# Background in Diophantine Approximation

§1. Classical Theory

§2. Diophantine Approximation on Manifolds

How dense is $\mathbb{Q}$ in $\mathbb{R}$? More precisly:

Q: Given $\alpha \in \mathbb{R}$, how small is $\left|\alpha - \frac{p}{q}\right|$ in terms of $q$?

Thm 1 (Dirichlet, 1842)

Let $Q \geq 1$ be an integer, and $\alpha \in \mathbb{R}$. There ex. $q, p \in \mathbb{Z}$, $1 \leq q \leq Q$

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qQ} < \frac{1}{q^2}.$$

Proof: Consider the $Q+1$ numbers $0, \langle\alpha\rangle, \ldots, \langle Q\alpha\rangle$, (fractional part)

and the intervals

$$I_b = \left[\frac{b}{Q}, \frac{b+1}{Q}\right) \qquad (b = 0, \ldots, Q-1).$$

By the pigenhole principle, there are $0 \leq r_2 < r_1 \leq Q$ s.t. $\langle r_1\alpha\rangle, \langle r_2\alpha\rangle \in I_b$ for some $b \leq Q-1$. Hence,

$$\frac{1}{Q} \geq \left|\langle r_1\alpha\rangle - \langle r_2\alpha\rangle\right| = \Big|\underbrace{(r_1 - r_2)}_{=q}\alpha + \underbrace{\lfloor r_2\alpha\rfloor - \lfloor r_1\alpha\rfloor}_{=p}\Big|$$

floor function

$$= |q\alpha - p|.$$

Since $1 \leq q \leq Q$, the proof is complete. □

Thm 1' (Dirichlet "multidimensional", 1842)

Suppose $(\alpha_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ are real numbers. Then there exist integers $q_1, \ldots, q_m$ and $p_1, \ldots, p_n$ s.t.

$$|\alpha_{i1}q_1 + \ldots + \alpha_{im}q_m - p_i| \leq \frac{1}{Q} \quad (1 \leq i \leq n),$$

and $1 \leq \max\{|q_1|, \ldots, |q_m|\} \leq Q^{n/m}$.

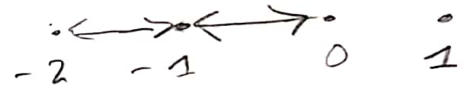
Q: Can you improve Thm 1?

A: Not really: it is optimal up to constants:

There ex $\alpha$ for which

dist. neares integer

$$\forall_{q \geq 1} \quad \frac{1}{q} \geq \|q\alpha\| \geq \frac{c(\alpha)}{q}$$



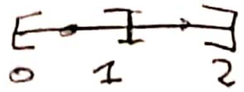
for some $c(\alpha) > 0$. E.g. $\alpha = \sqrt{2}$: $\quad \mathbb{Z}\setminus\{0\}$

$$|q\sqrt{2}+p| \ \frac{1}{2} > |q\sqrt{2}-p| \qquad |q\sqrt{2}+p| = |q^2 2 - p^2| \geq 1.$$

$$\Rightarrow |q\sqrt{2}-p| \geq \frac{1}{|q\sqrt{2}+p|} \geq \frac{1}{q \max\{\sqrt{2}, p/q\}}$$

**Def** (badly approximable numbers)

$\alpha$ is badly approx if $\inf_{q \geq 1} q \|\alpha q\| > 0$



**Q2:** Can you a.s. improve Dirichlet?

§1.2 Khintchine

**Def** ($\psi$-approx.) Let $\psi : \mathbb{Z}_{\geq 1} \to \mathbb{R}_{\geq 0}$ be non-increasing.

$\alpha$ is called $\psi$-approx. if $\|q\alpha\| < \psi(q)$ for $\infty$-many $q$

Let $W(\psi)$ be the set of $\psi$-approx. nbs. in $[0,1]$.

