Combining high and low entropy

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Abstract

In this text, we explain how to combine the high and low entropy methods to complete the classification of ergodic invariant measures under the action of the group of diagonal matrices on the space of 3-lattices. It corresponds to the last session of the learning seminar homogeneous dynamics and applications run by Barak Weiss at the university of Tel Aviv

1 Introduction

Denote by $G = SL_3(\mathbb{R})$, by $\Gamma = SL_3(\mathbb{Z})$, by $X = G/\Gamma$ and by m_X the probability measure on X induced by the Haar measure on G. Denote also by A the subgroup comprised of diagonal matrices in G. We want to prove the following:

Theorem 1 (Einsiedler-Katok-Lindenstrauss)**.** *Let ν be a A-invariant and A-ergodic probability measure on X. If ν has positive entropy with respect to some $a \in A$, then ν is the Haar measure.*

We will combine the low and high entropy explained during the previous sessions of the seminar. In particular, we will need the following:

Theorem 2 (High entropy method)**.** *Assume ν has high entropy, i.e there is some $a \in A$ such that $h_\nu(a) > \frac{1}{2}h_{m_X}(a)$, then ν is the haar measure m_X.*

This is for instance theorem 9.5 in [2]. One of the important inputs needed to prove theorem 2 is the commutator trick:

Lemma 1 (Commutator trick)**.** *Suppose there is some pairwise distinct i, j, k such that $\nu_x^{i,k}$ and $\nu_x^{k,j}$ are ν-almost everywhere non trivial, then ν is invariant by U_{ij}.*

The last ingredient we will need from the previous sessions is the following:

Theorem 3 (Low entropy method)**.** *Suppose there is a pair i, j such that $\nu_x^{i,j}$ is non trivial ν-almost everywhere but $\nu_x^{i,k}$ and $\nu_x^{k,j}$ are ν-almost everywhere trivial. Then ν is invariant by U_{ij}*

This is theorem 2.3 in [4]. Notice that for general lattices, there is an other alternative to invariance, dubbed "exceptional returns" by the authors, but it is shown in section 5 of the aforementioned paper that this alternative does not occur when $\Gamma = SL_3(\mathbb{Z})$ (or more generally $SL_n(\mathbb{Z})$ for arbitrary $n \geq 3$).

2 The proof of Theorem 1

If ν has high entropy, Theorem 1 is a consequence of Theorem 2. Our goal is thus to show that there is an element of A that acts with high entropy. From now on, we suppose to a contradiction that ν has low entropy, $i.e$ for any $a \in A$, we have:

$$h_\nu(a) \leq \frac{1}{2} h_{m_X}(a)$$

2.1 Exploiting low entropy

In this subsection, we collect some properties of the leaf wise measures of ν that follows from the low entropy assumption and show that we can invoke theorem 3. By assumption on ν, there is some $a \in A$ such that $h_\nu(a) > 0$. This means in particular that there is some pair i, j such that ν_x^{ij} is ν-almost everywhere non trivial.

Proposition 1. *The leaf wise measures ν_x^{kl} are a.s trivial for any k, l such that $\{k, l\} \neq \{i, j\}$ and the ν_x^{ji} are non trivial a.s.*

Proof. Let's consider first the measures ν_x^{jk} and suppose to a contradiction that they are a.s non trivial. Then by the the commutator trick of lemma 1 we deduce that ν is U_{ik} invariant. Let α be any element in A whose stable subgroup is $U_{ik}U_{ij}$. We claim that α has high entropy. To see this, recall the entropy formula:

$$h_\nu(\alpha) = h_\nu(\alpha, U_{ij}) + h_\nu(\alpha, U_{ik})$$

If we denote by λ the i-th diagonal coefficient of α, the fact that ν is U_{ik}-invariant implies that $h_\nu(\alpha, U_{ik}) = \ln(\lambda)$. Notice that this quantity is also equal to half the entropy of α with respect to the measure m_X. Because we assumed that the ν_x^{ij} are a.s non trivial, we also know that $h_\nu(\alpha, U_{ij})$ is strictly positive and thus α acts with high entropy with respect to ν, which is a contradiction. We prove likewise that the ν_x^{ki} are trivial almost surely. Now, let β be any element of A whose contracting subgroup is $U_{ji}U_{jk}$. We have:

$$h_\nu(\beta) = h_\nu(\beta, U_{ji}) + h_\nu(\beta, U_{jk}) = h_\nu(\beta, U_{ji})$$

The second equality is due to the fact that the ν_x^{jk} are trivial a.s. By the symmetry of entropy, we know that this quantity is also equal to the entropy $h_\nu(\beta^{-1})$. Using once more the entropy formula for β^{-1}, whose contracting subgroup is $U_{ij}U_{kj}$, we know that the entropy $h_\nu(\beta^{-1})$, and hence $h_\nu(\beta)$, is strictly positive because we assumed that that the ν_x^{ij} are non trivial a.s. As a consequence we deduce that the ν_x^{ji} are non trivial a.s. Finally, can now argue as before that the ν_x^{ki} and the ν_x^{kj} are trivial a.s. □

