there are $\theta \in [0, 2\pi]$ and $q' \in \mathcal{N}$ such that $g_t r_\theta q = q' \mathbin{+\!\!\!\!\!\bigcirc} v$. Let $w \overset{\mathrm{def}}{=} -r_{-\theta} g_{-t} v$. If $\|g_{-t} v\| < \tilde{\delta}$ then we also have $\|w\| < \tilde{\delta}$, and thus $q \mathbin{+\!\!\!\!\!\bigcirc} w$ is well-defined. But by (6), $q \mathbin{+\!\!\!\!\!\bigcirc} w = r_{-\theta} g_{-t} q' \in \mathcal{N}$ and we get a contradiction with the definition of $\tilde{\delta}$. This implies that $\|g_{-t} v\| \geqslant \tilde{\delta}$. □

Let n be a positive integer. We consider the diagonal action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{C}^n$ where $\mathrm{GL}_2^+(\mathbb{R})$ acts on each copy of $\mathbb{C} \simeq \mathbb{R}^2$ by linear transformations. For any $v \in \mathbb{C}^n$, we let

$$E_{t,v} = \{g_t r_\theta v : \theta \in [0, 2\pi]\}. \tag{49}$$

We will need the following:

Lemma 5.10. *For any $0 \leqslant \eta_1 \leqslant \eta_2 \leqslant 2$ and $0 < \varepsilon < 1$ we have*

$$\arccos(\varepsilon^2\eta_1 - 1) - \arccos(\varepsilon^2\eta_2 - 1) \leqslant \varepsilon(\arccos(\eta_1 - 1) - \arccos(\eta_2 - 1)). \tag{50}$$

Proof. Use $2\varepsilon x - \varepsilon^2 x^2 > \varepsilon(2x - x^2)$ in conjunction with the following computation:

$$\arccos(\varepsilon^2\eta_1 - 1) - \arccos(\varepsilon^2\eta_2 - 1) = \int_{\varepsilon^2\eta_1 - 1}^{\varepsilon^2\eta_2 - 1} \frac{1}{\sqrt{1 - x^2}}\, dx$$
$$= \int_{\varepsilon^2\eta_1}^{\varepsilon^2\eta_2} \frac{1}{\sqrt{2x - x^2}}\, dx = \varepsilon^2 \int_{\eta_1}^{\eta_2} \frac{1}{\sqrt{2\varepsilon^2 x - \varepsilon^4 x^2}}\, dx$$
$$\leqslant \varepsilon \int_{\eta_1}^{\eta_2} \frac{1}{\sqrt{2x - x^2}}\, dx = \varepsilon(\arccos(\eta_1 - 1) - \arccos(\eta_2 - 1)).$$

□

For any $r > 0$, let $B(r)$ be the ball of radius r in $\mathbb{R}^n$. Let $T, \delta > 0$ and define

$$B(T, \delta) \stackrel{\text{def}}{=} \{x + \mathbf{i} \cdot y \in \mathbb{C}^n : (x, y) \in B(T) \times B(\delta)\}.$$

Proposition 5.11. *Let $v \in \mathbb{C}^n$, let $T, \delta > 0$ and let $0 < \varepsilon < 1$. If $E_{t,v}$ is not contained in $B(T, \delta)$ then*

$$|\{\theta \in [0, 2\pi] : g_t r_\theta v \in B(\varepsilon T, \varepsilon\delta)\}| < \varepsilon |\{\theta \in [0, 2\pi] : g_t r_\theta v \in B(T, \delta)\}|.$$

Proof. We first observe that, up to replacing T and δ by $e^{-t}T$ and $e^t\delta$, we can assume that $t = 0$ and up to replacing v by $r_{\frac{\pi}{2}} v$ we can assume that $T \leqslant \delta$. Write $v = (x_1 + \mathbf{i}y_1, \ldots, x_n + \mathbf{i}y_n)$ and $r_\theta v = (x_\theta, y_\theta)$. Let $R = \|v\|$ and let $\theta_0 \in [0, 2\pi]$ be such that

$$\cos(2\theta_0)\left(\sum_{i=1}^n x_i y_i\right) - \sin(2\theta_0)\left(\sum_{i=1}^n \frac{x_i^2 - y_i^2}{2}\right) = 0$$
$$\cos(2\theta_0)\left(\sum_{i=1}^n \frac{x_i^2 - y_i^2}{2}\right) + \sin(2\theta_0)\left(\sum_{i=1}^n x_i y_i\right) \geqslant 0.$$

Let $r \geqslant 0$ such that

$$r^2 = 2\left(\cos(2\theta_0)\left(\sum_{i=1}^n \frac{x_i^2 - y_i^2}{2}\right) + \sin(2\theta_0)\left(\sum_{i=1}^n x_i y_i\right)\right).$$

Notice that $r \leqslant R$ and

$$\|x_{\theta+\theta_0}\|^2 = \frac{r^2}{2}\cos(2\theta) + \frac{R^2}{2}$$
$$\|y_{\theta+\theta_0}\|^2 = -\frac{r^2}{2}\cos(2\theta) + \frac{R^2}{2}.$$

The degenerate case where $r = 0$ is trivial so we can assume $r > 0$ and up to replacing v by $r_{\theta_0} v$ we can assume that $\theta_0 = 0$. It follows that for any $T', \delta' > 0$, we have that $r_\theta v \in B(T', \delta')$ if and only if

$$-\frac{2\delta'^2 - R^2}{r^2} \leqslant \cos(2\theta) \leqslant \frac{2T'^2 - R^2}{r^2}.$$

Let $\eta_1 \overset{\text{def}}{=} -\frac{2\delta^2-(R^2+r^2)}{r^2}$ and $\eta_2 \overset{\text{def}}{=} \frac{2T^2-(R^2-r^2)}{r^2}$. The assumption that $E_{0,v}$ is not contained in $B(T,\delta)$ implies that $\eta_2 \leqslant 2$. Notice that $\left[-\frac{2(\varepsilon\delta)^2-R^2}{r^2}, \frac{2(\varepsilon T)^2-R^2}{r^2}\right] \subset [\varepsilon^2\eta_1 - 1, \varepsilon^2\eta_2 - 1]$. Using Lemma 5.10,

$$\begin{aligned}|\{\theta \in [0,2\pi] : g_t r_\theta v \in B(\varepsilon T, \varepsilon\delta)\}| &< \arccos(\max(0,\varepsilon^2\eta_1) - 1) - \arccos(\varepsilon^2\eta_2 - 1)\\ &< \varepsilon(\arccos(\max(0,\eta_1) - 1) - \arccos(\eta_2 - 1))\\ &< \varepsilon|\{\theta \in [0,2\pi] \ : \ g_t r_\theta v \in B(T,\delta)\}|.\end{aligned}$$

□

For any $r, \delta > 0$ and any invariant subvariety $\mathcal{N} \subset \mathcal{M}$, we denote

$$B^\perp_{\mathcal{N}}(r,\delta) \overset{\text{def}}{=} \{x + \mathbf{i}\cdot y \in \mathfrak{R}_{\mathcal{M}} : (x,y) \in B^\perp_{\mathcal{N}}(r) \times B^\perp_{\mathcal{N}}(\delta)\}$$

Proposition 5.12 (Proportion of large circles near singular sets)**.** *Let $\mathcal{N} \subsetneq \mathcal{M}$ be an invariant subvariety, let $r > 0$, let K be a compact subset of $\mathcal{N}_\infty$, let K_0 be a compact subset of $\mathcal{M} \smallsetminus \mathcal{N}$ and let $\varepsilon > 0$. There is a neighborhood $\mathcal{U}$ of $Z(K,r)$ such that for any $q \in K_0$, there is $t_0 > 0$ such that for any $t > t_0$,*

$$\frac{1}{2\pi}|\{\theta \in [0,2\pi] : g_t r_\theta q \in \mathcal{U}\}| < \varepsilon.$$

Proof. Let $R > 0$ be large enough so that

$$R > \frac{8}{\varepsilon} r,$$

and let $\tilde{\delta}$ be as in Proposition 5.9. A straightforward extension of [BSW16, Prop 12.5] to arbitrary rank one invariant subvarieties gives the following analogue of Proposition 5.5: There exist $t_0 > 0$, $\delta < \frac{\tilde{\delta}}{\sqrt{2}}$, a neighborhood $\mathcal{W}$ of K and a compact set $\widehat{\mathcal{W}} \subset \mathcal{N}$ such that

