Bourgain's Projection Method as a Black Box

Bootstrap for Bénard–He–Zhang

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Goal of this talk

- The paper we are studying uses a Bourgain-type projection theorem as a black box.
- I want to explain what this black box does, not prove the full theorem.
- The main idea is:

$$\text{positive dimension} \quad \Longrightarrow \quad \text{larger dimension after projection/slicing.}$$

- This dimension growth is then used to prove effective equidistribution.

T. Bénard, W. He, and H. Zhang.
Khintchine dichotomy for self-similar measures.
arXiv:2409.08061.

T. Bénard and W. He.
Multislicing and effective equidistribution for random walks on some homogeneous spaces.
arXiv:2409.03300.

W. O'Regan, P. Shmerkin, and H. Wang.
Simple proofs of discretised projection theorems.
arXiv:2511.21656.

J. Bourgain.
The discretized sum-product and projection theorems.
Journal d'Analyse Mathématique, 112:193–236, 2010.

More about projection theory:
https://ocw.mit.edu/courses/18-156-projection-theory-spring-2025/
P. Shmerkin's lecture 1 of 3:
https://www.youtube.com/watch?v=0UYCJ0cjG9k

The paper we are studying

Bénard–He–Zhang

Khintchine dichotomy for self-similar measures

Its main Diophantine result is a Khintchine-type theorem for all self-similar probability measures on $\mathbb{R}$.

The dynamical input is an effective equidistribution theorem for expanding fractal measures on

$$X = \mathrm{SL}_2(\mathbb{R})/\Lambda.$$

The main dynamical theorem

Let σ be a self-similar probability measure on $\mathbb{R}$. For

$$a(t) = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}, \qquad u(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},$$

Bénard–He–Zhang prove that there exists $c > 0$ such that

$$\int_{\mathbb{R}} f(a(t)u(s)x)\, d\sigma(s) = \int_X f\, dm_X + O\big(\mathrm{inj}(x)^{-1} S_{\infty,1}(f) t^{-c}\big).$$

The expanded fractal measure converges exponentially fast towards Haar measure on a class of C^∞-functions.

Random-walk version

They prove the dynamical theorem through a random-walk statement. Let μ be a probability measure on the upper triangular group

$$\{a(t)u(s) : t > 0,\ s \in \mathbb{R}\} \subset \mathrm{SL}_2(\mathbb{R}).$$

Assume:

- the support is not simultaneously diagonalizable;
- the walk expands in one direction:

$$\int_G \log \|ge_1\| \, d\mu(g) > 0.$$

Then there exists $c > 0$ such that

$$\mu^{*n} * \delta_x(f) = m_X(f) + O\left(\mathrm{inj}(x)^{-1} S_{\infty,1}(f) e^{-cn}\right).$$

Where the projection method enters

The proof has three phases:

1. **Positive dimension:** the random walk first gains a small amount of dimension.
2. **Dimension bootstrap:** this small dimension is repeatedly improved until it is almost full dimension.
3. **Spectral gap:** once the measure has high dimension, spectral methods give equidistribution.

The Bourgain-type projection theorem is used in Phase 2: the dimension bootstrap.

"pulling yourself up by your bootstraps"

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"To succeed or solve a problem using only your own efforts and tools, without any outside help."

Positive dimension: first input

The first step says that for some $\kappa > 0$ and $A > 0$, for every small $\rho > 0$,

$$n \geq |\log \rho| + A|\log \mathrm{inj}(x)|$$

implies

$$\mu^{*n} * \delta_x(B_\rho y) \lesssim \rho^\kappa \qquad \text{for all } y \in X.$$

After enough time, the random walk distribution is not concentrated in a tiny ball. It has some positive dimension at scale ρ.

Dimension bootstrap: the black box

The goal is to upgrade

$$\mu^{*n} * \delta_x(B_\rho y) \lesssim \rho^\kappa$$

to a much stronger estimate, roughly

$$\mu^{*n} * \delta_x(B_\rho y) \lesssim \rho^{3-\varepsilon}.$$

Since

$$\dim \mathrm{SL}_2(\mathbb{R}) = 3,$$

this means the measure becomes almost full-dimensional.

The improvement in the dimension comes from Bourgain projection ideas.

Multislicing: the Bourgain-type black box

The exact tool used is a **multislicing theorem**.
Very roughly:

Black Box (Multislicing)

Suppose a measure has dimension α at scale ρ.
If a family of nonlinear "rectangles" is sufficiently non-concentrated in direction, then for most directions, the measure does not concentrate too much inside those rectangles.
Consequently, after applying the random walk, the dimension improves from α to $\alpha + \varepsilon$.

