FROBENIUS NUMBERS

Definition 1. Denote

$$\widehat{\mathbb{Z}}^d = \left\{a = (a_1, \ldots, a_d) \in \mathbb{Z}^d : \gcd(a_1, \ldots, a_d) = 1\right\}$$

the set of primitive lattice points in $\mathbb{R}^d$.

Given $a = (a_1, \ldots, a_d) \in \widehat{\mathbb{Z}}^d_{\geq 2}$, any sufficiently large integer $N > 0$ can be represented as $N = m \cdot a$ with $m \in \widehat{\mathbb{Z}}^d_{\geq 0}$. The largest integer which is not a **non-negative** integer combination of $a_1, \ldots, a_d$ is the Frobenius number of a

$$F(a) = \max \mathbb{Z} \backslash \left\{m \cdot a > 0 : m \in \mathbb{Z}^d_{\geq 0}\right\}.$$

We also define

$$f(a) = F(a) + a_1 + \cdots + a_d,$$

the largest integer which is not a **positive** integer combination of $a_1, \ldots, a_d$.

Theorem 1. *(Sylvester) For $d = 2$*

$$F(a_1, a_2) = a_1 a_2 - a_1 - a_2.$$

For $d \geq 3$ no explicit formula is known.

Results

Definition 2. Let L be a lattice in $\mathbb{R}^{d-1}$ and let

$$\Delta = \left\{x \in \mathbb{R}^{d-1}_{\geq 0} : \sum_{i=1}^{d-1} x_i \leq 1\right\}$$

be the $d-1$-dimensional simplex . The covering radius of Δ with respect to L is

$$Q_0(L) = Q(\Delta, L) = \inf\left\{t \in \mathbb{R}^+ : L + t\Delta_{d-1} = \mathbb{R}^{d-1}\right\}.$$

Notation. Let $G_0 = \mathrm{SL}_{d-1}(\mathbb{R})$ and $\Gamma_0 = \mathrm{SL}_{d-1}(\mathbb{Z})$, then $\Omega_0 = G_0/\Gamma_0$ is the space of unimodular lattices in $\mathbb{R}^{d-1}$. Let μ_0 be a left G_0 invariant probability measure on Ω_0.

Theorem 2. *(Marklof 2010) Let $d \geq 3$.*

(1) *There exists a continuous non-increasing function $\Psi_d(R) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\Psi_d(0) = 1$, such that for any bounded set $\mathcal{D} \subset \mathbb{R}^d_{>0}$ with boundary of Lebesgue measure zero, and any $R \geq 0$,*

$$\lim_{T\to\infty} \frac{1}{T^d} \# \left\{ a \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D} : \frac{f(a)}{(a_1 \cdots a_d)^{1/(d-1)}} \leq R \right\} = \frac{\mathrm{vol}\mathcal{D}}{\zeta(d)} (1 - \Psi_d(R))$$

In fact

$$1 - \Psi_d(R) = \mu_0 (\{L \in \Omega_0 : Q_0(L) \leq R\}).$$

(2) *Q_0 is a continuous function on Ω_0.*

(3) *$\mu_0(\{L \in \Omega_0 : Q_0(L) \leq R\})$ is continuous in R, that is $\mu_0(\{L \in \Omega_0 : Q_0(L) = R\}) = 0$ for any $R > 0$.*

Steps in the proof of (1)**:**

- For $a \in \widehat{\mathbb{Z}}^d_{\geq 2}$ there exists $L_a \in \Omega_0$ such that

$$\frac{f(a)}{(a_1 \cdots a_d)^{1/(d-1)}} = Q_0(L_a).$$

- For every bounded connected non-empty open subset $\mathcal{D} \subset \left\{ x \in \mathbb{R}^d : 0 < x_i < x_d < 1 \right\}$ with boundary of Lebesgue measure zero

$$\left| T\mathcal{D} \cap \widehat{\mathbb{Z}}^d \right| \sim \frac{T^d \mathrm{vol}\mathcal{D}}{\zeta(d)}.$$

- The set of lattices $\left\{ L_a : a \in T\mathcal{D} \cap \widehat{\mathbb{Z}}^d \right\}$ becomes equidistributed in Ω_0 as $T \to \infty$, that is, for any bounded continuous function ϕ on Ω_0

$$\lim_{T\to\infty} \frac{1}{T^d} \sum_{a \in T\mathcal{D}\cap\widehat{\mathbb{Z}}^d} \phi(L_a) = \frac{\mathrm{vol}\mathcal{D}}{\zeta(d)} \int_{\Omega_0} \phi d\mu_0.$$

- Since $E_{\leq R} \{L \in \Omega_0 : Q_0(L) \leq R\}$ has boundary of measure zero we can apply the above to $\phi = \chi_{E_{\leq R}}$.