- the map
$$\bar{\Phi} : \widehat{\mathcal{W}} \times B^\perp_{\mathcal{N}}(R) \times B^\perp_{\mathcal{N}}(\delta) \to \mathcal{M}, \quad \bar{\Phi}(q,x,y) \overset{\text{def}}{=} \mathrm{Rel}_{x+\mathbf{i}\cdot y}(q)$$
is well-defined, continuous and injective;
- for any $t > t_0$ and any interval $I \subset J \overset{\text{def}}{=} \left[\frac{\pi}{2} - \frac{\varepsilon\pi}{4}, \frac{\pi}{2} + \frac{\varepsilon\pi}{4}\right] \cup \left[\frac{3\pi}{2} - \frac{\varepsilon\pi}{4}, \frac{3\pi}{2} + \frac{\varepsilon\pi}{4}\right]$, if $q' \in \mathcal{N}$ and $\theta \in I$ satisfy $g_t r_\theta q' \in \mathcal{W}$, then

$$|\{\alpha \in I : g_t r_\alpha g_{-t} q' \notin \widehat{\mathcal{W}}\}| < \frac{\varepsilon}{8}|I|. \tag{51}$$

Let

$$\mathcal{V} \overset{\text{def}}{=} \mathcal{W} \times \operatorname{int}\left(B^\perp_{\mathcal{N}}\left(\frac{\varepsilon}{8}R\right) \times B^\perp_{\mathcal{N}}\left(\frac{\varepsilon}{8}\delta\right)\right) \quad \text{and} \quad \mathcal{U} = \bar{\Phi}(\mathcal{V}).$$

Since $\mathcal{M}$ has rank one, and since $\bar{\Phi}$ is a homeomorphism onto its image, $\mathcal{U}$ is open in $\mathcal{M}$, and contains $Z(K,r)$. Let $t > t_0$ and let

$$I_{\mathcal{U},q} \overset{\text{def}}{=} \{\theta \in [0,2\pi] : g_t r_\theta q \in \mathcal{U}\}.$$

We need to verify that $|I_{\mathcal{U},q}| < 2\pi\varepsilon$. Since $|J| < \varepsilon\pi$, it is enough to show that $|I_{\mathcal{U},q} \cap J^c| < \pi\varepsilon$. For any $\theta \in I_{\mathcal{U},q}$, let $(q_\theta, x_\theta, y_\theta) \in \mathcal{V}$ such that $g_t r_\theta q = \bar{\Phi}(q_\theta, x_\theta, y_\theta)$ and define

$$\mathcal{I}_\theta \overset{\text{def}}{=} \theta + \{\alpha \in [0,2\pi] : g_t r_\alpha g_{-t}(x_\theta + \mathrm{i}y_\theta) \in B^\perp_{\mathcal{N}}(\frac{\varepsilon}{8}R, \frac{\varepsilon}{8}\delta)\},$$

$$\mathcal{J}_\theta \overset{\text{def}}{=} \theta + \{\alpha \in [0,2\pi] : g_t r_\alpha g_{-t}(x_\theta + \mathrm{i}y_\theta) \in B^\perp_{\mathcal{N}}(R,\delta)\}.$$

By definition, we know that $\|g_{-t}(x_\theta+\mathrm{i}y_\theta)\| > \tilde{\delta}$. In particular, there is $\alpha \in [0, 2\pi]$ such that $r_\alpha g_{-t}(x_\theta+\mathrm{i}y_\theta) = x+\mathrm{i}y$ with $\|y\|^2 > \frac{\tilde{\delta}^2}{2} > \delta^2$ and thus $\alpha \notin \mathcal{J}_\theta$. Identifying $\mathbb{C}^n$ and $\mathfrak{R}^{\perp}_{\mathcal{N}}$ with a $\mathrm{GL}^+_2(\mathbb{R})$-equivariant isometry, it follows from Proposition 5.11 that

$$|\mathcal{I}_\theta| < \frac{\varepsilon}{8}|\mathcal{J}_\theta|. \tag{52}$$

We claim that if $\alpha \in \mathcal{J}_\theta \smallsetminus \mathcal{I}_\theta$ then either $g_t r_{\alpha-\theta} g_{-t} q_\theta \notin \widehat{\mathcal{W}}$ or $g_t r_\alpha q \notin \mathcal{U}$. Indeed, let $\alpha \in \mathcal{J}_\theta$ and suppose $g_t r_{\alpha-\theta} g_{-t} q_\theta \in \widehat{\mathcal{W}}$. It follows from (6) that

$$\begin{aligned} g_t r_\alpha q &= (g_t r_{\alpha-\theta} g_{-t}) g_t r_\theta q \\ &= (g_t r_{\alpha-\theta} g_{-t})(q_\theta + (x_\theta + \mathbf{i} y_\theta)) \\ &= g_t r_{\alpha-\theta} g_{-t} q_\theta + g_t r_{\alpha-\theta} g_{-t}(x_\theta + \mathbf{i} y_\theta), \end{aligned}$$

and thus by injectivity of $\bar{\Phi}$, it follows that $g_t r_\alpha q \notin \mathcal{U}$, which proves our claim. Using (51) and (52), we obtain:

$$\begin{aligned} |I_{\mathcal{U},q} \cap \mathcal{J}_\theta \cap J^c| &= |I_{\mathcal{U},q} \cap \mathcal{I}_\theta \cap J^c| + |I_{\mathcal{U},q} \cap \mathcal{J}_\theta \smallsetminus \mathcal{I}_\theta \cap J^c| \\ &\leqslant |\mathcal{I}_\theta| + \left|\left\{\alpha \in \mathcal{J}_\theta \cap J^c : g_t r_{\alpha-\theta} g_{-t} q_\theta \notin \widehat{\mathcal{W}}\right\}\right| \\ &< \frac{\varepsilon}{8}|\mathcal{J}_\theta| + \frac{\varepsilon}{8}|\mathcal{J}_\theta \cap J^c| \leqslant \frac{\varepsilon}{4}|\mathcal{J}_\theta|. \end{aligned}$$

We have shown that the set $I_{\mathcal{U},q} \cap J^c$ is covered by the $(\mathcal{J}_\theta)_{\theta \in I_{\mathcal{U},q}}$, and from this cover we can extract a sub-cover $(\mathcal{J}_{\theta_l})_l$ such that any point is covered at most twice. This yields

$$|I_{\mathcal{U},q} \cap J^c| \leqslant \sum_l |I_{\mathcal{U},q} \cap \mathcal{J}_l \cap J^c| < \frac{\varepsilon}{2} \sum_l |\mathcal{J}_l| \leqslant \varepsilon\pi.$$

□

6. From measure classification to equidistribution of orbits

For any rank-one invariant subvariety $\mathcal{M}$, we denote by $\mathcal{S}_{\mathcal{M}}$ the set of surfaces $q \in \mathcal{M}$ without horizontal saddle connections for which there is a proper invariant subvariety $\mathcal{N} \subsetneq \mathcal{M}$ and $x \in Z^{\perp}_{\mathcal{N}}$ such that $\mathrm{Rel}_x(q) \in \mathcal{N}$. By Proposition 1.3 in [EM18], there are at most countably many invariant subvarieties of $\mathcal{M}$. We assume there are countably many, and denote them by $\left(\mathcal{N}^{(i)}\right)_{i \in \mathbb{N}}$ (the reader will have no difficulty in adapting our arguments to the case in which there are finitely many). With this notation,

$$\mathcal{S}_{\mathcal{M}} = \bigcup_{(i,j) \in \mathbb{N} \times \mathbb{N}} Z\left(\mathcal{N}^{(i)}, j\right);$$

in particular, the set $\mathcal{S}_{\mathcal{M}}$ is measurable.

The following result is independent of the weak classification of U-invariant measure and holds in arbitrary rank-one invariant subvarieties. It contrasts with a surprising phenomenon in $\mathcal{H}(1,1)$ discovered in [CSW20, Theorem 1.2].

