The high entropy method for higher rank torus actions – the $\mathrm{SL}(3, \mathbb{R})$ case

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The setup

- $G = \mathrm{SL}(3, \mathbb{R})$.
- $\Gamma < G$ a discrete subgroup.
- $X = G/\Gamma$.
- m the Haar measure on X.
- $A < G$ the full diagonal group.
- μ an A-invariant and A-ergodic probability measure on X.
- For any $t, s \in \mathbb{R}$ denote

$$a(t,s) := \begin{pmatrix} t & & \\ & s & \\ & & t^{-1}s^{-1} \end{pmatrix}.$$

The Einsiedler–Katok 2003 measure rigidity theorem

Theorem

If $t, s \in \mathbb{R}$ *satisfy*

$$h_\mu\left(a(t,s)\right) > \frac{1}{2}\left(\left|\log\left|ts^{-1}\right|\right| + \left|\log\left|t^2 s\right|\right| + \left|\log\left|ts^2\right|\right|\right).$$

then $\mu \propto m$.

Ideas from the proof

- Show that μ is invariant under upper and lower triangular matrices.
- Work with leaf-wise measures along stable subgroups of a.
- The entropy assumption implies that a has enough stable subgroups with nontrivial leaf-wise measure.
- The product formula implies invariance of the leaf-wise measures along commutators of such groups.
- Every one-parameter triangular subgroup of G is the commutator of two other one-dimensional triangular directions.

One-dimensional groups of triangular matrices

Define the groups

$$G^{[1]} = \begin{pmatrix} 1 & * & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad G^{[2]} = \begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix}, \quad G^{[3]} = \begin{pmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{pmatrix},$$

and

$$G^{[-1]} = \begin{pmatrix} 1 & & \\ * & 1 & \\ & & 1 \end{pmatrix}, \quad G^{[-2]} = \begin{pmatrix} 1 & & \\ & 1 & \\ * & & 1 \end{pmatrix}, \quad G^{[-3]} = \begin{pmatrix} 1 & & \\ & 1 & \\ & * & 1 \end{pmatrix}.$$

The one-dimensional groups of triangular matrices

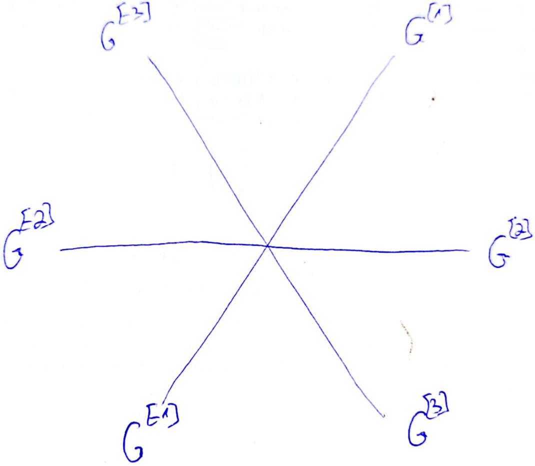

Figure: The commutator of any two groups which are separated by exactly one group is that group in the middle

The (standard) Heisenberg group

- $G^- := \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$
- $\left[G^{[1]}, G^{[3]}\right] = G^{[2]}$.
-

$$\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & -z \\ & & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & x & xz \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & xz \\ & 1 & -z \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & 2xz \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Proof of the Einsiedler–Katok Theorem

The proof splits into two cases:

- The singular case: two eigenvalues are equal in absolute value.
- The regular case: three eigenvalues with different absolute values.

The *singular* case

- The stable horospherical group of a, G_a^-, is a product of two commuting one-dimensional triangular subgroups.
- WLOG $0 < |t| = |s| < 1$ and $G_a^- = G^{[2]}G^{[3]}$.
- For every integer i with $1 \leq |i| \leq 3$ denote $\mu_x^{[i]} := \mu_x^{G^{[i]}}$.