Theorem 3. *(Sylvester):* $\Psi_2(R) = \begin{cases} 1 & R < 1 \\ 0 & R \geq 1 \end{cases}.$

Indeed here $\Delta = [0, 1]$ and the covering radius with respect to the lattice $\mathbb{Z}$, which is the unique element in the space of one-dimensional lattices of unit covolume, and so

$$\mu_0(\{L \in \Omega_0 : Q_0(L) > R\}) = \begin{cases} 1 & R < 1 \\ 0 & R \geq 1 \end{cases} = \Psi_2(R).$$

Theorem 4. *(Strombergsson 2012) : For* $d \geq 3$, $\Psi_d(R) = \frac{d}{2\zeta(d-1)} R^{-(d-1)} + O_d\left(R^{-d-\frac{1}{d-2}}\right)$, *as* $R \to \infty$.

For $d = 3$ there is an explicit formula for $\Psi_3(R)$ by Shur, Sinai and Ustinov (2009).

Theorem 5. *(Han Li 2014) There exists* $k > 0$ *such that for any* $R > 0$, *any non-empty connected open subset* $\mathcal{D} \subset \left\{x \in \mathbb{R}^d : 0 < x_d < 1,\, 0 < x_i < d\right\}$ *which has thin boundary (boundary contained in a union of finitely many connected smooth submanifolds, each with dimension* $< d$*) , there exists* $C_{R,\mathcal{D}}$*such that for every* $T \geq 1$

$$\left|\frac{1}{T^d} \# \left\{a \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D} : \frac{f(a)}{(a_1 \cdots a_d)^{1/(d-1)}} \leq R\right\} - \frac{\mathrm{vol}\mathcal{D}}{\zeta(d)}\left(1 - \Psi_d(R)\right)\right| < \frac{C_{R,\mathcal{D}}}{T^k}.$$

Frobenius numbers and covering radius

Lemma 1. *(Kannan 1992) Let* $a \in \widehat{\mathbb{Z}}^d$. *Then*

$$F := F(a_1, \ldots, a_d) = \max_{l \in \{1, \ldots, a_d - 1\}} t_l - a_d$$

where $t_l \in \mathbb{N}$ *smallest integer* $\equiv l \mod a_d$ *which is a non-negative integer combination of* $a_1, \ldots, a_{d-1}$.

Proof. Let $N \in \mathbb{N}$. If $N \equiv 0 \mod a_d$ then $N = 0 \cdot a_1 + \ldots + 0 \cdot a_{d-1} + k \cdot a_d$ and so $F \geq N$.

Claim. Assume $N \equiv l \mod a_d$. N is a non-negative integer combination of $a_1, \ldots, a_d \iff N \geq t_l$.

Proof of claim.($\Leftarrow$) $N = t_l + k \cdot a_d$. ($\Rightarrow$) Definition of t_l.
So $t_l - a_d$ is the largest integer congruent to $l \mod a_d$ which is not a non-negative integer combination. □

Definition 3. Let $a \in \widehat{\mathbb{Z}}^d$.

$$M_a = \left\{x \in \mathbb{Z}^{d-1} : \sum_{i=1}^{d-1} a_i x_i \equiv 0 \mod a_d\right\}, \quad S_a = \left\{x \in \mathbb{R}^{d-1}_{\geq 0} : \sum_{i=1}^{d-1} a_i x_i \leq 1\right\}$$

note that for $\alpha > 0$ we have $\alpha S_a = \left\{x \in \mathbb{R}^{d-1}_{\geq 0} : \sum_{i=1}^{d-1} a_i x_i \leq \alpha\right\}$, and that

$$\mathrm{diag}(a_1, \ldots, a_{d-1}) \cdot S_a = \Delta.$$

Theorem 6. *(Kannan) Let* $a \in \widehat{\mathbb{Z}}^d$ *and* M, S *as above. Then* $f(a) = Q(S, L)$.

