AN OVERVIEW OF THE PROOF OF THE VARIATIONAL PRINCIPLE, AND REDUCTION
– HANDOUT –

Templates

$d = m + n$, $d_+ = m$, $d_- = n$.

Definition. A $m \times n$-*template* is a continuous piecewise linear function $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ with the following properties:

(I) $f_1 \le f_2 \le \ldots \le f_d$
(II) $-\frac{1}{n} \le f_i' \le \frac{1}{m}$ for all i
(III) For all $j = 0, \ldots, d$ and for every interval I such that $f_j < f_{j+1}$ on I, the function

$$F_j = \sum_{i=1}^{j} f_i$$

is convex (*convexity condition*) and piecewise linear on I with slopes in

$$Z(j) = \{\tfrac{L_+}{m} - \tfrac{L_-}{n} : L_\pm \in \{1, \ldots, d_\pm\}, L_+ + L_- = j\}$$

(*quantized slope condition*). Here $f_0 = -\infty$, $f_{d+1} = \infty$.

For a template $\mathbf{f}$

$$\Delta(\mathbf{f}, T) := \tfrac{1}{T} \int_0^T \delta(\mathbf{f}, t)\, \mathrm{d}t, \quad \delta(\mathbf{f}, t) = \delta(\mathbf{f}, I) = \#\{(i_+, i_-) \in S_+ \times S_- : i_+ < i_-\}$$

for $t \in I$ where $f|_I$ is linear and

$$S_+ := \bigcup_{(p,q]_{\mathbb{Z}}} (p, p + M_+(p,q)]_{\mathbb{Z}}, \quad S_- := \bigcup_{(p,q]_{\mathbb{Z}}} (p + M_+(p,q), q]_{\mathbb{Z}}$$

where $(p,q]_{\mathbb{Z}}$ runs over intervals of equality and $M_\pm = M_\pm(p,q)$ are determined by

$$M_+ + M_- = q - p, \quad f_{p+1}' + \ldots + f_q' = \tfrac{M_+}{m} - \tfrac{M_-}{n}.$$

Moreover,

$$\overline{\delta}(f) = \limsup_{T \to \infty} \Delta(\mathbf{f}, T), \quad \underline{\delta}(f) = \liminf_{T \to \infty} \Delta(\mathbf{f}, T)$$

Definition. A function $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ is a *pseudo-template* (local notation!) if

(I) $f_1 \le f_2 \le \ldots \le f_d$
(II) $-\frac{1}{n} \le \frac{f_j(t_2) - f_j(t_1)}{t_2 - t_1} \le \frac{1}{m}$ for all j and all $t_1 \neq t_2$
(III) For every interval I such that $f_j < f_{j+1}$ on I, there exists a convex, piecewise-linear function $F_{j,I}$ on I with slopes in $Z(j)$ so that with

$$F_j = \sum_{i=1}^{j} f_i \asymp_+ F_{j,I}$$

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Varia

$\beta \in (0,1)$ small, $\gamma = -\frac{mn}{m+n}\log(\beta)$. If $(X_k)_{k\geq 1}$ are the outcomes of a Hausdorff game with parameters β, δ, set

$$\Lambda_{k+1} = g_{-k\gamma - c} u_{X_k} \mathbb{Z}^d$$

where c is some constant determined by the initial separation parameter ρ_0.

Definition. Let $\mathbf{g}$ be a γ-integral template. A lattice Λ is a *C-match* for time $t \in \gamma\mathbb{N}$ if

(1) $\|h(\Lambda) - \mathbf{g}(t)\| \leq C$

(2) There is a family of Λ-nested subspaces $(V_q)_{q\in Q(t)}$ where $Q(t) = \{q : g_q(t) < g_{q+1}(t)\}$ such that $\dim(V_q) = q$, for all i

$$|\log(\lambda_i(\Lambda \cap V_q)) - h_i(\Lambda)| \leq C$$

and $\dim(V_q \cap (\{0\} \times \mathbb{R}^n)) \geq L_-(\mathbf{g}, I, q)$ where I is an interval of linearity for $\mathbf{g}$ whose left endpoint is t.