Leafwise Measures

1 Reminder – Conditional measures

Theorem 1.1. *Let $(X, \mathcal{B}, \mu)$ be a probability space, with $(X, \mathcal{B})$ being a lcsc metric space, and let $\mathcal{A} \subseteq \mathcal{B}$ a sub-σ-algebra. Then there exists an $\mathcal{A}$-measurable $X' \subseteq X$ with $\mu(X \setminus X') = 0$, and a system $\{\mu_x^{\mathcal{A}} : x \in X'\}$ of probability measures on X, referred to as* ***conditional measures****, such that:*

$$\forall f \in L^1(X, \mathcal{B}, \mu) \quad E(f|\mathcal{A})(x) = \int f(y)\, d\mu_x^{\mathcal{A}}(y) \quad \textit{for } \mu\textit{-a.e } x$$

and $\mu_x^{\mathcal{A}}$ is uniquely determined for μ-a.e $x \in X$. In other words, for any $f \in L^1(X, \mathcal{B}, \mu)$ the map $x \mapsto \int f(y)\, d\mu_x^{\mathcal{A}}(y)$ is $\mathcal{A}$-measurable, and for all $A \in \mathcal{A}$

$$\int_A \int f(y)\, d\mu_x^{\mathcal{A}}(y)\, d\mu(x) = \int_A f d\mu$$

Definition 1.2. Let $\mathcal{A}$ be a σ-algebra of subsets of X. The **atom** of $x \in X$ is defined by:

$$[x]_{\mathcal{A}} = \bigcap_{x \in A \in \mathcal{A}} A$$

Note that for countably generated σ-algebra, all atoms are measurable.

Claim 1.3. Under the same assumptions as in the theorem, if $\mathcal{A}$ is countably generated, then:

1. $\forall x \in X' \quad \mu_x^{\mathcal{A}}([x]_{\mathcal{A}}) = 1$.
2. For every $x, y \in X'$ with $[x]_{\mathcal{A}} = [y]_{\mathcal{A}}$ we have $\mu_x^{\mathcal{A}} = \mu_y^{\mathcal{A}}$.

2 Leafwise Measures

2.1 Assumptions

- G is a locally compact, second countable group, equipped with a right-invariant metric such that any ball of finite radius has compact closure.

- X is a lcsc metric space, G acts continuously on X.
- μ is locally finite measure on X, meaning that $\mu(K) < \infty$ for any compact $K \subset X$.
- $H < G$ is a closed subgroup, such that for μ-a.e $x \in X$ the map $h \in H \mapsto h.x$ is injective.

Notation: For two measures μ, ν on X, denote $\mu \simeq \nu$ if μ, ν are proportional, i.e if there exists some $c > 0$ s.t $\mu = c\nu$.

2.2 Definition of Leafwise Measures

Theorem 2.1. *Under these assumptions, there exists a H-invariant set $X' \subseteq X$ with $\mu(X \setminus X') = 0$, and a system $\{\mu_x^H\}_{x \in X'}$ of locally-finite measures on H, called the **leafwise measures**, which are determined uniquely, up to proportionality and outside a set of measure zero, by the following properties:*

1. *For every $f \in C_c(H)$, the map $x \mapsto \int f d\mu_x^H$ is Borel measurable.*
2. *"μ_x^H describes μ along the H-orbit of x": Suppose $Y \subset X$ is measurable with $\mu(Y) < \infty$ and that there exists a countably generated σ-algebra $\mathcal{A}$ on Y that is H-**subordinate**, meaning that for μ-a.e $y \in Y$ the atom has the form $[y]_{\mathcal{A}} = V_y.y$, for some bounded neighborhood $V_y \subset H$ of the identity in H (referred to as the shape of the atom). Then for μ-a.e $y \in Y$*
$$\mu_y^{\mathcal{A}} \simeq \left(\mu_y^H|_{V_y}\right).y$$
i.e. we restrict the leafwise measure to the shape of the atom, push the restriction forward under the orbit map $h \in H \mapsto h.y$ and obtain the proportionality class of the conditional measure.
3. *Compatibility Formula: For every $x \in X'$ and $h \in H$, if $h.x \in X'$ then*
$$\mu_x^H \simeq (T_h)_* \mu_{h.x}^H$$
where $(T_h)_* \mu_{h.x}^H$ is the push-forward of $\mu_{h.x}^H$ by the right translation $T_h : H \to H$, $g \mapsto gh$ (Also denoted $\left(\mu_{h.x}^H\right) h$).