The product structure of leaf-wise measures

Corollary 8.8

If $a' \in A$, $U < G_{a'}^{-}$ normalised by a', $T < G_{a'}^{0}$ normalises U and $H = TU$, then μ-$\forall x \in X$ satisfies

$$\mu_x^H \propto \iota\left(\mu_x^T \times \mu_x^U\right),$$

where

$$\begin{aligned}\iota\ : T \times U &\to H \\ (t, u) &\mapsto tu\,.\end{aligned}$$

Applying Lemma 8.8

- Take e.g. $a' = \begin{pmatrix} 2 & & \\ & 1/4 & \\ & & 2 \end{pmatrix}$.
- $T := G^{[2]} < G^0_{a'}$.
- $U := G^{[3]} < G^-_{a'}$.
- $H := G^{[2]} G^{[3]} = G^-_a$.
- a' normalises U.
- H is commutative, and in particular T normalises U.
- Then μ-$\forall x \in X$ satisfies $\mu_x^{G_a^-} \propto \iota\left(\mu_x^{[2]} \times \mu_x^{[3]}\right)$.

The entropy contribution of stable subgroups normalised by a

Theorem 7.6

Assume $a \in G$, μ an a-invariant probability measure on X and $U < G_a^-$ is normalised by a.

1. *The entropy contribution of U at x*

$$D_\mu(a, U)(x) := \lim_{n \to \infty} \frac{\log \mu_x^U\left(a^{-n} B_1^U a^n\right)}{n}$$

is well defined.

2. *For μ-$\forall x$*

$$D_\mu(a, U)(x) \leq h_{\mu_x^{\mathcal{E}}}(a)$$

with equality if $U = G_a^-$. Here $\mathcal{E}$ is the σ-algebra of a-invariant sets.

3. *For μ-$\forall x \in X$, $D_\mu(a, U)(x) = 0$ iff μ_x^U is finite iff μ_x^U is trivial.*

The entropy contribution of stable subgroups normalised by a (cntd.)

Definition

The entropy contribution of U is

$$h_\mu(a, U) := \int D_\mu(a, U)\, d\mu \, .$$

Applying Theorem 7.6(i)

- $G^{[2]}$ and $G^{[3]}$ are subgroups of G_a^- and are normalised by a.
- Theorem 7.6(i) implies that the entropy contributions $h_\mu\left(a, G^{[2]}\right)$ and $h_\mu\left(a, G^{[3]}\right)$ are defined.
- a contracts G_a^-, therefore μ-$\forall x$ satisfies

$$\lim_{n\to\infty}\frac{\log\mu_x^{G_a^-}\left(a^{-n}B_1^{G_a^-}a^n\right)}{n}=\lim_{n\to\infty}\frac{\log\mu_x^{G_a^-}\left(a^{-n}\left(B_1^{G^{[2]}}B_1^{G^{[3]}}\right)a^n\right)}{n}$$
$$=\lim_{n\to\infty}\frac{\log\mu_x^{G^{[2]}}\left(a^{-n}B_1^{G^{[2]}}a^n\right)}{n}+$$
$$\lim_{n\to\infty}\frac{\log\mu_x^{G^{[3]}}\left(a^{-n}B_1^{G^{[3]}}a^n\right)}{n}.$$

- The definition of the entropy contribution implies that

$$h_\mu\left(a, G_a^-\right)=h_\mu\left(a, G^{[2]}\right)+h_\mu\left(a, G^{[3]}\right).$$

Applying Theorem 7.6(ii)

- On the other hand, Theorem 7.6(ii) with $U = G_a^-$ implies that

$$h_\mu\left(a, G_a^-\right) = \int h_{\mu_x^{\mathcal{E}}}(a) d\mu(x) .$$

- Section 5.14 "The ergodic decomposition revisited" implies that

$$h_\mu\left(a, G_a^-\right) = h_\mu\left(a\right) .$$

Using the entropy assumption

- The entropy assumption is

$$h_\mu\left(a(t,s)\right) > \frac{1}{2}\left(\left|\log\left|ts^{-1}\right|\right| + \left|\log\left|t^2 s\right|\right| + \left|\log\left|ts^2\right|\right|\right).$$

- Recall that $0 < |t| = |s| < 1$.
- Conclude that

$$h_\mu\left(a, G^{[2]}\right) + h_\mu\left(a, G^{[3]}\right) > -3\log|t|.$$

An upper bound on the entropy

Theorem 7.9

Assume $a \in G$, μ an a-invariant probability measure on X and $U < G_a^-$ is normalised by a. Then

$$h_\mu(a, U) \leq -\log \left|\det \mathrm{Ad}_a |_{\mathfrak{u}}\right|$$

and equality holds if and only if μ is U-invariant. Here $\mathfrak{u}$ is the Lie algebra of U.