Proof. Set $M := M_a$, $S := S_a$, $f := f(a)$ and $Q := Q(S, M)$.
($\leq$) Let $y \in \mathbb{Z}^{d-1}$ such that $\sum_{i=1}^{d-1} a_i x_i \equiv l \mod a_d$. By definition of t_l there exist $x_1, \ldots, x_d \in \mathbb{Z}_{\geq 0}$ such that

$$\sum_{i=1}^{d-1} a_i x_i = t_l = l + x_d \cdot a_d.$$

Denote $x' = (x_1, \dots, x_{d-1})$, then $y - x' \in M$. Since $x' \in t_l S$ then $(y - x') + t_l S$ contains $y = (y - x') + x'$. Since $t_l \leq F + a_d$ then

$$\mathbb{Z}^{d-1} \subset (F + a_d) S + M.$$

Let $z \in \mathbb{R}^{d-1}$, then $\lfloor z \rfloor \in \mathbb{Z}^{d-1}$ and $z - \lfloor z \rfloor \in I^{d-1}$. Since $I^{d-1} \subset \alpha S \iff \alpha \geq \sum a_i \cdot 1$, then

$$z - \lfloor z \rfloor \in (a_1 + \cdots + a_{d-1}) S$$

and so

$$\mathbb{R}^{d-1} \subset \mathbb{Z}^{d-1} + (a_1 + \cdots + a_{d-1}) S \subset (a_1 + \cdots + a_d + F) S + M = fS + M$$

therefore $Q \leq f$.

($\geq$) We begin with the following claim:

Claim. $F + a_d$ is the smallest t such that $\mathbb{Z}^{d-1} \subset tS + M$.

Proof of claim. Let $t' < F + a_d$ and assume $\mathbb{Z}^{d-1} \subset t'S + M$. Let $l \in \{1, \dots, a_d - 1\}$ and $y \in \mathbb{Z}^{d-1}$ so that $\sum a_i y_i \equiv l \mod a_d$. There exists $x \in M$ such that $y \in t'S + x$, and so $y - x \in t'S$. Therefore $y - x \in \mathbb{R}^{d-1}_{\geq 0}$ and $\sum a_i (y_i - x_i) \leq t'$. Since $\sum a_i (y_i - x_i) \equiv l \mod a_d$ we get $t_l \leq t'$. since this is true for all l we have by the previous theorem and our assumption

$$F \leq t' - a_d < F$$

which is a contradiction. We have seen in the first part of the proof that $\mathbb{Z}^{d-1} \subset (F + a_d) S + M$ and so

$$F + a_d = \min \left\{ t > 0 : \mathbb{Z}^{d-1} \subset tS + L \right\}.$$

Back to the proof: There exists $y \in \mathbb{Z}^{d-1}$ such that for every $x \in M$ with $y - x \in \mathbb{R}^{d-1}_{\geq 0}$

$$\sum a_i (y_i - x_i) \geq F + a_d$$

because otherwise we contradict the minimality proved above. Let $0 < \varepsilon < 1$ and define $p = (p_1, \dots, p_{d-1}) \in \mathbb{R}^{d-1}$ by $p_i = y_i + (1 - \varepsilon)$. Suppose $q \in M$ such that $p - q \in \mathbb{R}^{d-1}_{\geq 0}$. Since $q \in \mathbb{Z}^{d-1}$ then $y - q \in \mathbb{Z}^{d-1}_{\geq 0}$.

$$\sum a_i (p_i - q_i) = \sum a_i (p_i - y_i) + \sum a_i (y_i - q_i) \geq (1 - \varepsilon) \sum a_i + (F + a_d)$$

This is true for every ε and so if $p \in QS + M$ then

$$Q \geq \sum a_i (p_i - q_i) \geq F + (a_1 + \cdots + a_d) = f$$

□

Dynamics and lattices

Definition 4. For $T>0$, $x\in\mathbb{R}^{d-1}$ and $y\in\mathbb{R}^{d-1}$ with $y_i\neq 0$, define

$$D(T)=\begin{pmatrix} T^{-1/d-1}I_{d-1} & 0\\ 0 & T\end{pmatrix}\quad n(x)=\begin{pmatrix} I_{d-1} & 0\\ x^T & 1\end{pmatrix}\quad m(y)=\begin{pmatrix} m'(y) & 0\\ 0 & 1\end{pmatrix}$$

where $m'(y)=(y_1\cdots y_{d-1})^{-\frac{1}{d-1}}\operatorname{diag}(y_1,\ldots,y_{d-1})$. Clearly $D(T)\,,\,n(x)\,,\,m(y)\in G=\mathrm{SL}_d(\mathbb{R})$.