Proposition 2.2. *Let $H < G$. If μ_x^G is left H-invariant for μ-a.e $x \in X$ then μ is H-invariant.*

2.3 Example

Let $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and $G = \mathbb{R}$ acts on X by $r.x = x + r\vec{v} \mod \mathbb{Z}^2$, for some irrational vector $\vec{v}$. If $\mu = \lambda$ is the Lebesgue measure on $\mathbb{T}^2$, then we can take $\mu_x^G = \lambda_{\mathbb{R}}$ to be Lebesgue measure on $\mathbb{R}$.

Note that:

- Even though the space X is compact, none of the leaf-wise measures are finite.
- The naive approach to constructing these measures would be to look at conditional measures for the sub-σ-algebra $\mathcal{A}$ of G-invariant Borel sets. Unfortunately, this σ-algebra is not countably generated, and is equivalent to the trivial σ-algebra. Instead, we define the leaf-wise measures on small pieces of G-orbits and then glue them together.

2.4 Fubini-construction of Leafwise measures

Consider the product space $X \times H$ and define the σ-algebra:

$$\mathcal{C}_H = \Psi_H^{-1}\mathcal{B}_X$$

where $\Psi_H : X \times H \to X$ is the map defined by $\Psi_H(x_0, h_0) = h_0^{-1}.x_0$ for $(x_0, h_0) \in X \times H$, and $\mathcal{B}_X$ is the Borel σ-algebra of X. Since $\mathcal{B}_X$ is countably generated, the σ-algebra $\mathcal{C}_H$ is a countably generated algebra of Borel subsets of $X \times H$.

Define $\Delta_H \coloneqq \{(h,h) \mid h \in H\} \cong H$, and let Δ_H act on $X \times H$ by setting

$$\forall x_0 \in X,\ h_0, h \in H \quad (h,h).(x_0,h_0) = (h.x_0, hh_0)$$

Also, for $h \in H$, let $\Delta(h) = (h,h) \in \Delta_H$.

Let us calculate the atoms in $\mathcal{C}_H$. By definition of $\mathcal{C}_H$,

$$\begin{aligned} [(x_0,h_0)]_{\mathcal{C}_H} &= \{(x_1,h_1) \in X \times H \mid \Psi_H(x_1,h_1) = \Psi_H(x_0,h_0)\} \\ &= \left\{(x_1,h_1) \in X \times H \mid h_1^{-1}.x_1 = h_0^{-1}.x_0\right\} \end{aligned}$$

Any (x_1, h_1) in this atom belongs to $\Delta_H.(x_0, h_0)$: Indeed, set $h \coloneqq h_1 h_0^{-1}$, then $h_1 = hh_0$ and

$$x_1 = e.x_1 = hh^{-1}.x_1 = hh_0h_1^{-1}.x_1 = hh_0\left(h_1^{-1}.x_1\right) = hh_0h_0^{-1}.x_0 = h.x_0$$

So

$$\Delta(h).(x_0,h_0) = (h.x_0, hh_0) = (x_1,h_1)$$

And we get $[(x_0,h_0)]_{\mathcal{C}_H} \subseteq \Delta_H.(x_0,h_0)$. The other direction is also true: for all $h \in H$ we have

$$\Psi_H(\Delta(h).(x_0,h_0)) = \Psi_H(h.x_0, hh_0) = (hh_0)^{-1} h.x_0 = h_0^{-1}.x_0 = \Psi_H(x_0,h_0)$$

Thus $[(x_0,h_0)]_{\mathcal{C}_H} = \Delta_H.(x_0,h_0)$. In other words the σ-algebra countably generated $\mathcal{C}_H$ on $X \times H$ has atoms that consist of orbits of the group Δ_H.

2.4.1 Fibre measures

We now define fibre measures – a generalization of conditional measures (which are defined for probability measures) to locally finite measures.

