THE KLEINBOCK–LINDENSTRAUSS–WEISS NONDIVERGENCE OF FRACTALS

1. Quantitative nondivergence

Let X be a metric space.

Definition 1. X is called *Besicovitch* if there exists an integer $N > 0$ such that for any bounded set $A \subseteq X$ and every collection of balls $\mathcal{B}$ such that every point in A is a center of some ball in $\mathcal{B}$, there exists a subcollection $\Omega \subseteq \mathcal{B}$ which satisfies

$$A \subseteq \bigcup_{B \in \Omega} B,$$
$$\max_{x \in X} \# \{B \in \Omega \,:\, x \in B\} \leq N.$$

Example 2. *$\mathbb{R}^n$ is Besicovitch, hence, every $X \subseteq \mathbb{R}^n$ is Besicovitch.*

Let μ be a Borel measure on X. For any measurable function $f : X \to \mathbb{R}$ and any measurable set $B \subseteq X$ denote the *essentail supremum of f on B with respect to μ* by

$$\|f\|_{\mu,B} = \sup \{c \,:\, \mu(\{x \in B \,:\, |f(x)| > c\}) > 0\}.$$

If f is continuous then

$$\|f\|_{\mu,B} = \sup_{x \in B \cap \operatorname{supp} \mu} |f(x)|.$$

Definition 3. For $C, \alpha > 0$, $f : X \to \mathbb{R}$ and $U \subseteq X$, f is *(C,α)-good on U with respect to μ* if any $z \in \operatorname{supp} \mu$, $r, \varepsilon > 0$ and $B = B(z,r)$ satisfy

$$\mu(\{x \in B \,:\, |f(x)| < \varepsilon\}) \leq C \left(\frac{\varepsilon}{\|f\|_{\mu,B}} \right)^{\alpha} \mu(B).$$

Definition 4. For $U \subseteq X$ open and $D > 0$ say that μ is *D-Federer on U* if every $x \in \operatorname{supp} \mu \cap U$ and every $r > 0$ for which $B(x, 3r) \subseteq U$ satisfy

$$\frac{\mu(B(x,3r))}{\mu(B(x,r))} \leq D.$$

Example 5. *The Lebesgue measure on $\mathbb{R}^n$. The Cantor–Lebesgue measure on the middle third Cantor set, as well as any pushforward of it to $\mathbb{R}^n$ by a bi-Lipschitz function.*

Theorem 6. *Let X be a Besicovitch space. Given $C, D, \alpha > 0$ there exists $C' > 0$ with the following property. Suppose $U \subseteq X$ is open, μ is D-Federer on U, $h : X \to SL_{n+1}(\mathbb{R})$ is continuous, $0 < \rho \le 1$, $z \in \operatorname{supp}\mu \cap U$, $r_0 > 0$, $B = B(z, r_0)$ satisfies $3^n B = B(z, 3^n r_0) \subseteq U$, and every $0 \le j \le n$ and every $0 \neq \mathbf{v} = \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_j \in \bigwedge^j \mathbb{Z}^{n+1}$ satisfy*

1. *$\|h(x)\mathbf{v}\|$ is (C, α)-good on $3^n B$ with respect to μ,*
2. *$\|h(x)\mathbf{v}\|_B \ge \rho$.*

Then any $0 < \varepsilon \le \rho$ satisfies

$$\mu\left(\left\{x \in B : \exists 0 \neq \mathbf{v} \in \mathbb{Z}^{n+1} \text{ such that } \|h(x)\mathbf{v}\| < \varepsilon\right\}\right) \le C' \left(\frac{\varepsilon}{\rho}\right)^{\alpha} \mu(B).$$

Theorem 6 follows by induction from a more general statement about rational flags in $\mathbb{R}^{n+1}$. However, the induction argument requires passing to a possibly bigger set.