This is a nonlinear, higher-rank version of Bourgain's discretized projection theorem.

The rectangles

In the $\mathrm{SL}_2(\mathbb{R})$ setting, the relevant rectangles have side lengths

$$1, \qquad \rho^{1/2}, \qquad \rho.$$

So the model rectangle is

$$R_\rho = [0,1]e_1 + [0,\rho^{1/2}]e_2 + [0,\rho]e_3 \subset \mathbb{R}^3.$$

Here there are several different scales at once. That is why it is called *multislicing*. This is an estimate on the number of occurrences between balls and thin tubes

Simplified statement of the black box

Let ν be a measure on a unit ball in $\mathbb{R}^3$.
Assume ν has dimension α at scales between ρ and ρ^{ε}:

$$\nu(Q_r) \leq r^{3\alpha}.$$

Then, outside a small exceptional set of parameters, there is a large subset A_θ such that

$$\nu|_{A_\theta}\big(\varphi_\theta^{-1}(Q + R_\rho)\big) \leq \rho^{\frac{3}{2}\alpha+\varepsilon}.$$

Most nonlinear slices/rectangles do not capture too much mass. This is the mechanism that forces dimension growth.

How this gives dimension growth

The random walk sends balls backward to distorted rectangles.
So to bound

$$\mu^{*n} * \nu(B_\rho y),$$

we look at

$$\nu(g^{-1} B_\rho y).$$

For typical g, the set $g^{-1} B_\rho y$ looks like a nonlinear rectangle.

A slicing estimate for rectangles gives an upper bound for ball masses after applying the random walk.

The role of Bénard–He

Bénard–He developed a multislicing theorem for effective equidistribution of random walks on certain homogeneous spaces.

Their theorem generalizes Bourgain's discretized projection theorem and its later nonlinear extensions.

Bénard–He–Zhang use this multislicing input in the $\mathrm{SL}_2(\mathbb{R})/\Lambda$ setting to prove effective equidistribution for expanding self-similar measures.

Talk roadmap

1. State the black box used in Bénard–He–Zhang.
2. Explain why the proof needs a projection/slicing theorem.
3. Review the classical projection philosophy:

   a set is understood through its shadows.

4. Present Bourgain's discretized projection method in a simple model case.

Setup for the multislicing theorem

We work with a measurable parameter space Θ and a family of C^2-embeddings

$$\varphi_\theta : B_1^{\mathbb{R}^3} \to \mathbb{R}^3, \qquad \theta \in \Theta.$$

Each chart φ_θ has controlled distortion: there is a constant $L_\theta \geq 1$ such that

$$\frac{1}{L_\theta}\|x-y\| \leq \|\varphi_\theta(x)-\varphi_\theta(y)\| \leq L_\theta\|x-y\|, \quad \forall x, y \in B_1^{\mathbb{R}^3}$$

and

$$\|\varphi_\theta(x+h)-\varphi_\theta(x)-D_x\varphi_\theta(h)\| \leq L_\theta\|h\|^2, \quad \forall x, h \in B_1^{\mathbb{R}^3}$$

Setup for the multislicing theorem

For $\rho > 0$, define $\mathcal{D}_\rho = \{\text{translates of the cube } [0,\rho]^3\}$, and $\mathcal{R}_\rho$ the translates of the rectangles

$$R_\rho = [0,1]e_1 + [0,\rho^{1/2}]e_2 + [0,\rho]e_3.$$

For subspaces $V, W \subset \mathbb{R}^3$,

$$d_\angle(V,W) = \|v \wedge w\|,$$

where v, w are unit vectors spanning $\bigwedge^{\dim V} V$ and $\bigwedge^{\dim W} W$.

