LEAFWISES MEASURES

TAU SEMINAR ON HOMOGENEOUS DYNAMICS AND APPLICATIONS

1. Conditional measures of a probability measure

In this section, we review the notion of conditional measures. We will only state definitions and properties that are going to be used and refer the readers to Chapter 5 in the book [EW11] for proofs.

We say a σ-algebra $\mathcal{A}$ on a set X is countably generated, if it is generated as a σ-algebra by a countable subset $\mathcal{A}_0 \subset \mathcal{A}$. Let $\mathcal{A}$ be countably generated σ-algebra on X and $\mathcal{A}_0$ a generating subset. The atom of a point $x \in X$ is the smallest element of $\mathcal{A}$ containing x, that is

$$[x]_{\mathcal{A}} = \bigcap_{A \in \mathcal{A} : x \in A} A = \bigcap_{A \in \mathcal{A}_0 : x \in A} A.$$

A measurable space $(X, \mathcal{B})$ is a called a (standard) Borel space if X can be equipped with a locally compact second countable topology such that $\mathcal{B}$ is the corresponding Borel σ-algebra.

For a measurable space $(X, \mathcal{B})$, $\mathcal{L}^\infty(X, \mathcal{B})$ denote the space of bounded measurable functions on X.

For a measured space $(X, \mathcal{B}, \mu)$, let $\mathcal{L}^1(X, \mathcal{B}, \mu)$ denote the space of $\mathcal{B}$-measurable μ-integrable functions on X. Let $L^1(X, \mathcal{B}, \mu)$ denote the usual Banach space obtained as the space of equivalence classes in $\mathcal{L}^1(X, \mathcal{B}, \mu)$ for the relation "being equal μ-almost everywhere". Most of the time, we can safely identify a function in $\mathcal{L}^1(X, \mathcal{B}, \mu)$ and its equivalence class as an element in $L^1(X, \mathcal{B}, \mu)$. (An execption being in the definition of conditional measures, a few lines below.)

Now let $(X, \mathcal{B}, \mu)$ be a probability space. Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{B}$. Let $f \in L^1(X, \mathcal{B}, \mu)$. The conditional expectation of f knowing $\mathcal{A}$ is the unique function $\mathbb{E}(\phi \mid \mathcal{A}) \in L^1(X, \mathcal{A}, \mu)$ satisfying the following property. For any $A \in \mathcal{A}$,

$$\int_A \mathbb{E}(f \mid \mathcal{A}) \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu.$$

The conditional measure of μ knowing $\mathcal{A}$ is a collection of probability measures $(\mu_x^{\mathcal{A}})_{x \in X}$ on the measured space $(X, \mathcal{B})$ satisfying for all $f \in \mathcal{L}^1(X, \mathcal{B}, \mu)$, for almost all $x \in X$,

$$\mathbb{E}(f \mid \mathcal{A})(x) = \int_X f \, \mathrm{d}\mu_x^{\mathcal{A}}.$$

The collection $(\mu_x^{\mathcal{A}})_{x \in X}$ is unique up to a null set. Note that for any $f \in \mathcal{L}^\infty(X, \mathcal{B})$, for any $A \in \mathcal{A}$,

$$\int_A f \, \mathrm{d}\mu = \int_A \int f \, \mathrm{d}\mu_x^{\mathcal{A}} \, \mathrm{d}\mu(x).$$

Hence, by the monotone convergence theorem, the same holds for $\mathcal{B}$-measurable non-negative function $f \colon X \to \mathbb{R}_+$. In particular with $A = X$, we write

$$\mu = \int_X \mu_x^{\mathcal{A}} \, \mathrm{d}\mu(x).$$

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In the case where $\mathcal{A}$ is countably generated, then for μ-almost all $x \in X$, $\mu_x^{\mathcal{A}}([x]_{\mathcal{A}}) = 1$. The uniqueness can be reformulated as follows. Let $(\nu_x)_{x\in X}$ be a collection of probability measures on $(X, \mathcal{B})$. Assume

(1) For any $f \in \mathcal{L}^\infty(X, \mathcal{B})$, the function $x \mapsto \int f \,\mathrm{d}\nu_x$ is $\mathcal{B}$-measurable.
(2) For μ-almost all x and x', If $[x]_{\mathcal{A}} = [x']_{\mathcal{A}}$ then $\nu_x = \nu_{x'}$.
(3) For μ-almost all x, $\nu_x([x]_{\mathcal{A}}) = 1$.
(4) For any $f \in \mathcal{L}^\infty(X, \mathcal{B})$, $\int f \,\mathrm{d}\mu = \int_X \int f \,\mathrm{d}\nu_x \,\mathrm{d}\mu(x)$.

Then for μ-almost all x, $\nu_x = \mu_x^{\mathcal{A}}$.

A special situation is the following. Let $(Y, \mathcal{C})$ be a Borel space and let $\pi \colon X \to Y$ be a measurable map. Let $\eta = \pi_*\mu$ be the pushforword of μ on Y. Consider $\mathcal{A} = \pi^{-1}(\mathcal{C})$, which is countably generated since $\mathcal{C}$ is. Atoms in $\mathcal{A}$ are fibers of the map π, that is, for all $x \in X$, $[x]_{\mathcal{A}} = \pi^{-1}(\{\pi(x)\})$. For almost all $x, x' \in X$, if they have the same atom, i.e. $\pi(x) = \pi(x')$, then $\mu_x^{\mathcal{A}} = \mu_{x'}^{\mathcal{A}}$. If this is the case we define $\mu_{\pi(x)}^\pi = \mu_x^{\mathcal{A}}$. Thus μ_y^π is defined for η-almost all $y \in Y$ (we can complete arbitrarily to make (μ_y^π) a collection indexed by Y). We have

(1) For any $f \in \mathcal{L}^\infty(X, \mathcal{B})$, the function $y \mapsto \int f \,\mathrm{d}\mu_y^\pi$ is $\mathcal{C}$-measurable.
(2) For η-almost all y, $\mu_y^\pi(\pi^{-1}\{y\}) = 1$.
(3) For any $f \in \mathcal{L}^\infty(X, \mathcal{B})$, $\int f \,\mathrm{d}\mu = \int_Y \int f \,\mathrm{d}\mu_y^\pi \,\mathrm{d}\eta(y)$.