Applying Theorem 7.9 and Theorem 7.6(iii)

- Both $U = G^{[2]}$ and $U = G^{[3]}$ satisfy $\mathrm{Ad}_a|_{\mathfrak{u}} = t^3$.
- Theorem 7.9 implies that

$$h_\mu\left(a, G^{[2]}\right), h_\mu\left(a, G^{[3]}\right) \leq -3\log|t|.$$

- Conclude that

$$h_\mu\left(a, G^{[2]}\right), h_\mu\left(a, G^{[3]}\right) \neq 0\,.$$

- Theorem 7.6(iii) implies that $\mu_x^{G^{[2]}}$ and $\mu_x^{G^{[3]}}$ are nontrivial on a set of positive measure.
- In fact, $\mu_x^{G^{[2]}}$ and $\mu_x^{G^{[3]}}$ are nontrivial on a set of full measure.

Invariance properties of leaf-wise measures

Theorem 6.3(iii)

For every $s \in U$, μ-$\forall x \in X$ satisfies,

$$\mu_{sx}^{U} \propto (R_s)_* \left(\mu_x^U\right).$$

Lemma 7.16 (part 5 in Weikun's talk)

Assume $U < G$ is normalised by a and let μ be an a-invariant probabilty measure on X. Denote by i_a the conjugation by a. Then μ-$\forall x \in X$ satisfies

$$\mu_{ax}^{U} \propto (i_a)_* \, \mu_x^U.$$

Applying Lemma 7.16

- $G^{[2]}$ and $G^{[3]}$ are normalised by A.
- Theorem 7.16 implies that

$$\left\{x \in X \, : \, \mu_x^{[i]} = \delta_e\right\}$$

  is A-invariant μ-$\forall x \in X$ for $i \in \{2, 3\}$.
- The ergodicity of the A action implies that $\mu_x^{[i]}$ are nontrivial μ-$\forall x \in X$.

Using the backwards direction

- Repeat the same argument with a replaced with a^{-1}.
- Since $h_\mu\left(a^{-1}\right) = h_\mu(a)$ the assumptions hold.
- Conclude that $\mu_x^{[i]}$ is nontrivial μ-$\forall x \in X$ for $i \in \{1, 3\}$.

Getting invariance

- Note that $\left[G^{[-3]}, G^{[2]}\right] = G^{[1]}$.
- The group generated by $G^{[-3]}$, $G^{[1]}$ and $G^{[2]}$ is

$$\begin{pmatrix} 1 & * & * \\ & 1 & \\ & * & 1 \end{pmatrix}.$$

- This equals to $G^-_{a'}$ for $a' = \begin{pmatrix} 1/2 & & \\ & 2 & \\ & & 1 \end{pmatrix}$
- We'll now show that this implies that μ is $G^{[1]}$ invariant.

Getting invariance in the commutator direction

The theorem of Section 9.11

Let μ be an A-invariant measure on X and assume that $\mu_x^{[1]}$ and $\mu_x^{[3]}$ are nontrivial μ-$\forall x \in X$. Then μ is $G^{[2]}$ invariant.

The commutator argument

- If $g_i \in G^{[i]}$ for $1 \leq i \leq 3$ then

$$g_3 g_2 g_1 = g_2 g_3 g_1 = (g_2 [g_3, g_1]) g_1 g_3 = g_1 (g_2 [g_3, g_1]) g_3$$