Theorem 4.3: Multislicing [BHZ]/Corollary [BH]

Let $\kappa \in (0, 1/2)$. Then there exist

$$\varepsilon = \varepsilon(\kappa) > 0, \qquad \rho_0 = \rho_0(\kappa) > 0$$

such that the following holds for all $\rho \in (0, \rho_0]$.
Let ν be a Borel measure on $B_1^{\mathbb{R}^3}$ satisfying, there exists $\alpha \in (\kappa, 1-\kappa)$,

$$\sup_{Q \in \mathcal{D}_r} \nu(Q) \leq r^{3\alpha} \qquad \text{for all } r \in [\rho, \rho^{\varepsilon}].$$

Let Ξ be a probability measure on Θ such that:

$$\Xi\{\theta \in \Theta : L_\theta \leq \rho^{-\varepsilon}\} = 1,$$

and for every $k \in \{1, 2\}$, $x \in B_1^{\mathbb{R}^3}$, $r \in [\rho, \rho^{\varepsilon}]$, and $W \in \mathrm{Gr}(\mathbb{R}^3, 3-k)$,

$$\Xi\left\{\theta \in \Theta : d_\angle\big((D_x\varphi_\theta)^{-1} V_k, W\big) \leq r\right\} \leq r^{\kappa},$$

where

$$V_k = \mathrm{span}_{\mathbb{R}}(e_1, \ldots, e_k).$$

Theorem 4.3: Conclusion

There exists a set

$$F \subseteq \Theta$$

such that

$$\Xi(F) \geq 1 - \rho^{\varepsilon}.$$

For every $\theta \in F$, there exists a set

$$A_\theta \subseteq B_1^{\mathbb{R}^3} \quad \text{such that} \quad \nu(A_\theta) \geq 1 - \rho^{\varepsilon}$$

and

$$\sup_{Q \in \mathcal{R}_\rho} \nu|_{A_\theta}\left(\varphi_\theta^{-1} Q\right) \leq \rho^{\frac{3}{2}\alpha + \varepsilon}.$$

For most parameters θ, after removing a small exceptional part of the measure, no thin rectangle captures mass.

We need special charts

The multislicing theorem is a statement about measures on $\mathbb{R}^3$ and thin rectangles of the form

$$R_\rho = [0,1]e_1 + [0,\rho^{1/2}]e_2 + [0,\rho]e_3.$$

But our random walk lives on

$$X = G/\Lambda, \qquad G = \mathrm{SL}_2(\mathbb{R}).$$

To apply multislicing, we need local coordinates on G in which sets of the form

$$g^{-1}B_\rho h$$

look like thin rectangles.

Define charts that "straighten" preimages of balls under the random walk into rectangles.

Coordinates on $\mathfrak{sl}_2(\mathbb{R})$

Recall the basis

$$e_1 = e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad e_2 = e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_3 = e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The Lie algebra decomposes as

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+,$$

where

$$\mathfrak{g}_- = \mathbb{R}e_-, \qquad \mathfrak{g}_0 = \mathbb{R}e_0, \qquad \mathfrak{g}_+ = \mathbb{R}e_+.$$

These three directions will become the three sides of the rectangle:

$$1, \qquad \rho^{1/2}, \qquad \rho.$$

The chart Ψ

Section 4.2 defines a map

$$\Psi : \mathfrak{g} \to G$$

by writing a vector as

$$v = v_- + v_0 + v_+, \qquad v_- \in \mathfrak{g}_-, \quad v_0 \in \mathfrak{g}_0, \quad v_+ \in \mathfrak{g}_+,$$

and setting

$$\Psi(v) = \exp(v_-)\exp(v_0)\exp(v_+).$$

The product

$$\exp(v_-)\exp(v_0)\exp(v_+)$$

is adapted to the stable, central, and unstable directions of the diagonal flow.

Straightening balls into rectangles

Recall

$$a(t) = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}.$$

The key statement is Lemma 4.4.

Lemma 4.4

There is a constant $r_0 > 0$ such that if $t, \rho \in (0,1)$ and $|t^{-1}\rho| \leq r_0$, then for every $h \in G$, there exists $w \in \mathfrak{g}$ such that

$$\{v \in B^{\mathfrak{g}}_{r_0} : \Psi(v) \in a(t)B_\rho h\} \subset \mathrm{Ad}(a(t))B^{\mathfrak{g}}_{10\rho} + w.$$

In the Ψ-coordinates, the set $a(t)B_\rho h$ is roughly contained in a translated box.

Shape of the box

The adjoint action of $a(t)$ scales the directions $\mathfrak{g}_-,\ \mathfrak{g}_0,\ \mathfrak{g}_+$.
So $\mathrm{Ad}(a(t))B^{\mathfrak{g}}_{10\rho}$ is roughly a rectangular box

$$(0, 10t^{-1}\rho)e_1 + (0, 10\rho)e_2 + (0, 10t\rho)e_3.$$

In the dimension bootstrap, one chooses t so that this becomes comparable to

$$(0, 1)e_1 + (0, \rho^{1/2})e_2 + (0, \rho)e_3.$$

This is the type of rectangle that appears in Theorem 4.3.