**Thm 2** (Khintchine 1926)

Let $\lambda$ be Lebesgue measure on $\mathbb{R}$.

$$\lambda(W(\psi)) = \begin{cases} 1, & \text{if } \sum_{q \geq 1} \psi(q) = \infty, \\ 0, & \sum_{q \geq 1} \psi(q) < \infty. \end{cases}$$

**Rmk 1** a) convergence–divergence threshold.

$$\psi(q) = \frac{1}{q (\log q)^{1+\varepsilon}}$$

b) (2) $\sum_{q \geq 1} \psi(q) < \infty \to \lambda(W(\psi)) = 0$ is simple:

**Lemma 1** (Borel-Cantelli)

$(X, \mathcal{B}, P)$ a probability space, $\Omega_q \in \mathcal{B}$.

Then
$$\Omega = \limsup_{q \to \infty} \Omega_q = \{x \in X : x \in \Omega_q \text{ for } \infty\text{-many } q\} = \bigcap_{q \geq 1} \bigcup_{q' \geq q} \Omega_{q'}$$

has $P(\Omega) = 0$ if $\sum_{q \geq 1} P(\Omega_q) < \infty$.

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Proof: $\tilde{\Omega}_q = \bigcup_{q' \geq q} \Omega_{q'}$ is nested. $\tilde{\Omega}_1 \supseteq \tilde{\Omega}_2 \supseteq \dots$

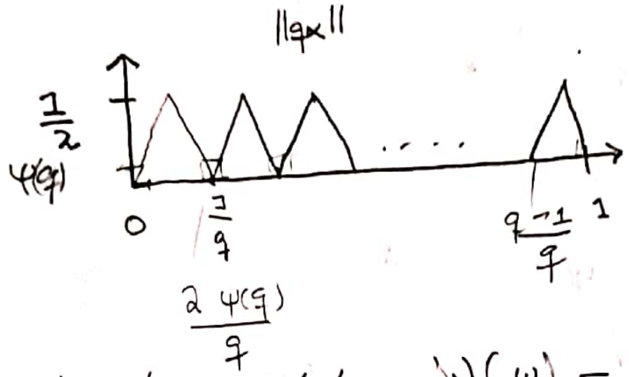
and $P(\tilde{\Omega}_q) \leq \sum_{q' \geq q} P(\Omega_{q'}) < \varepsilon$ for $q$ large. Hence,

$$P(\Omega) \leq P(\tilde{\Omega}_q) \leq \varepsilon.$$

$\square$

To prove (2) note that $\Omega_q = \{ x \in [0,1) : \|qx\| < \psi(q) \}$ is (Borel) measurable and $\lambda(\Omega_q) = 2\psi(q)$ :

$$= \frac{2\psi(q)}{q}(q-1) + 2\frac{\psi(q)}{q}$$




Now apply Lemma 1 to $W(\psi) = \limsup_{q \to \infty} \Omega_q$.

## §1.3 Definitions

**Def** $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ is called <u>$\psi$-approximable</u> if $\exists\, \underline{p} \in \mathbb{Z}^n$ s.t.

$$(1) \qquad \|q\underline{\alpha} - \underline{p}\|_\infty \leq \psi(q)$$

for $\infty$-many $q$; $\underline{\alpha}$ is <u>badly approximable</u> if

$$(2) \qquad \forall_{q \geq 1}\ \forall_{\underline{p} \in \mathbb{Z}^n}\ \|q\underline{\alpha} - \underline{p}\|_\infty \geq \frac{c(\underline{\alpha})}{q}$$

for some constant $c(\underline{\alpha}) > 0$.

Let $\psi_\varepsilon(q) = q^{-1-\varepsilon}$, $\varepsilon > 0$ fixed, and $\Pi_+(x) = \prod \max\{1$ ... is n

Then $\underline{\alpha}$ is called <u>very well approximable</u> if $\underline{\alpha}$ is $\psi_\varepsilon$-approximable (for some $\varepsilon > 0$), and

$\underline{\alpha}$ is <u>multiplicatively very well approximable</u> if

$$\prod_{i\le n} |q\alpha_i - p_i| < q^{-1-\varepsilon}$$

has $\infty$-many solutions $p_1, \dots, p_n, q$.