Suppose that ν is a locally finite measure on a σ-compact locally compact metric space Y and that $\mathcal{C}$ is a countably generated σ-algebra. We can choose a strictly positive continuous integrable function f_0 in Y with $\int f_0 d\nu = 1$ (Exercise: how?), and set $\nu_{\text{prob}} := f_0\nu$. Notice that ν_{prob}, ν are absolutely continuous to each other, meaning that they have the same null-sets. ν_{prob} is a probability measure, so there are conditional measures $(\nu_{\text{prob}})_y^{\mathcal{C}}$ for ν_{prob} with respect to $\mathcal{C}$ for a.e $y \in Y$.

Definition 2.3. **(Fibre measures)** $\nu_y^{\mathcal{C}} = \frac{1}{f_0}(\nu_{\text{prob}})_y^{\mathcal{C}}$ for a.e $y \in Y$.

From this definition we see that $\nu_{y_1}^{\mathcal{C}} = \nu_{y_2}^{\mathcal{C}}$ whenever $[y_1]_{\mathcal{C}} = [y_2]_{\mathcal{C}}$, and each $\nu_y^{\mathcal{C}}$ is a locally finite measure on Y.

A different choice f_0' of such a strictly positive continuous function instead f_0 will give a new system of fiber measures ν_y' that a.e. are in the same proportionality class as the the original system of fiber measures. Since we will mostly care only about the proportionality class, we keep the dependence of $\nu_y^{\mathcal{C}}$ on the choice of f_0 implicit.

Properties of fibre measures:

1. If $[y_1]_{\mathcal{C}} = [y_2]_{\mathcal{C}}$ then $\nu_{y_1}^{\mathcal{C}} = \nu_{y_2}^{\mathcal{C}}$.
2. Each $\nu_y^{\mathcal{C}}$ is a locally finite measure on Y.
3. If $K \subseteq Y$ is a subset of finite ν-measure, then the conditional measure of $\nu|_K$ with respect to $\mathcal{C}$ can be obtained by taking the normalized restriction of the fibre measures, i.e.
$$(\nu|_K)_y^{\mathcal{C}} \simeq \nu_y^{\mathcal{C}}|_K \quad \text{for a.e } y \in K$$
4. If $T : Y \to Y$ is preserving ν, or more generally if $T_*\nu \simeq \nu$, then
$$T_*\nu_y^{T^{-1}\mathcal{C}} \simeq \nu_{Ty}^{\mathcal{C}} \quad \text{for a.e } y$$

Proposition 2.4. *If $\mathcal{C} \subseteq \tilde{\mathcal{C}}$ are two countably generated σ-algebras, then for ν-a.e $y \in Y$ and for $\nu_y^{\mathcal{C}}$-a.e z it holds that $\left(\nu_y^{\mathcal{C}}\right)_z^{\tilde{\mathcal{C}}} \simeq \nu_z^{\tilde{\mathcal{C}}}$. In particular, if $\tilde{\mathcal{C}}$ satisfies that the $\mathcal{C}$-atoms are countable unions of $\tilde{\mathcal{C}}$-atoms, then the fibre measures $\nu_y^{\tilde{\mathcal{C}}}$ can be obtained by restricting the fibre measures $\nu_y^{\mathcal{C}}$ to the atoms for $\tilde{\mathcal{C}}$.*

2.4.2 Back to our product space

Assume μ is a probability measure on X (we can replace locally-finite μ by $\mu' = f_1\mu$, where f_1 is such that $\int_X f_1 d\mu = 1$, and μ, μ' have exactly the same null-sets). Choose $Y = X \times H$,

$\nu = \mu \times m_H$ (where m_H is the right Haar measure on H), we obtain the fibre measures $(\mu \times m_H)^{\mathcal{C}_H}_{(x_0,h_0)}$ for $(\mu \times m_H)$-a.e $(x_0, h_0) \in X \times H$.

Also, we can make a choice of f_0 in the construction of fibre measures: Take f_0 to be a strictly positive continuous integrable function on H, and implicitly identify it with the corresponding function on $X \times H$ depending only on the second coordinate.