Theorem 6.1. *Let $\mathcal{M}$ be a rank-one invariant subvariety, let $q \in \mathcal{M}_\infty \smallsetminus \mathcal{S}_{\mathcal{M}}$ and*

$$\mu_T \stackrel{\mathrm{def}}{=} \frac{1}{T}\int_0^T u_{s*}\delta_q \, ds. \tag{53}$$

Then any weak-$*$ limit of $(\mu_{T_n})_{n>0}$, taken along a sequence $T_n \to \infty$, is a U-invariant Radon probability measure that assigns zero measure to $\mathcal{M} \smallsetminus \mathcal{M}_\infty$ and to $\mathcal{S}_{\mathcal{M}}$. The same result holds true if in (53) *we replace the integral over* $[0,T]$ *with an integral over* $[-T,0]$.

Here and in the rest of this section, we will with the interval $[0,T]$ in the statements (for instance, in (53)). The reader will have no difficulty adapting the proofs to the second case $[-T,0]$.

Proof. Let ν be a weak-$*$ limit of a convergent subsequence $(\mu_{T_n})_{n>0}$, where $T_n \to \infty$.

Step 1. We show that $\nu(\mathcal{M}) = 1$. It is enough to show that for any $\varepsilon > 0$, there is a compact $K \subset \mathcal{M}$ such that $\nu(K) \geqslant 1 - \varepsilon$. We take K as in item (II) in Theorem 5.1 for $r = 1$. By assumption q does not have any horizontal saddle connections and in particular, no horizontal saddle connection shorter than r. Thus there is $T_0 > 0$ such that for any $T > T_0$,

$$\frac{1}{T}|\{s \in [0,T] : u_s q \notin K\}| < \varepsilon.$$

It follows that for n for which $T_n > T_0$, we have $\mu_{T_n}(K) \geqslant 1 - \varepsilon$. Since K is compact, by a well-known property of weak-* convergence, we obtain $\nu(K) \geqslant 1-\varepsilon$.

Step 2. We now show that $\nu(\mathcal{M} \smallsetminus \mathcal{M}_\infty) = 0$. For any $i > 0$, let $K_i \subset \mathcal{M} \smallsetminus \mathcal{M}_\infty$ be the set of surfaces whose saddle connections have length bounded below by $\frac{1}{i}$, and which have horizontal saddle connections, all of whose lengths are bounded above by i. By construction the sets $(K_i)_{i>0}$ are compact and exhaust $\mathcal{M} \smallsetminus \mathcal{M}_\infty$ and thus it is enough to show that for any $i > 0$ and any $\varepsilon > 0$, $\nu(K_i) < \varepsilon$. By a well-known property of weak-* convergence, it suffices to show that for each $\varepsilon > 0$ and each i, there is an open set $\mathcal{U}$ that contains K_i, such that $\mu_T(\mathcal{U}) < \varepsilon$ for any T large enough. Once again, let K be as in item (II) in Theorem 5.1 for $r = 1$. Since the lengths of all saddle connection are bounded below on the compact set K, there is $t > 0$ such that $g_{-t}(K_i) \subset K^c$. We denote by $\mathcal{U}$ the set $g_t(K^c)$ and let $q_t = g_{-t}q$. Since q_t does not have horizontal saddle connection either, there is a $T_0 > 0$ such that for any $T > T_0$,

$$\frac{1}{T}|\{s \in [0,T] : u_s q_t \notin K\}| < \varepsilon.$$

Now, let T_1 be large enough so that $e^{-2t}T_1 > T_0$ and let $T > T_1$. Then, using the commutation relation $g_{-t}u_s q = u_{s'} g_t q$, where $s' \stackrel{\text{def}}{=} e^{-2t}s$, we obtain

$$\begin{aligned}
\mu_T(\mathcal{U}) &= \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{U}}(u_s q)\, ds \\
&= \frac{1}{T}\int_0^T \mathbf{1}_{K^c}(g_{-t}u_s q)\, ds \\
&= \frac{1}{Te^{-2t}}\left|\left\{s' \in \left[0, Te^{-2t}\right] : u_{s'}q_t \notin K\right\}\right| < \varepsilon.
\end{aligned}$$

Step 3. We now show that for any invariant subvariety $\mathcal{N} \subsetneq \mathcal{M}$ and $r > 0$ we have $\nu(Z(\mathcal{N}, r)) = 0$. By Step 2, it is enough to show that for any $\varepsilon > 0$ and any compact $K \subset \mathcal{N}_\infty$, there is an open set $\mathcal{U}$ that contains $Z(K, r)$ such that $\mu_T(\mathcal{U}) < \varepsilon$ for any T large enough. This follows from Proposition 5.6, applied to $L = \{q\}$. $\square$

Theorem 6.2 (Genericity)**.** *Suppose $\mathcal{M}$ satisfies the weak classification of U-invariant measures. The orbit of any surface $q \in \mathcal{M}$ without horizontal saddle connections is generic for some measure as in §2.5. That is, for any $q \in \mathcal{M}_\infty$, there are $x \in Z_{\mathcal{M}}$, an invariant subvariety $\mathcal{N} \subset \mathcal{M}$ and $q' \in \mathcal{N}_\infty$ such that $q = \mathrm{Rel}_x(q')$ and*

$$\frac{1}{T}\int_0^T u_{s*}\delta_q \, ds \underset{T\to\infty}{\longrightarrow} \mathrm{Rel}_{x*} m_{\mathcal{N}} \qquad \textit{(w.r.t. the weak-* topology).} \tag{54}$$

The same conclusion holds replacing $[0,T]$ with $[-T,0]$.

Remark 6.3. *It is instructive to compare Theorem 6.2 with* [CSW20, Thm. 1.4]*, which gives examples of surfaces in the stratum $\mathcal{H}(1,1)$ which are not generic for any measure. Note that $\mathcal{H}(1,1)$ is not of rank one. It should also be noted that the genericity result in* [BSW16, Thm. 11.1] *is stronger than Theorem 6.2, as it does not assume that q has no horizontal saddle connections. This is possible since it relies on a stronger measure classification result.*

Proof. Let $\mathcal{N}$ be an invariant subvariety of smallest dimension amongst the collection of all invariant subvarieties $\mathcal{O}$ such that there is $\xi \in Z_{\mathcal{M}}$ satisfying $\mathrm{Rel}_\xi(q) \in \mathcal{O}$. For any $T > 0$, we define

$$\nu_T \overset{\mathrm{def}}{=} \frac{1}{T}\int_0^T u_{s*}\delta_{\mathrm{Rel}_\xi(q)} \, ds.$$

The measure ν_T is supported on $\mathcal{N}$. By definition of $\mathcal{N}$, we have that $\mathrm{Rel}_\xi(q) \notin \mathcal{S}_{\mathcal{N}}$ and since q does not have horizontal saddle connections we have that $\mathrm{Rel}_\xi(q) \in \mathcal{N}_\infty$. It follows from Theorem 6.1 (applied to $\mathcal{N}$ and $\mathrm{Rel}_\xi(q)$) that any limit ν of a convergent subsequence of ν_T is a Radon probability measure that assigns zero measure to the set of surfaces with horizontal saddle connections, as well as to the singular set $\mathcal{S}_{\mathcal{N}}$. We decompose the measure ν into U-ergodic components; that is,

$$\nu = \int_{\mathcal{M}} \nu_p \, d\nu(p)$$

where there is a ν-conull set E comprised of surfaces $p \in \mathcal{M}$ for which ν_p is an ergodic U-invariant probability Radon measure that assigns zero measure to $\mathcal{N} \smallsetminus \mathcal{N}_\infty$ and to $\mathcal{S}_{\mathcal{N}}$. Let $p \in E$. Since $\mathcal{M}$ satisfies the weak classification of U-invariant measures, ν_p is a measure obtained by pushing a measure $m_{\mathcal{N}'}$ on an invariant subvariety $\mathcal{N}'$ by an element of real Rel. Since ν_p is supported in $\mathcal{N}$ and $\nu_p(\mathcal{S}_{\mathcal{N}}) = 0$, it must be that $\nu_p = m_{\mathcal{N}}$. Therefore also $\nu = m_{\mathcal{N}}$, and since ν was an arbitrary subsequential limit of ν_T, we have that $\nu_T \longrightarrow_{T\to\infty} m_{\mathcal{N}}$ (with respect to the weak-* topology).