The collection $(\mu_y^\pi)_{y\in Y}$ is characterized up to a ν-null set by these properties.

2. Conditional measures of a σ-finite measure

Now assume no longer μ is a probability measure. Instead only assume that μ is σ-finite.

Theorem 2.1. *Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be a Borel spaces. Let $\pi \colon X \to Y$ be a measurable map. Let μ be a σ-finite measure on $(X, \mathcal{B})$. Then there is a finite measure η on $(Y, \mathcal{C})$ and a collection $(\mu_x^\pi)_{y\in Y}$ of nonzero σ-finite measures on $(X, \mathcal{B})$ such that*

(1) *For any $\mathcal{B}$-measuralbe function $f \colon X \to \mathbb{R}_+$, the function $y \mapsto \int f \,\mathrm{d}\mu_y^\pi$ is $\mathcal{C}$-measurable.*
(2) *For η-almost all y, $\mu_y^\pi(X \setminus \pi^{-1}\{y\}) = 0$.*
(3) *For any $\mathcal{B}$-measuralbe function $f \colon X \to \mathbb{R}_+$,*
$$\int f \,\mathrm{d}\mu = \int_Y \int f \,\mathrm{d}\mu_y^\pi \,\mathrm{d}\eta(y).$$

Moreover, these properties characterizes the collection $(\mu_x^\pi)_{y\in Y}$ up to proportionality and up to an η-null set. That is, if η' is a finite measure on $(Y, \mathcal{C})$ and $(\nu_y)_{y\in Y}$ is a collection of σ-finite measures on $(X, \mathcal{B})$ satisfying

(1) *For any $\mathcal{B}$-measuralbe function $f \colon X \to \mathbb{R}_+$, the function $y \mapsto \int f \,\mathrm{d}\nu_y$ is $\mathcal{C}$-measurable.*
(2) *For η-almost all y, $\nu_y(X \setminus \pi^{-1}\{y\}) = 0$.*
(3) *For any $\mathcal{B}$-measuralbe function $f \colon X \to \mathbb{R}_+$,*
$$\int f \,\mathrm{d}\mu = \int_Y \int f \,\mathrm{d}\nu_y \,\mathrm{d}\eta'(y).$$

Then for η-almost all $y \in Y$, there exists $c(y) \in (0, +\infty)$ such that $\nu_y = c(y)\mu_y^\pi$.

Proof. Since μ is sigma finite, there is an increasing sequence of measurable sets $K_n \in \mathcal{B}, n \geq 1$ such that
$$\forall n \geq 1,\ \mu(K_n) < +\infty, \quad \text{and} \quad X = \bigcup_{n\geq 1} K_n.$$

Set
$$f_0 = \sum_{n\geq 1} \frac{\mathbb{1}_{K_n}}{2^n \mu(K_n)}$$
so that $f_0\mu$ is a probability measure on $(X, \mathcal{B})$ equivalent to μ. Let $\eta = \pi_*(f_0\mu)$ and let $(f_0\mu)^\pi_y$, $y \in Y$ be the conditional measures of $f_0\mu$ with respect to the sub-σ-algebra $\pi^{-1}(\mathcal{C})$, constructed above. It is then straightforward to check that the collection $\mu^\pi_y := f_0^{-1}(f_0\mu)^\pi_y$, $y \in Y$ satisfies the required properties. This proves the existence.

To prove the uniqueness, assume η' and $(\nu_y)_{y\in Y}$ are as in the statement. First observe that from the second point, it follows that, for any or any $\mathcal{B}$-measuralbe function $f\colon X \to \mathbb{R}_+$, any $B \in \mathcal{C}$ and η'-almost all y,
$$\int_{\pi^{-1}(B)} f \, \mathrm{d}\nu_y = \mathbb{1}_B(y) \int f \, \mathrm{d}\nu_y.$$

By the Lebesgue decomposition theorem and the Radon-Nikodym theorem there exists a function $f_1\colon Y \to \mathbb{R}_+$ and a measure η'' singular to η such that
$$\eta' = f_1\eta + \eta''.$$
Let $E \in \mathcal{C}$ be such that $\eta''(E) = 0$ and $\eta(Y \setminus E) = 0$. On the one hand, by the third point, the observation above and the choice of B,
$$\begin{aligned}\int_{\pi^{-1}(E)} f \, \mathrm{d}\mu &= \int_Y \int_{\pi^{-1}(E)} f \, \mathrm{d}\nu_y \, \mathrm{d}\eta'(y) \\ &= \int_E \int f \, \mathrm{d}\nu_y \, \mathrm{d}\eta'(y) \\ &= \int_E \left(\int f \, \mathrm{d}\nu_y \right) f_1(y) \, \mathrm{d}\eta(y) + \int_E \int f \, \mathrm{d}\nu_y \, \mathrm{d}\eta''(y) \\ &= \int_Y f_1(y) \int f \, \mathrm{d}\nu_y \, \mathrm{d}\eta(y).\end{aligned}$$
One the other hand, for the same reason
$$\int_{\pi^{-1}(E)} f \, \mathrm{d}\mu = \int_Y \int f \, \mathrm{d}\mu^\pi_y \, \mathrm{d}\eta(y) = \int f \, \mathrm{d}\mu.$$
Therefore, for all $\mathcal{B}$-measurable $f\colon X \to \mathbb{R}_+$,
$$\int f \, \mathrm{d}\mu = \int_Y f_1(y) \int f \, \mathrm{d}\nu_y \, \mathrm{d}\eta(y).$$
In particular, for all $\mathcal{B}$-measurable $f\colon X \to \mathbb{R}_+$,
$$\int f f_0 \, \mathrm{d}\mu = \int_Y f_1(y) \int f f_0 \, \mathrm{d}\nu_y \, \mathrm{d}\eta(y). \tag{2.1}$$
In particular, for any $B \in \mathcal{C}$, using again the observation,
$$\begin{aligned}\eta(B) = \int_{\pi^{-1}(B)} f_0 \, \mathrm{d}\mu &= \int_Y \int_{\pi^{-1}(B)} f_0 \, \mathrm{d}\big(f_1(y)\nu_y\big) \, \mathrm{d}\eta(y) \\ &= \int_B f_1(y) \int f_0 \, \mathrm{d}\nu_y \, \mathrm{d}\eta(y).\end{aligned}$$
This implies that for η-almost all $y \in Y$, the measure $f_1(y)f_0\nu_y$ is a probability measure. Morevoer it gives full measure to $\pi^{-1}\{y\}$ by the second point and remember (2.1). Thus, by the uniqueness of conditional measures in the probability case applied to $f_0\mu$, we obtain, for η-almost all $y \in Y$,
$$f_1(y) f_0 \nu_y = (f_0\mu)^\pi_y = f_0 \mu^\pi_y$$