- We'll first show that this implies that for μ-$\forall x \in X$, $\mu_x^{[1]}$-$\forall g_1 \in G_1$ and $\mu_x^{[1]}$-$\forall g_3 \in G_3$ satisfy

$$\mu_x^{[2]} \propto \left(R_{[g_3, g_1]^{-1}}\right)_* \mu_x^{[2]}.$$

Applying Corollary 8.8 again

- Apply Corollary 8.8 with $a' = \begin{pmatrix} 1/2 & & \\ & 1/2 & \\ & & 4 \end{pmatrix}$,
- $T = G^{[1]} < G^0_{a'}$.
- $U = G^{[2]}G^{[3]} = G^-_{a'}$.
- $G^- = TU$ the group of upper triangular matrices.
- T normalises U.
- Conclude that μ-$\forall x \in X$ satisfies $\mu_x^{G^-} \propto \iota\left(\mu_x^{[1]} \times \mu_x^{G^{[2]}G^{[3]}}\right)$.
- We already saw that μ-$\forall x \in X$ satisfies

$$\mu_x^{G^{[2]}G^{[3]}} \propto \iota\left(\mu_x^{[2]} \times \mu_x^{[3]}\right).$$

- Therefore, μ-$\forall x \in X$ satisfies

$$\mu_x^{G^-} \propto \iota\left(\mu_x^{[1]} \times \mu_x^{[2]} \times \mu_x^{[3]}\right).$$

Applying Corollary 8.8 again (cntd.)

- Similarly, apply Corollary 8.8 again, this time with

$$a' = \begin{pmatrix} 1/4 & & \\ & 2 & \\ & & 2 \end{pmatrix},$$

- This gives that μ-$\forall x \in X$ satisfies

$$\mu_x^{G^-} \propto \iota\left(\mu_x^{[3]} \times \mu_x^{[2]} \times \mu_x^{[1]}\right).$$

Using the commutator argument

- Any $f \in C_c\left(G_a^-\right)$ satisfies μ-$\forall x \in X$

$$\int f(g) d\mu_x^{G_a^-} = \int f\left(g_1 g_2 g_3\right) d\mu_x^{[1]} d\mu_x^{[2]} d\mu_x^{[3]}$$
$$= \int f\left(g_1\left(g_2\left[g_3, g_1\right]\right) g_3\right) d\mu_x^{[1]} d\mu_x^{[2]} d\mu_x^{[3]}.$$

- For any g_1, g_3 and any $f_2 \in C_c\left(G^{[2]}\right)$ denote

$$F_{f_2}\left(g_1, g_3\right) := \int f_2\left(g_2\left[g_3, g_1\right]\right) - f_2\left(g_2\right) d\mu_x^{[2]}.$$

- Use the equality above with a compactly supported continuous approximation of the function $f = f_2 F_{f_2}$ to deduce that $F_{f_2}\left(g_1, g_3\right) = 0$ $\mu_x^{[1]}$-$\forall g_1 \in G_1$ and $\mu_x^{[3]}$-$\forall g_3 \in G_3$.
- Therefore, μ-$\forall x \in X$ satisfies

$$\mu_x^{[2]} \propto \left(R_{\left[g_3, g_1\right]^{-1}}\right)_* \mu_x^{[2]}.$$

- In fact, proportionality may be upgraded to equality:

An upper bound on the growth rate of the measures of balls

Theorem 6.30

Fix any sequence $r_n \nearrow \infty$. Then μ-$\forall x \in X$ satisfies

$$\lim_{n\to\infty} \frac{\mu_x^{[2]}\left(B_{r_n}^{G^{[2]}}\right)}{n^2 m\left(B_{r_n+5}^{G^{[2]}}\right)} = 0\,.$$

Remark

In the notes published on Elon's webpage (which are linked to in the seminar webpage) this is Theorem 6.29. Compared to the official lecture notes published by the Clay Mathematics Institute, all subsections in section 6 after section 6.7 are shifted by one, since Problem 6.8 about existence of r-cross sections in the homogeneous setup appears in the official version but not in the version that's on Elon's webpage.

Applying Theorem 6.30 to get a strict invariance

- Assume $[g_3, g_1] = \begin{pmatrix} 1 & & r \\ & 1 & \\ & & 1 \end{pmatrix}$ with $r \neq 0$.
- Let $c > 1$ be such that $\mu_x^{[2]} = c \left(R_{[g_3,g_1]^{-1}} \right)_* \mu_x^{[2]}$.
- Then for every $n \in \mathbb{N}$,

$$\mu_x^{[2]} = c^n \left(R_{[g_3,g_1]^{-1}} \right)_* \mu_x^{[2]}.$$

- On the other hand, for any r_0, $[g_3, g_1]^n B_{r_0} \subseteq B_{r_0+nr}$.
- This implies that

$$\mu_x^{[2]} \left(B_{r_0+nr} \right) \geq c^n \mu_x^{[2]} \left(B_{r_0} \right).$$

- This contradicts Theorem 6.30.