This amounts to saying that $q' \overset{\mathrm{def}}{=} \mathrm{Rel}_\xi(q)$ is generic for $m_{\mathcal{N}}$. From Proposition 2.6 we obtain that $q = \mathrm{Rel}_{-\xi}(q')$ is generic for $\mathrm{Rel}_{-\xi*} m_{\mathcal{N}}$, which is exactly what we wanted with $x = -\xi$. □

Using that the U-action commutes with real Rel deformations, we can extend the Theorem 6.2, replacing $\mathcal{M}$ with a slightly more general locus.

Corollary 6.4. *Suppose $\mathcal{M}$ satisfies the weak classification of U-invariant measures. Let $x_0 \in Z \smallsetminus Z_{\mathcal{M}}$, and let $\mathcal{M}_0 \overset{\mathrm{def}}{=} \mathrm{Rel}_{x_0}(\mathcal{M})$. Then for any $q_0 \in (\mathcal{M}_0)_\infty$ there are $x \in Z$, and an invariant subvariety $\mathcal{N} \subset \mathcal{M}$ such that the conclusion of Theorem 6.2 holds.*

Proof. Using [BSW16, Prop. 4.27], we can replace q_0 and $\mathcal{M}_0$ with $q = \mathrm{Rel}_{-x_0}(q_0)$ and $\mathcal{M}$, and apply Theorem 6.2. $\square$

Remark 6.5. *Although we do not use this fact, it is interesting to note that the invariant subvariety $\mathcal{N}$ used in the proof of Theorem 6.2 is unique. To see this, suppose $\mathcal{N}_1, \mathcal{N}_2$ are both invariant subvarieties of $\mathcal{M}$ and ξ_1, ξ_2 are elements of $Z_{\mathcal{M}}$, such that $q \in \mathrm{Rel}_{\xi_i}(\mathcal{N}_i)$ for $i = 1, 2$ and $\dim \mathcal{N}_1 = \dim \mathcal{N}_2$ is the smallest possible. Then the proof of Theorem 6.2 shows that the measures in* (54) *converge to $\mathrm{Rel}_{\xi_i *} m_{\mathcal{N}_i}$, and thus these two measures are the same. This implies that*

$$\mathrm{Rel}_{\xi'}(\mathcal{N}_2) = \mathcal{N}_1, \quad \textit{where} \quad \xi' \stackrel{\mathrm{def}}{=} \xi_1 - \xi_2.$$

Applying the geodesic flow g_t, the invariance of both $\mathcal{N}_i$ under g_t, and the relation (6)*, we see that $\mathrm{Rel}_{e^t \xi'}(\mathcal{N}_2) = \mathcal{N}_1$ for all $t \in \mathbb{R}$, and taking the limit as $t \to -\infty$ we obtain that $\mathcal{N}_1 = \mathcal{N}_2$.*

The next result shows that the convergence in Theorem 6.2 holds uniformly on compact sets, outside finitely many obvious exceptions. The statement is modeled on [DM93, Thm. 3] and will be useful in [CWY]. It is new even in the case of eigenform loci in $\mathcal{H}(1,1)$ studied in [BSW16].

Theorem 6.6 (Uniform equidistribution)**.** *Suppose $\mathcal{M}$ satisfies the weak classification of U-invariant measures, let $K \subset \mathcal{M}$ be compact, let $\varepsilon > 0$ and let $f \in C_c(\mathcal{M})$. There are finitely many invariant subvarieties $\mathcal{N}^{(1)}, \ldots, \mathcal{N}^{(k)}$ contained in $\mathcal{M}$ and $r > 0$ such that for any compact $L \subset K \cap \mathcal{M}_r \smallsetminus \bigcup_{i=1}^{k} Z\left(\mathcal{N}^{(i)}, r\right)$, there is $T_0 > 0$ such that for all $T > T_0$ and all $q \in L$, we have*

$$\left| \frac{1}{T} \int_0^T f(u_s q)\, ds - \int_{\mathcal{M}} f \; dm_{\mathcal{M}} \right| < \varepsilon.$$

The same conclusion holds if we replace $[-T, 0]$ with $[0, T]$.

Proof. Let $\left(\mathcal{N}^{(i)}\right)_{i \in \mathbb{N}^*}$ be the collection of all invariant subvarieties properly contained in $\mathcal{M}$ (if the collection is finite we index it so that $\mathcal{N}^{(i)}$ is eventually constant). Suppose by way of contradiction that there are a compact set $K \subset \mathcal{M}$, a function $f \in C_c(\mathcal{M})$ and $\varepsilon_0 > 0$ such that the following holds:

Contradicting assumption: *For all $n \geqslant 1$ and $r > 0$, there is a compact $L \subset K \cap \mathcal{M}_r \smallsetminus \bigcup_{i=1}^{n} Z\left(\mathcal{N}^{(i)}, r\right)$ such that for all $S > 0$, there is $T > S$ and $q \in L$ such that:*

$$\left| \frac{1}{T} \int_0^T f(u_s q)\; ds - \int_{\mathcal{M}} f \; dm_{\mathcal{M}} \right| \geqslant \varepsilon_0. \tag{55}$$

For any $i \in \mathbb{N}$, let $(K_{i,j})_{j \in \mathbb{N}}$ be an exhaustion by compact subsets of $\mathcal{N}_{\infty}^{(i)}$ (the set of surfaces without horizontal saddle connections in $\mathcal{N}^{(i)}$), let

$$K_n \stackrel{\mathrm{def}}{=} \bigcup_{i \leqslant n} Z(K_{i,n}, n),$$

and let L_n be the compact subset comprised of surfaces that have a horizontal saddle connection of length at most n and whose shortest saddle connection is longer than $\frac{1}{n}$. We claim that there is a sequence of surfaces $(q_n)_{n \in \mathbb{N}}$ in K, a sequence of open sets $(\mathcal{U}_n)_{n \in \mathbb{N}}$, and a sequence of times $(T_n)_{n \in \mathbb{N}}$ with $T_n > n$ such that:

(i) for any n, $K_n \cup L_n \subset \mathcal{U}_n$;

(ii) for any n,

$$\left|\frac{1}{T_n}\int_0^{T_n} f(u_s q_n)\,ds - \int_{\mathcal{M}} f\ dm_{\mathcal{M}}\right| \geqslant \varepsilon_0; \tag{56}$$

(iii) for any $n > 0$, for all $0 < i \leqslant n$,

$$\frac{1}{T_n}|\{s \in [0, T_n] : u_s q_n \in \mathcal{U}_i\}| \leqslant \frac{1}{i}. \tag{57}$$

Indeed, for any $n \geqslant 1$ and any $0 < i \leqslant n$, let $\mathcal{U}_{i,n}$ and $R_{i,n} > n$ be as in Proposition 5.6 for $K = K_{i,n}$, $r = n$ and $\varepsilon = \frac{1}{2n^2}$. Let also $\mathcal{W}_n$ and R'_n be be the $\mathcal{U}$ and R given by Proposition 5.8 for $\varepsilon = \frac{1}{2n}$ and $r = n$. Now, define $R_n = \max(R_{1,n}, \ldots, R_{n,n}, R'_n)$ and $\mathcal{U}_n = \bigcup_{i\leqslant n} \mathcal{U}_{i,n} \cup \mathcal{W}_n$. By construction, $\mathcal{U}_n$ contains $K_n \cup L_n$ and satisfies the following property: for any compact subset L of $\mathcal{M}_{R_n} \smallsetminus \bigcup_{i\leqslant n} Z\left(\mathcal{N}^{(i)}, R_n\right)$, there is $S_n > 0$ such that

$$\forall q \in L,\ \forall T > S_n,\ \frac{1}{T}|\{t \in [0, T] : u_s q \in \mathcal{U}_n\}| \leqslant \frac{1}{n}. \tag{58}$$

Now, let $n \geqslant 1$, let $r_n = \max(R_1, \ldots, R_n)$ and let L be as in the contradicting assumption for n and r_n. For any $i \leqslant n$, the set L is a compact subset of $\mathcal{M}_{R_i} \smallsetminus \bigcup_{j\leqslant n} Z\left(\mathcal{N}^{(j)}, R_i\right)$ and let S_i be as in (58). We define $S = \max(S_1, \ldots, S_n, n)$. By the contradicting assumption, there is $q_n \in L$ together with $T_n > S$ such that (56) is satisfied. By definition of T_n, the surface q_n satisfies (57). Since $r_n > n$, this concludes the construction of $(q_n)_{n\in\mathbb{N}}$.