and hence $f_1(y)\nu_y = \mu_y^\pi$. To conclude, remark that $f_1(y) > 0$ for η-almost all $y \in Y$ simply because $f_1(y)f_0\nu_y$ is a probability measure. $\square$

Below is a technical lemma about measurability.

Lemma 2.2. *Let $\pi(X, \mathcal{B}) \to (Y, \mathcal{C})$ be a measurable map between Borel spaces. Let $(Z, \mathcal{D})$ be another Borel space. Let $(\nu_y)_{y\in Y}$ be a collection of measures on $(X, \mathcal{B})$ satisfying the first point in Theorem 2.1. Then for any non-negative measurable function f on $X \times Z$, the map $Y \times Z \to \mathbb{R}$*

$$(y, z) \to \int_X f(x, z)\,\mathrm{d}\nu_y(x)$$

is measurable.

Proof. Let $\mathcal{A}$ be the set of all measurable subsets $A \subset X \times Z$ such that $(y, z) \to \int_X \mathbb{1}_A(x, z)\,\mathrm{d}\nu_y(x)$ is measurable. It is a monotone class, by linearity and the monotone convergence theorem. It contains all subsets of the form $B \times D$ with $B \in \mathcal{B}$ and $D \in \mathcal{D}$, because

$$(y, z) \mapsto \int_X \mathbb{1}_{B\times C}(x, z)\,\mathrm{d}\nu_y(x) = \mathbb{1}_C(z)\nu_y(B)$$

is the product of two measurable functions, using the assumption. By the monotone class theorem, $\mathcal{A}$ is equal to the product σ-algebra $\mathcal{B} \times \mathcal{D}$.

By linearity, for any simple function f on $X \times Z$, $(y, z) \to \int_X f(x, z)\,\mathrm{d}\nu_y(x)$ is measurable. Writing any non-negative measurable function as the increasing limit of non-negative simple functions, we conclude the proof of the lemme with the monotone convergence theorem. $\square$

3. Leafwise measures

Here we present the approach in [BQ11, Section 4], with more details.

3.1. Construction.

Let R be a locally compact group acting measurably on a Borel space Z. Let λ be a Borel probability measure on Z.

Example 3.1. Let G be a Lie group and Γ a lattice in G. Let R be a subgroup of G. Then R acts on $Z = G/\Gamma$.

A measurable subset $\Sigma \subset Z$ is called a *discrete section* for the action of R if for every $z \in \Sigma$, there exists a neighbourhood $U \subset R$ of the identity 1_R in R such that for $r \in U$, $r.z \in \Sigma$ if and only if $r = 1_R$. Note that this is equivalent to require that for any $z \in Z$, the set $\{ r \in R \mid r.z \in \Sigma \}$ is discrete and closed in R. A discrete section is always equipped with the trace σ-algebra from Z. A discrete section is said to be *complete* if moreover $R.\Sigma = Z$

From now on assume that the stabiliser $\mathrm{Stab}_R(z)$ of each point $z \in Z$ is discrete in R. By a result of Kechris [Ke92], complete discrete sections exist.

Example 3.2. In the setting of Example 3.1, for any $g\Gamma \in Z$, $\mathrm{Stab}_R(g\Gamma) = R \cap g\Gamma g^{-1}$ is discrete. One can construct easily discrete sections using a linear complement of the Lie algebra of R in the Lie algebra of G.

For a discrete section Σ, let $a\colon R \times \Sigma \to Z$ denote the map

$$a(r, z) = r.z.$$

Define $a^*\lambda$ to be the measure on $R \times \Sigma$ satisfying for any measurable function $f\colon R \times \Sigma \to \mathbb{R}_+$,

$$\int f\,\mathrm{d}a^*\lambda = \int_Z \left(\sum_{(r,w)\in a^{-1}\{z\}} f(r, w) \right) \mathrm{d}\lambda(z). \tag{3.1}$$

First remark that $a^*\lambda$ is σ-finite. Indeed, let $(U_m)_{m\geq 1}$ be a countable basis of neighbourhoods of 1_R in R. Consider the subsets

$$L_m = \{\, z \in \Sigma \mid \forall r \in U_m,\, r.z \in \Sigma \Rightarrow r = 1_R \,\}. \tag{3.2}$$

These sets are measurable, by the assumption that Σ is a discrete section and a general fact from descriptive set theory [Ke95, Lemma 18.12]. Using again the assumption that Σ is a discrete section, we obtain

$$\Sigma = \bigcup_{m\geq 1} L_m.$$

Take $(K_n)_{n\geq 1}$ to be a sequence of compact sets in R such that the interiors of $K_n, n \geq 1$ cover R. Then for $n, m \geq 1$ and any $z \in Z$, the intersection $(K_n \times L_m) \cap a^{-1}\{z\}$ is finite and with cardinality bounded independently of z. (More precisely the cardinality is less than the maximal cardinality of U_m-separated subsets in K_n.) It follows that $a^*\lambda(K_n \times L_m)$ is finite. Since $\bigcup_{n,m\geq 1} K_n \times L_m = R \times \Sigma$, this shows that $a^*\lambda$ is σ-finite.