Proposition 2. *The measure ν is invariant by the groups U_{ij} and U_{ji}*

Proof. By assumption and proposition 1, we know that the leaf wise measures ν_x^{ij} and ν_x^{ji} are a.s non trivial. The statement is then a consequence of theorem 3 together with proposition 1. $\square$

Let H denotes the closure of the group generated by U_{ij} and U_{ji} and by $\tilde{H}$ the group generated by H and A.

Conclusion: The measure ν is invariant under the action of H, which is isomorphic to $SL_2(\mathbb{R})$, and all the leaf wise measures associated to the groups U_{kl} for $\{k,l\} \neq \{i,j\}$ are trivial a.s.

2.2 Exploiting the invariance

Let $\nu = \int_X \nu_x \, d\nu(x)$ be the ergodic decomposition of ν with respect to H. The following proposition relates the stabilizers of the different measures ν_x.

Proposition 3. *There is a A-normalized, closed and connected subgroup L that contains H such that for ν a.e $x \in X$, we have $L = Stab(\nu_x)^o$.*

Proof. Denote by $\mathfrak{g}$ the Lie algebra of G, by S_x the connected component of the identity of the stabilizer of ν_x and by $\mathfrak{s}_x$ the corresponding Lie sub-algebra. We know by uniqueness of the ergodic decomposition and A-invariance of ν that for any $a \in A$ we have $a_*\nu_x = \nu_{a\cdot x}$ a.s. This implies in particular that $S_{a\cdot x} = aS_xa^{-1}$, or equivalently that $Ad_a(\mathfrak{s}_x) = \mathfrak{s}_x$ and thus the map $x \mapsto \dim(\mathfrak{s}_x)$ is constant on A-orbits and thus is constant a.e by ergodicity. We denote by d this constant value. Let $(s_1^x, \cdots, s_d^x)$ be a basis of $\mathfrak{s}_x$ and define the following map:

$$\Phi : \begin{array}{ccc} X & \to & \mathbb{P}(\bigwedge_1^d \mathfrak{g}) \\ x & \mapsto & [s_1^x \wedge \cdots \wedge s_d^x] \end{array}$$

Notice that this does not depend on the choice of the basis chosen for the $\mathfrak{s}_x$. In particular, since $Ad_a(s_1^x), \cdots, Ad_a(s_d^x)$ is also a basis of $\mathfrak{s}_{a\cdot x}$, we have $\Phi(a \cdot x) = a \cdot \Phi(x)$ where the action of G on $\mathbb{P}(\bigwedge_1^d \mathfrak{g})$ is induced by the adjoint. We thus define a A-invariant measure on $\mathbb{P}(\bigwedge_1^d \mathfrak{g})$ by setting $\tilde{\nu} = \Phi_*\nu$. Let $P = [s_1 \wedge \cdots \wedge s_d]$ be a point in the support of the measure $\tilde{\nu}$. By Poincare recurrence there is an increasing and unbounded sequence n_k such that $a^{n_k} \cdot P$ converges to P. Since Ad_a is diagonalizable, so is the induced action of a on $\bigwedge_1^d \mathfrak{g}$ and thus it implies that $s_1 \wedge \cdots \wedge s_d$ is an eigenvector. In particular, $a \cdot P = P$ and Ad_a preserves the subspace spanned by the s_i. As a consequence, we have that a.s $\Phi(a \cdot x) = \Phi(x)$. By ergodicity, it means that Φ is almost everywhere constant to some $[s_1 \wedge \cdots \wedge s_d]$. We denote by $\mathfrak{l}$ the linear subspace spanned by the s_i and by L the corresponding closed and connected subgroup. We deduce the following consequences :

1. a.s $\mathfrak{s}_x$ is equal to $\mathfrak{l}$

2. L is normalized by A

For item 1 we use that the exponential map is a local diffeomorphism and the fact that the S_x are connected and for the second item we use the fact that a closed connected group J is normalized by some $g \in G$ if and only if Ad_g preserves $Lie(J)$. This concludes the proof. $\square$