Define right action of H on $X \times H$ by

$$R_h (x_0, h_0) = \left(x_0, h_0 h^{-1}\right)$$

for every $(x_0, h_0) \in X \times H$ and $h \in H$. Then $R_h : X \times H \to X \times H$ is measure-preserving, and commutes with the the left action of Δ_H. Moreover, for all $h \in H$ we have

$$\Psi_H (R_h (x_0, h_0)) = \Psi_H \left(x_0, h_0 h^{-1}\right) = h h_0^{-1}.x_0 = h.\Psi_H (x_0, h_0)$$

for all $(x_0, h_0) \in X \times H$, which implies $R_h^{-1}\mathcal{C}_H = \mathcal{C}_H$, so we get

$$(R_h)_* (\mu \times m_H)^{\mathcal{C}_H}_{(x_0,h_0)} \simeq (\mu \times m_H)^{\mathcal{C}_H}_{(x_0,h_0 h^{-1})} \tag{2.1}$$

for any fixed $h \in H$ and $(\mu \times m_H)$-a.e (x_0, h_0), where the null set may depend on h.

We also denote by R_h the right action on H: $R_h (h_0) = h_0 h^{-1}$.

Note that when identifying a function f on H with functions on $X \times H$ depending only on the second coordinate, both possible interpretations of $f \circ R_h$ coincide.

A cleaner version of the fibre measures. Recall that by construction

$$\int f_0 d (\mu \times m_H)^{\mathcal{C}_H}_{(x_0,h_0)} = 1 \quad \text{a.e}$$

Define

$$f_H (h) = \inf_{h' \in B_1^H} f_0 \left(h h'\right)$$

Since the metric on G is proper (any ball has a compact closure) and by the assumptions on f_0 it follows that f_H is also a strictly positive integrable and continuous function on H:

$$f_H (h) = \inf_{h' \in B_1^H} f_0 \left(h h'\right) \geq \inf_{h' \in \overline{B_1^H}} f_0 \left(h h'\right) = \min_{h' \in \overline{B_1^H}} f_0 \left(h h'\right) > 0$$

Let $p : X \times H \to H$ be the projection $p (x, h) = h$ for $(x, h) \in X \times H$, and define

$$\mu_{(x_0,h_0)} = (R_{h_0} \circ p)_* (\mu \times m_H)^{\mathcal{C}_H}_{(x_0,h_0)}$$

Since p is a homeomorphism when restricted to the Δ_H-orbit, the measure $\mu_{(x_0,h_0)}$ is a locally finite measure on H. Moreover, for all $(x_0, h_0) \in X \times H$ and $(x_1, h_1) \in \Delta_H.(x_0, h_0)$:

$$\Delta\left(R_{h_0} \circ p\left(x_1, h_1\right)\right).(x_0, h_0) = \Delta\left(h_1 h_0^{-1}\right).(x_0, h_0) = (x_1, h_1)$$

So the push-forward of $\mu_{(x_0,h_0)}$ under $h \in H \mapsto \Delta(h).(x_0, h_0)$ is again $(\mu \times m_H)^{\mathcal{C}_H}_{(x_0,h_0)}$.

Also note that $p \circ R_h = R_h \circ p$ for $h \in H$. Thus, applying $R_{h_0 h^{-1}} \circ p$ to both measures in (2.1) we get:

$$\left(p \circ R_{h_0 h^{-1}}\right)_* (R_h)_* (\mu \times m_H)^{\mathcal{C}_H}_{(x_0,h_0)} \simeq \left(R_{h_0 h^{-1}} \circ p\right)_* (\mu \times m_H)^{\mathcal{C}_H}_{(x_0,h_0 h^{-1})}$$

Thus

$$\mu_{(x_0,h_0)} \simeq \mu_{(x_0,h_0 h^{-1})} \tag{2.2}$$

for any fixed $h \in H$ and $(\mu \times m_H)$-a.e $(x_0, h_0) \in X \times H$.