[WH] by U_m-separated I mean for r, r' in the subset $r \in U_m r'$ implies $r = r'$.

Then, apply Theorem 2.1 to the coordinate projection $\pi_\Sigma \colon R \times \Sigma \to \Sigma$. We obtain a finite measure η on Σ and conditional measures $(a^*\lambda)_z^{\pi_\Sigma}$, $z \in \Sigma$. Note that $(a^*\lambda)_z^{\pi_\Sigma}$ is null outside $R \times \{z\}$. Thus we can identify it with a Borel measure written σ_z on R in the obvious way. Thus we have, for any non-negative measurable function f on $R \times \Sigma$,

$$\int f \,\mathrm{d}a^*\lambda = \int_\Sigma \int_R f(r,z)\,\mathrm{d}\sigma_z(r)\,\mathrm{d}\eta(z). \tag{3.3}$$

In particular, using the notation from the previous paragraph,

$$a^*\lambda(K_n \times L_m) = \int_{L_m} \sigma_z(K_n)\,\mathrm{d}\eta(z),$$

which, by the above, is finite for any $n, m \geq 1$. If follows that for η-almost all $z \in \Sigma$, σ_z is a Radon measure on R. By modifying σ_z for z in a η-null set, we may safely assume that σ_z is a nonzero Radon measure for all $z \in \Sigma$.

The following observation is going to be useful,

Lemma 3.3. *Let N be measurable subset of Σ. If $\eta(N) = 0$, then $\lambda(R.N) = 0$.*

Here $R.N = a(R \times N) = \{\, r.z \mid r \in R, z \in N \,\}$. Since the map $a \colon R \times \Sigma \to Z$ is countable-to-1, by [Ke95, Exercise 18.14], $R.N$ is indeed measurable.

Proof. This follows from (3.1) and (3.3),

$$\lambda(R.N) \leq (a^*\lambda)(R \times N) = \int_N \sigma_z(R)\,\mathrm{d}\eta(z) = 0. \qquad \square$$

The next step is to extend the definition of σ_z from Σ to the whole Z. For $r \in R$, let $\tau_r \colon R \to R$ denote the right translation $t \mapsto t.r$. Note that

$$\forall r, r' \in R,\ \tau_r \circ \tau_{r'} = \tau_{r'r}.$$

For two measures σ and σ' on R, we write $\sigma \propto \sigma'$ if there exists $c > 0$ such that $\sigma' = c\sigma$.

Lemma 3.4. *For η-almost all $z \in \Sigma$ and for all $r \in R$ such that $r.z \in \Sigma$, we have*

$$\sigma_z \propto \tau_{r*}\sigma_{r.z}.$$

Proof. Because Σ is a discrete section, we can find measurable subsets $\Sigma_i \subset \Sigma$ and measuralbe maps $r_i \colon \Sigma_i \to R$, indexed by $i \in \mathbb{N}$ such that

$$\{\, (r,z) \in R \times \Sigma \mid r.z \in \Sigma \,\} = \bigcup_{i\in\mathbb{N}} \{\, (r_i(z), z) \mid z \in \Sigma_i \,\}.$$

It is possible to do so such that for each $i \in \mathbb{N}$, the map $\phi_i \colon \Sigma_i \to \Sigma$ defined by

$$\phi_i(z) = r_i(z).z$$

is injective and the inverse map $\phi_i^{-1} : \phi_i(\Sigma_i) \to \Sigma_i$ is measurable. For each $i \in \mathbb{N}$, let η_i' be the image measure of the restriction of η to $\phi_i(\Sigma_i)$ by ϕ_i^{-1}.

We claim that for any non-negative measurable function f on $R \times \Sigma_i$,

$$\int_{R\times\Sigma_i} f \,\mathrm{d}a^*\lambda = \int_{\Sigma_i}\int_R f(r,z)\,\mathrm{d}({\tau_{r_i(z)}}_*\sigma_{r_i(z).z})(r)\,\mathrm{d}\eta'(z).$$

Here, we can use Lemma 2.2 to see that $z \mapsto \int_R f(r,z)\,\mathrm{d}({\tau_{r_i(z)}}_*\sigma_{r_i(z).z})(r)$ is measurable on Σ_i. Then, an application of the uniqueness statment of Theorem 2.1 to the restriction of $a^*\lambda$ to $R \times \Sigma_i$ yields

$$\text{for } \eta\text{-almost all } z \in \Sigma_i, \quad \sigma_z \propto {\tau_{r_i(z)}}_*\sigma_{r_i(z).z},$$

which will finish the proof of the lemma.