Let $\mathcal{W}$ be the set of nonzero rational subspaces of $\mathbb{R}^{n+1}$. A *flag* $\mathcal{F}$ is a chain of rational subspaces (not necessarily maximal)

$$\{0\} =: V_0 < V_1 < V_2 < \ldots V_m < \mathbf{v}_{m+1} := R^{n+1}$$

with $V_i \in \mathcal{W}$ for every $1 \le i \le m$. In this case the *length* of $\mathcal{F}$ is m.

If $\mathcal{F}$ and $\mathcal{F}'$ are flags and $V \lesseqgtr V'$ for all nonzero proper subspaces $\mathbf{v} \in \mathcal{F}$ and $\mathbf{v}' \in \mathcal{F}'$ denote $\mathcal{F} \lesseqgtr \mathcal{F}'$, say that $\mathcal{F}$ and $\mathcal{F}'$ *subordinate* and set $\mathcal{F} + \mathcal{F}'$ to be the chain made by combining all the subspaces of $\mathcal{F}$ and $\mathcal{F}'$.

Recall that if $V \in \mathcal{W}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{Z}^{n+1}$ are linearily independent and satisfy

$$V \cap \mathbb{Z}^{n+1} = \operatorname{span}_{\mathbb{Z}}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$$

then

$$\operatorname{covol}\left(V/V \cap \mathbb{Z}^{n+1}\right) = \|\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k\|.$$

For any $g \in \mathrm{SL}_{n+1}(\mathbb{R})$ and any $V \in \mathcal{W}$ define $\ell_V(g) = \operatorname{covol}(gV)$ to be the covolume.

We will pass freely between rational subspaces and the wedge product of a generating set of it.

Definition 7. Let $\mathcal{F} \lesseqgtr \mathcal{F}'$ be flags and assume $0 < \varepsilon \le \rho \le 1$. A point $x \in X$ is *marked by $\mathcal{F}$ relative to $\mathcal{F}'$* if it satisfies the following:

(i) If $0 \neq V \in \mathcal{F}$ then $\varepsilon \le \ell_V(h(x)) < \rho$.
(ii) If $\mathcal{F}'' \lesseqgtr \mathcal{F}'$ satisfies $\ell_V(h(x)) < \rho$ for every $V \in \mathcal{F}''$ then $|\mathcal{F}''| \le |\mathcal{F}|$.

Let $\mathcal{F}_0 = \{\{0\}, \mathbb{R}^{n+1}\}$. For a flag $\mathcal{F}$ say that x is *marked by $\mathcal{F}$* if it is marked by $\mathcal{F}$ relative to $\mathcal{F}_0$. Say that x is *marked* if x is marked by some flag.

Exercise 8. *What does it mean to be marked by $\mathcal{F}_0$?*

Lemma 9. *If x is marked then $h(x) \in K_\varepsilon$.*

Proof. Denote $g = h(x)$. Assume that $0 \neq \mathbf{v} \in \mathbb{Z}^{n+1}$ satisfies $\|h(x)\mathbf{v}\| < \varepsilon$. Let $\mathcal{F}$ be any flag. Let $0 \le i \le m$ be such that $\mathbf{v} \in V_{i+1} \setminus V_i$. Let

$V = \operatorname{span}_{\mathbb{R}}\{V_i, \mathbf{v}\}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be linearly independent vectors such that $V_i \cap \mathbb{Z}^{n+1} = \operatorname{span}_{\mathbb{Z}}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$ and $V \cap \mathbb{Z}^{n+1} = \operatorname{span}_{\mathbb{Z}}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Then

$$\ell_V(g) = \|g\mathbf{v}_1 \wedge \ldots \wedge g\mathbf{v}_k\| \leq \|g\mathbf{v}_1 \wedge \ldots \wedge g\mathbf{v}_{k-1}\| \cdot \|g\mathbf{v}_k\| < \rho\varepsilon \leq \varepsilon.$$

In particular, $\ell_V(g) \leq \rho$. Since V is comparable to $\mathcal{F}$ this implies by definition of x being marked that $V = V_{i+1}$. But $\ell_V(g) \geq \varepsilon$, which is a contradiction. □

Say that x is marked relative to $\mathcal{F}'$ if there exists $\mathcal{F} \lneqq \mathcal{F}'$ such that x is marked by $\mathcal{F}$ relative to $\mathcal{F}'$.