Lemma 2.5. *There exists a measurable conull set $X' \subseteq X$ and a choice of the fibre measures $(\mu \times m_H)^{\mathcal{C}_H}_{(x_0,h_0)}$ on $X' \times H$ so that:*

1. *Equations* (2.1) ,(2.2) *hold for all $x_0 \in X'$ and all $h_0, h \in H$.*
2. *$\int_H f_H d\mu_{(x_0,h_0)} < \infty$ for all $x_0 \in X'$ and $h_0 \in H$.*

Proposition 2.6. *Let $H < G$,X, $\mathcal{C}_H$ and μ as above. Let $X' \subseteq X$ and $\mu_{(x,h)}$ for $(x, h) \in X' \times H$ be as in the lemma. Then the measures defined by*

$$\mu_x^H \coloneqq \mu_{(x,e)}$$

for all $x \in X'$ satisfy the characterising properties of leaf-wise measures.

Proof. **[of Proposition2.6]** First, let us show that the compatibility formula holds. If $x, h.x \in X'$ then (x, e) and $(h, h).(x, e) = (h.x, h)$ are in the same Δ_H-orbit, meaning that they are in the same atom of $\mathcal{C}_H$, so $(\mu \times m_H)^{\mathcal{C}_H}_{(x,e)} \simeq (\mu \times m_H)^{\mathcal{C}_H}_{(h.x,h)}$. Applying the push forward under p to this fibre measure we then obtain:

$$\begin{aligned}\mu^H_{h.x} \overset{\text{def}}{=} \mu_{(h.x,e)} \overset{(2.2)}{\simeq} \mu_{(h.x,h)} \overset{\text{def}}{=} (R_h \circ p)_* (\mu \times m_H)^{\mathcal{C}_H}_{(h.x,h)} \simeq (R_h \circ p)_* (\mu \times m_H)^{\mathcal{C}_H}_{(x,e)} \\ \simeq (R_h)_* \mu_{(x,e)} \overset{\text{def}}{=} \mu_{(x,e)} h^{-1} \overset{\text{def}}{=} \mu^H_x h^{-1}\end{aligned}$$

then $\mu^H_{h.x} \simeq \mu^H_x h^{-1}$.

For the other property: Suppose $Y \subset X$ is measurable and $\mathcal{A}$ is a countably generated σ-algebra on Y which is H-subrodinate. We may assume without loss of generality that the injectivity requirement $h \in H \mapsto h.y$ holds for all $y \in Y$. Then for all $y \in Y$ the atom has the form $[y]_{\mathcal{A}} = V_y.y$, for some bounded neighborhood $V_y \subset H$ of the identity in H.

We now consider four σ-algebras of subsets of $Y \times H$:

1. The restriction of $\mathcal{C}_H$ to $Y \times H$, for simplicity again denoted by $\mathcal{C}_H$, whose atoms are

$$[(y,h)]_{\mathcal{C}_H} = (\Delta_H.(y,h)) \cap (Y \times H)$$

2. $\mathcal{A}_{\times H} := \{A \times H : A \in \mathcal{A}\}$, whose atoms are

$$[(y,h)]_{\mathcal{A}_{\times H}} = [y]_{\mathcal{A}} \times H = (V_y.y) \times H$$

3. $\mathcal{A}_{\text{flat}} := \mathcal{A} \times \mathcal{B}_H$ (where $\mathcal{B}_H$ is the Borel σ-algebra on H), whose atoms are

$$[(y,h)]_{\mathcal{A}_{\text{flat}}} = [y]_{\mathcal{A}} \times \{h\} = (V_y.y) \times \{h\}$$

4. $\mathcal{A}_H := \sigma(\mathcal{C}_H \cup \mathcal{A}_{\times H})$, whose atoms are

$$[(y,h)]_{\mathcal{A}_H} = \Delta(V_y).(y,h)$$

We are now going to use Proposition 2.4 in two different ways in order to calculate the fibre measures $(\mu \times m_H)^{\mathcal{A}_H}_{(y,h)}$ on the $\mathcal{A}_H$-atoms for a.e $(y,h) \in Y \times H$.