Now define a sequence of measures by

$$\mu_n \stackrel{\text{def}}{=} \frac{1}{T_n}\int_0^{T_n} \delta_{u_s q_n}\,ds.$$

Our goal is to show that this sequence of measures weak-* converges to $m_{\mathcal{M}}$, which will contradict (56). By passing to a subsequence, we can suppose that $(\mu_n)_{n>0}$ converges to some measure ν, and we need to show $\nu = m_{\mathcal{M}}$. We follow the strategy used in the proof Theorem 6.1.

Step 1. We show that $\nu(\mathcal{M}) = 1$. It is enough to show that for any $\varepsilon > 0$, there is a compact $K_0 \subset \mathcal{M}$ such that $\mu_n(K_0) \geqslant 1 - \varepsilon$ for any n large enough. We can choose K_0 and T_0 as in item (I) in Theorem 5.1 for $r = 1$ and $L = K$. For any $n > T_0$ we have $T_n > T_0$ and thus

$$\mu_n(K_0) = \frac{1}{T_n}|\{t \in [0, T_n] : u_t q_n \in K_0\}| \geqslant 1 - \varepsilon.$$

Step 2. We now want to show that $\nu(\mathcal{M} \smallsetminus \mathcal{M}_\infty) = 0$. By construction, the $(L_i)_{i>0}$ exhaust $\mathcal{M} \smallsetminus \mathcal{M}_\infty$ and thus it is enough to show that for any $\varepsilon > 0$ and $i > 0$, there is an open set $\mathcal{U}$ that contains L_i and such that $\mu_n(\mathcal{U}) < \varepsilon$ for any n large enough. Let

$$N > \max\left(i, \frac{1}{\varepsilon}\right)$$

and set $\mathcal{U} = \mathcal{U}_N$. By (i) and the definition of $N, \mathcal{U}_N$ contains L_i and for any $n > N$ we have from (57) that

$$\mu_n(\mathcal{U}) = \frac{1}{T_n}|\{t \in [0, T_n] : u_s q_n \in \mathcal{U}_N\}| \leqslant \frac{1}{N} < \varepsilon.$$

Step 3. We now prove that if $\mathcal{N} \subsetneq \mathcal{M}$ is an invariant subvariety and $r > 0$, then $\nu(Z(\mathcal{N}, r)) = 0$. Let $i > 0$ such that $\mathcal{N} = \mathcal{N}^{(i)}$. We recall that we had an exhaustion by nested compact sets $(K_{i,j})_{j>0}$ of $\mathcal{N}_\infty^{(i)}$. It is enough to show that for any $\varepsilon > 0$ and $j > 0$, there is a open set $\mathcal{U}$ that contains $Z(K_{i,j}, r)$ and such that $\mu_n(\mathcal{U}) < \varepsilon$ for any n large enough. Let

$$N > \max\left(i, j, r, \frac{1}{\varepsilon}\right)$$

and set $\mathcal{U} = \mathcal{U}_N$. By (i) and the definition of N, $\mathcal{U}_N$ contains $Z(K_{i,j}, r)$ and for any $n > N$ we have from (57) that

$$\mu_n(\mathcal{U}) = \frac{1}{T_n}|\{t \in [0, T_n] : u_s q_n \in \mathcal{U}_N\}| \leqslant \frac{1}{N+1} < \varepsilon.$$

Step 4. Since for any $n > 0$, we have $T_n > n$, the measure ν is a U-invariant Radon measure. It follows from Step 1 that it is a probability measure, from Step 2 that it assigns measure zero to the set of surfaces that have a horizontal saddle connection and from Step 3 that it assigns measure zero to the set $\mathcal{S}_{\mathcal{M}}$. Let

$$\nu = \int_{\mathcal{M}} \nu_p \, d\nu(p)$$

be the decomposition of ν into U-invariant ergodic measures. There is a ν-conull set E of surfaces $p \in \mathcal{M}$ for which ν_p is an ergodic U-invariant probability Radon measure that assigns measure zero to the set of surfaces with horizontal saddle connections and such that $\nu_p(\mathcal{S}_{\mathcal{M}}) = 0$. By the weak classification of U-invariant measure, for every $p \in E$ we have $\nu_p = m_{\mathcal{M}}$, and hence $\nu = m_{\mathcal{M}}$. This is a contradiction to (56), concluding the proof. $\square$

7. Equidistribution results for sequences of measures

In this section we state and prove several equidistribution results regarding limits of sequences of measures. The first result extends [BSW16, Thm. 1.5].

Theorem 7.1. *Suppose $\mathcal{M}$ satisfies the weak classification of U-invariant measures, let $\mathcal{L} \subset \mathcal{M}$ be an invariant subvariety and let $x \in Z_{\mathcal{M}}$. There is an invariant subvariety $\mathcal{N} \subset \mathcal{M}$ that contains $\mathcal{L}$ and such that $x \in Z_{\mathcal{N}}$ and $\mathrm{Rel}_{tx*} m_{\mathcal{L}} \to m_{\mathcal{N}}$ as $t \to \infty$, in the weak-* topology.*

For the proof of Theorem 7.1 we will need the following:

Lemma 7.2. *Suppose $\mathcal{N}_1, \mathcal{N}_2 \subsetneq \mathcal{M}$ are two invariant subvarieties. If there is $r > 0$ such that $m_{\mathcal{N}_1}(Z(\mathcal{N}_2, r)) > 0$ then $\mathcal{N}_1 \subset \mathcal{N}_2$.*

Proof. Since $m_{\mathcal{N}_1}$ is U-invariant and ergodic, we have that $m_{\mathcal{N}_1}(Z(\mathcal{N}_2, r)) = 1$. For any t, since $m_{\mathcal{N}_1}$ is g_t-invariant and $g_{-t}(Z(\mathcal{N}_2, r)) = Z(\mathcal{N}_2, e^{-t}r)$, we have $m_{\mathcal{N}_1}(Z(\mathcal{N}_2, e^{-t}r)) = 1$. It follows that $\mathcal{N}_2 = \bigcap_{n>0} Z(\mathcal{N}_2, re^{-n})$ also has full measure for $m_{\mathcal{N}_1}$. This implies that $\mathcal{N}_2$ contains $\mathcal{N}_1 = \operatorname{supp} m_{\mathcal{N}_1}$. $\square$

Proof of Theorem 7.1. Let $\mathcal{C}(\mathcal{L}, x)$ be the collection of invariant subvarieties $\mathcal{N}$ of $\mathcal{M}$ that contain $\mathcal{L}$ and such that $x \in Z_{\mathcal{N}}$, and let $\mathcal{N} \in \mathcal{C}(\mathcal{L}, x)$ be of smallest dimension. We will show that $\mu_t = \mathrm{Rel}_{tx*} m_{\mathcal{L}}$ converges to $m_{\mathcal{N}}$ as $t \to \infty$.

It is enough to show that if $\mu_n \stackrel{\mathrm{def}}{=} \mu_{t_n}$ is a convergent subsequence then its limit is $m_{\mathcal{N}}$. Denote by ν the limit of μ_n. By construction, $\operatorname{supp} \nu \subset \mathcal{N}$.

Step 1. We show that ν is a probability measure. It is enough to show that for any $\varepsilon > 0$ there is compact $K \subset \mathcal{M}$ such that $\mu_n(K) \geqslant 1 - \varepsilon$ for n large enough. We choose K as in Theorem 5.1, item (II) for $r = 1$. The measure $m_{\mathcal{L}}$ assigns full measure to surfaces without horizontal saddle connections, and since this set is invariant under Rel_{tx} we have $\mu_n(\mathcal{M}_\infty) = 1$. Therefore there is $q_n \in \mathcal{M}_\infty$ that is Birkhoff generic for the indicator function $\mathbf{1}_K$ with respect to μ_n. By the choice of K, for any T large enough,

$$\frac{1}{T}|\{s \in [0,T] : u_s q_n \in K\}| > 1 - \varepsilon.$$

By Birkhoff genericity, the left-hand side tends to $\mu_n(K)$ as $T \to \infty$ and thus $\mu_n(K) \geqslant 1 - \varepsilon$.

