**Exercise sheet – Random walks on homogeneous spaces.**
**Tel Aviv University, Spring 2016**

**1.** Let $\Gamma$ be the semigroup generated by the two matrices

$$\begin{pmatrix} 1 & 10 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}.$$

Prove that for any $(x_1, x_2) \in \mathbb{T}^2 \smallsetminus \mathbb{Q}^2$, the trajectory $\Gamma x$ is dense in $\mathbb{T}^2$.

**2.** Let $X$ be a second countable topological space, let $\mathcal{B}$ be the Borel $\sigma$-algebra, let $\mu$ be a measure on $\mathcal{B}$ and suppose that for any non-empty open set $\mathcal{O}$, $\mu(\mathcal{O}) > 0$. Let $T : X \to X$ be an ergodic measure-class preserving transformation (not necessarily invertible, not necessarily measure preserving). Suppose $T$ is *conservative*, i.e., for any $A \in \mathcal{B}$ with $\mu(A) > 0$, there is a positive integer $n$ such that $\mu(A \cap T^{-n}(A)) > 0$. Prove that for $\mu$-a.e. $x$, the orbit $\{T^n x : n = 1, 2, \ldots\}$ is dense in $X$.

**3.** Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$. Let $T(x) = x + \alpha$ and $\phi_t(x) = x + t\alpha$ (addition on $\mathbb{T}^d$). Let $m$ be Haar measure on $\mathbb{T}^d$.

a) Prove that $(\mathbb{T}^d, T)$ is uniquely ergodic, with invariant measure $m$, if and only if $1, \alpha_1, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$.
b) Prove that the $\mathbb{R}$-action $(t, x) \mapsto \phi_t(x)$ is uniquely ergodic, with invariant measure $m$, if and only if $\alpha_1, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$.

**4.** Some things we left out of the proof of the pointwise ergodic theorem (you may assume what was proved in class): Let $(X, \mathcal{B}, \mu, T)$ be a probability-preserving system and suppose $T$ is invertible. Prove that for $\mu$-a.e. $x \in X$, the two limits $f^* = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(T^i x)$ and $f^{**} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(T^{-i} x)$ exist, coincide, and we have $f^* \in L^1(\mu)$ with $\int_X f d\mu = \int_X f^* d\mu$. Conclude that for $\mu$-a.e. $x$, $f^*(x) = \mathbb{E}(f|\mathcal{A})(x)$, where $\mathcal{A} = \{B \in \mathcal{B} : T^{-1}(B) = B\}$.

**5.** a. Let $G$ be a countable group which is the increasing union of finite groups $G_n$, that is $G = \bigcup G_n$, each $G_n$ is a finite subgroup of $G$, and $G_1 \subset G_2 \subset \cdots$. Let $(X, \mathcal{B}, \mu)$ be a probability space with a $G$-action, so that $G$ preserves the measure $\mu$ (that is $g_*\mu = \mu$ for each $g \in G$) and the action is ergodic (if $A \in \mathcal{B}$ with $gA = A$ for all $g \in G$, then $\mu(A) \in \{0, 1\}$). Prove that for all $f \in L^1(\mu)$, for $\mu$-a.e. $x \in X$,

$$\frac{1}{\#G_n} \sum_{g \in G_n} f(gx) \to_{n\to\infty} \int_X f d\mu.$$

b. State and prove a pointwise ergodic theorem for the additive group of $\mathbb{Q}_p$.

**6.** Let $\alpha \notin \mathbb{Q}$ and define $T : \mathbb{T} \to \mathbb{T}$ by $T(x) = x + \alpha$ (addition mod 1). Let $A$ be the projection of the interval $[0, 1/2]$ to $\mathbb{T}$. Prove that

$$\{x \in \mathbb{T} : \forall n \in \mathbb{N}, \#\{i \leqslant n : T^i x \in A\} > \#\{i \leqslant n : T^i x \notin A\}\}$$

has Haar measure 0.

**7.** In the following examples, $\mu$ is a probability measure on $G = \mathrm{SL}_2(\mathbb{R})$ supported on two matrices $a, b$, where $G$ acts on $V = \mathbb{R}^2$ by matrix multiplication. In each example, determine whether or not $\mu$ irreducible, strongly irreducible, proximal.

(a) $a, b$ are the two matrices in question 1.

(b) $a = \begin{pmatrix} 5 & 0 \\ 0 & 1/5 \end{pmatrix}$, $b = \begin{pmatrix} 5 & 1 \\ 0 & 1/5 \end{pmatrix}$.

(c) $a = \begin{pmatrix} 5 & 0 \\ 0 & 1/5 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 5 \\ -1/5 & 0 \end{pmatrix}$.

(d) $a = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

**8.** Let $G$ act on a vector space $V$, let $\mu$ be a compactly supported measure on $V$ which is irreducible (i.e. for any proper $W \subset V$, $\mu(\{g \in G : g(W) = W\}) < 1$), and let $\Gamma$ be the semigroup generated by $\operatorname{supp} \mu$. Prove that there is a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_r$ such that $\{V_1, \ldots, V_r\}$ are permuted by the action of $\mu$-a.e. $g$, and for each $i$, there is no proper $V_i' \subset V_i$ such that $\{gV_i' : g \in \Gamma\}$ is finite.

**9.** Let $G$ be an lcsc group and $\Gamma \subset G$ a closed semigroup. Suppose that $\Gamma$ is compact, and prove that $\Gamma$ is a subgroup. Show by example that $\Gamma$ might not be a group if one does not assume it is compact.