Ψ-chart and straightening charts

Recall that

$$\Psi : \mathfrak{g} \to G, \qquad \Psi(v_- + v_0 + v_+) = \exp(v_-)\exp(v_0)\exp(v_+).$$

Choose $r_1 > 0$ such that

$$\Psi\big|_{B^{\mathfrak{g}}_{r_1}}$$

is a smooth diffeomorphism onto a neighborhood

$$\mathcal{O} \subset G$$

of the identity.

The straightening charts are obtained by combining this local inverse of Ψ with conjugation by an upper unipotent element.

Definition of the straightening charts

Let

$$\Theta = u(\mathbb{R}), \qquad u(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

For each $\theta \in \Theta$, define

$$\varphi_\theta : \mathcal{O} \to \mathfrak{g}$$

by

$$\varphi_\theta(h) = \mathrm{Ad}(\theta^{-1}) \circ \left(\Psi|_{B^{\mathfrak{g}}_{r_1}}\right)^{-1}(h) \qquad h \in \mathcal{O}.$$

First write h in the special Ψ-coordinates, then rotate/shear the coordinates by $\mathrm{Ad}(\theta^{-1})$.

Equivalent formula

Equivalently

$$\varphi_\theta = \left(\Psi|_{\mathrm{Ad}(\theta^{-1})B^{\mathfrak{g}}_{r_1}}\right)^{-1} \circ C_{\theta^{-1}},$$

where

$$C_{\theta^{-1}} : G \to G, \qquad C_{\theta^{-1}}(h) = \theta^{-1} h \theta.$$

This follows from the fact that Ψ is compatible with conjugation:

$$\theta^{-1}\Psi(v)\theta = \Psi\big(\mathrm{Ad}(\theta^{-1})v\big).$$

The chart φ_θ is a local coordinate system centered around the direction determined by θ.

Distortion control

The straightening charts satisfy the distortion assumptions needed for multislicing.
For each $\theta \in \Theta$, there is a constant $L_\theta \geq 1$ such that φ_θ is L_θ-bi-Lipschitz:

$$\frac{1}{L_\theta}\|x - y\| \leq \|\varphi_\theta(x) - \varphi_\theta(y)\| \leq L_\theta\|x - y\|.$$

Moreover, the second-order error is controlled:

$$\|\varphi_\theta(x + h) - \varphi_\theta(x) - D_x\varphi_\theta(h)\| \leq L_\theta\|h\|^2.$$

We can take

$$L_\theta = L\|\theta\|^4,$$

for a constant $L > 1$ depending only on r_1.

Why these charts

For an element $g \in P$, write

$$g^{-1} = \theta_g a(r_g),$$

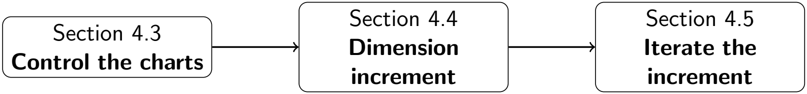
where $\theta_g = u(-b_g) \in \Theta$.
Associate to g the chart φ_{θ_g}.
Lemma 4.4 implies that, in this chart, $\varphi_{\theta_g}(g^{-1} B_\rho h)$ is essentially contained in a translate of

$$\mathrm{Ad}(a(r_g)) B_\rho^{\mathfrak{g}}.$$

After applying the right straightening chart, the preimage of a small ball looks like a thin rectangle. This is what allows the multislicing theorem to be applied.

How the black box is used

The rest of Section 4 explains how this black box is inserted into the proof.



The multislicing theorem is used once to improve dimension, and then that improvement is repeated until the measure is almost full-dimensional.

What Sections 4.3–4.5 do

Section 4.3: Control of the charts
To make the charts satisfy hypothesis of Theorem 4.3.
Section 4.4: Dimension increment
Show that if a measure has dimension α, then after applying the random walk it has dimension $\alpha + \varepsilon$.
Section 4.5: Iterating the increment
Upgrades positive dimension to almost full dimension:

$$\kappa \longrightarrow \kappa + \varepsilon \longrightarrow \cdots \longrightarrow 1 - \kappa.$$

From multislicing to Bourgain projection

The engine behind the dimension increment is a Bourgain-type projection theorem.
Now we explain the projection method itself.

Philosophy

A set is understood by its shadows.

In BHZ the "shadows" are not ordinary linear projections. They are thin slices adapted to the random walk.
But the underlying philosophy comes from Bourgain's discretized projection theorem.