Proposition 4. *The group L is contained in $\tilde{H}$.*

Proof. The measure ν is L-invariant. If L is not contained in $\tilde{H}$ we get extra invariance by some U_{kl} using the fact that L is A-normalized, in contradiction with proposition 1. More precisely, the lie algebra $\mathfrak{g}$ of G splits as follows:

$$\mathfrak{g} = \bigoplus_{k \neq l} \mathfrak{u}_{kl} \oplus \mathfrak{a}$$

where $\mathfrak{u}_{kl}$ is the Lie algebra of U_{ij} and $\mathfrak{a}$ is the Lie algebra of A. If L is not contained in $\tilde{H}$, then there is a $v = u_{kl} + u + a \in \mathfrak{l}$ where u_{kl} belongs to some $\mathfrak{u}_{kl}$ with $\{k, l\} \neq \{i, j\}$. Now, let $\alpha \in A$ whose k-th diagonal coefficient is equal to some $\lambda > 1$ and whose l-th diagonal coefficient is λ^{-1} so that $\mathfrak{u}_{kl}$ is the dominant eigenspace of Ad_α. In particular, this implies that $Ad_\alpha^n(v)$ converges projectively to u_{kl}. But since A normalizes L according to proposition 3, we deduce that $u_{kl} + a$ belongs $\mathfrak{l}$. Consequently, the measure ν is invariant by U_{kl}, in contradiction with proposition 1. $\square$

We now invoke a version of Ratner's measure classification for $SL_2(\mathbb{R})$-actions, as can be found for instance in [3].

Theorem 4 (Ratner, case of $SL_2(\mathbb{R})$)**.** *Let G be a Lie group, Γ be a discrete subgroup and let H be a subgroup of G that is isomorphic to $SL_2(\mathbb{R})$. Let ν be any H-invariant and ergodic probability measure on $X = G/\Gamma$ and denote by L_ν the connected component of the identity of its stabilizer. Then ν is supported on a closed orbit of L_ν. More precisely, there is a $\xi \in X$ such that $L_\nu \cdot \xi$ is closed and has full measure.*

We can now prove:

Proposition 5. *The group L is actually equal to H*

Proof. Theorem 4 tells us that for ν-almost every $x \in X$, there is a $\xi_x = g\Gamma \in X$ such that $L \cdot \xi_x$ is closed and $\nu_x(L \cdot \xi_x) = 1$. In particular $g^{-1}Lg \cap \Gamma$ is a lattice in $g^{-1}Lg$ and thus the latter is defined over $\mathbb{Q}$, as a consequence of Borel density theorem. See proposition 3.1.8 and theorem 3.2.5 in [5]. Since $L \subset \tilde{H}$ and L is normalized by A, the commutator subgroup $[g^{-1}Lg, g^{-1}Lg]$ is equal to $g^{-1}Hg$. It follows that the latter is also defined over $\mathbb{Q}$ as the image of a regular morphism between algebraic groups defined over $\mathbb{Q}$. See for instance corollary 3.1.2 in [5]. Since H is isomorphic to $SL_2(\mathbb{R})$, the group $g^{-1}Hg$ is also semi-simple. We conclude that $g^{-1}Hg \cap \Gamma$ is a lattice in $g^{-1}Hg$ thanks to a theorem of Borel, Harish and Chandra. See for instance theorem 3.1.7 in [5]. In particular, we get that $H \cdot \xi_x$ is closed and has finite measure. This implies that $L = H$ or else the ergodicity of ν_x is contradicted. $\square$

Conclusion: ν-almost every point has a closed orbit under the action of H

2.3 The final contradiction

Let denote by a_t^{ij} the diagonal matrix whose diagonal coef i and j are e^{-t}. We will be able to conclude using the following:

Proposition 6. *If $H \cdot x$ is closed then $t \mapsto a_t^{ij} \cdot x$ is divergent.*

Proof. We denote $x = g\Gamma$. Notice that for every $h \in g^{-1}Hg$, we have $e_k^\intercal gh = e_k^\intercal g$. Since $g^{-1}Hg \cap \Gamma$ is a lattice in $g^{-1}Hg$, in particular, there is a $h_0 \in \Gamma$ such that $e_k^\intercal gh_0 = e_k^\intercal g$. In particular, the line D spanned by $g^\intercal e_k$ is rational and so is its orthogonal. Let $y \in \mathbb{Z}^3 \cap D^\perp$ be a non vanishing vector. We have thus have $(g^\intercal e_k)^\intercal y = e_k^\intercal gy = 0$. Since the canonical basis is orthogonal, this implies that gy does not have any component on e_k. In particular:

$$a_{ij}^t gy \underset{t\to\infty}{\to} 0$$

We conclude that $t \mapsto a_t^{ij}$ is divergent using Mahler's compactness criterion. See for instance page 418 of [1]. $\square$

We saw at the end of the previous subsection that ν-almost every point $x \in X$ has a closed orbit under the action of H. By proposition 6, this implies that a.s the trajectories under the flow given by the action of a_t^{ij} are divergent. This is in contradiction with Poincare recurrence theorem since the measure ν is by assumption A-invariant.

Conclusion: There is at least one element of A that acts with high entropy with respect to ν and we deduce that the measure ν is the haar measure using theorem 2

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