Step 2. We now show that $\nu(\mathcal{M} \smallsetminus \mathcal{M}_\infty) = 0$. For any $i > 0$, let K_i be the closure of the set of surfaces whose saddle connection lengths are bounded below by $\frac{1}{i}$, and which have at least one horizontal saddle connection, with the lengths of all horizontal saddle connections bounded above by i. By construction the $(K_i)_{i>0}$ exhaust $\mathcal{M} \smallsetminus \mathcal{M}_\infty$ and thus it is enough to show that for any $i > 0$ and any $\varepsilon > 0$, there is an open set $\mathcal{U}$ that contains K_i and such that $\mu_n(\mathcal{U}) \leqslant \varepsilon$ for any n large enough. Once again, let K be as in item (II) in Theorem 5.1 for r as above. Since the length of saddle connections is bounded below on compact sets and K is compact, there is a $t > 0$ such that $g_{-t}(K_i) \subset K^c$, so that

$$K_i \subset \mathcal{U} \overset{\mathrm{def}}{=} g_t(K^c).$$

Let n be large enough so that $t_n > t$. We have $\mu_n(\mathcal{M} \smallsetminus \mathcal{M}_{re^{t_n}}) = 0$ and thus there is a surface q_n without horizontal saddle connections shorter than re^{t_n} that is Birkhoff generic for $\mathbf{1}_{K^c}$ with respect to μ_n. The surfaces $q_n' \overset{\mathrm{def}}{=} g_{-t} q_n$ do not have horizontal saddle connections shorter that r and thus there is $T_0 > 0$ such that for any $T > T_0$,

$$\frac{1}{T}|\{s \in [0,T] : u_s q_n' \notin K\}| < \varepsilon.$$

Now, let T_1 be large enough so that $e^{-2t} T_1 > T_0$. For any $T > T_1$ we have

$$\begin{aligned} \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{U}}(u_s q_n)\, ds &= \frac{1}{T}\int_0^T \mathbf{1}_{K^c}(g_{-t} u_s q_n)\ ds \\ =& \frac{1}{Te^{-2t}}\left|\left\{s \in \left[0, Te^{-2t}\right] : u_s q_n' \notin K\right\}\right| < \varepsilon. \end{aligned}$$

By Birkhoff genericity the quantity on the left-hand side converges to $\mu_n(\mathcal{U})$ as $T \to \infty$ and thus $\mu_n(\mathcal{U}) \leqslant \varepsilon$.

Step 3. Let $\mathcal{N}' \subsetneq \mathcal{N}$ be an invariant subvariety and let $r > 0$. We want to show that $\nu(Z(\mathcal{N}', r) = 0$. It is enough to show that for any compact subset $K \subset \mathcal{N}'_\infty$ and $\varepsilon > 0$ there is an open set $\mathcal{U}$ that contains $Z(K, r)$ such that $\mu_n(\mathcal{U}) \leqslant \varepsilon$ for n large enough. Let $\mathcal{U}$ and $R > r$ be as in Proposition 5.6.

Suppose first that $\mathcal{L}$ is not contained in $\mathcal{N}'$. If $\mu_n(Z(\mathcal{N}', R)) > 0$ then $m_{\mathcal{L}}(Z(\mathcal{N}', R + t_n\|x\|)) > 0$ and thus by Lemma 7.2, we get that $\mathcal{L} \subset \mathcal{N}'$, which is a contradiction. It follows that $\mu_n(Z(\mathcal{N}', R)) = 0$ and thus by Birkhoff's Theorem, one can find a point $q_n \in \mathcal{M} \smallsetminus Z(\mathcal{N}', R)$ such that

$$\mu_n(\mathcal{U}) = \lim_{T\to\infty} \frac{1}{T}|\{s \in [0,T] : u_s q_n \in \mathcal{U}\}|.$$

Applying Proposition 5.6 to $L = \{q_n\}$ shows that for all T large enough,

$$\frac{1}{T}|\{s \in [0,T] : u_s q_n \in \mathcal{U}\}| < \varepsilon.$$

This shows that $\mu_n(\mathcal{U}) \leqslant \varepsilon$.

Suppose now that $\mathcal{L} \subset \mathcal{N}'$. By definition of $\mathcal{N}$, we know that $x \notin Z_{\mathcal{N}'}$. It thus follows from Proposition 5.3 that the map

$$\Phi : \mathcal{N}'_\infty \times \mathbb{R} \to \mathcal{N}, \qquad \Phi(q,x) \overset{\text{def}}{=} \mathrm{Rel}_{tx}(q)$$

is injective. This means that for any t_n large enough such that $t_n\|x\| > R$, the surface $q_n = \mathrm{Rel}_{t_n x}(q'_n)$ does not belong to $Z(\mathcal{N}, R)$ and thus applying once more Proposition 5.6 to $L = \{q_n\}$ shows that $\mu_n(\mathcal{U}) \leqslant \varepsilon$ for n large enough.

To deduce now that $\nu = m_{\mathcal{N}}$, we use the ergodic decomposition argument which we used in Step 4 of the Proof of Theorem 6.6. □

Theorem 7.3. *Suppose $\mathcal{M}$ satisfies the weak classification of U-invariant measures and let μ be a U-invariant ergodic Radon probability measure on $\mathcal{M}$. Then there is an invariant subvariety $\mathcal{N} \subset \mathcal{M}$ such that $g_{t*}\mu$ converges to $m_{\mathcal{N}}$ as $t \to \infty$. If we furthermore assume that $\mu(\mathcal{M}_\infty) = 1$, then there is an invariant subvariety $\mathcal{L} \subset \mathcal{N}$ such that $g_{t*}\mu$ converges to $m_{\mathcal{L}}$ as $t \to -\infty$.*

Remark. If μ is an ergodic U-invariant measure with $\mu(\mathcal{M}_\infty) < 1$ then by ergodicity μ-a.e $q \in \mathcal{M}$ has a horizontal saddle connection and $g_{t*}\mu$ converges to the zero measure as $t \to -\infty$.

Proof of Theorem 7.3. We first prove the first assertion. We assume first that for any invariant subvariety $\mathcal{N} \subsetneq \mathcal{M}$, we have $\mu(\mathcal{N}) = 0$. In that case, we are going to show that $g_{t*}\mu$ converges to $m_{\mathcal{M}}$ as $t \to \infty$. Let $\mu_n = g_{t_n*}\mu$ be a convergent subsequence when $t \to \infty$ and let ν be its limit. If generic surfaces for μ have some horizontal saddle connections, let r be the length of the shortest of them. Otherwise, let $r = 1$.

Step 1. We show that $\nu(\mathcal{M}) = 1$. Let $\varepsilon > 0$. It is enough to show that there is a compact K such that $\mu_n(K) \geqslant 1 - \varepsilon$. Let K be a compact as in item (II) of Theorem 5.1 with r, ε as above. The measure μ_n assigns measure zero to surfaces with a horizontal saddle connection shorter than r and there is a surface q_n that is Birkhoff generic for the indicator function $\mathbf{1}_K$ with respect to μ_n. By definition of K, for any T large enough,

$$\frac{1}{T}|\{s \in [0,T] : u_s q_n \in K\}| \geqslant 1 - \varepsilon.$$

By Birkhoff genericity, the quantity on the left-hand side converges to $\mu_n(K)$ as $T \to \infty$ and thus $\mu_n(K) \geqslant 1 - \varepsilon$.