First example: finite-field projections

Let $X \subset \mathbb{F}_p^2$, and let $D \subset \mathbb{F}_p$ contain at least two distinct elements θ, θ'. For each $\theta \in D$, define

$$\pi_\theta(x, y) = \langle (1, \theta), (x, y) \rangle, \qquad (x, y) \in X.$$

Then, we get the so-called *trivial bound*

$$\max_{\theta \in D} |\pi_\theta(X)| \geq |X|^{1/2}.$$

The map

$$z \longmapsto \big(\pi_\theta(z), \pi_{\theta'}(z)\big), \qquad z \in X,$$

is injective, because $\theta \neq \theta'$.
Hence

$$|X| \leq |\pi_\theta(X)|\,|\pi_{\theta'}(X)|.$$

Notation: we write π_θ for projections, following O'Regan–Shmerkin–Wang.

Over $\mathbb{R}$

The finite-field example is too easy because two different projections determine the original point.
Over $\mathbb{R}^2$, Bourgain's theorem is about a more delicate question:

If a set is spread out in the plane, must many of its projections be large?

The answer is yes, provided the set and the family of directions satisfy non-concentration assumptions.

We no longer only count points. We must work at a *small scale* $\delta > 0$.

Small scale

Observed at scale δ: For a bounded set $A \subset \mathbb{R}^d$, write $|A|_\delta$ for the smallest number of balls of radius δ needed to cover A.

$|A|_\delta \approx$ number of visible δ-scale pieces of A.

If A is a union of δ-balls, then

$$|A|_\delta \asymp \delta^{-d}|A|,$$

where $|A|$ denotes Lebesgue measure.

Non-concentration condition for Bourgain-type projection theorems

A set $A \subset \mathbb{R}^d$ is called a (δ, κ, C)-set if:

- A is a finite union of balls of radius δ;
- for every $x \in A$ and every $r \geq \delta$,

$$|A \cap B(x, r)| \leq Cr^\kappa |A|.$$

No ball of radius r captures more than an r^κ-proportion of the set.

A Radon probability measure μ on $\mathbb{R}^d$ is called a (κ, C)-measure

$$\text{if} \quad \mu(B(x,r)) \leq Cr^\kappa \quad \text{for every} \quad x \in \operatorname{spt}\mu \qquad \text{and} \qquad r > 0.$$

The measure μ cannot put too much mass inside any small ball.
At scale r, every ball captures at most an $O(r^\kappa)$-amount of mass. So μ behaves like a measure of dimension at least κ.

Bourgain projection theorem: informal version

Let $E \subset \mathbb{R}^2$ be a discretized set at scale δ.
Assume:

- E has size roughly $\delta^{2-\alpha}$;
- E is non-concentrated;
- the set of directions is also non-concentrated.

Then for most directions θ,

$$|\pi_\theta(E)|_\delta$$

is larger than the trivial lower bound.

A two-dimensional set cannot have small shadows in many well-distributed directions.

Bourgain's discretized projection theorem [3, Thm. 1.3]

Let $0 < \alpha < 2$, $\beta, \kappa > 0$. There exist $\eta > 0$, $\varepsilon, \delta_0 \in (0, 1]$, depending only on α, β, κ, such that the following holds for all $0 < \delta < \delta_0$.
Let μ be a $(\kappa, \delta^{-\varepsilon})$-measure on S^1, and let

$$E \subset B^2(0, \delta^{-\varepsilon}) \quad \text{be a } (\delta, \beta, \delta^{-\varepsilon})\text{-set with} \quad |E| \leq \delta^{2-\alpha}.$$

Then there exists

$$\Theta \subset \operatorname{spt} \mu \quad \text{with} \quad \mu(\Theta) > 1 - \delta^{\varepsilon}$$

such that for all $\theta \in \Theta$,

$$|\pi_\theta(G)|_\delta > \delta^{-\eta} |E|_\delta^{1/2} \quad \text{for every } G \subset E \text{ with} \quad |G| > \delta^{\varepsilon} |E|.$$

Since $|E| \leq \delta^{2-\alpha}$, then $|E|_\delta \leq C\delta^{-\alpha}$ for $C > 0$. We obtain that

$$|\pi_\theta(G)|_\delta \geq C |E|_\delta^{1/2+\eta/\alpha}.$$

Bourgain's theorem gives an improvement over the trivial lower bound after ignoring the leeway constant C and replacing G by E.