- Applying on $\mathcal{C}_H \subseteq \mathcal{A}_H$: Recall that the group H is second countable and that for any $y \in Y$ the set V_y contains an open neighborhood of the identity. Hence each atom of $\mathcal{C}_H$ contains only countably many atoms of $\mathcal{A}_H$. By (the second statement of) Proposition 2.4, it follows that

$$(\mu \times m_H)^{\mathcal{A}_H}_{(y,h)} \simeq (\mu \times m_H)^{\mathcal{C}_H}_{(y,h)}|_{\Delta(V_y).(y,h)} \quad \text{for a.e } (y,h) \in Y \times H \tag{2.3}$$

- Applying on $\mathcal{A}_{\times H} \subseteq \mathcal{A}_{\text{flat}}$: Notice that the definition of fibre measures implies for $(\mu \times m_H)$-a.e $(y_0,h_0) \in Y \times H$ that

$$(\mu \times m_H)^{\mathcal{A}_{\times H}}_{(y_0,h_0)} \simeq \mu^{\mathcal{A}}_{y_0} \times m_H$$

  Fix such a $(y_0,h_0) \in Y \times H$ and look conditionally on its atom in $\mathcal{A}_{\times H}$: $[y_0]_{\mathcal{A}} \times H = (V_{y_0}.y_0) \times H$, with respect to this fibre measure. Consider the map

$$\xi : \left(h'.y_0, h'h\right) = \Delta\left(h'\right).(y_0,h) \mapsto \left(h'.y_0, h\right)$$

  - ξ is a map from the atom $[y_0]_{\mathcal{A}} \times H$ to itself, preserving $(\mu \times m_H)^{\mathcal{A}_{\times H}}_{(y_0,h_0)}$.
  - On $[y_0]_{\mathcal{A}} \times H$, ξ maps atoms of $\mathcal{A}_H$ precisely to atoms of $\mathcal{A}_{\text{flat}}$, equivalently: $\xi^{-1}\mathcal{A}_{\text{flat}} = \mathcal{A}_H$.

  Also notice that

$$(\mu \times m_H)^{\mathcal{A}_{\text{flat}}}_{(y,h)} \simeq \mu^{\mathcal{A}}_y \times \delta_h$$

for a.e $(y,h)\in Y\times H$, where δ_h denotes the delta measure at h. Thus, by Proposition 2.4, it follows that for $(\mu\times m_H)^{\mathcal{A}\times H}_{(y_0,h_0)}$-a.e $(y,h)=(h'.y_0,h)$ with $h'\in V_{y_0}$

$$\xi_*\left((\mu\times m_H)^{\mathcal{A}_H}_{(y,h)}\right)\simeq\mu^{\mathcal{A}}_y\times\delta_{(h')^{-1}h} \tag{2.4}$$

Now take a look at (2.3) again:

$$(\mu\times m_H)^{\mathcal{A}_H}_{(y,h)}\simeq(\mu\times m_H)^{\mathcal{C}_H}_{(y,h)}|_{\Delta(V_y).(y,h)}\quad\text{for a.e }(y,h)\in Y\times H$$

By construction of $\mu^H_y=\mu_{(y,\cdot)}$ for $y\in X'\cap Y$, the fibre measure on the right can be obtained by restricting μ^H_y to V_y and pushing it forward under the map $D:h_1\in V_y\mapsto\Delta(h_1).(y,h)$, meaning that

$$(\mu\times m_H)^{\mathcal{C}_H}_{(y,h)}|_{\Delta(V_y).(y,h)}\simeq D_*\left(\mu^H_y|_{V_y}\right)$$

which means, by (2.3), that $D_*\left(\mu^H_y|_{V_y}\right)\simeq(\mu\times m_H)^{\mathcal{A}_H}_{(y,h)}$. Applying the push forward by ξ, we get by (2.4)

$$\xi_*D_*\left(\mu^H_y|_{V_y}\right)\simeq\mu^{\mathcal{A}}_y\times\delta_{(h')^{-1}h}$$

Finally, applying the projection to X we obtain a measure proportional to $\mu^{\mathcal{A}}_y$, and composing these maps we obtain that the push-forward of $\mu^H_y|_{V_y}$ under the map $h\in V_y\mapsto h.y\in[y]_{\mathcal{A}}$ is proportionate to $\mu^{\mathcal{A}}_y$, meaning that

$$\mu^{\mathcal{A}}_y\simeq\left(\mu^H_y|_{V_y}\right).y$$

as we wished. $\square$

3 References

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