Theorem 10. *Let X be a Besicovitch space. Given $C, D, \alpha > 0$ and $0 \leq m \leq n$ there exists $C' > 0$ with the following property: For $U \subseteq X$ open, μ D-Federer on U, $h : X \to SL_{n+1}(\mathbb{R})$ continuous, $0 < \rho \leq 1$, $z \in \operatorname{supp}\mu \cap U$, $r_0 > 0$, $B = B(z, r_0)$, if $\mathcal{F}$ a flag of length m such that $3^{n-m}B \subseteq U$ and every $V \in \mathcal{W}$ satisfies*

- (1') *$\ell_V \circ h$ is (C, α)-good on $3^{n-m}B$ with respect to μ,*
- (2') *If $V \lneqq \mathcal{F}$ then $\|\ell_V \circ h\|_B \geq \rho$,*

then every $0 < \varepsilon \leq \rho$ satisfies

$$\mu\left(\{x \in B \,:\, x \textit{ is not marked relative to } \mathcal{F}\}\right) \leq C' \left(\frac{\varepsilon}{\rho}\right)^{\alpha} \mu(B).$$

This theorem with $\mathcal{F} = \mathcal{F}_0$ implies Theorem 6.

Proof. By induction. For $m = n$ the conclusion is trivial since there are no subspaces that are comparable to a complete flag. Assume the theorem hold for $1 \leq m+1 \leq n$, and let $\mathcal{F}$ be any flag of length m such that $3^{n-m}B \subseteq U$ and (1') and (2') are satisfies. For any $x \in B$ that is not marked relative to $\mathcal{F}$, define

$$r_x = \max_{V \lneqq \mathcal{F}} \sup\left\{r \,:\, \|\ell_V(h(x))\|_{B(x,r)} < \rho\right\}, \tag{1}$$

This is well defined. Indeed, $\mathcal{W}$ is discrete and the set $\{V \in \mathcal{W} \,:\, \ell_V(h(x)) < \rho\}$ is bounded, and if $\ell_V(h(x)) \geq \rho$ for all $V \lneqq \mathcal{F}$ then x is marked by $\mathcal{F}_0$ relative to $\mathcal{F}$, which is a contradiction to the assumption that x is not marked by $\mathcal{F}$. Also, r_x is positive since h is continuous. Let $V_0(x) \lneqq \mathcal{F}$ be any subspace that attaines the maximum in (1). By (2'), $B(x, r)$ does not contain B for any $r < r_x$, therefore

$$r_x \leq r_0 + \mathrm{d}(z, x) \leq 2r_0. \tag{2}$$

$$\mu\left(\{x\in B\,:\,x \text{ is not marked relative to } \mathcal{F}\}\right)\leq$$

$$\mu\left(\bigcup_{x\in\Omega}\{y\in B\left(x,r_x\right)\,:\,\ell_{V_0(x)}(h(y))<\varepsilon\}\right)$$

$$+\,\mu\left(\bigcup_{x\in\Omega}\{y\in B\left(x,r_x\right)\,:\,\ell_{V_0(x)}(h(y))\geq\varepsilon \text{ and } y \text{ is not marked relative to } \mathcal{F}\}\right)\leq$$

$$\sum_{x\in\Omega}C\left(\frac{\varepsilon}{\rho}\right)^{\alpha}\mu\left(B(x,r_x)\right)\quad(\|\ell_{V_0(x)}\circ h\|_{B(x,r_x)}\geq\rho + (C,\alpha)\text{-good on } U \text{ with respect to } \mu)$$