The proof of the claim is straightforward:

$$\begin{aligned}
&\int_{\Sigma_i}\int_R f(r,z)\,\mathrm{d}({\tau_{r_i(z)}}_*\sigma_{r_i(z).z})(r)\,\mathrm{d}\eta'(z)\\
&= \int_{\Sigma_i}\int_R f(rr_i(z),z)\,\mathrm{d}\sigma_{\phi_i(z)}(r)\,\mathrm{d}((\phi_i^{-1})_*\eta)(z)\\
&= \int_{\phi_i(\Sigma_i)}\int_R f\big(rr_i(\phi_i^{-1}(w)),\phi_i^{-1}(w)\big)\,\mathrm{d}\sigma_w(r)\,\mathrm{d}\eta(w)\\
&= \int_Z\left(\sum_{(r,w')\in R\times\phi_i(\Sigma_i):r.w'=z} f\big(rr_i(\phi_i^{-1}(w')),\phi_i^{-1}(w')\big)\right)\mathrm{d}\lambda(z)\\
&= \int_Z\left(\sum_{(r,z')\in R\times\Sigma_i:r.\phi_i(z')=z} f\big(rr_i(z'),z'\big)\right)\mathrm{d}\lambda(z)\\
&= \int_Z\left(\sum_{(s,z')\in R\times\Sigma_i:s.z'=z} f\big(s,z'\big)\right)\mathrm{d}\lambda(z)\\
&= \int_{R\times\Sigma_i} f\,\mathrm{d}a^*\lambda.
\end{aligned}$$

The second equality is a change of variable $w = \phi_i(z)$. The third equality is (3.3) and (3.1). The fourth equality is a change of vaiable $w' = \phi_i(z')$. The fifth equality is a change of variable $s = rr_i(z')$. $\square$

With Lemma 3.4 at hand, we can extend the domain of definition of σ_z to Z. Assume the section is complete (i.e. $R.\Sigma = Z$). Then we can choose (by the Lusin-Novikov uniformization theorem [Ke95, Theorem 18.10]) a measurable map $r\colon Z \to R$ such that $t(z) = 1_R$ if $z \in \Sigma$ and that for all $z \in Z$, $t(z).z \in \Sigma$. Now define for every $z \in Z$,

$$\sigma_z = {\tau_{t(z)}}_*\sigma_{t(z).z}.$$

The measurability of $z \mapsto \sigma_z$ is going to be useful, when combined with Egorov's theorem or Lusin's theorem.

[WH] One might also give a topological/measurable structure to the set of Radon measures on R and say that $z \mapsto \sigma_z$ is measurable.

Lemma 3.5. *For any non-negative measurable function f on R, the map*

$$Z \to [0,+\infty], \quad z \mapsto \int_R f\,\mathrm{d}\sigma_z$$

is measurable.

Proof. This map is the composition of the measurable map

$$Z \to R \times \Sigma, \quad z \mapsto (t(z), t(z).z)$$

and the map

$$R \times \Sigma \to \mathbb{R} \cup \{+\infty\}, \quad (s, z) \to \int_R f(rs)\, \mathrm{d}\sigma_z(r)$$

which is also measurable thanks to Lemma 2.2. □

At this stage, note that in the construction of (σ_z) we made the following 3 choices:

(1) the complete discrete section Σ,
(2) the conditional measures of the σ-finite measure $a^*\lambda$,
(3) and the measurable map t making the map $z \mapsto t(z).z$ a measurable retract from Z to Σ.

The next lemma shows that the proportionality class of σ_z up a null set is independent of these choices.

Lemma 3.6. *Let $(\sigma'_z)_{z\in Z}$ be a family of leafwise measures constructed in the same way but subject to the choice of another discrete section Σ', the choice the conditional measures, and the choice of measurable map t'. Then*

$$\text{for } \lambda\text{-almost all } z \in Z, \quad \sigma'_z \propto \sigma_z.$$

Proof. Observe that $\Sigma \cup \Sigma'$ is also a discrete section. This allows us to reduce to the case where $\Sigma' \subset \Sigma$.

Write

$$D = \{\, z \in Z \mid \sigma'_z \not\propto \sigma_z \,\}.$$

and

$$N = \{\, z \in \Sigma \mid \exists r \in R, r.z \in \Sigma \text{ but } \sigma_z \not\propto \tau_{r*}\sigma_{r.z} \,\}.$$

By Lemma 3.4, $\eta(N) = 0$. Hence $\lambda(R.N) = 0$ by Lemma 3.3. Write also

$$N' = \{\, z \in \Sigma' \mid \sigma'_z \not\propto \sigma_z \,\}.$$

Let η' denote the finite measure on Σ' in the construction of (σ'_z). Using (3.1) and (3.3) each twice, we have for any non-negative measurable function f on $R \times \Sigma$,

$$\int_{R\times\Sigma'} f\, \mathrm{d}(a^*\lambda) = \int_\Sigma \int_R f(r, z)\, \mathrm{d}\sigma_z(r)\, \mathrm{d}\eta(z) = \int_{\Sigma'} \int_R f(r, z)\, \mathrm{d}\sigma'_z(r)\, \mathrm{d}\eta'(z)$$

By the uniquness in Theorem 2.1 applied to the restriction of $a^*\lambda$ to $R \times \Sigma'$,

$$\eta'(N') = 0.$$

By Lemma 3.3 again, $\lambda(R.N') = 0$.

For any $z \in Z$, on the one hand,

$$\sigma_z = \tau_{t(z)_*}\sigma_{t(z).z} \propto \tau_{t(z)_*}\tau_{t'(z)t(z)^{-1}{}_*}\sigma_{t'(z).z} = \tau_{t'(z)_*}\sigma_{t'(z).z}$$

unless $t'(z).z \in N$. On the other hand,

$$\sigma'_z = \tau_{t'(z)_*}\sigma'_{t'(z).z} \propto \tau_{t'(z)_*}\sigma_{t'(z).z}$$

unless $t'(z).z \in N'$. We have shown $D \subset R.N \cup R.N'$. Hence $\lambda(D) = 0$. □

A consequence of (the proof of) the previous lemma is the following statement which summarises the above construction.

