Ratner's theorems for dynamics on homogeneous spaces, exercises

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Let $G_1 = G_2 = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, $G_3 = \mathrm{SL}_3(\mathbb{R})$, $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$, $\Gamma_3 = \mathrm{SL}_3(\mathbb{Z})$, and define $\Gamma_2 \subset G_2$ as follows. Let $k = \mathbb{Q}(\sqrt{2})$, $(a+b\sqrt{2})' = a - b\sqrt{2}$ (where $a, b \in \mathbb{Q}$), for a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in k$, let $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ and let

$$\Gamma_2 = \{(A, A') : A \in \mathrm{SL}_2(k)\}.$$

1. Prove that Γ_i is a lattice in G_i for $i = 1, 2, 3$.

Hint: In the first semester we outlined a proof for $i = 3$, using nondivergence of unipotent flows. Explain this proof and adapt this proof for cases $i = 1, 2$.

2. Let Id be the identity element (of any group). Define

$$u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \ g(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

and

$$u_1(t) = (u(t), u(t)), \ g_1(t) = (g(t), g(t)), \ u_2(t) = (u(t), \mathrm{Id}), g_2(t) = (g(t), \mathrm{Id}).$$

Also let

$$u_3(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \ g_3(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}.$$

Using Ratner's orbit-closure theorem, identify all possible orbit-closures for the action of $\{u_i(t)\}$ on G_i/Γ_i.

4. Let $\pi : G_i \to G_i/\Gamma_i$ be the natural projection. Give a diophantine condition that is equivalent to the statement that $\{g_1(t)\pi(u(x), u(y)) : t \geq 0\}$ is divergent in G_1/Γ_1. State this in terms of the continued fraction expansions of x and y. Show that there are uncountably many pairs (x, y) such that $\{g_1(t)\pi(u(x), u(y)) : t \geq 0\}$ is divergent.

5. A trajectory $\{g_i(t)\pi(x) : t \geq 0\}$ in G_i/Γ_i is *obvious* if there is a representation $\rho : G \to \mathrm{GL}(V)$ defined over $\mathbb{Q}$, and $v \in V(\mathbb{Q})$ such that $\rho(g_i(t)x)v \to_{t\to\infty} 0$ in V (the rationality conditions mean that we can fix a basis of V such that with respect to this basis, v has rational coefficients, and that for the map $g \mapsto \rho(g) \in \mathrm{AUT}(V)$, the matrix coefficients of $\rho(g)$ can be expressed by polynomials in the coefficients of g with rational coefficients). Show that obvious trajectories are divergent. Explain what the obvious trajectories are in G_1/Γ_1, and show that in both $G_1/\Gamma_1, G_3/\Gamma_3$ there are uncountably many non-obvious divergent trajectories.

6. Show that in G_2/Γ_2 all divergent trajectories are obvious.

7. Write down the steps of proof of the Ratner measure classification result completely in one of the three cases $i = 1, 2, 3$. Explain which of the cases treated in class can be avoided in this case.