(3)

$$+\sum_{x\in\Omega}C'\left(\frac{\varepsilon}{\rho}\right)^{\alpha}\mu\left(B(x,r_x)\right)\leq\qquad(\text{induction} + (1))$$

$$N(C+C')\left(\frac{\varepsilon}{\rho}\right)^{\alpha}\mu(B(z,3r_0))\leq\qquad((2))$$

$$ND(C+C')\left(\frac{\varepsilon}{\rho}\right)^{\alpha}\mu(B)\qquad(\mathit{Federer})$$

The upper bound (3) is where the induction hypothesis is used (applied to a possibly larger set). Indeed, assume that $y\in B\left(x,r_x\right)$ is marked relative to $\mathcal{F}'(x):=\mathcal{F}+V_0(x)$ and satisfies $\ell_{V_0(x)}(h(y))\geq\varepsilon$. Then there exists $\mathcal{F}''\lneqq\mathcal{F}'(x)$ such that y is marked by $\mathcal{F}''$ relative to $\mathcal{F}'(x)$. Since $\varepsilon\leq\ell_{V_0(x)}(h(y))\leq\rho$ we also have that y is marked by $\mathcal{F}''+V_0(x)$ relative to $\mathcal{F}'(x)$. So, y is marked by $\mathcal{F}'(x)$. Working backwards with the negation we see that if $y\in B\left(x,r_x\right)$ is not marked relative to $\mathcal{F}$ and satisfies $\ell_{V_0(x)}(h(y))\geq\varepsilon$ then y is not marked relative to $\mathcal{F}'(x)$. The flag $\mathcal{F}'(x)$ has length $m+1$ so we're in shape to apply the inductive hypothesis. First, as already noticed, $B(x,r_x)\subseteq B(z,3r_0)$ so $B\left(x,3^{n-(m+1)}r_x\right)\subseteq B\left(z,3^{n-m}r_0\right)\subseteq U$, hence, (1') holds. Secondly, in order to verify (2') with $\mathcal{F}'(x)$, assume that $U\in\mathcal{W}$ satisfies $U\lneqq\mathcal{F}'(x)$. In particular, this implies that $U\lneqq\mathcal{F}$, and, therefore, necessarily, $\|\ell_U\circ h\|_{B(x,r_x)}\geq\rho$. Indeed, $\|\ell_U\circ h\|_{B(x,r_x)}<\rho$ would contradict the definition of r_x. This verifies the conditions of Theorem 10. □

2. Applications

Let μ be a Borel measure on $\mathbb{R}^n$.

Definition 11. μ is *nonplanar* if $\mu(H)$ for any affine hyperplane $H \subseteq \mathbb{R}^n$.

Definition 12. For $C, \alpha > 0$, $U \subseteq \mathbb{R}^n$, μ is (C,α)-*decaying* if every $r > 0$, $z \in \operatorname{supp}\mu$, $B = B(x,r) \subseteq U$, $\varepsilon > 0$ satisfy

$$\mu(B \cap B(H,\varepsilon)) \leq C \left(\frac{\varepsilon}{\|d_H\|_{\mu,B}} \right)^\alpha \mu(B).$$

It is (C,α)-*absolutely decaying* if

$$\mu(B \cap B(H,\varepsilon)) \leq C \left(\frac{\varepsilon}{r} \right)^\alpha \mu(B).$$

Definition 13. μ is *friendly* if it is nonplanar, decaying and Federer. It is *absolutely friendly* if it is absolutely decaying and Federer.

Theorem 14 (KLW 2004)**.** *Every friendly measure μ on $\mathbb{R}^n$ satisfies $\mu(VWA) = 0$.*

Theorem 15 (KW 2005)**.** *Every absolutely friendly measure μ on $\mathbb{R}^n$ satisfies* $\operatorname{supp}\mu \cap BA \neq \varnothing$.