Definition 5. Let $a\in\mathbb{Z}^d$ with $a_d\neq 0$, then

$$\widehat{a}=\left(\frac{a_1}{a_d},\ldots,\frac{a_{d-1}}{a_d}\right)^T\in\mathbb{R}^{d-1}.$$

Theorem 7. *(Han Li) Let $a\in\widehat{\mathbb{Z}}^d$, then*

$$\frac{f(a)}{(a_1\cdots a_d)^{\frac{1}{d-1}}}=Q_0\left(m(\widehat{a})\,D(a_d)\,n(\widehat{a})\,\mathbb{Z}^d\cap e_d^{\perp}\right).$$

Proof. Notice that

$$m(\widehat{a})\,D(a_d)\,n(\widehat{a})=\begin{pmatrix} m'(\widehat{a})\,a_d^{-\frac{1}{d-1}} & 0\\ 0 & a_d\end{pmatrix}n(\widehat{a})=\begin{pmatrix} m'(\widehat{a})\,a_d^{-\frac{1}{d-1}} & 0\\ a_1\,\cdots\,a_{d-1} & a_d\end{pmatrix}.$$

We identify $e_d^{\perp}\cap\mathbb{R}^d$ with $\mathbb{R}^{d-1}$in the natural way, and so

$$\begin{aligned} m(\widehat{a})\,D(a_d)\,n(\widehat{a})\,\mathbb{Z}^d\cap e_d^{\perp} &=\left\{\begin{pmatrix} m'(\widehat{a})\,a_d^{-\frac{1}{d-1}} & 0\\ a_1\,\cdots\,a_{d-1} & a_d\end{pmatrix}z|\; z\in\mathbb{Z}^d,\ \text{last entry of } m(\widehat{a})\,D(a_d)\,n(\widehat{a})\,z \text{ is zero}\right\}.\\ &=m'(\widehat{a})\,a_d^{-\frac{1}{d-1}}\left\{x\in\mathbb{Z}^{d-1}|\;\sum_{i=1}^{d-1}x_ia_i\equiv 0 \mod a_d\right\}=m'(\widehat{a})\,a_d^{-\frac{1}{d-1}}M_a\\ &=\frac{1}{(a_1\cdots a_d)^{\frac{1}{d-1}}}\operatorname{diag}(a_1,\ldots,a_d)\,M_a \end{aligned}$$

and so

$$\begin{aligned} Q_0\left(m(\widehat{a})\,D(a_d)\,n(\widehat{a})\,\mathbb{Z}^d\cap e_d^{\perp}\right) &=Q_0\left(\frac{1}{(a_1\cdots a_d)^{\frac{1}{d-1}}}\operatorname{diag}(a_1,\ldots,a_{d-1})\,M_a\right)\\ &=\frac{1}{(a_1\cdots a_d)^{\frac{1}{d-1}}}Q_0\left(\operatorname{diag}(a_1,\ldots,a_{d-1})\,M_a\right) \end{aligned}$$

But we have proved

$$\begin{aligned} f(a)&=Q\left((\operatorname{diag}(a_1,\ldots,a_{d-1}))^{-1}\Delta,M_a\right)\\ &=Q\left(\Delta,\operatorname{diag}(a_1,\ldots,a_{d-1})\,M_a\right)=Q_0\left(\operatorname{diag}(a_1,\ldots,a_{d-1})\,M_a\right) \end{aligned}$$

finishing the proof. ∎

Definition 6. Let $\mathcal{D}_0 = \left\{x \in \mathbb{R}^d : 0 < x_d < 1,\, 0 < x_i < x_d\right\}$ and denote

$$\mathcal{M}_{\mathcal{D}_0} = \{(\boldsymbol{x}, y, z) : (\boldsymbol{x}, y) \in \mathcal{D}_0,\, z \in \Omega_0\}$$

and the product measure $d_{\mathcal{D}_0} = d\boldsymbol{x} dy d\mu_0(z)$.