Proof of the Theorem of Section 9.11 (cntd.)

- The stabiliser $\left\{g_2 \in G^{[2]} \,:\, g_2\mu_x^{[2]} = \mu_x^{[2]}\right\}$ is a closed group.
- Conclude that $\mu_x^{[2]}$ is invariant under $\left[\operatorname{supp}\mu_x^{[1]}, \operatorname{supp}\mu_x^{[3]}\right]$.
- Claim: $\operatorname{supp}\mu_x^{[i]}$ contains arbitrarily small elements for $i \in \{1, 3\}$.

Proving that supp $\mu_x^{[i]}$ contains arbitrarily small elements

- By assumption, $\mu_x^{[i]} \neq \{e\}$ μ-$\forall x \in X$, so the μ measure of

$$Y = \left\{ x \in X \; : \; \operatorname{supp} \mu_x^{[i]} \cap B_R^{G^{[i]}} \neq \{e\} \right\}$$

  approaches to 1 as R gets larger.
- By Poincare reccurence, for μ-$\forall x \in Y$ there exists arbitrarily large $n \in \mathbb{Z}$ for which $a^n x \in Y$.
- I.e.,

$$\operatorname{supp} \mu_{a^n x}^{[i]} \cap B_R^{G^{[i]}} \neq \{e\}\,.$$

- By Theorem 7.16 we know that $\mu_{a^n x}^{[i]} \propto a^n \mu_x^{[i]} a^{-n}$.
- So $\operatorname{supp} \mu_{a^n x}^{[i]} = a^n \operatorname{supp} \mu_x^{[i]} a^{-n}$.
- Therefore,

$$\operatorname{supp} \mu_x^{[i]} \cap a^{-n} B_R^{G^{[i]}} a^n \neq \{e\}\,.$$

- Conclude that $\mu_x^{[2]}$ is $G^{[2]}$ invariant.
- Hence, $\mu_x^{[2]}$ is Haar μ-$\forall x \in X$.
- This implies that μ itself is $G^{[2]}$ invariant (Problem 6.28).

Concluding the singular case

- Recall we wanted to show that μ is Haar.
- Using the entropy assumption we verified that $\mu_x^{[-3]}$ and $\mu_x^{[2]}$ are nontrivial.
- The commutator argument shows that μ is $G^{[1]}$ invariant.
- In particular, $\mu_x^{[1]}$ is Haar, so nontrivial.
- Verify that μ is invariant under $G^{[i]}$ for all $1 \leq |i| \leq 3$, by presenting each as a commutator of two other triangular groups for which we already know that the leaf-wise measures are nontrivial a.e.
- Finally, $\mu \propto m$.

The one-dimensional groups of triangular matrices

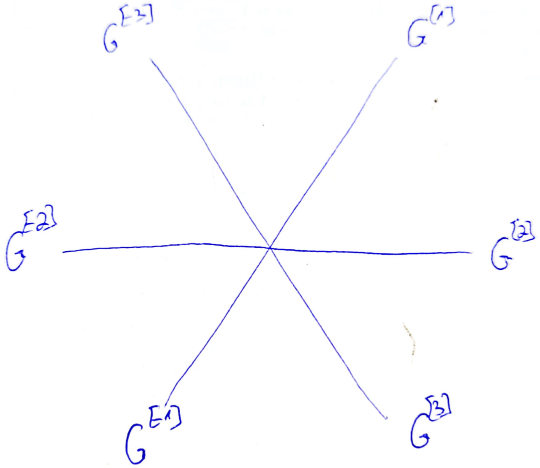

Figure: Each group is the commutator of its adjacent groups

The *regular* case

- The stable horospherical group of *a* G_a^- is a product of three commuting one-dimensional triangular subgroups.
- WLOG $0 < |t| < |s| < \left|t^{-1}s^{-1}\right|$ and $G_a^- = G^{[1]}G^{[2]}G^{[3]}$.