Step 2. We now show that $\nu(\mathcal{M} \smallsetminus \mathcal{M}_\infty) = 0$. For any $i > 0$, let K_i be the closure of the set of surfaces whose saddle connection lengths are bounded below by $\frac{1}{i}$, and which have at least one horizontal saddle connection, with the lengths of all horizontal saddle connections bounded above by i. By construction the $(K_i)_{i>0}$ exhaust $\mathcal{M} \smallsetminus \mathcal{M}_\infty$ and thus it is enough to show that for any $i > 0$ and any $\varepsilon > 0$, there is an open set $\mathcal{U}$ that contains K_i and such that $\mu_n(\mathcal{U}) \leqslant \varepsilon$ for any n large enough. Once again, let K be as in item (II) in Theorem 5.1 for r as above. Since the length of saddle connections is bounded below on compact sets and K is compact, there is a $\tau > 0$ such that $g_\tau(K_i) \subset K^c$, so that

$$K_i \subset \mathcal{U} \stackrel{\mathrm{def}}{=} g_\tau(K^c).$$

Let n be large enough so that $t_n > \tau$. We have $\mu_n(\mathcal{M}_{re^{t_n}}) = 0$ and thus there is a surface q_n without horizontal saddle connections shorter than re^{t_n} that is Birkhoff generic for $\mathbf{1}_{K^c}$ with respect to μ_n. The surfaces $q_n' \stackrel{\mathrm{def}}{=} g_{-\tau} q_n$ do not have horizontal saddle connections shorter that r and thus there is $T_0 > 0$ such that for any $T > T_0$,

$$\frac{1}{T} |\{s \in [0, T] : u_s q_n' \notin K\}| < \varepsilon.$$

Now let T_1 be large enough so that $e^{-2t} T_1 > T_0$. For any $T > T_1$ we have

$$\begin{aligned} \frac{1}{T} \int_0^T \mathbf{1}_{\mathcal{U}}(u_s q_n)\, ds &= \frac{1}{T} \int_0^T \mathbf{1}_{K^c}(g_{-\tau} u_s q_n)\ ds \\ = & \frac{1}{Te^{-2\tau}} \left| \left\{ s \in \left[0, Te^{-2\tau}\right] : u_s q_n' \notin K \right\} \right| < \varepsilon. \end{aligned}$$

By Birkhoff genericity the quantity on the left-hand side converges to $\mu_n(\mathcal{U})$ as $T \to \infty$ and thus $\mu_n(\mathcal{U}) \leqslant \varepsilon$.

Step 3. We now show that for any invariant subvariety $\mathcal{N} \subsetneq \mathcal{M}$ and $r > 0$ we have $\nu(Z(\mathcal{N}, r)) = 0$. It is enough to show that for any compact subset K of $\mathcal{N}_\infty$ and $\varepsilon > 0$, there is a open set $\mathcal{U}$ that contains $Z(K, r)$ and such that $\mu_n(\mathcal{U}) < \varepsilon$ for any n large enough. We claim that there is $r_0 > 0$ such that $\mu(Z(\mathcal{N}, r_0)) = 0$. Indeed, suppose that for any $n > 1$, we have $\mu(Z(\mathcal{N}, \frac{1}{n})) > 0$. Since μ is ergodic and the sets $Z(\mathcal{N}, r)$ are U-invariant, we actually have $\mu\left(Z(\mathcal{N}, \frac{1}{n})\right) = 1$ and thus $\mathcal{N} = \bigcap_{n>1} Z(\mathcal{N}, \frac{1}{n})$ also has full measure, contradicting the assumption on μ made at the beginning of the proof. This proves the claim.

According to Proposition 5.6, there is $R > r$ and a neighborhood $\mathcal{U}$ such that for any compact subset L of $\mathcal{M} \smallsetminus Z(\mathcal{N}, R)$, there is $T_0 > 0$ such that for any $T > T_0$,

$$\frac{1}{T}\ |\{s \in [0, T] : u_s q \in \mathcal{U}\}| < \varepsilon.$$

Let n be large enough so that $e^{-t_n} R < r_0$ and let q_n be a generic point for the measure μ_n. If $q_n \in Z(\mathcal{N}, R)$ then the measure μ is supported on $Z(\mathcal{N}, e^{-t_n} R)$ and thus $\mu(Z(\mathcal{N}, r_0)) = 1$, a contradiction. Thus we can apply Proposition 5.6 to $L = \{q_n\}$ to find T_0 such that for any $T > T_0$, we have

$$\frac{1}{T}\ |\{s \in [0, T] : u_s q_n \in \mathcal{U}\}| < \varepsilon.$$

By the Birkhoff Ergodic Theorem, the left-hand side converges to $\mu_n(\mathcal{U})$ when T tends to $+\infty$ and thus $\mu_n(\mathcal{U}) < \varepsilon$ for any n large enough. We prove as in the final step of the proof of Theorem 6.6 that $\nu = m_{\mathcal{M}}$.

General case. Suppose now μ is any U-invariant ergodic Radon probability measure and consider the collection $\mathcal{C}(\mu)$ of all the invariant subvarieties that have positive measure for μ. Then in fact, by ergodicity, $\mu(\mathcal{N}) = 1$ for any $\mathcal{N} \in \mathcal{C}(\mu)$. Let $\mathcal{N}_0 \in \mathcal{C}(\mu)$ of smallest dimension. We claim that there is no $\mathcal{N}_1 \in \mathcal{C}(\mu)$ which is properly contained in $\mathcal{N}_0$. Indeed, if $\mathcal{N}_1 \in \mathcal{C}(\mu)$ with $\mathcal{N}_1 \subset \mathcal{N}_0$ then $\dim \mathcal{N}_1 = \dim \mathcal{N}_0$ by choice of $\mathcal{N}_0$, so that $\mathcal{N}_1$ is open in $\mathcal{N}_0$. This implies that $m_{\mathcal{N}_0}(\mathcal{N}_1) > 0$, which implies by ergodicity and G-invariance that $m_{\mathcal{N}_0} = m_{\mathcal{N}_1}$ and hence $\mathcal{N}_0 = \mathcal{N}_1$. Thus we can now apply the first part of the proof to $\mathcal{N}_0$ in place of $\mathcal{M}$, which proves that $g_{t*}\mu$ converges to $m_{\mathcal{N}_0}$.

We now prove the second assertion of the statement. We assume that $\mu(\mathcal{M}_\infty) = 1$. Since $\mathcal{M}$ satisfies the weak classification of U-invariant measures, it follows that there is an invariant subvariety $\mathcal{L} \subset \mathcal{M}$ and $x \in Z_{\mathcal{M}}$ such that $\mu = \mathrm{Rel}_{x*} m_{\mathcal{L}}$. Notice that for any $t \in \mathbb{R}$, we have $g_{t*}\mu = \mathrm{Rel}_{e^t x *} m_{\mathcal{L}}$. It follows from Theorem 7.1 and the first part of the proof that $\mathcal{L} \subset \mathcal{N}$. Finally, it is clear that $g_{t*}\mu$ converges to $m_{\mathcal{L}}$ when $t \to -\infty$. □

Theorem 7.4. *Suppose $\mathcal{M}$ satisfies the weak classification of horocycle invariant measures. Let $q \in \mathcal{M}$, $\mathcal{N} \stackrel{\mathrm{def}}{=} \overline{Gq} \subset \mathcal{M}$,*

$$\mu_q \stackrel{\mathrm{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} r_{\theta *}\delta_q \, d\theta,$$

and let $\mu_{t,q}$ denote the 'pushed circle average' $g_{t*}\mu$. Then $\mu_{t,q} \underset{t\to\infty}{\longrightarrow} m_{\mathcal{N}}$.

Proof. Up to replacing $\mathcal{M}$ by $\overline{Gq}$, we may assume that q is not contained in any invariant subvariety $\mathcal{N} \subsetneq \mathcal{M}$. Let ν be a weak-$*$ limit of a convergent subsequence $(\mu_{t_n,q})_{n>0}$. We want to show that $\nu = m_{\mathcal{M}}$.

Step 1. We want to show that $\nu(\mathcal{M}) = 1$. It is enough to show that for any $\varepsilon > 0$, there is a compact $K \subset \mathcal{M}$ such that $\nu(K) > 1 - \varepsilon$. This follows from [EM01, Theorem 5.2].