**10.** Suppose $G$ is lcsc, $\mu$ is a probability measure on $\mu$, $\Gamma_u$ is the closure of the semigroup generated by $\operatorname{supp} \mu$, $X$ is a compact $G$ space and $\nu$ is a $\mu$-stationary Borel probability measure on $X$. Prove that:

(a) If $G$ is abelian then $\nu$ is $\Gamma_\mu$-invariant.

(b) If $X$ if finite then $\nu$ is $\Gamma_\mu$-invariant.

**Prove or disprove:** If $\nu$ is purely atomic (i.e., there is a sequence of points $x_1, x_2, \ldots$ in $X$ such that $\nu(X \smallsetminus \bigcup_i \{x_i\}) = 0$) then $\nu$ is $\Gamma_\mu$-invariant.

**11.** Let $G = \mathrm{SL}_2(\mathbb{R})$, $V = \mathfrak{g}$ be the Lie algebra of $G$ (i.e. $2 \times 2$ real matrices with trace zero), where $G$ acts on $V$ by conjugation. Let $W$ be the span of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and for $a > 1$, $b_1 \neq b_2$, let

$$g_i = \begin{pmatrix} a & b_i \\ 0 & 1/a \end{pmatrix}, \quad i = 1, 2.$$

Let $p_1, p_2 > 0$ with $p_1 + p_2 = 1$ and let $\mu = p_1\delta_{g_1} + p_2\delta_{g_2}$. Let $B = (\operatorname{supp}\mu)^{\mathbb{N}}$ and $\beta = \mu^{\otimes\mathbb{N}}$. Let dist denote the metric on the projective space, where the distance between two lines is the minimal Euclidean distance of a norm one vector in one line, from the other line. Prove:

(i) For any $\alpha > 0$ there are $n_0 \geqslant 1$, $\varepsilon > 0$ such that for any $v \in V \setminus \{0\}$,

$$\beta\left(\{b \in B : \forall n \geqslant n_0, \|b_n \cdots b_1 v\| \geqslant \varepsilon \|b_n \cdots b_1\| \, \|v\|\}\right) \geqslant 1 - \alpha.$$

(ii) For any $\alpha > 0$ and $\eta > 0$ there is $n_0 \geqslant 1$, such that for any $v \in V \setminus \{0\}$,

$$\beta\left(\{b \in B : \forall n \geqslant n_0, \operatorname{dist}(\mathbb{R} b_n \cdots b_1 v, W) \leqslant \eta\}\right) \geqslant 1 - \alpha.$$

**12.** In this exercise we will carry out 'Step I' of the proof of the Benoist-Quint theorem, in the case that $X = \mathbb{R}^d/\mathbb{Z}^d$ is a torus. Here $\mu$ is a compactly supported measure on $G = \mathrm{SL}_d(\mathbb{R})$, such that the group generated by $\operatorname{supp}\mu$ is Zariski dense in $G$, $\nu$ is a non-atomic $\mu$-stationary measure, and the goal is to prove that $\mu = m$ where $m$ is the Haar measure on $X$. We assume that $\mu$ is proximal (in fact this follows from our other hypotheses but we did not prove this in class). For any $b = (b_1, b_2, \ldots) \in B = (\operatorname{supp}\mu)^{\mathbb{N}}$, let $W_b$ denote the limit of the images of $\left(\frac{b_1 \cdots b_n}{\|b_1 \cdots b_n\|}\right)$ and let $\nu_b = \lim_{n\to\infty}(b_1 \cdots b_n)_* \nu$. Let $\beta = \mu^{\otimes\mathbb{N}}$ and $\beta^X = \int_B \delta_b \otimes \nu_b d\beta(b)$.

Show that: if for $\beta$-a.e. $b$, $\nu_b$ is $W_b$-invariant, then $\nu = m$.

*Hints.* a. Let $\mathcal{F}$ denote the set of measures on $X$ which are invariant and ergodic under a line in $\mathbb{R}^d$. Use Ex. 3 to show that $\mathcal{F}$ is a countable union of compact sets of measures.

b. Let $\nu_{b,x}$ be the disintegration of $\beta^X$ along the map $(b, x) \mapsto (b, W_b)$ (that is, into conditional measures for the $\sigma$-algebra obtained by pulling back the Borel $\sigma$-algebra on $B \times \mathbb{P}(\mathbb{R}^d)$ using this measurable map). Write the ergodic decomposition of $\nu_{b,x}$ into $W_b$-ergodic components as $\nu_{b,x} = \int_X \zeta(b, x') \, d\nu_{b,x}(x')$. Prove that $\eta = \zeta_* \beta_X$ is a $\mu$-stationary measure on $\mathcal{F}$. Apply Ex. 10.

**13.** Suppose $\{g_n\}$ is an unbounded sequence in $\mathrm{SL}_d(\mathbb{R})$ for some $d \geqslant 2$, and let $\nu_1, \nu_2$ be two Borel probability measures on the projective space $\mathbb{P}(\mathbb{R}^d)$ such that $(g_n)_*\nu_1 \to_{n\to\infty} \nu_2$. Prove that there are two proper linear subspaces $V_1, V_2$ of $\mathbb{R}^d$ such that $\nu_2$ is supported on $[V_1] \cup [V_2]$, where $[V]$ denotes the image in $\mathbb{P}(\mathbb{R}^d)$ of $V$.