EXERCISE - SCHMIDT'S GAME

The game is played between two players - Alice and Bob. We first fix two parameters $0 < \alpha, \beta < 1$ and a set $S \subset \mathbb{R}^d$. Bob and Alice successively choose nested closed balls in $\mathbb{R}^d$:

$$B_1 \supseteq A_1 \supseteq B_2 \supseteq A_2 ...$$

with radii satisfying:

$$\rho(A_i) = \alpha\rho(B_i), \quad \rho(B_{i+1}) = \beta\rho(A_i)$$

where $\rho(B)$ denotes the radius of the ball B.

Then, we look at the unique point of intersection $p = \bigcap_{i=1}^{\infty} B_i$. The set S is called:

- (α, β) - winning set if Alice has a startegy guaranteeing that $p \in S$.
- α - winning if it is (α, β) - winning for every $0 < \beta < 1$.
- winning if it is α- winning for some α.

Exercise. Prove the following statements:

(1) If S is a winning set then it is dense.
(2) If S is a winning set then it is uncountable.
(3) The set of numbers in $[0, 1]$ which are not normal in base 2 is a winning set. In this case assume that Bob and Alice can only choose balls contained in $[0, 1]$. (a number x is normal in base $b \in \mathbb{N}$ if it's expansion in base b - $(b_1, b_{2,}...)$ contains every word of length i in the alphabet $(0, 1, ..., b-1)$ with asymptotic frequency $\frac{1}{b^i}$).