Step 2. We now want to show that $\nu(\mathcal{M} \smallsetminus \mathcal{M}_\infty) = 0$. Let $L_i \subset \mathcal{M} \smallsetminus \mathcal{M}_\infty$ be the set of surfaces whose saddle connections have length bounded below by $\frac{1}{i}$, and which have horizontal saddle connections, all of whose lengths are bounded above by i. The $(L_i)_{i>0}$ are compact and exhaust $\mathcal{M} \smallsetminus \mathcal{M}_\infty$, and thus it is enough to show that for any $\varepsilon > 0$ and $i > 0$, there is a open set $\mathcal{U}$ that contains L_i and such that $\mu_{t,q}(\mathcal{U}) < \varepsilon$ for any t large enough. It follows once again from [EM01, Theorem 5.2] that there are $t_1 > 0$ and $\delta > 0$ such that for any $t > t_1$, we have $\mu_{t,q}(C_\delta) < \varepsilon$, where C_δ is the open set of surfaces that have a saddle connection strictly shorter than δ. Let $t_2 = \log(\frac{\delta}{2}) - \log(i)$ and notice that $g_{t_2} L_i \subset C(\delta)$. Let $\mathcal{U} = g_{-t_2}(C(\delta))$. For any $t > t_1 - t_2$, we have

$$\mu_{t,q}(\mathcal{U}) = \mu_{t+t_2,q}(C(\delta)) < \varepsilon.$$

Step 3. We now want to show that for any invariant subvariety $\mathcal{N} \subsetneq \mathcal{M}$ and any $r > 0$, we have $\nu(Z(\mathcal{N}, r)) = 0$. It is enough to show that for any compact subset K of $\mathcal{N}_\infty$ and $\varepsilon > 0$, there is a open set $\mathcal{U}$ that contains $Z(K, r)$ and such that $\mu_{t,q}(\mathcal{U}) < \varepsilon$ for any t large enough. This is exactly Proposition 5.12.

It follows that ν is a U-invariant probability measure that assigns measure zero to the set of surfaces with horizontal saddle connections as well as to any Rel push of any invariant subvariety $\mathcal{N} \subsetneq \mathcal{M}$. It follows by ergodic decomposition, and using the weak classification of U-invariant measures, that $\nu = \mu_{\mathcal{M}}$. □

By a well-known argument developed in [EM01] (see also [Esk06]), this implies

Theorem 7.5. *Suppose $\mathcal{M}$ satisfies the weak classification of horocycle invariant measures. Let $q \in \mathcal{M}$ and for any $L > 0$, denote by $N_L(q)$ the number of saddle connection of length at most L on q. Then there is a constant $C > 0$, depending only on $\overline{Gq}$, such that*

$$\frac{1}{L^2} N_L(q) \underset{L\to\infty}{\longrightarrow} C.$$

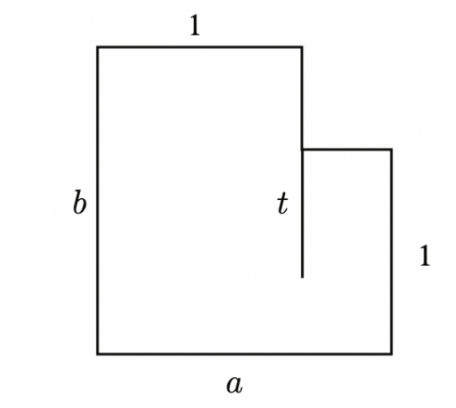

FIGURE 1. A Bainbridge billiard table (reprinted with permission from [Bai10]).

Remark 7.6. *The results of §5, §6 and §7 all hold for products of strata, with the same proofs.*

8. A counting problem

Let a, b be two real irrationals, both greater than one, and belonging to the same quadratic field $\mathbb{K}$. Suppose also that $a + b' = 1$, where b' is the conjugate of b (that is, the image of b under the nontrivial field automorphism of $\mathbb{K}$). Let $t \in (0, 1)$, and consider the billiard table $\mathcal{P} = \mathcal{P}(a, b, t)$ depicted in Figure 1. Using the strategy developed in [EM01], and using a suitable version of Theorem 7.4, it was shown in [Bai10] (see also [BSW16]) that the holonomy vectors of generalized diagonals and periodic trajectories in $\mathcal{P}$ satisfy quadratic asymptotics. Using the results of this paper, we are able to obtain the following more refined counting result:

Corollary 8.1. *Let a, b, t be as above and let $\mathcal{P} = \mathcal{P}(a, b, t)$. Let $p_0 \in \mathcal{P}$ be any point besides the bottom of the slit, and let $N_L(\mathcal{P}, p_0)$ denote the periodic trajectories for the billiard flow on P, passing through p_0 and of length at most L. Then there is a constant $C = C(p_0) > 0$ such that*

$$\frac{1}{L^2} N_L(\mathcal{P}, p_0) \underset{L\to\infty}{\longrightarrow} C. \tag{59}$$

The constant $C(p_0)$ can take only one of finitely many values.

Remark 8.2. *There are different conventions in the literature regarding continuing billiard trajectories through vertices. In some papers, a billiard trajectory ends at any vertex, whereas in others, it is continued continuously when it passes through a vertex of angle π/n for some positive integer n. In the above statement of Corollary 8.1 we have followed the latter convention (as was also done in* [Bai10]*): billiard trajectories are continued through all the points of $\mathcal{P}$, except for the point at the bottom of the slit. If one uses the other convention, a similar results hold, but for finitely many p_0 (e.g., the vertices of $\mathcal{P}$) we will have $C(p_0) = 0$.*

Remark 8.3. *One of the main contributions of* [Bai10] *is the computation of the quadratic growth constant $C' = C'(a, b, t)$ for the growth of saddle connections and* *cylinders. With some additional effort, which we omit, it is also possible to compute the constant C which arises in Corollary 8.1 (for periodic trajectories through p_0), for any possible choice of p_0. This requires an understanding of periodic sets on the surface obtained by unfolding $\mathcal{P}$; such periodic sets were completely classified in* [Api].

Proof. Let M be the unfolding of $\mathcal{P}$ (see e.g. [MT02]). Recall that M is obtained by gluing to each other the images of $\mathcal{P}$ under the action of the dihedral group Δ generated by reflections in the sides of P, which is a group of cardinality 4. Thus there is a projection $\pi_{\mathcal{P}} : M \to \mathcal{P} = M/\Delta$, and p_0 has either one, two or four pre-images, depending on whether it is a vertex of $\mathcal{P}$, belongs to the interior of $\mathcal{P}$ or lies on an edge. Assume in what follows that p_0 has four pre-images $\bar{p}_1, \ldots, \bar{p}_4$, the other cases being similar, and let M_0 denote the translation surface obtained from M by marking the points $\bar{p}_i$. A billiard trajectory on $\mathcal{P}$ is periodic and passes through p_0, if and only if its preimage under $\pi_{\mathcal{P}}$ is a union of four saddle connections from some $\bar{p}_i$ to some $\bar{p}_j$, which are images of each other under the action of Δ. Thus our counting problem on $\mathcal{P}$ reduces to a counting problem of a configuration of saddle connections on M_0, and following the strategy of [EM01] and Theorem 7.4, it is enough to show that M_0 belongs to an invariant subvariety of rank one satisfying the weak classification of horocycle invariant measures.

The technique of [EM01] will show that $C > 0$ provided the set of saddle connections we are counting is not empty, or equivalently, that there are indeed periodic trajectories on $\mathcal{P}$ passing through p_0. We leave it to the reader to check that such periodic trajectories indeed exist, through any p_0 except the bottom of the slit.

It is easy to see that $M \in \mathcal{H}(1,1)$, and hence (in the case in which p_0 has four pre-images), that

$$M_0 \in \mathcal{H}(1^2, 0^4) = \mathcal{H}(1,1,0,0,0,0).$$

It was shown in [Bai10], using results of McMullen [McM05], that M belongs to an eigenform locus $\mathcal{E}_D$ of discriminant D, where D is a positive integer satisfying $\mathbb{K} = \mathbb{Q}(\sqrt{D})$. Note that $\mathcal{E}_D$ is a rank-one locus of Rel-dimension one. Also note that surfaces obtained by marking points on surfaces in rank-one loci, also belong to rank-one loci; that is, with the notation of §4, $\mathcal{M}$ is of rank one if and only if so is $p_{\Sigma_0}(\mathcal{M})$. This implies via Corollary 4.5 that M_0 belongs to an invariant subvariety of rank one satisfying the weak classification of horocycle invariant measures. This completes the proof. □

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