Theorem 3.7. *Given a measurable action of a locally compact second countable group R on a Borel space Z with discrete stabilisers, given a Borel probability measure λ on Z, there is a collection $(\sigma_z)_{z\in Z}$ of nonzero Radon measures on R satisfying the following properties.*

(1) *For any non-negative measurable function f on R, the map*
$$Z \to [0, +\infty], \quad z \mapsto \int_R f \, \mathrm{d}\sigma_z$$
is measurable.

(2) *For any discrete section $\Sigma \subset Z$ for the action of R, there exists a finite measure η on Σ such that for any non-negative measurable function f on $R \times \Sigma$,*

$$\int_Z \left(\sum_{(r,w) \in R \times \Sigma : r.w = z} f(r, w) \right) \mathrm{d}\lambda(z) = \int_\Sigma \int_R f(r, z) \, \mathrm{d}\sigma_z(r) \, \mathrm{d}\eta(z). \tag{3.4}$$

Moreover, these properties characterise the measures (σ_z) up to proportionality class and up to a null set.

We call the collection $(\sigma_z)_{z \in Z}$ the *leafwise measures* of λ along the action of R.

In particular, if $\Sigma \subset Z$ is a measurable subset and U is an open neighbourhood of 1_R in R such that the map $a \colon U \times \Sigma$, $a(r, z) = r.z$ is injective. Then (3.4) becomes
$$\int_{U \times \Sigma} f \, \mathrm{d}a_*^{-1} \lambda_{|a(U \times \Sigma)} = \int_\Sigma \int_U f(r, z) \, \mathrm{d}\sigma_z(r) \, \mathrm{d}\eta(z),$$
Which is saying that $(\sigma_{z|U})_{z \in \Sigma}$, up to renormalisations, is the conditional measure of the finite measure $a_*^{-1}\lambda_{|a(U \times \Sigma)}$ on $U \times \Sigma$ along the projection to Σ. The approach in [EL10, Chapter 6] to define the leafwise measures is to define $\sigma_{z|U}$ in this way and then let U grow bigger and bigger and then patch they together.

[WH] This somehow explains why the EL approach needs the extra assumption that $r \mapsto r.z$ is injective for almost all z.

3.2. **Further properties.** Let R, Z, λ and $(\sigma_z)_{z \in Z}$ be as in Theorem 3.7.

Lemma 3.8. *For λ-a.e. $z \in Z$, for all $r \in R$,*
$$\sigma_z \propto \tau_{r*} \sigma_{r.z}.$$

Proof. Let D denote the set of $z \in Z$ such that there exists $r \in R$, such that $\sigma_z \not\propto \tau_{r*}\sigma_{r.z}$. In view of Lemma 3.4 and Lemma 3.3, it suffices to prove that $D \subset R.N$, where
$$N = \{\, z \in \Sigma \mid \exists r \in R, r.z \in \Sigma \text{ but } \sigma_z \not\propto \tau_{r*}\sigma_{r.z} \,\}.$$
Indeed, if $z \notin R.N$ then $t(z).z \notin N$. Hence for any $r \in R$,
$$\sigma_z = \tau_{t(z)_*} \sigma_{t(z).z} = \tau_{t(z)_*} \tau_{t(r.z) r t(z)^{-1}{}_*} \sigma_{t(r.z).r.z} = \tau_{r*} \tau_{t(r.z)_*} \sigma_{t(r.z).r.z} = \tau_{r*} \sigma_{r.z}.$$
Hence $z \notin D$. $\square$

Lemma 3.9. *Let (Z', λ') be another measured Borel space on which R acts measurably. Let $\phi \colon (Z, \lambda) \to (Z', \lambda')$ be an isomorphism of measured spaces. Assume that there is an automorphism α of R as topological group such that for all $r \in R$, for all $z \in Z$,*
$$r.(\phi(z)) = \phi(\alpha(r).z).$$
Let $(\sigma'_{z'})_{z' \in Z'}$ denote the leafwise measures of λ' with respect to the action of R. Then for λ-a.e. $z \in Z$,
$$\sigma_z \propto \alpha_* \sigma'_{\phi(z)}.$$

Proof. The proof is straightforward. Let $\Sigma \subset Z'$ be a discrete section. Then $\Sigma' = \phi(\Sigma)$ is a discrete section in Z'. By the construction of $\sigma'_{z'}$ there is a finite measure η' on Σ' such that for any non-negative measurable function f on $R \times \Sigma'$,
$$\int_{Z'} \left(\sum_{(r,w') \in R \times \Sigma' : r.w' = z'} f(r, w') \right) \mathrm{d}\lambda'(z') = \int_{\Sigma'} \int_R f(r, z') \, \mathrm{d}\sigma'_{z'}(r) \, \mathrm{d}\eta'(z').$$

For any non-negative measurable function f on $R \times \Sigma$,

$$\begin{aligned}
\int_\Sigma \int_R f(\alpha(r), z)\, \mathrm{d}\sigma'_{\phi(z)}(r)\, \mathrm{d}(\phi_*^{-1}\eta')(z) &= \int_\Sigma \int_R f(\alpha(r), \phi^{-1}(z'))\, \mathrm{d}\sigma'_{z'}(r)\, \mathrm{d}\eta'(z') \\
&= \int_{Z'} \left(\sum_{(r,w') \in R \times \Sigma' : r.w' = z'} f(\alpha(r), \phi^{-1}(w')) \right) \mathrm{d}\lambda'(z') \\
&= \int_Z \left(\sum_{(s,w) \in R \times \Sigma : \alpha^{-1}(s).\phi(w) = \phi(z)} f(s,w) \right) \mathrm{d}\lambda(z)
\end{aligned}$$

In the last equality we did the changes of variables $s = \alpha(r)$, $w' = \phi(w)$ and $z' = \phi(z)$. But $\alpha^{-1}(s).\phi(w) = \phi(z)$ if only if $s.w = z$ by the assumption. Hence

$$\int_\Sigma \int_R f(\alpha(r), z)\, \mathrm{d}\sigma'_{\phi(z)}(r)\, \mathrm{d}(\phi_*^{-1}\eta')(z) = \int_Z \left(\sum_{(s,w) \in R \times \Sigma : s.w = z} f(s,w) \right) \mathrm{d}z.$$