Theorem 8. *(Han Li) There exists $\alpha > 0$ so that for $\mathcal{C}$ a relatively compact, open subset of Ω_0, and $\mathcal{C}'$ a compact subset of $\mathcal{C}$, for any $\psi \in C^\infty(\mathcal{M}_{\mathcal{D}_0})$ with $\operatorname{supp}(\psi) \subset \{(x, y, z) \in \mathcal{M}_{\mathcal{D}_0} : z \in \mathcal{C}'\}$, and any $T > 1$*

$$\left| \frac{1}{T^d} \sum_{a \in T\mathcal{D}_0 \cap \widehat{\mathbb{Z}}^d} \psi\left(\frac{a}{T}, m(\widehat{a}) D(a_d) n(\widehat{a}) \Gamma \cap e_d^{\perp}\right) - \frac{1}{\zeta(d)} \int\limits_{\mathcal{M}_{\mathcal{D}_0}} \psi(x, y, z)\, d_{\mathcal{D}_0} \right| \ll |\mathcal{C}|^d \|\psi\|_{C^1} T^{-\alpha},$$

Corollary 1. *For any bounded continuous function ϕ on Ω_0*

$$\lim_{T \to \infty} \frac{1}{T^d} \sum_{a \in T\mathcal{D} \cap \widehat{\mathbb{Z}}^d} \phi(L_a) = \frac{\mathrm{vol}\mathcal{D}}{\zeta(d)} \int\limits_{\Omega_0} \phi d\mu_0.$$

Proof. Let ϕ_0 be the function on $\mathcal{M}_{\mathcal{D}_0}$ defined by $\phi_0(x, y, z) := \chi_{\mathcal{D}}(x, y) \phi(z)$. Recall $L_a = m(\widehat{a}) D(a_d) n(\widehat{a}) \Gamma \cap e_d^{\perp}$, then

$$\frac{1}{T^d} \sum_{a \in T\mathcal{D} \cap \widehat{\mathbb{Z}}^d} \phi(L_a) - \frac{\mathrm{vol}\mathcal{D}}{\zeta(d)} \int\limits_{\Omega_0} \phi d\mu_0 = \frac{1}{T^d} \sum_{a \in T\mathcal{D} \cap \widehat{\mathbb{Z}}^d} \phi_0\left(\frac{a}{T}, L_a\right) - \frac{1}{\zeta(d)} \int\limits_{\mathcal{M}_{\mathcal{D}_0}} \phi_0 d_{\mathcal{D}_0}$$

We can take $\mathcal{D} \subset \mathcal{D}_0$ and $\mathcal{D}$ has boundary of Lebesgue measure zero and so ϕ_0 can be approximated by smooth functions and the result follows from the theorem. □

Remark 1. The constants in the theorem are defined as follows: for a smooth function f on $\mathcal{M}_{\mathcal{D}_0}$

$$\|f\|_{C^1} := \|f\|_{L^\infty} + \sum_{i=1}^{d-1} \left\| \frac{\partial}{\partial x_i} f \right\|_{L^\infty} + \left\| \frac{\partial}{\partial y} f \right\|_{L^\infty} + \sum_X \|\partial X(f)\|_{L^\infty},\; X \in \mathcal{X} \cap \mathrm{Lie}(G_0),$$

where $\mathcal{X} \cap \mathrm{Lie}(G_0)$ is a basis of $\mathrm{Lie}(G_0)$. Let $\pi : G \to \Omega$ be the natural projection given by $\pi(g) = g\Gamma$. For $g \in G$ and $x \in \Omega$ we set

$$|g|_\infty := \max\{|a_{ij}| : g = (a_{ij})\},\;\; |x|_\infty := \inf\{|g|_\infty : \pi(g) = x\}.$$

For $\mathcal{C} \subset \Omega$ a Borel subset, we define

$$|\mathcal{C}| := \max(1, \sup\{|x|_\infty : x \in \mathcal{C}\}).$$