You can add weights to generalize things:

Take $\underline{r} = (r_1, \dots, r_n) \in [0,1]^n$ s.t. $r_1 + \dots + r_n = 1$

Put $\|\underline{\alpha}\|_{\underline{r}} = \max\limits_{i=1,\dots,n} |\alpha_i|^{1/r_i} \leftarrow$ (quasi-norm).

<u>Thm3</u> (Dirichlet weighted version)

For any $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $N \in \mathbb{N}$ ex. $q \in \mathbb{Z}$
$\underline{p} \in \mathbb{Z}^n$ s.t. $\|q\underline{\alpha} - \underline{p}\|_{\underline{r}} < \frac{1}{Q}$ and $1 \le q \le Q$.

Q: How to construct badly approx. vectors?

Let $\alpha \in \mathbb{R}$ be algebraic of degree $n+1$

1) Then $\underline{\alpha} = (\alpha, \alpha^2, \dots, \alpha^n)$ is badly approx due to Perron.

<u>Thm.</u> (Schmidt 196?)

The set of badly approx vectors has full Hausdorff dimension.

<u>Rmk</u> This is deduced by geometric games, called Schmidt games.

<u>Def</u> (singular vector)

$\underline{\alpha}$ is called <u>singular</u> if for any $\delta > 0$ there ex. $Q_0$ s.t. for $Q \ge Q_0$ there is $\underline{p} \in \mathbb{Z}^n, q \in \mathbb{Z}_{\ge 1}$ s.t.

$$\|q\underline{\alpha} - \underline{p}\|_\infty < \frac{\delta}{Q} \quad \text{and} \quad q \le \delta Q.$$

<u>Examples:</u> $n=1$: $\frac{p}{q}$ is singular

$n \ge 2$

$$\dim \underset{\text{set of singular}}{S_n} \underset{\text{Cheung-Chevallier}}{=} \frac{n^2}{n+1} = n+1 - \frac{2n+1}{n+1} = n-1 + \frac{1}{n+1}$$

$\Rightarrow S_n$ is not contained in any affine hyperplane.

Def (Dirichlet improvable)

$\underline{\alpha}$ is called Dirichlet improvable if there $\exists c \in (0,1)$ s.t. for any $Q \in \mathbb{Z}_{\geq 1}$ large enough there is $\underline{p} \in \mathbb{Z}^n$ s.t.

$$\|q\underline{\alpha} - \underline{p}\|_\infty < c \cdot \frac{1}{Q}$$

for with $q \leq Q$.

Thm (Davenport - Schmidt)

The set of Dirichlet improvable vectors has measure zero. Moreover, any badly approx vector is Dirichlet improvable.

Rmk Dirichlet improvable vectors have full Hausdorff dimension.

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We can also consider $\underline{r} = (r_1, \ldots, r_n) \in [0,1]^n$, $r_1 + \ldots + r_n = 1$ singular vectors: $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is called $r$-singular if for any $\delta > 0$ there is a $T_0 > 0$ s.t. for all $T \geq T_0$ one can find $p \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ with

$$\|q\underline{\alpha} - \underline{p}\|_{\underline{r}} < \frac{\delta}{T} \quad \text{and} \quad q < \delta T.$$

Rmk For any $\underline{r}$, the set of $\underline{r}$-singular vectors is know to have zero Lebesgue measure, see Cassel's (book on DA) Ch. V. §7.

Similarly, we can define $\underline{r}$-badly approximable, $\underline{r}$-very well approximable, replacing $\|\cdot\|_\infty$ in (1), (2) by $\|\cdot\|_{\underline{r}}$.