The product structure

$$\mu_x^{G_a^-} \propto \iota\left(\mu_x^{G^{[1]}} \times \mu_x^{G^{[2]}} \times \mu_x^{G^{[3]}}\right) .$$

The entropy contribution of stable subgroups noramalised by a

- The definition of the entropy contribution implies that

$$h_\mu\left(a, G_a^-\right) = h_\mu\left(a, G^{[1]}\right) + h_\mu\left(a, G^{[2]}\right) + h_\mu\left(a, G^{[3]}\right).$$

- Theorem 7.6(ii) and the ergodic decomposition imply that

$$h_\mu(a) = h_\mu\left(a, G^{[1]}\right) + h_\mu\left(a, G^{[2]}\right) + h_\mu\left(a, G^{[3]}\right).$$

Using the entropy assumption

- The entropy assumption is

$$h_\mu\left(a(t,s)\right) > \frac{1}{2}\left(\log\left|t^{-1}s\right| - \log\left|t^2 s\right| - \log\left|ts^2\right|\right).$$

An upper bound on the entropy

- $U = G^{[1]}$, $U = G^{[2]}$ and $U = G^{[3]}$ satisfy $\mathrm{Ad}_a|_{\mathfrak{u}} = ts^{-1}$, $\mathrm{Ad}_a|_{\mathfrak{u}} = t^2 s$ and $\mathrm{Ad}_a|_{\mathfrak{u}} = ts^2$ respectively.
- Theorem 7.9 implies that

$$h_\mu\left(a, G^{[1]}\right) \leq -\log\left|ts^{-1}\right|$$
$$h_\mu\left(a, G^{[2]}\right) \leq -\log\left|t^2 s\right|$$
$$h_\mu\left(a, G^{[3]}\right) \leq -\log\left|ts^2\right| .$$

- Note that the second upper bound is the sum of the other two:

$$\log\left|ts^{-1}\right| + \log\left|ts^2\right| = \log\left|t^2 s\right| .$$

- Conclude that at least two indices $1 \leq i \leq 3$ satisfy

$$h_\mu\left(a, G^{[i]}\right) \neq 0 .$$

The invariance property of leaf-wise measures

- Theorem 7.6(iii) implies that for these two indices $\mu_x^{G^{[i]}}$ is nontrivial on a set of positive measure.
- Ergodicity of the A action implies that

$$\left\{x \in X \,:\, \mu_x^{G^{[i]}} = \delta_e\right\}$$

  has zero measure.
- I.e., μ_x^U is nontrivial a.e. for both $U = G^{[2]}$ and $U = G^{[3]}$.

Using the backwards direction

- Repeat the same argument with a replaced with a^{-1}.
- Since $h_\mu\left(a^{-1}\right) = h_\mu(a)$ the assumptions hold.
- Conclude that μ_x^U is nontrivial a.e. for some pair of non-commuting triangular subgroups.

Getting invariance

- Use the commutator trick to verify that the leaf-wise measure in of the commutator subgroup is Haar.
- Use this extra information and repeat with all directions.

Problem 9.13

Theorem (Einsiedler–Katok 2003)

Suppose μ is A-invariant and all elements have positive entorpy. Then μ is Haar.

The proof in one picture

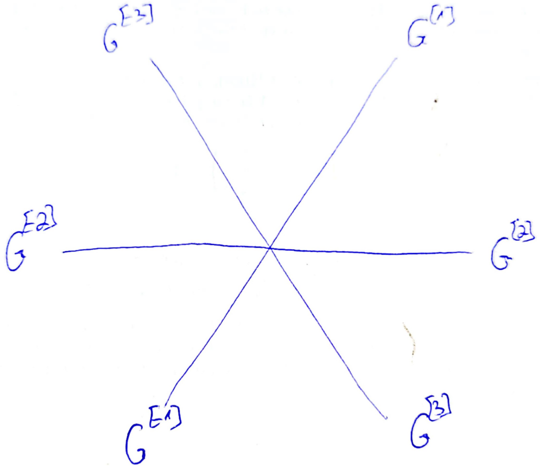

Figure: Every pair of adjacent directions is the stable subgroup of a singular $a \in A$