This concludes the proof of the lemma the in view of the uniqueness of (σ_z). □

Lemma 3.10. *Let $\Sigma \subset Z$ be a discrete section. Let η be a measure on Σ for which the property* (3.4) *holds. Let $\mathcal{P}(\sigma)$ be a predicate on the set of proportionality class of Radon measures on R that is invariant for the right translation, i.e. for any Radon measures σ, σ' on R and any $r \in R$, if $\sigma' \propto \tau_{r*}\sigma$ then $\mathcal{P}(\sigma) \Leftrightarrow \mathcal{P}(\sigma')$. If $\mathcal{P}(\sigma_z)$ holds for λ-almost every $z \in Z$, then it holds for η-almost every $z \in \Sigma$.*

Proof. Assume that $\mathcal{P}(\sigma_z)$ holds for λ-almost every $z \in Z$. Then together with Lemma 3.8, there is a subset $E \subset Z$ of full measure such that for all $z \in E$ and all $r \in R$, $\mathcal{P}(\sigma_z)$ and moreover $\sigma_z \propto \tau_{r*}\sigma_{r.z}$, hence $\mathcal{P}(\sigma_{r.z})$. Writing

$$N_1 = \{\, z \in \Sigma \mid \mathcal{P}(\sigma_z) \text{ is false} \,\}.$$

From the above, $R.N_1$ is disjoint from E. By (3.4) applied to $f = \mathbb{1}_{R \times N_1}$, after restricting the integral on Z to E,

$$\int_{N_1} \sigma_z(R)\, \mathrm{d}\eta(z) = \int_E \#\{(r,w) \in R \times N_1 \mid r.w = z\,\}\, \mathrm{d}\lambda(z) = 0,$$

Since $\sigma_z(R) > 0$ for all $z \in \Sigma$, this implies $\eta(N_1) = 0$. □

Lemma 3.11. *If for λ-almost every $z \in Z$, σ_z is left-invariant on R, then λ is invariant under the action of R.*

Proof. Let $\Sigma \subset Z$ be a complete discrete section. Let η be a positive measure on Σ for which the property (3.4) holds. First, observe that a right translate of a left-invariant measure on R is also left-invariant. Thus by Lemma 3.10, for η-almost every $z \in Z$, σ_z is left-invariant.

Then, the property (3.4) implies that for all $s \in R$ and all non-negative measurable functions f on $R \times \Sigma$,

$$\int_Z \left(\sum_{(r,w) \in R \times \Sigma : r.w = z} f(r,w) \right) \mathrm{d}\lambda(z) = \int_Z \left(\sum_{(r,w) \in R \times \Sigma : r.w = z} f(r,w) \right) \mathrm{d}(s_*\lambda)(z).$$

Therefore, it is enough to show that for any measurable set $A \subset Z$, there is a measurable set B on $R \times \Sigma$, such that for all $z \in Z$,

$$\mathbb{1}_A(z) = \#\{\, (r,w) \in B \mid r.w = z \,\}.$$

Let $(U_m)_{m \geq 1}$ be a countable basis of neighbourhoods of 1_R in R and let L_m be defined as in (3.2). Let $A_1 = A \cap R.L_1$ and $A_{m+1} = (A \setminus A_m) \cap R.L_{m+1}$ for $m \geq 1$ so that $(A_m)_{m \geq 1}$ is a countable measurable partition of A. Thus, we may assume without loss of generality that $A \subset R.L_m$ for some $m \geq 1$. Let V_m be open neighbourhoods of 1_R in R such that $V_m^{-1}V_m \subset U_m$. Partition R into countably many measurable subsets $X_i, i \geq 1$ with the property that each X_i is contained in a left translate of V_m. By these choices, we have that for each $i \geq 1$, the map

$$a_{i,m}\colon X_i \times L_m \to Z, \quad (r, z) \mapsto r.z$$

is injective. By [Ke95, Corollary 15.2] $X_i.L_m$ is measurable and the map $a_{i,m}$ is a Borel isomorphism from $X_i \times L_m$ to $X_i.L_m$. Without loss of generality we may assume $A \subset X_i.L_m$ for some i. In this case, $B = a_{i,m}^{-1}(A)$ satisfy trivially the required property. □

We say λ is *R-recurrent* if for every measurable set $B \subset Z$ of positive measure, for μ-almost every $z \in B$, the set of return times $\{\, r \in R \mid r.z \in B \,\}$ is unbounded.

Lemma 3.12. *In the setting of Theorem 3.7, if the probability measure λ is R-recurrent then σ_z is infinite for λ-almost every $z \in Z$. The converse is true if we assume moreover that*

[WH] I don't know if (1) is necessary

(1) *Z has a locally compact second countable topology compatible with its structure of Borel space such that the action $R \times Z \to Z$ is continuous; and*
(2) *for λ-almost every $z \in Z$, $\mathrm{Stab}_R(z) = \{1_R\}$ or*
(2') *R does not have nontrivial compact subgroup.*

Proof. Assume for a contradiction that λ is R-recurrent but the set

$$Y = \{\, z \in Z \mid \sigma_z(R) < +\infty \,\}$$

satisfies $\lambda(Y) > 0$. Taking a large enough compact set $K \subset R$, we have

$$Y' = \{\, y \in Y \mid \sigma_y(K) > \frac{1}{2}\sigma_y(R) \,\}$$

also has $\lambda(Y') > 0$. Using Lemma 3.8 and restricting Y' again, we may assume that for all $y \in Y$ and all $r \in R$, $\sigma_y \propto \tau_{r*}\sigma_{r.y}$. By the recurrence, there exists $r \notin K^{-1}K$ such that there exists $y \in Y'$ such that $r.y \in Y'$. Hence we have both

$$\frac{\sigma_y(K)}{\sigma_y(R)} > \frac{1}{2} \quad \text{and} \quad \frac{\sigma_y(Kr)}{\sigma_y(R)} = \frac{\sigma_{r.y}(K)}{\sigma_{r.y}(R)} > \frac{1}{2}$$

This is impossible because Kr and K are disjoint.

As for the converse, assume for a contradiction that σ_z is infinite for λ-almost every $z \in Z$ but λ is not R-recurrent. Note that the subset $Z_0 = \big\{\, z \in Z \mid \mathrm{Stab}_R(z) \neq \{1_R\} \,\big\}$ is R-invariant. In case of (2), Z_0 is a null set. In case of (2') the restriction of λ to Z_0 is obviously R-recurrent. Thus in both cases, we may further assume that the action is free.

Being not R-recurrent implies the existence of a measurable subset $B \subset Z$ of positive measure and of a compact set $K_0 \subset R$ such that

$$\forall y \in B, \forall r \in R, \quad r.y \in B \Rightarrow r \in K_0.$$

Let Σ be a complete discrete section. Then there exists a compact subset $K_1 \in R$ such that $\lambda(B \cap K_1.\Sigma) > 0$. Shrinking B again, we may assume that $B \subset K_1.\Sigma$. Now apply the following lemma to $K = K_1 K_0 K_1^{-1}$.

Lemma 3.13. *Given any compact set K, there is a countable measurable cover of $\Sigma = \bigcup_{m \geq 1} \Sigma_m$ such that for each $m \geq 1$, the action map $K \times \Sigma_m \to Z$ is injective.*

[WH] It is in this lemma that we use continuity and freedom.

Proof. Using the countable cover Σ by the sets (3.2), we may assume that there is an open neighborhood $U \subset R$ of 1_R such that

$$\forall r \in U, \forall z \in \Sigma, \quad r.z \in \Sigma \Rightarrow r = 1_R.$$

Let $z \in \Sigma$, we claim there is $n \geq 1$ such that $\Sigma_{z,n} = \Sigma \cap B(z, \frac{1}{n})$ satisfies $K \times \Sigma_{z,n} \to Z$ is injective. For otherwise, there would be sequences $k_n, k'_n \in K$ and $z_n, z'_n \in \Sigma_{z,n}$ such that $(k_n, z_n) \neq (k'_n, z'_n)$ and $k_n.z_n = k'_n.z'_n$. Since the action is free, we have $z_n \neq z'_n$. Then $k_n^{-1}k'_n.z'_n \in \Sigma \setminus \{z'_n\}$, implying that $k_n^{-1}k'_n \in K^{-1}K \setminus U$, which is a compact set not containing 1_R. Extracting a subsequence and using the continuity, we find some $k \in K^{-1}K \setminus U$ such that $k.z = z$, contradicting the freedom of the action. This proves the claim.

Because Z is second countable, we can cover Σ using countably many such sets $\Sigma_{z,n}$. □

Pick $m \geq 1$ such that $\lambda(B \cap K_1.\Sigma_m) > 0$ and shrink once more B to $B \cap K_1.\Sigma_m$. Consider $D = \{ z \in \Sigma_m \mid \exists r \in K_1,\, r.z \in B \}$, which is measurable by [Ke95, Lemma 18.12]. We claim that $R \times D \to Z$ is injective. Indeed, if $r_1.z_1 = r_2.z_2$ and $t_1.z_1 = b_1 \in B$ and $t_2.z_2 = b_2 \in B$ for some $z_1, z_2 \in \Sigma_m$, $r_1, r_2 \in R$ and $t_1, t_2 \in K_1$, then $b_2 = t_2 r_2^{-1} r_1 t_1^{-1}.b_1$, implying that $t_2 r_2^{-1} r_1 t_1^{-1} \in K_0$, hence $r_2^{-1} r_1 \in K_1 K_0 K_1^{-1} = K$. Then the construction of Σ_m and the equality $r_2^{-1} r_1.z_1 = z_2$ forces $z_1 = z_2$ and $r_1 = r_2$. This proves the claim.

Now let η be the finite measure on Σ given by Theorem 3.7. Applying (3.4) to $f = \mathbb{1}_{R \times D}$, we obtain

$$\int_D \sigma_z(R)\, \mathrm{d}\eta(z) = \lambda(R.D) \leq 1.$$

By Lemma 3.10, $\sigma_z(R)$ is infinite for η-almost every $z \in \Sigma$. Therefore $\eta(D) = 0$ and

$$0 < \lambda(B) \leq \lambda(R.D) = \int_D \sigma_z(R)\, \mathrm{d}\eta(z) = 0,$$

which is absurd. □

References

BQ11. Y. Benoist and J.-F. Quint. Mesures stationnaires et fermés invariants des espaces homogènes. *Ann. of Math.* (2) 174 (2011), no. 2, 1111–1162.

EL10. M. Einsiedler and E. Lindenstrauss. Diagonal actions on locally homogeneous spaces. *Homogeneous flows, moduli spaces and arithmetic*, 155–241, Clay Math. Proc., 10, *Amer. Math. Soc.*, *Providence, RI*, 2010.

EW11. M. Einsiedler and T. Ward. Ergodic theory with a view towards number theory. Graduate Texts in Mathematics, 259. *Springer-Verlag London, Ltd., London*, 2011.

Ke92. A. Kechris. Countable sections for locally compact group actions. *Ergodic Theory Dynam. Systems* 12 (1992), no. 2, 283–295.

Ke95. A. Kechris. Classical descriptive set theory. Graduate Texts in Mathematics, 156. *Springer-Verlag, New York*, 1995.